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# Mean ergodic composition operators on Banach spaces of holomorphic functions

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## ABSTRACT

Given a symbol  $\varphi$ , i.e., a holomorphic endomorphism of the unit disc, we consider the composition operator  $C_\varphi(f) = f \circ \varphi$  defined on the Banach spaces of holomorphic functions  $A(\mathbb{D})$  and  $H^\infty(\mathbb{D})$ . We obtain different conditions on the symbol  $\varphi$  which characterize when the composition operator is mean ergodic and uniformly mean ergodic in the corresponding spaces. These conditions are related to the asymptotic behavior of the iterates of the symbol. Finally, we deal with some particular case in the setting of weighted Banach spaces of holomorphic functions.

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## 1. Introduction, preliminaries and notation

### 1.1. Introduction

We study when a composition operator defined on the disc algebra  $A(\mathbb{D})$  or on the space  $H^\infty(\mathbb{D})$  of bounded holomorphic functions on the unit disc is mean ergodic or uniformly mean ergodic. We refer to the following subsection for the basic definitions.

The following statements are a summary of the main results of our work. In the first one we give a complete characterization for (uniformly) mean ergodic composition operators on  $H^\infty(\mathbb{D})$ . It is a consequence of [Theorems 2.2, 3.3 and 3.6](#).

**Theorem A.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  belong to  $H(\mathbb{D})$ . The following are equivalent:*

- (i)  $C_\varphi : H^\infty(\mathbb{D}) \rightarrow H^\infty(\mathbb{D})$  is (uniformly) mean ergodic.
- (ii)  $(\varphi^n)_n$  converges uniformly to an interior Denjoy–Wolff point  $z_0 \in \mathbb{D}$  or  $\varphi$  is a periodic elliptic automorphism.

In the disc algebra the situation is different. For  $\varphi$  in the unit ball of  $A(\mathbb{D})$ , we consider the following properties:

- (\*) The density of the orbit  $(\varphi^n(z))_n$  on each neighborhood of the Denjoy–Wolff point  $z_0$  is 1 for every  $z \in \overline{\mathbb{D}}$  (see [Section 3.2](#) for the precise definition).
- (\*\*)  $(\varphi^n(z))_n$  converges to the Denjoy–Wolff point  $z_0$  for every  $z \in \overline{\mathbb{D}}$ .

A priori (\*) is weaker than (\*\*), but we do not have an example separating both properties. The mean ergodicity of the composition operator on  $A(\mathbb{D})$  is completely characterized using these conditions. The case with symbol which does not have an interior Denjoy–Wolff point follows by [Theorems 2.2 and 3.7](#).

**Theorem B1.** *Let  $\varphi : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ ,  $\varphi \in A(\mathbb{D})$ , be a symbol which does not have an interior Denjoy–Wolff point. The following are equivalent:*

- (i)  $C_\varphi : A(\mathbb{D}) \rightarrow A(\mathbb{D})$  is mean ergodic.
- (ii)  $\varphi$  is an elliptic automorphism or  $\varphi$  satisfies (\*).

*In this case,  $\varphi$  is uniformly mean ergodic if and only if  $\varphi$  is a periodic elliptic automorphism.*

As a consequence we get that composition operators associated to parabolic automorphisms are mean ergodic, but when the symbol is a hyperbolic automorphism, they are not. The interior Denjoy–Wolff case follows by [Theorem 3.4](#).

**Theorem B2.** Let  $\varphi : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ ,  $\varphi \in A(\mathbb{D})$ , be a symbol with interior Denjoy–Wolff point. The following are equivalent:

- (i)  $C_\varphi : A(\mathbb{D}) \rightarrow A(\mathbb{D})$  is mean ergodic.
- (i)  $C_\varphi : A(\mathbb{D}) \rightarrow A(\mathbb{D})$  is uniformly mean ergodic.
- (ii)  $\varphi$  satisfies (\*\*).

We remark that in [Theorem B1](#), if we restrict the symbol to be a finite Blaschke product or a linear fractional transformation (LFT) then (\*) can be replaced by the much more natural property (\*\*). In particular, we get that for  $\varphi$  a LFT, mean ergodic composition operators are just those whose symbol is not a hyperbolic automorphism, and in the case of finite Blaschke products, the mean ergodic ones are just those associated to a parabolic automorphism (see [Propositions 3.10 and 3.11](#)). Also if  $\varphi$  is hyperbolic and differentiable in a neighborhood of its Denjoy–Wolff point, properties (\*) and (\*\*) are equivalent ([Theorem 3.9](#)).

From the work of Fonf, Lin and Wojstascyck [\[14\]](#) it follows that every Banach space which is not reflexive and has a Schauder basis admits:

- (a) an operator which is power bounded but not mean ergodic,
- (b) an operator which is power bounded and mean ergodic but not uniformly mean ergodic.

From our results it follows that  $C_\varphi : A(\mathbb{D}) \rightarrow A(\mathbb{D})$  is a concrete operator satisfying (a) when  $\varphi$  is a hyperbolic automorphism, and it is an example of (b) when  $\varphi$  is a parabolic or a non-periodic elliptic automorphism.

Finally, in the appendix we solve in the negative a problem posed by Wolf [\[30\]](#) regarding the mean ergodicity of the composition operator  $C_\varphi$  in the weighted Banach spaces of analytic functions  $H_v^\infty(\mathbb{D})$  when the symbol  $\varphi$  is an elliptic automorphism similar to a non-periodic rotation.

## 1.2. Preliminaries and notation

Let  $\mathbb{D}$  denote the open unit disc in the complex plane and  $H(\mathbb{D})$  the set of analytic functions on  $\mathbb{D}$ . A symbol  $\varphi$ , that is, an analytic self-map of  $\mathbb{D}$ , induces through composition the linear composition operator

$$C_\varphi : H(\mathbb{D}) \rightarrow H(\mathbb{D}), \quad f \rightarrow f \circ \varphi.$$

Obviously,  $C_\varphi^n f = C_{\varphi^n} f$  for every  $f \in H(\mathbb{D})$ , where  $\varphi^n := \varphi \circ \dots \circ \varphi$ . We denote by  $C_{z_0}$ ,  $z_0 \in \mathbb{D}$ , the composition operator defined by

$$C_{z_0} : H(\mathbb{D}) \rightarrow H(\mathbb{D}), \quad C_{z_0} f(z) = f(z_0), \quad z \in \mathbb{D}.$$

We are interested in the composition operator defined on the Banach space

$$H^\infty(\mathbb{D}) = \{f \in H(\mathbb{D}), \|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)| < \infty\}$$

or on the disc algebra

$$A(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f \text{ continuous on } \bar{\mathbb{D}}\}.$$

When  $C_\varphi$  acts on  $A(\mathbb{D})$ , we consider that the symbol also belongs to the disc algebra. In this paper we study when the composition operator  $C_\varphi$  is (uniformly) mean ergodic on the spaces  $H^\infty(\mathbb{D})$  and  $A(\mathbb{D})$ .

Let  $X$  be a (real or complex) Banach space and  $T : X \rightarrow X$  a continuous and linear operator on  $X$ . The iterates of  $T$  are denoted by  $T^n := T \circ \dots \circ T$ ,  $n \in \mathbb{N}$ . For  $x \in X$  we denote by  $\text{Orb}(T, x) := \{T^n x, n \in \mathbb{N}_0\}$  the *orbit* of  $x$  under  $T$ . If  $\sup_{n \geq 0} \|T^n\| < \infty$ ,  $T$  is called *power bounded*. Given any  $T \in L(X)$ , we introduce the notation

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^n T^m, \quad n \in \mathbb{N},$$

for the *Cesàro means* of the sequence  $(T^n)_n$  formed by the iterates of  $T$ . The operator  $T$  is called *mean ergodic* precisely when  $\{T_{[n]}\}_{n=1}^\infty$  converges for the strong operator topology. If  $\{T_{[n]}\}_{n=1}^\infty$  converges for the norm topology, then  $T$  is called *uniformly mean ergodic*.

The study of mean ergodicity in the space of linear operators defined on a Banach space goes back to von Neumann. In 1931 he proved that if  $H$  is a Hilbert space and  $T$  is a unitary continuous operator on  $H$ , then  $T$  is mean ergodic. A power bounded operator  $T$  is mean ergodic precisely when  $X = \text{Ker}(I-T) \oplus \overline{\text{Im}(I-T)}$ . Moreover,  $\text{Im}P = \text{Ker}(I-T)$  and  $\text{Ker}P = \overline{\text{Im}(I-T)}$ . In [25, Theorem] Lin proved that a continuous operator on a Banach space  $X$  satisfying  $\|T^n/n\| \rightarrow 0$  is uniformly mean ergodic if and only if  $\text{Im}(I-T)$  is closed.

A Banach space  $X$  is said to be *mean ergodic* if each power bounded operator is mean ergodic. Riesz proved that all  $L_p$  spaces are mean ergodic for  $1 < p < \infty$ . Lorch extended this last result to all reflexive Banach spaces. Given a power bounded operator  $T \in L(X)$  in a Banach space  $X$ , Yosida gave a characterization for the convergence of the Cesàro means in the strong operator topology. This was shown to be equivalent to the convergence of the sequence of these Cesàro means in the weak operator topology. From this clearly follows the following fact that we will use repeatedly during the work: *if  $(T^n)_n$  is convergent in the weak operator topology, then  $T$  is mean ergodic* (see [28, Theorem 1.3, p. 26]). For a Grothendieck Dunford–Pettis (GDP) space  $X$  satisfying  $\|T^n/n\| \rightarrow 0$ , Lotz proved that  $T \in L(X)$  is mean ergodic if and only if it is uniformly mean ergodic [26].  $H^\infty(\mathbb{D})$  is a Grothendieck Banach space with the Dunford–Pettis property.

More recently, Fonf, Lin and Wojtaszczyk showed in [14] that the converse of Lorch theorem above is true whenever  $X$  is a Banach space with a Schauder basis. That is, if  $X$  is a Banach space with Schauder basis in which every power bounded operator is mean ergodic, then  $X$  is reflexive. More precisely, from this work it follows also that in a Banach space with Schauder basis which is not reflexive there exists an operator  $T \in L(X)$  which is power bounded but not mean ergodic [14, Theorem 1] and an operator  $T \in L(X)$  which is power bounded and mean ergodic but not uniformly mean ergodic [14, Theorem 2]. In our work we show that composition operators in  $A(\mathbb{D})$  present examples of these two situations.

Composition operators on various spaces of analytic functions have been the object of intense study in recent years, especially the problem of relating operator-theoretic properties of  $C_\varphi$  to function theoretic properties of the symbol  $\varphi$ . See the books of Cowen and MacCluer [13] and Shapiro [29] for discussions of composition operators on classical spaces of holomorphic functions. Several authors have studied dynamical properties. In this paper, we focus on mean ergodicity. This dynamical property has been studied by Bonet and Domański when the operator acts on the space  $H(U)$  of holomorphic functions defined on a domain  $U$  in a Stein manifold [7]. They characterized those composition operators which are power bounded and proved that this condition is equivalent to the operator being mean ergodic or uniformly mean ergodic. In [30], the author studied when the composition operator is power bounded or uniformly mean ergodic on the weighted Bergman spaces of infinite order  $H_v^\infty(\mathbb{D})$ . Bonet and Ricker also studied when multiplication operators are power bounded or uniformly mean ergodic on these spaces [8].

There is a huge literature about the dynamical behavior of various linear continuous operators on Banach, Fréchet and more general locally convex spaces; see the survey paper by Grosse-Erdmann [17] and the recent books by Bayart and Matheron [5] and by Grosse-Erdmann and Peris [18]. For more details of mean ergodic operators on locally convex spaces, see [1,2,31] and the references therein.

Some research on the spectra of weighted composition operators acting on spaces of holomorphic functions can be found in [3,22,23].

Composition operators are always power bounded on the spaces  $A(\mathbb{D})$  and  $H^\infty(\mathbb{D})$ . In fact, it is easy to see that  $\|C_\varphi^n\| = 1$  for every  $n \in \mathbb{N}$ . In this paper we get a characterization of the (uniform) mean ergodicity of the operator looking at the type of the symbol  $\varphi$  in terms of its Denjoy–Wolff point in case this point exists. The Denjoy–Wolff theorem [9, Theorem 0.2] is stated below.

**Theorem 1.1** (*Denjoy–Wolff*). *Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . If  $\varphi$  is not the identity and not an automorphism with exactly one fixed point, then there is a unique point, called the “Denjoy–Wolff point”,  $z_0 \in \overline{\mathbb{D}}$  such that  $(\varphi^n)_n$  converges to  $z_0$  uniformly on the compact subsets of  $\mathbb{D}$ .*

In the case  $\varphi$  is an automorphism (i.e., a bijective holomorphic self-map of the disc) with exactly one fixed point in  $\mathbb{D}$ , known as an *elliptic automorphism*,  $C_\varphi$  is similar to a

rotation of the disc centered at zero, that is, there is an automorphism  $\phi$  interchanging some  $z_0$  and 0 such that  $C_\varphi = C_\phi C_{\varphi_\lambda} C_{\phi^{-1}}$ , where  $\varphi_\lambda(z) = \lambda z$ ,  $z \in \mathbb{D}$ ,  $|\lambda| = 1$ . Hence  $z_0$  is a fixed point of  $\varphi$  (but not a Denjoy–Wolff one) and

$$\sum_{m=1}^n C_\varphi^m = C_\phi \sum_{m=1}^n (C_{\varphi_\lambda}^m) C_{\phi^{-1}}.$$

As a consequence, for this case, we only need to study the mean ergodicity of  $C_{\varphi_\lambda}$  (see [19]).

If the Denjoy–Wolff point  $z_0$  belongs to the boundary of  $\mathbb{D}$  and  $\varphi \in A(\mathbb{D})$ , we have two cases: we say that the symbol  $\varphi$  is *hyperbolic* if  $\varphi'(z_0) < 1$  and *parabolic* if  $\varphi'(z_0) = 1$  (see [9, Definition 0.3]). Here we are referring to the angular derivative of  $\varphi$ , which is nothing but the usual derivative when the symbol admits a holomorphic extension at  $z_0$  (see [29, Chapters 4 and 5]).

## 2. Elliptic automorphism symbol

In this section we focus on composition operators  $C_\varphi$  associated to an elliptic automorphism symbol  $\varphi$ . We have seen that in this case it is enough to consider that the symbol is a rotation of the disc centered at zero. The following lemma is well-known and we add it without proof for the sake of clarity:

**Lemma 2.1.** *Let  $(T_n)_n$  be a sequence of equicontinuous operators on a locally convex space  $E$ . If  $(T_n)_n$  is pointwise convergent to a continuous operator  $T$  on some dense set  $D \subseteq E$ , then  $(T_n)_n$  is pointwise convergent to  $T$  in  $E$ .*

In the following theorem we obtain an example of a power bounded and mean ergodic operator that is not uniformly mean ergodic. This result should be compared with the general result in [14, Theorem 2]:

**Theorem 2.2.** *Consider the elliptic automorphism  $\varphi(z) = \lambda z$ ,  $z \in \overline{\mathbb{D}}$ ,  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$ . The composition operator  $C_\varphi$  satisfies:*

- (i) *If there exists  $k \in \mathbb{N}$  such that  $\lambda^k = 1$  (consider the smallest  $k$ ), then  $C_\varphi$  is periodic, and thus, uniformly mean ergodic on  $A(\mathbb{D})$  and on  $H^\infty(\mathbb{D})$  with*

$$\lim_n (C_\varphi)_{[n]}(f) = \frac{1}{k} \sum_{m=0}^{k-1} f(\lambda^m z) = \sum_{l=0}^{\infty} a_{lk} z^{lk}$$

*for every  $f(z) = \sum_{j=0}^{\infty} a_j z^j \in A(\mathbb{D})$ .*

- (ii) *If  $\lambda$  is not a root of unity, then  $C_\varphi$  is mean ergodic on  $A(\mathbb{D})$  with  $\lim_n (C_\varphi)_{[n]} f = C_0(f)$  for every  $f \in A(\mathbb{D})$ , but not uniformly mean ergodic on  $A(\mathbb{D})$  and not mean ergodic on  $H^\infty(\mathbb{D})$ .*

**Proof.** (i) Since  $C_\varphi^k(f) = f$  for every  $f \in H(\mathbb{D})$ , the operator is periodic with period  $k$ . This implies that the operator is uniformly mean ergodic. In fact, a standard procedure yields

$$\lim_n (C_\varphi)_{[n]}(f) = \frac{1}{k} \sum_{m=0}^{k-1} f(\lambda^m z)$$

(see for example the proof of [30, Proposition 18]). Moreover, as

$$\sum_{m=0}^{k-1} \lambda^{jm} = \begin{cases} k & \text{if } j = lk, l \in \mathbb{N}_0 \\ \frac{1-\lambda^{kj}}{1-\lambda^j} & = 0 \text{ otherwise,} \end{cases}$$

we easily get  $\frac{1}{k} \sum_{m=0}^{k-1} f(\lambda^m z) = \sum_{l=0}^{\infty} a_{lk} z^{lk}$  for every  $f(z) = \sum_{j=0}^{\infty} a_j z^j \in A(\mathbb{D})$ .

(ii) Observe that  $C_\varphi(1)$  is the function identically 1 for each symbol  $\varphi$ . Using the formula

$$\left\| \frac{1}{n} \sum_{j=1}^n \lambda^{kj} z^k \right\|_\infty = \frac{|\lambda^k - \lambda^{k(n+1)}|}{n|1 - \lambda^k|} \|z^k\|_\infty \leq \frac{2}{n|1 - \lambda^k|} \|z^k\|_\infty$$

for  $k \in \mathbb{N}$ ,  $k \geq 1$ , we get  $\lim_n (C_\varphi)_{[n]} = C_0$  on the monomials. Since the polynomials are dense in  $A(\mathbb{D})$  and  $C_\varphi$  is power bounded, the operator is mean ergodic with  $\lim_n (C_\varphi)_{[n]}(f) = C_0(f)$  for every  $f \in A(\mathbb{D})$  (Lemma 2.1 for  $T_n = (C_\varphi)_{[n]}$ ). Assume now that  $C_\varphi$  is uniformly mean ergodic. By [25, Theorem],

$$\text{Im}(I - C_\varphi) = \{f \in A(\mathbb{D}) : \lim_{n \rightarrow \infty} (C_\varphi)_{[n]}(f) = 0\} = \{f \in A(\mathbb{D}) : f(0) = 0\}.$$

The sequence  $(\lambda^n)_n$  is uniformly distributed in  $\partial\mathbb{D}$ . Hence we can take a sequence  $(n_k)_k$ ,  $n_k \geq k$ , such that  $|1 - \lambda^{n_k}| \leq \frac{1}{2^k}$  for every  $k \in \mathbb{N}$ . Take  $f(z) = \sum_{k=1}^{\infty} (1 - \lambda^{n_k}) z^{n_k}$ . Since

$$\left\| \sum_{k=1}^{\infty} (1 - \lambda^{n_k}) z^{n_k} \right\|_\infty \leq \sum_{k=1}^{\infty} |1 - \lambda^{n_k}| \|z^{n_k}\|_\infty < \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty,$$

it follows that  $f \in A(\mathbb{D})$  with  $f(0) = 0$ . But  $f \notin \text{Im}(I - C_\varphi)$ . Observe that if there exists  $g \in A(\mathbb{D})$  such that  $f(z) = g(z) - g(\lambda z)$ , then

$$\sum_{k=1}^{\infty} (1 - \lambda^{n_k}) z^{n_k} = \sum_{j=1}^{\infty} \frac{g^{(j)}(0)}{j!} (1 - \lambda^j) z^j,$$

which yields  $g(z) = \sum_{k=1}^{\infty} z^{n_k}$ . But  $g \notin H^2(\mathbb{D})$ , which is a contradiction since  $H^\infty(\mathbb{D}) \subseteq H^2(\mathbb{D})$ .  $\square$



By an application of [Theorem 1.1](#) or [Theorem 2.2](#) to  $f(z) = z$ , we also obtain a Denjoy–Wolff point result for any analytic self-map  $\varphi$  of  $\mathbb{D}$  in a Cesàro sense:

**Corollary 2.3** (*Cesàro Denjoy–Wolff theorem*). *Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . If  $\varphi$  is not the identity, then there is a unique fixed point  $z_0 \in \overline{\mathbb{D}}$  such that  $\frac{1}{n} \sum_{m=1}^n \varphi^m$  converges to  $z_0$  uniformly on the compact subsets of  $\mathbb{D}$ .*

### 3. Symbol with Denjoy–Wolff point

In this section we consider the case in which the symbol  $\varphi$  has a Denjoy–Wolff point  $z_0 \in \overline{\mathbb{D}}$ . By the Denjoy–Wolff theorem,  $(\varphi^n)_n$  converges to  $z_0$  uniformly on the compact subsets of  $\mathbb{D}$ . As a direct consequence we get the following remark:

**Remark 3.1.** Let  $\varphi \in A(\mathbb{D})$  and  $C_\varphi : A(\mathbb{D}) \rightarrow A(\mathbb{D})$ . If the composition operator  $C_\varphi$  is mean ergodic, then  $\lim_n \frac{1}{n} \sum_{m=1}^n C_\varphi^m f = f(z_0)$  for every  $z \in \overline{\mathbb{D}}$ .

If we assume that  $(\varphi^n(z))_n$  converges to  $z_0$  for every  $z \in \overline{\mathbb{D}}$ , we obtain that the composition operator  $C_\varphi$  is mean ergodic on  $A(\mathbb{D})$ :

**Proposition 3.2.** *Let  $\varphi \in A(\mathbb{D})$  and  $C_\varphi : A(\mathbb{D}) \rightarrow A(\mathbb{D})$ . If  $\lim_n \varphi^n(z) = z_0$  for all  $z \in \overline{\mathbb{D}}$ , then  $C_\varphi$  is mean ergodic with  $\lim_n \frac{1}{n} \sum_{m=1}^n C_\varphi^m f = f(z_0)$ .*

**Proof.**  $C_{\varphi^n} f(z)$  converges to  $f(z_0)$  for each  $z \in \overline{\mathbb{D}}$ . Since  $A(\mathbb{D})$  is a closed subspace of the space of continuous functions on the disc  $C(\overline{\mathbb{D}})$ , pointwise convergence in bounded sequences implies weak convergence. This is a standard argument using the Riesz representation theorem and Lebesgue theorem. Consequently,  $(C_{\varphi^n})_n$  is convergent for the weak operator topology to  $C_{z_0} : A(\mathbb{D}) \rightarrow A(\mathbb{D})$ ,  $f \mapsto f(z_0)$  and hence, also the Cesàro means are weakly convergent. Yosida’s theorem [[28, Theorem 1.3, p. 26](#)] yields the mean ergodicity of the operator.  $\square$

In what follows we prove that the converse of [Proposition 3.2](#) holds for some symbols. We also obtain a complete characterization about the (uniformly) mean ergodicity of  $C_\varphi$ .

#### 3.1. Symbol with interior Denjoy–Wolff point

In this section we can assume without loss of generality that the Denjoy–Wolff point of  $\varphi$  is  $z_0 = 0$ . Otherwise,  $C_\varphi$  is similar to a composition operator  $C_\phi$  with  $\phi$  a symbol with 0 as Denjoy–Wolff point.

**Theorem 3.3.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  holomorphic be a symbol with Denjoy–Wolff point 0. The following are equivalent on  $H^\infty(\mathbb{D})$ :*

- (i)  $C_\varphi$  is mean ergodic.
- (ii)  $C_\varphi$  is uniformly mean ergodic.
- (iii)  $\|\varphi^n\|_\infty$  converges to 0, as  $n$  tends to infinity.

**Proof.** (i) and (ii) are equivalent, since  $H^\infty(\mathbb{D})$  is a Grothendieck Banach space with the Dunford–Pettis property. Let us see (iii) $\Rightarrow$ (ii): For  $f \in H^\infty(\mathbb{D})$ ,  $\|f\|_\infty < 1$ , the Schwarz lemma applied to  $(1/2)(f(z) - f(0))$  implies  $|f(z) - f(0)| \leq 2|z|$ . More precisely,

$$|f(\varphi^n(z)) - f(0)| \leq 2|\varphi^n(z)|,$$

and so, by (iii),  $C_{\varphi^n}(f)(z) \rightarrow f(0)$  uniformly on  $\mathbb{D}$ . Since the estimate is valid for any  $f$  in the unit ball of  $H^\infty(\mathbb{D})$ , we have in fact that  $C_{\varphi^n}$  tends to  $C_0$  in the norm topology.

(ii) $\Rightarrow$ (iii): By the Schwarz lemma  $(|\varphi^n(z)|)_n$  is a decreasing sequence for each  $z \in \mathbb{D}$ . Proceeding by contradiction, if  $\|\varphi^n\|_\infty$  does not converge to 0 as  $n$  goes to infinity, there exists  $r > 0$  such that for each  $n$  there is  $a_n \in \mathbb{D}$  such that  $|\varphi^n(a_n)| > r$ . Let  $z_j^n = \varphi^j(a_n)$ ,  $0 \leq j \leq n$ ,  $n \in \mathbb{N}$ . We have  $|z_j^n| > r$  for  $0 \leq j \leq n$ ,  $n \in \mathbb{N}$ . By [12, Lemma 13], there exists  $M > 0$  such that for each  $n \in \mathbb{N}$  there exists  $g_n \in H^\infty(\mathbb{D})$  with  $g_n(z_j^n) = \overline{z_j^n}$ , where  $\overline{z_j^n}$  denotes the complex conjugate of  $z_j^n$ , and  $\|g_n\|_\infty \leq M$ . Let  $f_n(z) := zg_n(z)$ . We have  $f_n(z_j^n) = |z_j^n|^2 > r^2$  and  $\|f_n\|_\infty \leq M$ . If  $C_{\varphi^n}$  were Cesàro convergent the convergence would be to  $C_0 : H^\infty(\mathbb{D}) \rightarrow H^\infty(\mathbb{D})$ ,  $f \mapsto f(0)$ . Now  $(f_n)_n$  is a sequence in the ball of radius  $M$  of  $H^\infty(\mathbb{D})$ ,  $f_n(0) = 0$  for all  $n \in \mathbb{N}$  and

$$M\|(C_\varphi)_{[n]} - C_0\|_\infty \geq |(C_\varphi)_{[n]}(f_n)(a_n)| = \left| \frac{\sum_{j=1}^n f_n(z_j^n)}{n} \right| > r^2 \quad \text{for each } n \in \mathbb{N}.$$

Hence  $((C_\varphi)_{[n]})_n$  is not (uniformly) convergent.  $\square$

A *finite Blaschke product* is a map of the form

$$B(z) = e^{i\lambda} \prod_{j=1}^n \frac{z - a_j}{1 - z\bar{a}_j},$$

where  $n \geq 1$ ,  $a_j \in \mathbb{D}$ ,  $j = 1, \dots, n$  and  $\lambda \in \mathbb{R}$ .  $B$  is a rational function that is analytic on the closed unit disc  $\overline{\mathbb{D}}$  (whose poles  $\frac{1}{\bar{a}_j}$  are outside the closed unit disc) and  $B$  maps  $\partial\mathbb{D}$  onto itself [16].  $B$  is an automorphism when  $n = 1$ . In case  $n > 1$ ,  $B$  is called *non-trivial Blaschke product*.

**Theorem 3.4.** Let  $\varphi : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ ,  $\varphi \in A(\mathbb{D})$  be a symbol with Denjoy–Wolff point 0. The following are equivalent on  $A(\mathbb{D})$ :

- (i)  $C_\varphi$  is mean ergodic.
- (ii)  $C_\varphi$  is uniformly mean ergodic.
- (iii)  $\lim_n \varphi^n(z) = 0$  for all  $z \in \overline{\mathbb{D}}$ .

**Proof.** (iii) $\Rightarrow$ (ii): By the Schwarz lemma, we get that the sequence  $(|\varphi^n|)_n$  is monotonically decreasing, i.e.,  $|\varphi^{n+1}(z)| \leq |\varphi^n(z)|$  for every  $z \in \overline{\mathbb{D}}$ . So, since  $\lim_n \varphi^n(z) = 0$  for all  $z \in \overline{\mathbb{D}}$ , by Dini's theorem we get that  $\|\varphi^n\|_\infty$  converges to 0, as  $n$  tends to infinity. Applying now [Theorem 3.3](#) we obtain that  $C_\varphi$  is uniformly mean ergodic on  $H^\infty(\mathbb{D})$ , and so, it is uniformly mean ergodic on  $A(\mathbb{D})$ . (ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (iii): Assume there exists  $\omega \in \partial\mathbb{D}$  such that  $(\varphi^n(\omega))_n \subseteq \partial\mathbb{D}$  and  $C_\varphi$  is mean ergodic. First we observe that  $(\varphi^n(\omega))_n$  must be uniformly distributed in  $\partial\mathbb{D}$ . Otherwise, by Weyl's criterion [[24, Theorem 2.1](#)], there would exist  $j \in \mathbb{N}$  such that, for  $f(z) = z^j$ , we have

$$\lim_n \frac{1}{n} \sum_{m=1}^n C_\varphi^m f(\omega) \neq 0 = f(0).$$

Thus,  $\varphi(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$ . Otherwise, there would exist  $a \in \partial\mathbb{D}$  such that  $\varphi(a) \in \mathbb{D}$ , and from the density of  $(\varphi^n(\omega))_n$  in the boundary we would get some  $n_0$  such that  $\varphi^{n_0}(\omega)$  is close enough to  $a$  to conclude that  $\varphi^{n_0+1}(\omega) \in \mathbb{D}$ , a contradiction. Hence, since  $\varphi \in A(\mathbb{D})$  with  $\varphi(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$ , the symbol must be a finite Blaschke product of degree  $\geq 2$  and  $\varphi(\partial\mathbb{D}) = \partial\mathbb{D}$  (see for example [[15, p. 265](#)]). Since the Julia set of  $\varphi$  (the closure of its repelling periodic points) is  $\partial\mathbb{D}$  [[4](#)] (see also [[11](#)] and [[21](#)]), we get that  $\varphi$  has periodic points on  $\partial\mathbb{D}$ . Let  $z_0$  be a periodic point of  $\varphi$  with period  $k$ . We get the contradiction evaluating the Cesàro means of the orbit of  $C_\varphi$  at a polynomial  $p$  which satisfies  $p(0) = 0$  and  $p(\varphi^j(z_0)) = 1$ ,  $1 \leq j \leq k$ .  $\square$

### Example 3.5.

- (i) For  $\varphi(z) = \lambda z$ ,  $|\lambda| < 1$ , the operator  $C_\varphi$  is uniformly mean ergodic on  $H^\infty(\mathbb{D})$ , thus on  $A(\mathbb{D})$ , since  $(\|\varphi^n\|_\infty)_n$  converges to 0 ([Theorem 3.3](#)).
- (ii) For  $\varphi(z) = z^2$  or  $\varphi(z) = \lambda z + (1 - \lambda)z^2$ ,  $0 < \lambda < 1$ , the operator  $C_\varphi$  is not mean ergodic on  $A(\mathbb{D})$ , thus neither on  $H^\infty(\mathbb{D})$ , since 1 is a fixed point in the boundary ([Theorem 3.4](#)).

### 3.2. Symbol with boundary Denjoy–Wolff point

In this section we will assume that the Denjoy–Wolff point of  $\varphi$  is  $z_0 = 1$  if needed. There is no loss of generality since  $C_\varphi$  is similar to a composition operator  $C_\phi$ , with  $\phi$  a symbol of the same type with 1 as Denjoy–Wolff point.

**Theorem 3.6.** *Consider  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  a symbol with Denjoy–Wolff point  $z_0 \in \partial\mathbb{D}$ . Then the operator  $C_\varphi$  is not (uniformly) mean ergodic on  $H^\infty(\mathbb{D})$ . Moreover, if  $\varphi \in A(\mathbb{D})$  then  $C_\varphi : A(\mathbb{D}) \rightarrow A(\mathbb{D})$  is not uniformly mean ergodic.*

**Proof.** Assume that  $C_\varphi$  is uniformly mean ergodic. Since  $(\varphi^n)_n$  is pointwise convergent to  $z_0$  on  $\mathbb{D}$ , then  $\lim_n \frac{1}{n} \sum_{m=1}^n C_\varphi^m(f)$  must converge to  $f(z_0)$  for each  $f \in A(\mathbb{D}) \subseteq H^\infty(\mathbb{D})$ .

Put  $g(z) := \frac{z+z_0}{2} \in A(\mathbb{D})$ . Observe that  $|g(z_0)| = 1$  and  $|g(z)| < 1$  for every  $z \in \bar{\mathbb{D}} \setminus \{z_0\}$ . Fix  $n \in \mathbb{N}$  and take  $r > 0$  such that  $\{\varphi^j(0)\}_{j=0}^n \cap B(z_0, r) = \emptyset$ . Consider  $\rho < 1$  such that  $|g(z)| < \rho$  for every  $z \in \bar{\mathbb{D}} \setminus B(z_0, r)$  and  $k \in \mathbb{N}$  such that  $|g(z)|^k < \rho^k < \frac{1}{2}$  for every  $z \in \bar{\mathbb{D}} \setminus B(z_0, r)$ . Observe that  $g(z)^k \in A(\mathbb{D})$ ,  $|g(z_0)^k| = 1$  and  $|g(z)^k| < 1$  for every  $z \in \bar{\mathbb{D}} \setminus \{z_0\}$ . We get

$$\left| g(z_0)^k - \frac{1}{n} \sum_{m=1}^n g(\varphi^m(0))^k \right| \geq 1/2.$$

Since this holds for every  $n \in \mathbb{N}$ ,  $C_\varphi$  is not uniformly mean ergodic restricted to  $A(\mathbb{D})$ , thus, neither on  $H^\infty(\mathbb{D})$ . By [Theorem 3.3](#),  $C_\varphi$  cannot be mean ergodic on  $H^\infty(\mathbb{D})$ .  $\square$

Now, we will use the following notation: given a set  $J$  of natural numbers we denote the *lower density* of the set as

$$\underline{\text{dens}} J = \liminf_{N \rightarrow +\infty} \frac{\#\{J \cap [0, N]\}}{N}.$$

If the limit when  $N$  tends to infinity of the fraction  $\frac{\#\{J \cap [0, N]\}}{N}$  above exists, we denote it by  $\text{dens } J$  and it is called the *density* of  $J$ . It is clear that  $\text{dens } J = 1$  if and only if  $\underline{\text{dens}} J = 1$ . Let  $\varphi \in A(\mathbb{D})$  and  $z_0$  be the Denjoy–Wolff point in  $\partial\mathbb{D}$ . For any given neighborhood  $U$  of  $z_0$  and  $z \in \bar{\mathbb{D}}$  we write

$$\mathbb{N}_U^{z\varphi} := \{n \in \mathbb{N} : \varphi^n(z) \in U\}.$$

For a fixed  $N \in \mathbb{N}$  we also write

$$(\mathbb{N}_U^{z\varphi})^N := \{n \in \mathbb{N}_U^{z\varphi} : n \leq N\}.$$

Therefore, in this case we obtain

$$\underline{\text{dens}} \mathbb{N}_U^{z\varphi} = \liminf_{N \rightarrow +\infty} \frac{\#(\mathbb{N}_U^{z\varphi})^N}{N}.$$

If the limit above exists, we denote it by  $\text{dens } \mathbb{N}_U^{z\varphi}$ . We call it *the density of the orbit*  $(\varphi^n(z))_n$  on the neighborhood  $U$ .

**Theorem 3.7.** *Let  $\varphi \in A(\mathbb{D})$  be a symbol with Denjoy–Wolff point  $z_0 \in \partial\mathbb{D}$ . The following are equivalent:*

- (i)  $C_\varphi$  is mean ergodic on  $A(\mathbb{D})$ .
- (ii)  $\text{dens } \mathbb{N}_U^{z\varphi} = 1$  for all  $z \in \partial\mathbb{D}$  and for all neighborhoods  $U$  of  $z_0$ .
- (iii)  $\lim_n \frac{1}{n} \sum_{m=1}^n (\varphi^m(z))^j = z_0^j$  for every  $z \in \bar{\mathbb{D}}$  and for every  $j \in \mathbb{N}$ .

**Proof.** (i) $\Rightarrow$ (ii) We proceed by contradiction. Assume there exists  $z \in \partial\mathbb{D}$  and  $U$  such that  $\text{dens}\mathbb{N}_U^{z,\varphi} < \delta < 1$ . Take  $\varepsilon > 0$  such that  $\delta < 1 - \varepsilon$  and  $\alpha > 0$  satisfying  $\delta + \alpha < 1 - \varepsilon$ .

Consider  $f(z) = (\frac{z+z_0}{2})^n \in A(\mathbb{D})$ , with  $n$  big enough to satisfy  $|f(z)| < \alpha$  on  $\overline{\mathbb{D}} \setminus U$ . By hypothesis there is an increasing sequence  $(N_k)_k$  of natural numbers such that

$$\frac{\#(\mathbb{N}_U^{z,\varphi})^{N_k}}{N_k} < \delta, \quad k \in \mathbb{N}.$$

If we denote  $\ell_k := \#(\mathbb{N}_U^{z,\varphi})^{N_k}$  we obtain that for every  $k \in \mathbb{N}$ ,

$$|(C_\varphi)_{[N_k]}(f)(z)| \leq \frac{\ell_k}{N_k} + \frac{N_k - \ell_k}{N_k} \cdot \alpha < \delta + \alpha < 1 - \varepsilon.$$

Therefore, since  $|f(z_0)| = 1$ , we conclude

$$\lim_{k \rightarrow \infty} (C_\varphi)_{[N_k]}(f)(z) \neq f(z_0).$$

(ii) $\Rightarrow$ (i) Let  $f \in A(\mathbb{D})$ . Given  $\varepsilon > 0$  we take  $0 < \delta < 1$  such that  $\|f\|_\infty < \frac{\varepsilon}{4\delta}$ . Let  $U$  be a neighborhood of  $z_0$  such that  $|f(z) - f(z_0)| < \frac{\varepsilon}{2}$  for all  $z \in U$ .

Fix  $z \in \partial\mathbb{D}$ . From  $\text{dens}\mathbb{N}_U^{z,\varphi} = 1$  we get that there is  $N_\delta \in \mathbb{N}$  such that for every  $N \geq N_\delta$  we have

$$\frac{\#\{n \in \mathbb{N} : \varphi^n(z) \in U, n \leq N\}}{N} \geq 1 - \delta.$$

Now, let  $\ell_N := \#\{n \in \mathbb{N} : \varphi^n(z) \in U, n \leq N\}$  for all  $N \in \mathbb{N}$ . For all  $N \geq N_\delta$  we have that  $\frac{N - \ell_N}{N} \leq \delta$ , and then

$$\begin{aligned} |(C_\varphi)_{[N]}f(z) - f(z_0)| &\leq \frac{\ell_N \sup_{z \in U} |f(z) - f(z_0)|}{N} + \frac{N - \ell_N}{N} \cdot 2 \sup_{z \in \mathbb{D}} |f(z)| \\ &\leq \frac{\varepsilon}{2} + 2\delta \|f\|_\infty < \varepsilon. \end{aligned}$$

Thus  $((C_\varphi)_{[n]}f)_n$  is a bounded sequence which is pointwise convergent to  $f(z_0)$ , and then we have weak convergence. By the same argument used in [Proposition 3.2](#) the operator is mean ergodic.

(i) $\Rightarrow$ (iii) follows from [Remark 3.1](#) and the definition of mean ergodicity, considering the functions  $f(z) = z^j$ ,  $j \in \mathbb{N}$ .

(iii) $\Rightarrow$ (i) The hypothesis implies that  $\lim_{n \rightarrow \infty} (C_\varphi)_{[n]} = C_{z_0}$  on the monomials. Therefore, as  $C_\varphi$  is power bounded and the polynomials are dense in  $A(\mathbb{D})$ , the composition operator is mean ergodic.  $\square$

**Remark 3.8.** According to the definition of density of an orbit, it is clear that for  $\varphi \in A(\mathbb{D})$  with Denjoy–Wolff point  $z_0 \in \partial\mathbb{D}$  and such that there exists a point in the boundary whose orbit does not intersect a neighborhood of  $z_0$ , then  $C_\varphi$  is not mean ergodic. This certainly happens when the symbol has another fixed or periodic point in the boundary.

**Theorem 3.9.** Let  $\varphi : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ ,  $\varphi \in A(\mathbb{D})$ , be a hyperbolic symbol with  $z_0 \in \partial\mathbb{D}$  as Denjoy–Wolff point. Assume  $\varphi$  is holomorphic in a neighborhood of  $z_0$ . The following are equivalent on  $A(\mathbb{D})$ :

- (i)  $C_\varphi$  is mean ergodic on  $A(\mathbb{D})$ .
- (ii)  $\lim_n \varphi^n(z) = z_0$  for all  $z \in \overline{\mathbb{D}}$ .

**Proof.** (ii) $\Rightarrow$ (i) follows from [Proposition 3.2](#).

(i) $\Rightarrow$ (ii) Since  $\varphi$  is a hyperbolic symbol with  $z_0 \in \partial\mathbb{D}$  as Denjoy–Wolff point it follows that  $|\varphi'(z_0)| < 1$ , then, by the hypothesis there exists  $0 < \rho < 1$  and  $r > 0$  such that  $\varphi$  is holomorphic in  $B(z_0, r)$  and

$$|\varphi(z) - z_0| < \rho|z - z_0|$$

for every  $z \in B(z_0, r)$ . This implies first  $\varphi(B(z_0, r)) \subseteq B(z_0, r)$ , and inductively we get  $\varphi^n(B(z_0, r)) \subseteq B(z_0, r)$  and

$$|\varphi^n(z) - z_0| < \rho^n r$$

for each  $z \in B(z_0, r)$  and for each  $n \in \mathbb{N}$ . Therefore  $(\varphi^n(z))_n$  converges to  $z_0$  for every  $z \in B(z_0, r)$ . Now, assume that there exists  $z_1 \in \partial\mathbb{D}$  such that  $(\varphi^n(z_1))_n$  does not converge to  $z_0$ . We have that  $\{\varphi^n(z_1) : n \in \mathbb{N}\} \cap B(z_0, r) = \emptyset$ . Hence, by [Remark 3.8](#),  $C_\varphi$  cannot be mean ergodic.  $\square$

We find examples of different character, also for the parabolic case. In the next propositions we obtain that condition (\*) is equivalent to (\*\*) in the case that the symbol is a linear fractional transformation or a finite Blaschke product.

Recall that a *linear fractional transformation* (LFT) is a transformation of the extended complex plane  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  of the form

$$\zeta \mapsto \frac{a\zeta + b}{c\zeta + d}, \quad \zeta \in \hat{\mathbb{C}}, \quad ad - cb \neq 0, \quad a, b, c, d \in \mathbb{C}.$$

It is well-known that the linear fractional transformations are precisely the conformal mappings of  $\hat{\mathbb{C}}$  and that every linear fractional transformation except the identity has one or two fixed points.

**Proposition 3.10.** Let  $\varphi : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ ,  $\varphi \in A(\mathbb{D})$ , be a LFT different from an elliptic automorphism. The following are equivalent:

- (i)  $C_\varphi$  is mean ergodic on  $A(\mathbb{D})$ .
- (ii)  $\varphi$  is not a hyperbolic automorphism.

**Proof.** (i) $\Rightarrow$ (ii) If  $\varphi$  is a hyperbolic automorphism, the symbol has a repulsive fixed point different from the Denjoy–Wolff point  $z_0$ . So, by [Remark 3.8](#),  $C_\varphi$  is not mean ergodic.

(ii) $\Rightarrow$ (i) If  $\varphi$  is a parabolic automorphism we have that  $(\varphi^n(z))_n$  converges to  $z_0$  for every  $z \in \overline{\mathbb{D}}$  [[18, Proposition 4.47](#)]. So, by [Proposition 3.2](#) we get that  $C_\varphi$  is mean ergodic. We also have this situation if  $\varphi$  is not an automorphism. Observe that in this case we have that  $\varphi$  is not a Blaschke product, and so,  $\varphi(\partial\mathbb{D}) \not\subseteq \partial\mathbb{D}$ . Since  $\varphi$  is a conformal map, it maps  $\partial\mathbb{D}$  to a circle different from  $\partial\mathbb{D}$ . This implies that the circle  $\varphi(\partial\mathbb{D})$  must intersect  $\partial\mathbb{D}$  only in the Denjoy–Wolff point, otherwise  $\varphi(\partial\mathbb{D}) = \partial\mathbb{D}$  (see [[20, p. 71–72](#)]). Applying now the Denjoy–Wolff theorem, we get that  $\varphi^n(z)$  converges to  $z_0$  for every  $z \in \overline{\mathbb{D}}$ . Again [Proposition 3.2](#) yields the mean ergodicity.  $\square$

Thanks to the Linear-Fractional Model Theorem [[9, Theorem 0.4](#)] we get that for every univalent symbol  $\varphi$  which is not a hyperbolic automorphism,  $C_\varphi$  is mean ergodic. Observe that in this case,  $\varphi$  is conjugated to an LFT.

**Proposition 3.11.** *For  $\varphi : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ ,  $\varphi \in A(\mathbb{D})$  a finite Blaschke product different from an elliptic automorphism, the following are equivalent:*

- (i)  $C_\varphi$  is mean ergodic on  $A(\mathbb{D})$ .
- (ii)  $\varphi$  is a parabolic automorphism.

**Proof.** In [Proposition 3.10](#) we have seen that if  $\varphi$  is a parabolic automorphism, then  $C_\varphi$  is mean ergodic, and if  $\varphi$  is a hyperbolic automorphism, then it is not. So, it is enough to prove that if  $\varphi$  is a finite Blaschke product which is not an automorphism,  $C_\varphi$  is not mean ergodic. By [[10, Example p. 58](#)] we get that for symbols of this type, the Julia set is  $\partial\mathbb{D}$  or a Cantor set of  $\partial\mathbb{D}$ . Since the Julia set is the closure of the repelling periodic points,  $\varphi$  must have a periodic point different from  $z_0$ . We conclude now by [Remark 3.8](#).  $\square$

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## Appendix A

In this section, we will consider composition operators defined in the weighted Banach spaces of analytic functions  $H_v^\infty$  defined as follows:

$$H_v^\infty = H_v^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) : \|f\|_v = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty\},$$

$$H_v^0 = H_v^0(\mathbb{D}) := \{f \in H_v^\infty(\mathbb{D}) : \lim_{|z| \rightarrow 1} v(z)|f(z)| = 0\},$$

endowed with the norm  $\|\cdot\|_v$ , where  $v : \mathbb{D} \rightarrow \mathbb{R}_+$  is an arbitrary *weight*, that is, a bounded continuous positive function. If the weight is radial (that is,  $v(z) = v(|z|)$ ) for all  $z \in \mathbb{D}$ , non-increasing with respect to  $|z|$  and  $\lim_{|z| \rightarrow 1} v(z) = 0$ , the weight is called *typical*. From now on, let us assume that  $v$  is a typical weight. By [27, Theorem 1.1],  $H_v^\infty$  is isomorphic to  $\ell^\infty$  or to  $H^\infty(\mathbb{D})$ , both Grothendieck Banach spaces with the Dunford–Pettis property. Then, by Lotz [26] we get that  $C_\varphi$  is mean ergodic if and only if it is uniformly mean ergodic.

In [30], Wolf shows Theorem A.1(i) below and asks if for  $\lambda \in \mathbb{C}$ ,  $|\lambda| = 1$ , not a root of unity, the composition operator  $C_\varphi$ ,  $\varphi(z) = \lambda z$ ,  $z \in \mathbb{D}$ , is (uniformly) mean ergodic on  $H_v^\infty$ , where  $v$  is a typical weight on  $\mathbb{D}$ . In Theorem A.1(ii) we solve this question in the negative by proving that this is not true in general for every weight  $v$ .

**Theorem A.1.** *Let  $v$  be a typical weight on  $\mathbb{D}$ . The composition operator  $C_\varphi$  associated to  $\varphi(z) = \lambda z$ ,  $z \in \mathbb{D}$ ,  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$ , is power bounded on  $H_v^\infty$  and satisfies:*

- (i) *If there exists  $k \in \mathbb{N}$  such that  $\lambda^k = 1$  (consider the smallest  $k$ ), then  $C_\varphi$  is uniformly mean ergodic on  $H_v^\infty$  with  $\lim_n (C_\varphi)_{[n]}(f) = \frac{1}{k} \sum_{m=0}^{k-1} f(\lambda^m z)$  for every  $f \in H_v^\infty$ .*
- (ii) *If  $\lambda$  is not a root of unity,  $C_\varphi$  is mean ergodic on  $H_v^0$  with  $\lim_{n \rightarrow \infty} (C_\varphi)_{[n]} = C_0$  for every weight  $v$ .*

*Fix  $0 < \alpha < 1$ ,  $R > 1$  and  $0 < r_0 < 1$ . Take  $(n_k)_k \subseteq \mathbb{N}$ ,  $n_k \geq k$ , such that  $|1 - \lambda^{n_k}| \leq \frac{1}{R^k}$  for every  $k \in \mathbb{N}$ . For the typical weight*

$$v_\alpha(r) = \begin{cases} C (\sum_{k=1}^{\infty} r^{n_k})^{-\alpha}, & r_0 \leq r < 1, \\ 1, & r \leq r_0, \end{cases}$$

*where  $C = (\sum_{k=1}^{\infty} r_0^{n_k})^\alpha$ ,  $C_\varphi$  is not uniformly mean ergodic on  $H_{v_\alpha}^0$ , thus not mean ergodic on  $H_{v_\alpha}^\infty$ .*

**Proof.** Since the weight is radial,  $\|C_\varphi^n f\|_v = \|f\|_v$  for every  $f \in H_v^\infty$ ,  $n \in \mathbb{N}$ , and thus,  $C_\varphi$  is power bounded. (i) follows since  $C_\varphi$  is periodic (see [30, Proposition 18] for the standard argument).

(ii) Proceeding as in the proof of Theorem 2.2(ii) we get that for  $\lambda$  not a root of unity,  $\lim_{n \rightarrow \infty} (C_\varphi)_{[n]}$  equals  $C_0$  on the monomials. As  $C_\varphi$  is power bounded and the polynomials are dense in  $H_v^0$  (see [6, Theorem 1.5]), the operator is mean ergodic on  $H_v^0$ .

Now, let us see that for the weight  $v_\alpha$ , the operator  $C_\varphi$  is not uniformly mean ergodic on  $H_{v_\alpha}^0$ . By [25, Theorem], it is enough to show that

$$\text{Im}(I - C_\varphi) \neq \{f \in H_{v_\alpha}^0 : \lim_{n \rightarrow \infty} (C_\varphi)_{[n]}(f) = 0\} = \{f \in H_{v_\alpha}^0 : C_0(f) = f(0) = 0\}.$$

Take  $f(z) = \sum_{k=1}^{\infty} (1 - \lambda^{n_k}) z^{n_k}$ . Since



$$\begin{aligned} v(z)|f(z)| &= v(z) \left| \sum_{k=1}^{\infty} (1 - \lambda^{n_k}) z^{n_k} \right| \leq v(z) \sum_{k=1}^{\infty} |1 - \lambda^{n_k}| |z|^{n_k} \\ &\leq v(z) \sum_{k=1}^{\infty} \frac{1}{R^k} = v(z) \left( \frac{1}{R-1} \right) \rightarrow 0 \text{ as } |z| \rightarrow 1, \end{aligned}$$

we get that  $f \in H_v^0$  with  $f(0) = 0$ . But  $f \notin \text{Im}(I - C_\varphi)$ . Observe that if there exists  $g \in H_{v_\alpha}^0$  such that  $f(z) = g(z) - g(\lambda z)$ , proceeding as in the proof of [Theorem 2.2\(ii\)](#), we get  $g(z) = \sum_{k=1}^{\infty} z^{n_k}$ . But  $g \notin H_{v_\alpha}^0$  since for  $r > r_0$ ,

$$v_\alpha(r)|g(r)| = C \left( \sum_{k=1}^{\infty} r^{n_k} \right)^{1-\alpha}$$

does not converge to 0 as  $r \rightarrow 1$ . As a consequence,  $C_\varphi$  cannot be uniformly mean ergodic on  $H_{v_\alpha}^0$ , neither on  $H_{v_\alpha}^\infty$ .  $\square$

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