Characterisation of the consistent completion of AHP comparison matrices using graph theory

J. Benítez[*], S. Carpitella[†], A. Certa[‡], J. Izquierdo[§]

Abstract

Decision-making is frequently affected by uncertainty and/or incomplete information, which turn decision-making into a complex task. It is often the case that some of the actors involved in decision-making are not sufficiently familiar with all of the issues to make the appropriate decisions. In this paper, we are concerned about missing information. Specifically, we deal with the problem of consistently completing an AHP comparison matrix, and make use of graph theory to characterise such a completion. The characterization includes the degree of freedom of the set of solutions, a linear manifold; and, in particular, characterizes the uniqueness of the solution, a result already known in the literature, for which we provide a completely independent proof. Additionally, in the case of non-uniqueness, we reduce the problem to the solution of non-singular linear systems. In addition to obtaining the priority vector, our investigation, also focus on building the complete pairwise comparison matrix, a crucial step in the necessary process (between synthetic consistency and personal judgment) with the experts. The performance of the obtained results is confirmed.

Keywords: Decision-making; incomplete information; AHP; graph theory; layout reorganisation

Mathematics Subject Classification: 90B50, 90C35

1 Introduction and literature review.

Decision-making is intimately linked to the human condition. The need to make decisions pervades human life at virtually any level (individual, social, entrepreneurial, political, etc.) and conditions human behaviour. In the literature (Homenda, Jastrzebska, & Pedrycz, 2016; Liu, Chan, & Ran, 2016) a decision-maker (DM) is defined as an actor who makes and influences decisions with his/her evaluation of arguments and his/her personal and professional background. Decisions usually derive from a combination of descriptive and experiential information (Weiss-Cohen, Konstantinidis, Speekenbrink, & Harvey, 2016).

Decision-making driven by a well-defined decision structure and integrated by objective elements may be relatively easy. However, when subjectivity permeates the decision-making environment, things become harder. If, in addition, the decision-making context is plagued with uncertainty and/or incomplete information, then decision-making may become a complex task.

[*]IMM, Universitat Politècnica de València (jbenitez@mat.upv.es)
[†]DIID, Università degli Studi di Palermo, Palermo, Italy (silvia.carpitella@unipa.it)
[‡]DIID, Università degli Studi di Palermo, Palermo, Italy (antonella.certa@unipa.it)
[§]FluIng-IMM, Universtitat Politècnica de València, Valencia, Spain, (jizquier@upv.es)

As underlined by Floricel, Michela, & Piperca, 2016, complexity is an intrinsic factor in any field and environment. The authors approach this factor both in its structural and dynamic shape, and stress the need to model complexity with the aim of better managing project planning and strategies. In fact, complexity is usually determined and impacted by the presence of uncertain or incomplete information regarding the process under analysis. Significant losses, especially in terms of costs and time (Qazi, Quigley, Dickson, & Kirytopoulos, 2016), may derive when the main complex aspects are not faced or considered. However, frequently, it is natural that some of the DMs are not sufficiently familiar with all the issues to make an appropriate judgment. There are several reasons for an actor to provide incomplete information. Three such reasons are provided by Harker (1987), namely, insufficient time to make a judgment, unwillingness to issue an opinion, and lack of certainty about an opinion.

In this paper we are concerned about the calculation of missing information. It is necessary to formulate decisional models with a solid scientific basis that is capable of managing the intrinsically subjective and partially informed nature of decisions. This formulation should aim to make decisions as objective as possible, even if the decision-making process cannot be totally objective. Flexible decision-making methods are required that consider a wide variety of aspects, *i.e.* various criteria and alternatives, since a decision on one alternative with the best objective value is affected by various and frequently conflicting criteria. The final selection of the alternative is usually made with the help of inter and intra-attribute comparisons, which may involve explicit or implicit trade-offs (Huang & Yung, 1981). Specific techniques may be needed that can quantify human behaviour related to perceptual and cognitive processes. Quantification is then fundamental in this endeavour and closely related to so-called mathematical psychology. Among various multi-criteria decision-making methods, one of the most used in decision-making is the Analytic Hierarchy Processes (AHP), developed by Saaty (1994), in which decisions are driven by eliciting judgments from the DMs about the importance of a given set of decision elements. As asserted by Vargas *et al.* (2017), the AHP is particularly suitable for group decision-making scenarios, as considered by the case study of the present paper. The authors also assert that this method provides an effective framework for structuring a decision problem, relating the main elements of the problem to the general goal, weighting criteria and alternatives, and resolving conflicts. In AHP, local comparison matrices at the various levels of a well-defined hierarchy are created (Saaty and Vargas, 1994). According to Saaty (1980, 2008), the eigenvector (EV) method is used for deriving weights from local matrices. That is, the EV is the prioritisation method used, and the computational procedure is thus called prioritisation. After calculating the local weights at all levels of the hierarchy, a priority aggregation process is performed by multiplying the criterion-specific weights of the alternatives with the corresponding weights for the criteria and then summing the results to obtain composite weights of the alternatives with respect to the objective. This procedure is unique for all alternatives and all criteria.

The AHP has been successfully applied in many fields and problems, especially to support industrial processes, as for instance shown by Lolli *et al.* (2017) in the sector of electrical resistor manufacturing and by Seiti *et al.* (2017) in the field of steel rolling production. Given the possibility of integrating the AHP with other techniques (Ho, 2008; Ortiz-Barrios *et al.*, 2017), a plethora of applications is discussed in the literature. Vaidya and Kumar (2006) present a wide literature review related to the AHP technique and, after revising a sample of 150 papers on AHP, provide a wide number of AHP applications. Referring to the uncertainty of data or judgments, a state-of-the-art survey was conducted by Kubler, Robert, Derigent, Voisin and Le Traon (2016) on the fuzzy development of the AHP – the

FAHP method (Van Laarhoven & Pedrycz, 1983). The authors reviewed 192 papers and classified them into the following categories: selection; evaluation; development; priority; decision-making; and resource allocation. The FAHP method is considered helpful in various applications, as shown by Hsu, Huang and Tseng (2016). However, as assumed by Wang and Chen (2008), this method presents some weaknesses in relation to the number of pairwise judgments expressed with respect to a given criterion, that is, the difficulty in obtaining consistent pairwise comparison matrices.

The literature presents multiple efforts to improve consistency (Pandeya & Kumar, 2016; Wang & Tong, 2016), and AHP is no exception. It is necessary to guarantee the coherence of judgments expressed by decision-makers in terms of consistency. As underlined by Karanik, Wanderer, Gomez-Ruiz and Pelaez (2016), this aspect is fundamental for reliably applying the AHP method. As many authors affirm (Massanet, Riera, Torrens, & Herrera-Viedma, 2016; Xu, Chen, Rodríguez, Herrera, & Wang, 2016; Zhang, 2016a), the lack of consistency is generally because decision-makers express their preferences by means of preference relations and sometimes fail to make judgments. According to Zhang (2016b), these relations may not satisfy reciprocity properties, especially when expressed by a decision-group. Moreover, Wang and Xu (2016) clarify that incomplete preference relations can be rarely avoided in group decision-making problems. For this reason, the aim is to support experts in expressing their preferences by means of consistency-based interactive algorithms to estimate the missing matrices entries (Benítez, Delgado-Galván, Gutiérrez-Pérez, & Izquierdo, 2011)

Many authors have expressed opinions regarding incomplete information characterising matrices in AHP applications. In (Srdjevic, Srdjevic, & Blagojevic, 2014) a method to complete gaps in matrices is proposed. Starting from the knowledge of two consolidated methodologies (Harker, 1987; van Uden, 2002) that are used to generate missing data in comparisons matrices, the authors propose the first-level transitive rule (FLTR) method. This consists in, firstly, screening matrix entries in the neighbourhood of a missing entry; and, secondly, the scaling and geometric averaging of screened entries to fill the gap. In (Bozóki, Fülöp, & Rónyai, 2010), for both the EV method (Saaty, 1977) and the logarithmic least squares method (LLSM) (Crawford & Williams, 1985), the uniqueness of the completion of a pairwise comparison matrix (PCM) is characterized in terms of the connectedness of a graph. In (Ergu, Kou, Peng, & Zhang, 2016) the need to improve the consistency ratio of matrices related to emergency management is stressed. To this end, they propose a model that quickly estimates missing comparisons in an incomplete matrix by extending the geometric mean induced bias matrix method (Ergu, Kou, Peng, Li, & Shi, 2012). In (Bozóki, Csató, & Temesi, 2016) the authors address a ranking of professional tennis players over the last 40 years using an obviously incomplete history of match results between top tennis players. The literature also proposes estimating incomplete judgments by focusing on uncertainty management. With this perspective, as emphasized in (Certa, Enea, Galante, & La Fata, 2013), Hua, Gong, and Xu (2008) propose an innovative approach to solve multi-attribute decision-making problems with incomplete information. They integrate the AHP method with the Dempster-Shafer (DS) theory of evidence (Shafer, 1976) using a mixed DS-AHP approach (Beynon, Curry, & Morgan, 2000). This method enables coping with the uncertainty of experts and determining preference relations among all the decision alternatives by comparing their belief intervals. Dong, Li and Zhang (2015) estimate missing preference information using a consistent recovery method. The authors focus on multi-criteria group decision-making problems in which preference alternatives are expressed using fuzzy triangular numbers.

Given the importance in the literature of the issue of incomplete judgments that could characterise AHP pairwise comparison matrices, in this paper, we follow the line initiated by the authors in (Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2015; Benítez, Carrión, Izquierdo, & Pérez-García, 2014) for building consistent information from an incomplete body of pairwise comparisons.

The purpose of this paper is to study the system obtained in Theorem 1 of (Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2015) in terms of a graph related to an incomplete pairwise comparison matrix. We compute the degree of freedom of the set of solutions, a linear manifold, in terms of the number of connected components of this graph. In particular, we will prove that the solution to the problem is unique if, and only if, this graph is connected. For this result, previously given in (Bozóki, Fülöp, & Rónyai, 2010), we provide a proof that follows a completely independent approach. Furthermore, when the solution is not unique, we always obtain non-singular linear systems, in contrast with the linear systems obtained in (Benítez, Delgado-Galván, Izquierdo, & R. Pérez-García, 2015). More importantly, in addition to get the priority vector and level of consistency based on the known entries, we are also interested in building the complete PCM, since optimal values of the unknown entries may be informative as well (Bozóki, Fülöp, & Rónyai, 2010). This step is crucial in the necessary trade-off process (between synthetic consistency and personal judgment) with the experts. Let us finally observe that even though the number of missing entries in an elicited comparison matrix is small in practical problems (frequently reduced to one or two above the main diagonal), we calculate the general situation and so obtain a result of wide generality. To show the performance of the results obtained we first use a theoretical matrix with a large number of missing entries and an associated graph with two non-connected components that exhibits the generality we claim, and various other matrices corresponding to a real case of decision-making with one or two missing entries. If possible, we compare the results obtained with other approaches found in the literature.

The paper is organised as follows. After this introduction and the literature review, Section 2 presents the necessary prerequisites. Section 3 develops the main results of this research – including proofs of various theorems, a synthetic example and two illustrative comparisons with other methods. Section 4 presents a case study and the solution obtained. Finally, conclusions close the work.

2 Prerequisites.

2.1 Notation and basic definitions.

One of the necessary steps in AHP theory is performing pairwise comparisons between n elements thus forming an $n \times n$ matrix $A = (a_{ij})$. The reader is encouraged to consult (Saaty, 1994; Saaty, 2008) to see the fundamentals of AHP theory. The entry a_{ij} measures the relative importance of element i over element j. To extract priority vectors from the comparison matrices, the eigenvector method, which was first proposed by Saaty (2008), is used in this paper.

A comparison matrix is always reciprocal. A positive $n \times n$ matrix A is *reciprocal* when $a_{ij}a_{ji} = 1$ for all $1 \leq i, j \leq n$. In addition to the reciprocity property, another property, consistency, should theoretically be desirable for a comparison matrix. A positive $n \times n$ matrix is *consistent* if $a_{ij}a_{jk} = a_{ik}$ for all $1 \leq i, j, k \leq n$.

Consistency expresses the coherence that may exist between judgements about the elements of a set. Since preferences are expressed in a subjective manner, it is reasonable for some kind of incoherence to exist. For a consistent matrix A, the leading eigenvalue and the principal (Perron) eigenvector of A provide information to deal with complex decisions (Saaty, 2008). In the general case, however, A is not consistent. For non-consistent matrices, the problem to solve is the eigenvalue problem $A\mathbf{w} = \lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the unique largest eigenvalue of A that gives the Perron eigenvector as an estimate of the priority vector. As a measurement of inconsistency, Saaty proposed the *consistency index*: $CI = (\lambda_{\max} - n)/(n-1)$ and the *consistency ratio*: $CR = CI/RI$, where RI is the *random index* (Saaty, 2008). If $CR < 0.1$, the estimation is accepted; otherwise, a new comparison matrix is solicited until $CR < 0.1$.

The set of $n \times m$ real matrices is denoted by $\mathscr{M}_{n,m}$. We write $\mathscr{M}^+_{n,m} = \{(a_{ij}) \in \mathscr{M}_{n,m} : a_{ij} > 0 \text{ for all } i,j\}$. If A is a matrix, then $\operatorname{tr}(A)$ and A^T will denote the trace and the transpose of A, respectively. We will write $0_{n,m}$ for the zero matrix in $\mathscr{M}_{n,m}$ and $\mathbf{0}_n$ for the zero vector in $\mathbb{R}^n$. When there is no danger of confusion, we will write simply 0 and $\mathbf{0}$ for the zero matrix and the zero vector, respectively.

2.2 Problem setting.

The following problem was solved in (Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2015): Given an incomplete reciprocal matrix $A \in \mathscr{M}^+_{n,n}$, find a reciprocal completion of A, say X, such that

$$\mathrm{d}(X, \mathscr{C}_n) \leq \mathrm{d}(X', \mathscr{C}_n)$$

for any $X' \in \mathscr{M}^+_{n,n}$ reciprocal completion of A, where $\mathscr{C}_n$ denotes the subset of $\mathscr{M}_{n,n}$ composed of consistent matrices. Here $\mathrm{d}(\cdot,\cdot)$ is the following distance defined in $\mathscr{M}^+_{n,n}$:

$$\mathrm{d}(X,Y) = \| \mathrm{LOG}(X) - \mathrm{LOG}(Y) \|_F,$$

where $\mathrm{LOG} : \mathscr{M}^+_{n,n} \to \mathscr{M}_{n,n}$ is such that if a_{ij} is the (i,j)-entry of A, then the (i,j)-entry of $\mathrm{LOG}(A)$ is $\log(a_{ij})$. Furthermore, $\|\cdot\|_F$ is the Frobenius norm (i.e., $\|A\|_F^2 = \operatorname{tr}(A^T A)$). Let us observe that A is reciprocal if and only if $\mathrm{LOG}(A)$ is skew-symmetric. Observe that the rule $\langle A, B\rangle = \operatorname{tr}(A^T B)$ defines an inner product in $\mathscr{M}_{n,n}$ and that the aforementioned Frobenius norm is induced by this inner product.

We shall be more precise: the stated problem can be formulated as follows.

Problem 1 Let $A \in \mathscr{M}_{n,n}$ be an incomplete reciprocal matrix. Let $(i_1, j_1), \ldots, (i_k, j_k)$ the unknown entries of A above the main diagonal of A. Let $X(\lambda_1, \ldots, \lambda_k) \in \mathscr{M}_{n,n}$ be a completion of A and such that $X_{i_r,j_r} = \exp(\lambda_r)$, $X_{j_r,i_r} = \exp(-\lambda_r)$ for $r = 1, \ldots, k$. Find $\lambda_1, \cdots, \lambda_k$ such that

$$\mathrm{d}(X(\lambda_1, \ldots, \lambda_k), \mathscr{C}_n) \leq \mathrm{d}(X(\lambda'_1, \ldots, \lambda'_k), \mathscr{C}_n)$$

for any $\lambda'_1, \ldots, \lambda'_k \in \mathbb{R}$.

The solution of Problem 1 was given in the next result, see Theorem 4 in (Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2015). From now on, we will consider any vector of $\mathbb{R}^n$ as a column and we will denote $\mathbb{1}_n = [1 \ \cdots \ 1]^T \in \mathscr{M}_{n,1}$. The standard basis of $\mathbb{R}^n$ will be denoted by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

Theorem 1 *Let $A \in \mathscr{M}_{n,n}^{+}$ be an incomplete reciprocal matrix and $(i_1, j_1), \dots, (i_k, j_k)$ its unknown entries above its main diagonal. Any solution of Problem 1 satisfies*

$$\boldsymbol{\lambda} = S\mathbf{m}, \qquad \left(D - \frac{1}{n} S^T S\right) \mathbf{m} = \mathbf{b}, \tag{1}$$

where $\boldsymbol{\lambda} = [\lambda_1 \ \cdots \ \lambda_k]^T$, $\mathbf{m} = [\mu_1 \ \cdots \ \mu_{n-1}]^T$, S is the $k \times (n-1)$ matrix whose (r,s)-entry is $\mathbf{d}_{i_r j_r}^T \mathbf{y}_s$, D is the diagonal $(n-1) \times (n-1)$ matrix whose (s,s)-entry is $\|\mathbf{y}_s\|^2$, and $\mathbf{b} = [\mathbf{w}^T \mathbf{y}_1 \ \cdots \ \mathbf{w}^T \mathbf{y}_{n-1}]^T$, being $\mathbf{w} = \frac{1}{n} \sum_{i<j} c_{ij} \mathbf{d}_{ij}$. Here

$$c_{ij} = \begin{cases} \log a_{ij} & \text{if we know the } (i,j)\text{-entry of } A, \\ 0 & \text{if we do not know the } (i,j)\text{-entry of } A, \end{cases} \tag{2}$$

$\{\mathbf{y}_1, \dots, \mathbf{y}_{n-1}\}$ is an orthogonal basis of $(\operatorname{span}\{\mathbb{1}_n\})^\perp$ and $\mathbf{d}_{ij} = \mathbf{e}_i - \mathbf{e}_j$.

The purpose of this paper is to study system (1) in terms of certain graph related to the incomplete matrix A. In particular, we will prove that the solution of Problem 1 is unique if, and only if, this graph is connected.

The meaning of the values $\lambda_1, \dots, \lambda_k$ in the above Theorem 1 is clear: the missing entry (i_r, j_r) of A must be filled with $\exp(\lambda_r)$. One can see $\mu_1, \dots, \mu_{n-1}$ as auxiliary values useful to find $\boldsymbol{\lambda}$. But, we will give here the meaning of $\boldsymbol{\mu}$.

If A is an incomplete reciprocal matrix, then

$$\mathscr{A} = \{\mathrm{LOG}(X) : X \text{ is a reciprocal completion of } A\}$$

is a linear manifold because if X is any reciprocal completion of A, then

$$\mathrm{LOG}(X) = \mathrm{LOG}(X_0) + \sum_{r=1}^{k} \lambda_r (\mathbf{e}_{i_r} \mathbf{e}_{j_r}^T - \mathbf{e}_{j_r} \mathbf{e}_{i_r}^T), \tag{3}$$

where in this last equality, X_0 is the reciprocal completion of A with 1s on its missing entries.

Also,

$$\mathscr{L}_n = \{\mathrm{LOG}(Z) : Z \in \mathscr{M}_{n,n}, Z \text{ is consistent}\}$$

is a linear subspace of $\mathscr{M}_{n,n}$. In fact, it can be proved (see Theorems 2.2 and 2.4 (Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2011)) that if we define the linear mapping $\phi_n : \mathbb{R}^n \to \mathscr{M}_{n,n}$ by $\phi_n(\mathbf{v}) = \mathbf{v}\mathbb{1}_n^T - \mathbb{1}_n \mathbf{v}^T$, then $\operatorname{im} \phi_n = \mathscr{L}_n$, $\ker \phi_n = \operatorname{span}\{\mathbb{1}_n\}$, and a basis of $\mathscr{L}_n$ is $\{\phi_n(\mathbf{y}_1), \dots, \phi_n(\mathbf{y}_{n-1})\}$. Here, as in Theorem 1, $\{\mathbf{y}_s\}_{s=1}^{n-1}$ is an orthogonal basis of $(\operatorname{span}\{\mathbb{1}_n\})^\perp$.

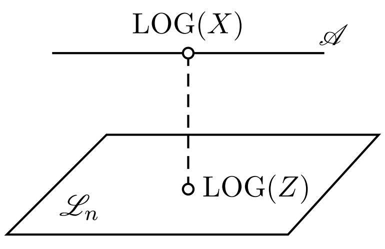


Figure 1: The matrices $\mathrm{LOG}(X)$ and $\mathrm{LOG}(Z)$ minimize the distance between $\mathscr{A}$ and $\mathscr{L}_n$.

With these preparatives, if $\mathrm{LOG}(X) \in \mathscr{A}$ and $\mathrm{LOG}(Z) \in \mathscr{L}_n$ are the matrices such that minimize $\mathrm{d}(X', Z') = \|\mathrm{LOG}(X') - \mathrm{LOG}(Z')\|_F$ for $\mathrm{LOG}(X') \in \mathscr{A}$ and $\mathrm{LOG}(Z') \in \mathscr{L}_n$, then

$$\mathrm{LOG}(Z) = \mu_1 \phi_n(\mathbf{y}_1) + \cdots + \mu_{n-1} \phi_n(\mathbf{y}_{n-1}) = \phi_n(\mu_1 \mathbf{y}_1 + \cdots + \mu_{n-1} \mathbf{y}_{n-1}).$$

See Theorem 4 in (Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2015) for a deeper explanation. Therefore, Theorem 1 also gives the consistent matrix closest to the best completion of A. Furthermore, if we define $Y = [\mathbf{y}_1 \ \cdots \ \mathbf{y}_{n-1}] \in \mathscr{M}_{n,n-1}$ and $\boldsymbol{\theta} = Y\mathbf{m}$, then $\mathrm{LOG}(Z) = \phi_n(\boldsymbol{\theta})$. In other words, vector $\boldsymbol{\theta}$ gives the consistent matrix closest to the best completion of A.

The following theorem is important because to fill matrix A, we can forget the scalars $\lambda_1, \dots, \lambda_r$ and fix our attention to $\boldsymbol{\theta}$.

Theorem 2 *Let $A \in \mathscr{M}^+_{n,n}$ be an incomplete reciprocal matrix and $(i_1, j_1), \dots, (i_k, j_k)$ its unknown entries above its main diagonal. Let X be a reciprocal completion of A and Y be a consistent matrix of order n such that $\mathrm{d}(X, Z) \le \mathrm{d}(X', Z')$ for all X' reciprocal consistent completion of A and Z' a consistent matrix. Then for $r = 1, \dots, k$, the entry (i_r, j_r) of X equals to the entry (i_r, j_r) of Z.*

Proof. Let us denote $B_{i,j} = \mathbf{e}_i\mathbf{e}_j^T - \mathbf{e}_j\mathbf{e}_i^T$. If $M = (m_{ij}) \in \mathscr{M}_{n,n}$, then by using that $\mathrm{tr}(PQ) = \mathrm{tr}(QP)$ holds for any pair of matrices P, Q such that PQ and QP are meaningful,

$$\langle B_{i,j}, M\rangle = \mathrm{tr}(B_{i,j}^T M) = \mathrm{tr}(\mathbf{e}_j\mathbf{e}_i^T M) - \mathrm{tr}(\mathbf{e}_i\mathbf{e}_j^T M) = \mathrm{tr}(\mathbf{e}_i^T M\mathbf{e}_j) - \mathrm{tr}(\mathbf{e}_j^T M\mathbf{e}_i) = m_{ij} - m_{ji}. \quad (4)$$

By (3), the support subspace of $\mathscr{A}$ is the subspace spanned by $B_{i_1,j_1}, \dots, B_{i_r,j_r}$ Since $\mathrm{LOG}(X) - \mathrm{LOG}(Z)$ is orthogonal to the support subspace of $\mathscr{A}$, by using (4) for $M = \mathrm{LOG}(X) - \mathrm{LOG}(Z)$, one has that the (i_r, j_r) entry of $L(X)$ equals to the (i_r, j_r) entry of $L(Z)$ for $r = 1, \dots, k$. □

2.3 Some review of graph theory.

Here we shall review some basic facts of graph theory. The reader is encouraged to consult (Bapat, 2011) for a further insight. In the forthcoming we shall assume that any graph has no loops.

We recall the concepts of the Laplacian matrix and the incidence matrix of a graph G with vertices $\{1, 2, \dots, n\}$, edges $\{e_1, e_2, \dots, e_m\}$ and no loops. The *Laplacian matrix* of G is the $n \times n$ matrix, denoted by $L(G)$, defined as follows: if $i \neq j$, then the (i, j)-entry of $L(G)$ is 0 if vertices i and j are not adjacent, and it is -1 if i and j are adjacent. The (i, i)-entry of $L(G)$ is the degree of vertex i (i.e., the number of edges incident to vertex i).

Suppose that each edge of G has assigned an orientation, which is arbitrary but fixed. The *incidence matrix* of G, denoted by $Q(G)$, is the $n \times m$ matrix defined as follows: the rows and the columns of $Q(G)$ are indexed by vertices and edges, respectively. The (i, j)-entry of $Q(G)$ is 0 if vertex i and edge e_j are not incident, and otherwise it is 1 or -1 depending if e_j begins or finishes at i, respectively. For a graph G one has the following equalities:

$$L(G) = Q(G)Q(G)^T, \qquad \mathbb{1}_n^T Q(G) = \mathbf{0}. \quad (5)$$

A basic property of the Laplacian and incidence matrices is that

$$\mathrm{rk}(L(G)) = \mathrm{rk}(Q(G)) = n - p,$$

where p is the number of connected components of G and n is the number of vertices of G.

If G is a graph with vertices $\{1,\ldots,n\}$, then the complement of G, denoted by $\overline{G}$, is the graph with the same vertices and the edges are defined by the following rule: i and j are adjacent in $\overline{G}$ if and only if i and j are not adjacent in G. It is easy to see that

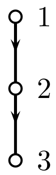
$$L(G) + L(\overline{G}) = nI_n - \mathbb{1}_n\mathbb{1}_n^T. \tag{6}$$

The proof is simple: if $i \neq j$, then only one of the two following possibilities can occur: "i and j are adjacent" or "i and j are not adjacent", hence $L(G)_{ij} + L(\overline{G})_{ij} = -1$, which equals the (i,j)-entry of $nI_n - \mathbb{1}_n\mathbb{1}_n^T$. Since vertex i can be adjacent to the $n-1$ remaining vertices, then $L(G)_{ii} + L(\overline{G})_{ii} = n-1$, which again equals the (i,i)-entry of $nI_n - \mathbb{1}_n\mathbb{1}_n^T$.

3 Main results.

Next, we shall study system (1) appearing in Theorem 1. To this end, we associate an incomplete reciprocal matrix $A = (a_{ij}) \in \mathscr{M}_{n,n}^+$ to a directed graph in the following way. We have $i \to j$ when $i < j$ and the entries a_{ij} and a_{ji} are known. This graph will be denoted G_A. Recall that the Laplacians of G_A and $\overline{G_A}$ are independent on the orientation of the edges. However, the incidence matrices of G_A and $\overline{G_A}$ depend on the chosen orientation and thus, we need to order the edges. To order the edges, we will use the lexicographical order, $(i_1 \to j_1) \prec (i_2 \to j_2)$ when $i_1 < i_2$ or $(i_1 = j_1)\&(j_1 < j_2)$. We can see an example in Figure 2.

$$A = \begin{bmatrix} 1 & a & * \\ a^{-1} & 1 & b \\ * & b^{-1} & 1 \end{bmatrix}$$



$$Q(G_A) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \quad L(G_A) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Figure 2: Example of an incomplete reciprocal matrix, its associated directed graph, the incidence matrix, and the Laplacian

To understand the third item of the next theorem, let us observe that by (3) and Theorem 1, the values $\lambda_1,\ldots,\lambda_k$ provide the set of solutions of Problem 1.

Theorem 3 *Let $A \in \mathscr{M}_{n,n}^+$ be an incomplete reciprocal matrix and G_A its associate graph. Let p be the number of connected components of G_A. Under the notation of Theorem 1, one has*

(i) *The rank of $nD - S^TS$ is $n-p$.*

(ii) *The solutions $[\boldsymbol{\lambda}^T\ \mathbf{m}^T]^T$ of system (1) is a linear manifold whose dimension is $p-1$.*

(iii) *The set*

$$\mathscr{S} = \left\{ S\mathbf{m} : \left(D - \frac{1}{n}S^TS\right)\mathbf{m} = \mathbf{b} \right\}$$

is a linear manifold whose dimension is $p-1$.

Proof. We express matrices D and S in another way. Define $Y = [\mathbf{y}_1 \ \cdots \ \mathbf{y}_{n-1}] \in \mathscr{M}_{n,n-1}$, where the meaning of the vectors $\mathbf{y}_i$ is written in Theorem 1: they form an orthogonal basis of $(\operatorname{span}\{\mathbb{1}_n\})^\perp$. Since $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ is an orthogonal system, we have

$$D = \begin{bmatrix} \|\mathbf{y}_1\|^2 & 0 & \cdots & 0 \\ 0 & \|\mathbf{y}_2\|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\mathbf{y}_{n-1}\|^2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_{n-1}^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_{n-1} \end{bmatrix} = Y^T Y. \tag{7}$$

Observe that the matrix $[\mathbf{d}_{i_1 j_1} \ \cdots \ \mathbf{d}_{i_k j_k}] \in \mathscr{M}_{n,k}$ is the incidence matrix of the graph $\overline{G_A}$. Therefore,

$$S = \begin{bmatrix} \mathbf{d}_{i_1 j_1}^T \mathbf{y}_1 & \cdots & \mathbf{d}_{i_1 j_1}^T \mathbf{y}_{n-1} \\ \vdots & \ddots & \vdots \\ \mathbf{d}_{i_k j_k}^T \mathbf{y}_1 & \cdots & \mathbf{d}_{i_k j_k}^T \mathbf{y}_{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{i_1 j_1}^T \\ \vdots \\ \mathbf{d}_{i_k j_k}^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_{n-1} \end{bmatrix} = Q(\overline{G_A})^T Y. \tag{8}$$

Hence,

$$D - \frac{1}{n} S^T S = \frac{1}{n} \left[n Y^T Y - Y^T Q(\overline{G_A}) Q(\overline{G_A})^T Y \right] = \frac{1}{n} Y^T \left[n I_n - L(\overline{G_A}) \right] Y. \tag{9}$$

Another useful equality is

$$\mathbb{1}_n^T Y = 0, \tag{10}$$

because the columns of Y are orthogonal to $\mathbb{1}_n$. Therefore, we obtain by (6), (9), and (10)

$$D - \frac{1}{n} S^T S = \frac{1}{n} Y^T \left(L(G_A) + \mathbb{1}_n \mathbb{1}_n^T \right) Y = \frac{1}{n} Y^T L(G_A) Y. \tag{11}$$

Let us define $Z = [Y \ \mathbb{1}_n] \in \mathscr{M}_{n,n}$. Obviously, Z is a nonsingular matrix because the $n-1$ first columns of Z form an orthogonal basis of $(\operatorname{span}\{\mathbb{1}_n\})^\perp$. Observe that from (5) we obtain

$$\begin{aligned} Z^T L(G_A) Z &= \begin{bmatrix} Y^T \\ \mathbb{1}_n^T \end{bmatrix} L(G_A) \begin{bmatrix} Y & \mathbb{1}_n \end{bmatrix} \\ &= \begin{bmatrix} Y^T L(G_A) Y & Y^T L(G_A) \mathbb{1}_n \\ \mathbb{1}_n^T L(G_A) Y & \mathbb{1}_n^T L(G_A) \mathbb{1}_n \end{bmatrix} = \begin{bmatrix} Y^T L(G_A) Y & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}. \end{aligned}$$

Since Z is nonsingular, by (11) and the previous computation,

$$\operatorname{rk}(nD - S^T S) = \operatorname{rk}(Y^T L(G_A) Y) = \operatorname{rk}(Z^T L(G_A) Z) = \operatorname{rk}(L(G_A)) = n - p, \tag{12}$$

where p is the number of connected components of G_A. This proves (i).

If d is the dimension of the manifold $\{[\boldsymbol{\lambda}^T \mathbf{m}^T]^T : \boldsymbol{\lambda}, \mathbf{m} \text{ satisfy (1)}\}$, then d is the dimension of the null space of the matrix

$$\begin{bmatrix} I_k & -S \\ 0 & D - \frac{1}{n} S^T S \end{bmatrix} \in \mathscr{M}_{k+n-1,k+n-1}.$$

Thus, by the previous item (i)

$$d = k + n - 1 - \operatorname{rk} \begin{bmatrix} I_k & -S \\ 0 & D - \frac{1}{n} S^T S \end{bmatrix} = k + n - 1 - (k + \operatorname{rk}(nD - S^T S)) = p - 1.$$

This proves (ii).

Let us prove (iii). The dimension of $\mathscr{S}$ equals $\dim \mathscr{S}_1$, where $\mathscr{S}_1 = \{S\mathbf{m} : (nD - S^T S)\mathbf{m} = \mathbf{0}\}$. But $\mathscr{S}_1$ is the image of the linear mapping $\Phi : \mathscr{N} \to \mathbb{R}^k$, where $\mathscr{N}$ is the null space of $nD - S^T S$ and $\Phi(\mathbf{v}) = S\mathbf{v}$. Thus,

$$\dim \mathscr{S}_1 = \dim \operatorname{im} \Phi = \dim \mathscr{N} - \dim \ker \Phi.$$

Since $nD - S^T S$ is a square $(n-1) \times (n-1)$ matrix, by using item (i), one obtains

$$\dim \mathscr{N} = n - 1 - \operatorname{rk}(nD - S^T S) = n - 1 - (n - p) = p - 1.$$

Thus, to finish the proof, we must prove $\ker \Phi = \{\mathbf{0}\}$. Let $\mathbf{x} \in \mathbb{R}^{n-1}$ such that $\Phi(\mathbf{x}) = \mathbf{0}$, *i.e.*, $S\mathbf{x} = \mathbf{0}$ and $(nD - S^T S)\mathbf{x} = \mathbf{0}$. Hence $D\mathbf{x} = \mathbf{0}$. The nonsingularity of D (as one can easily see from (7)), leads to $\mathbf{x} = \mathbf{0}$. □

We get the following two corollaries:

Corollary 1 *There exists at least one solution to Problem 1.*

Corollary 2 *Under the notation of Theorem 3, the following three conditions are equivalent:*

(i) *G_A is connected.*

(ii) *The matrix $nD - S^T S$ is nonsingular.*

(iii) *The solution of Problem 1 is unique.*

The equivalence of statements (i) and (iii) of Corollary 2 was proven in (Bozóki, Fülöp, Rónyai, 2010). Observe that Theorem 3 also characterizes the degree of freedom of the set of solutions.

Next, we shall express system (1) in a simpler way making more explicit the role of the graph G_A. We shall use the following lemma.

Lemma 1 *Let G be a graph with n vertices and m edges. Let $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ be any basis of $(\operatorname{span}\{\mathbb{1}_n\})^\perp$ and $Y = [\mathbf{y}_1 \ \cdots \ \mathbf{y}_{n-1}]$. If $\mathbf{v} \in \mathbb{R}^m$, then $Y^T Q(G)\mathbf{v} = \mathbf{0} \Leftrightarrow Q(G)\mathbf{v} = \mathbf{0}$.*

Proof. The '$\Leftarrow$' part is trivial. We will prove the '$\Rightarrow$' part: the vector $Q(G)\mathbf{v}$ is orthogonal to $\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}$. By the second equality of (5), also $Q(G)\mathbf{v}$ is orthogonal to $\mathbb{1}_n$. Hence $Q(G)\mathbf{v} \in \mathbb{R}^n$ is orthogonal to a basis of $\mathbb{R}^n$, and thus, $Q(G)\mathbf{v} = \mathbf{0}$. □

From now on, we will denote by m the number of edges of the graph G_A. Therefore, the incidence matrix of the graph G_A, namely $Q(G_A)$, is an $n \times m$ matrix.

Theorem 4 *Let $A \in \mathscr{M}_{n,n}^+$ be an incomplete reciprocal matrix and $(i_1, j_1), \ldots, (i_k, j_k)$ its unknown entries. Any solution $\boldsymbol{\lambda} = [\lambda_1 \ \cdots \ \lambda_k]^T$ of Problem 1 satisfies*

$$\boldsymbol{\lambda} = Q(\overline{G_A})^T \boldsymbol{\theta}, \quad L(G_A)\boldsymbol{\theta} = Q(G_A)\boldsymbol{\rho}, \tag{13}$$

where $\boldsymbol{\rho} = [\log(a_{i_1,j_1}) \ \cdots \ \log(a_{i_m,j_m})]^T$.

Proof. We will use the notation of Theorem 1. Also, we denote $Y = [\mathbf{y}_1 \ \cdots \ \mathbf{y}_{n-1}] \in \mathscr{M}_{n,n-1}$, and $\boldsymbol{\theta} = Y\mathbf{m}$. By (8), the first equality of (1) reduces to $\boldsymbol{\lambda} = Q(\overline{G_A})^T\boldsymbol{\theta}$. Let us prove the second equality of (13). We have

$$\mathbf{b} = \begin{bmatrix} \mathbf{y}_1^T\mathbf{w} \\ \vdots \\ \mathbf{y}_{n-1}^T\mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1^T \\ \vdots \\ \mathbf{y}_{n-1}^T \end{bmatrix} \mathbf{w} = Y^T\mathbf{w} = \frac{1}{n}Y^T\sum_{i<j} c_{ij}\mathbf{d}_{ij}. \tag{14}$$

Observe that by the definition of the numbers c_{ij} (see (2)), in the summation appearing in (14), the indices can be restricted with no problem to the edges of the graph G_A. Thus, we have

$$\sum_{i<j} c_{ij}\mathbf{d}_{ij} = Q(G_A)\boldsymbol{\rho}.$$

Therefore, $\mathbf{b} = \frac{1}{n}Y^TQ(G_A)\boldsymbol{\rho}$, and the second equality of (1) becomes $(nD - S^TS)\mathbf{m} = Y^TQ(G_A)\boldsymbol{\rho}$. Now, it is enough to recall expression (11) to get $Y^TL(G_A)\boldsymbol{\theta} = Y^TQ(G_A)\boldsymbol{\rho}$. From here and the first equality of (5), we get $Y^TQ(G_A)(Q(G_A)^T\boldsymbol{\theta} - \boldsymbol{\rho}) = \mathbf{0}$. From Lemma 1, we get $Q(G_A)(Q(G_A)^T\boldsymbol{\theta} - \boldsymbol{\rho}) = \mathbf{0}$. Therefore, the second equality of (13) has been proven. □

A drawback associated to the second equality of system (13) is that matrix $L(G_A)$ is always nonsingular since $L(G_A)$ is an $n \times n$ matrix and $\text{rk}(L(G_A)) = n - p$, where p is the number of connected components of G_A.

In (Benítez, Carrión, Izquierdo, & Pérez-García, 2014) it was characterised when an incomplete, positive, and reciprocal matrix can be completed to become a consistent matrix. Concretely, it was stated in Theorems 7 and 10 of (Benítez, Carrión, Izquierdo, & Pérez-García, 2014) that, under the notation of Theorem 1 of this paper, A can be completed to be consistent if and only if there exists $\mathbf{x} \in \mathbb{R}^n$ such that $Q(G_A)^T\mathbf{x} = \boldsymbol{\rho}$, and in this case, we have $\boldsymbol{\lambda} = Q(\overline{G_A})^T\mathbf{x}$. We can observe that, precisely, the second system in (13) corresponds to the least squares system related to $Q(G_A)^T\mathbf{x} = \boldsymbol{\rho}$.

Next, we study system (13) by decomposing it in simpler systems.

3.1 The structure of the system (13).

For the sake of readability, we provide Table 1 indicating the notation for some parameters of the graph G_A.

Table 1: Used notation for the parameters of a graph

n	No. of points
p	No. of connected components
m	No. of edges
s	No. of isolated points
$G_1, \ldots, G_q$	Connected components of G_A with more than 2 points
n_i	No. of points of the connected component G_i
m_i	No. of edges of the connected component G_i

Rearranging the points of G_A, the matrix $Q(G_A)$ has the following structure

$$Q(G_A) = \begin{bmatrix} 0_{s,m} \\ Q_1 \end{bmatrix} \in \mathscr{M}_{n,m},$$

where

$$Q_1 = Q(G_1) \oplus \cdots \oplus Q(G_q) \in \mathscr{M}_{n-s,m}, \qquad Q(G_i) \in \mathscr{M}_{n_i,m_i},$$

$G_1, \ldots, G_q$ being the connected components of G composed of more than two points. The ideas to study system (13) are: a) "Forget" the isolated points and b) Study each connected component separatedly.

Observe that the number of isolated points plus q equals the number of connected components of G_A, i.e., $s + q = p$. Since n_i is the number of points of G_i for $i = 1, \ldots, q$, evidently, we have

$$s + n_1 + \cdots + n_q = n.$$

Also, observe that $\mathrm{rk}(Q(G_i)) = n_i - 1$ because G_i is connected. This is in full agreement with the fact that $n - p = \mathrm{rk}(Q(G_A)) = \mathrm{rk}(Q_1) = \mathrm{rk}(Q(G_1)) + \cdots + \mathrm{rk}(Q(G_q))$.

Also, the Laplacian of G_A has a block structure:

$$L(G_A) = Q(G_A)Q(G_A)^T = \begin{bmatrix} 0_{s,m} \\ Q_1 \end{bmatrix} \begin{bmatrix} 0_{m,s} & Q_1^T \end{bmatrix} = \begin{bmatrix} 0_{s,s} & 0 \\ 0 & Q_1 Q_1^T \end{bmatrix} \tag{15}$$

and

$$Q_1 Q_1^T = L(G_1) \oplus \cdots \oplus L(G_q). \tag{16}$$

Let us study system (13). First, with the notation of Theorem 4, we shall simplify $Q(G_A)\boldsymbol{\rho}$.

$$Q(G_A)\boldsymbol{\rho} = \begin{bmatrix} 0_{s,m} \\ Q_1 \end{bmatrix} \boldsymbol{\rho} = \begin{bmatrix} \mathbf{0}_s \\ Q_1\boldsymbol{\rho} \end{bmatrix}.$$

Recall that we have denoted by m the number of edges of G_A and by m_i the number of edges of G_i for $i = 1, \cdots, q$. Let us note $m_1 + \cdots + m_q = m$. We partition $\boldsymbol{\rho} \in \mathscr{M}_{m,1}$ as follows:

$$\boldsymbol{\rho}^T = \begin{bmatrix} \boldsymbol{\rho}_1^T & \cdots & \boldsymbol{\rho}_q^T \end{bmatrix}, \qquad \boldsymbol{\rho}_i \in \mathscr{M}_{m_i,1}.$$

Therefore

$$Q_1\boldsymbol{\rho} = \begin{bmatrix} Q(G_1)\boldsymbol{\rho}_1 \\ \vdots \\ Q(Q_q)\boldsymbol{\rho}_q \end{bmatrix}.$$

Now, let us recall that $\boldsymbol{\theta} \in \mathscr{M}_{n,1}$. We decompose

$$\boldsymbol{\theta}^T = \begin{bmatrix} \boldsymbol{\theta}_0^T & \boldsymbol{\theta}_1^T & \cdots & \boldsymbol{\theta}_q^T \end{bmatrix},$$

where $\boldsymbol{\theta}_0 \in \mathscr{M}_{s,1}$ and $\boldsymbol{\theta}_i \in \mathscr{M}_{n_i,1}$ for $i = 1, \cdots, q$. Now, (13), (15), and (16) lead to

$$L(G_i)\boldsymbol{\theta}_i = Q(G_i)\boldsymbol{\rho}_i, \quad i = 1, \cdots, q. \tag{17}$$

To solve system (13), we must think on the connected components of G_A. However, let us note that the systems (17) are always singular since the Laplacian of any graph is always a singular matrix.

So, what is the general solution of (17)? First, the systems (17) are solvable because these systems are the least square systems of $Q(G_i)^T\boldsymbol{\theta}_i = \boldsymbol{\rho}_i$. Let $\widehat{\boldsymbol{\theta}}_i$ be a solution of (17). We know that the general solution of (17) is $\widehat{\boldsymbol{\theta}}_i + \mathscr{N}(L(G_i))$, where $\mathscr{N}(\cdot)$ stands for the null space of a matrix. Since $\mathrm{rk}(L(G_i)) = n_i - 1$ and $L(G_i) \in \mathscr{M}_{n_i,n_i}$ (recall that G_i is a *connected* component of the graph G_A), then

$$\dim \mathscr{N}(L(G_i)) = n_i - \mathrm{rk}(L(G_i)) = 1.$$

Thus, to find $\mathscr{N}(L(G_i))$, it is enough to find a nonzero vector in $\mathscr{N}(L(G_i))$. But from (5) one gets $L(G_i)\mathbb{1}_{n_i} = 0$. Hence

$$\mathscr{N}(L(G_i)) = \{\alpha\mathbb{1}_{n_i} : \alpha \in \mathbb{R}\}.$$

Therefore, the general solution of (17) is

$$\widehat{\boldsymbol{\theta}}_i + \alpha\mathbb{1}_{n_i}, \qquad \alpha \in \mathbb{R},$$

where $\widehat{\boldsymbol{\theta}}_i$ is a particular solution of (17).

Now, we will show how to find a particular solution of (17). Let Y_i be a matrix in $\mathscr{M}_{n_i,n_i-1}$ whose $n_i - 1$ columns form a basis of $(\mathrm{span}\{\mathbb{1}_{n_i}\})^\perp$ and let $\widehat{\mathbf{m}}_i$ be the unique solution of the linear system

$$Y_i^T L(G_i) Y_i \widehat{\mathbf{m}}_i = Y_i^T Q(G_i)\boldsymbol{\rho}_i. \tag{18}$$

This system has a unique solution because $Y_i^T L(G_i) Y_i \in \mathscr{M}_{n_i-1,n_i-1}$, (11), and (12) imply that $Y_i^T L(G_i) Y_i$ is nonsingular. Lemma 1 leads to $Y_i\widehat{\mathbf{m}}_i$ is a solution of (17). Hence the general solution of (17) is

$$Y_i\widehat{\mathbf{m}}_i + \alpha\mathbb{1}_{n_i}, \quad \alpha_i \in \mathbb{R}.$$

Hence, we can solve the right system in (13). Since $\boldsymbol{\theta}_0 \in \mathbb{R}^s$ is arbitrary, then if $\boldsymbol{\theta}$ is any solution of the right linear system in (13), then

$$\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\theta}_0 \\ Y_1\widehat{\mathbf{m}}_1 + \alpha_1\mathbb{1}_{n_1} \\ \vdots \\ Y_q\widehat{\mathbf{m}}_q + \alpha_q\mathbb{1}_{n_q} \end{bmatrix}, \qquad \boldsymbol{\theta}_0 \in \mathbb{R}^s, \alpha_1, \cdots, \alpha_q \in \mathbb{R} \text{ are arbitrary.} \tag{19}$$

We have arrived to the following theorem. Recall that the mapping $\phi_n : \mathbb{R}^n \to \mathscr{M}_{n,n}$ is defined by $\phi_n(\mathbf{v}) = \mathbf{v}\mathbb{1}_n^T - \mathbb{1}\mathbf{v}^T$. Also, it is useful to recall Theorem 2.

Theorem 5 *Let $A \in \mathscr{M}_{n,n}^+$ be an incomplete reciprocal matrix whose unspecified entries above its main diagonal are $(i_1, j_1), \ldots, (i_k, j_k)$. Let G_A be its associate graph whose parameters are specified in Table 1. Let $Y_i \in \mathscr{M}_{n_i,n_i-1}$ a matrix whose $n_i - 1$ columns form a basis of $(\mathrm{span}\{\mathbb{1}_{n_i}\})^\perp$, let $\widehat{\mathbf{m}}_i$ be the unique vector satisfying (18), and let $\boldsymbol{\theta}$ be any vector of $\mathbb{R}^n$ given by (19). If X is a reciprocal completion of A such that $\mathrm{d}(X, \mathscr{C}_n) \leq \mathrm{d}(X', \mathscr{C}_n)$ for any reciprocal completion X' of A, then the (i_r, j_r) entry of X is the (i_r, j_r) entry of Y, where $\mathrm{LOG}(Y) = \phi_n(\boldsymbol{\theta})$.*

3.2 Synthetic example.

Let A be the following incomplete reciprocal matrix:

$$A = \left[\begin{array}{cccc|cc} 1 & 2 & 4 & * & * & * \\ 1/2 & 1 & 5 & * & * & * \\ 1/4 & 1/5 & 1 & 2 & * & * \\ * & * & 1/2 & 1 & * & * \\ \hline * & * & * & * & 1 & 3 \\ * & * & * & * & 1/3 & 1 \end{array}\right].$$

Let us observe that if we delete the 4th, 5th, and 6th rows and columns of matrix A, we get a nonsingular matrix. Hence, $\mathrm{rk}(A) \geq 3$ and, in view of Theorem 3 of (Benítez, Delgado-Galván, Izquierdo, & Pérez-Garca, 2012), A cannot be completed to be consistent. It is easy to check that the associated graph G_A has two connected components, $G_1 = \{1, 2, 3, 4\}$ and $G_2 = \{5, 6\}$. Since G_A is not connected, by Corollary 2, the solution of problem 1 is not unique (in fact, the solutions of system (1) constitute a one-dimensional linear manifold, in view of Theorem 3).

Let us find $\widehat{\mathbf{m}}_1$: since the number of points of G_1 is $n_1 = 4$ and Y_1 is a matrix whose $n_1 - 1$ columns are a basis of $(\mathrm{span}\{\mathbb{1}_{n_1}\})^{\perp}$, then we can pick

$$Y_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}.$$

Furthermore, one can easily see that the Laplacian of G_1 is the following matrix:

$$L(G_1) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

To construct $Q(G_A)$ and $\boldsymbol{\rho}_1$ we employ the lexicographical order.

$$Q(G_1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \boldsymbol{\rho}_1 = \begin{bmatrix} \log a_{12} \\ \log a_{13} \\ \log a_{23} \\ \log a_{34} \end{bmatrix} = \begin{bmatrix} \log 2 \\ \log 4 \\ \log 5 \\ \log 2 \end{bmatrix}.$$

The solution of the system $Y_1^T L(G_1) Y_1 \widehat{\mathbf{m}}_1 = Y_1^T Q(G_1)\boldsymbol{\rho}_1$ is $\widehat{\mathbf{m}}_1 \simeq [0.194, 0.499, 0.423]^T$. Now, $Y_1\widehat{\mathbf{m}}_1 \simeq [1.116, 0.728, -0.576, -1.269]^T$. Let us now find $\widehat{\mathbf{m}}_2$ and $Y_2\widehat{\mathbf{m}}_2$. Since $n_2 = 2$ is the number of points of G_2,

$$Y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad L(G_2) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad Q(G_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{\rho}_2 = \begin{bmatrix} \log a_{35} \end{bmatrix} = \begin{bmatrix} \log 3 \end{bmatrix}.$$

The system $Y_2^T L(G_2) Y_2 \widehat{\mathbf{m}}_2 = Y_2^T Q(G_2)\boldsymbol{\rho}_2$ is $4\widehat{\mathbf{m}}_2 = 2\log 3$. Hence $\widehat{\mathbf{m}}_2 = (\log 3)/2$ and

$$Y_2\widehat{\mathbf{m}}_2 = \begin{bmatrix} (\log 3)/2 \\ -(\log 3)/2 \end{bmatrix} \simeq \begin{bmatrix} 0.549 \\ -0.549 \end{bmatrix}.$$

Therefore, by (19),

$$\boldsymbol{\theta} \simeq [1.116 + \alpha_1, 0.727 + \alpha_1, -0.5756 + \alpha_1, -1.2669 + \alpha_1, 0.5493 + \alpha_2, -0.5493 + \alpha_2]^T.$$

Since $\mathrm{LOG}(Y) = \phi_n(\boldsymbol{\theta})$ and Theorem 5, the (i_r, j_r) entry of the optimal completion of A is the (i_r, j_r) entry of Y, which is $\exp(\theta_{i_r})/\exp(\theta_{j_r}) = \exp(\theta_{i_r} - \theta_{i_j})$. Thus, if X_{i_r,j_r} is the (i_r, j_r) entry of the optimal completion of A, then $a_{14} = \exp(\theta_1 - \theta_4) \simeq 10.858$, $a_{15} = \exp(\theta_1 - \theta_5) \simeq 1.763\exp(\alpha_1 - \alpha_2)$, and so on. Finally, we get (we denote $K = \exp(\alpha_1 - \alpha_2)$) that the optimal completion of A is

$$\left[\begin{array}{cccc|cc} 1 & 2 & 4 & 10.858 & K1.763 & K5.288 \\ 1/2 & 1 & 5 & 7.368 & K1.196 & K3.588 \\ 1/4 & 1/5 & 1 & 2 & K0.325 & K0.974 \\ 0.092 & 0.136 & 1/2 & 1 & K0.162 & K0.487 \\ \hline K^{-1}0.567 & K^{-1}0.836 & K^{-1}3.080 & K^{-1}6.160 & 1 & 3 \\ K^{-1}0.189 & K^{-1}0.279 & K^{-1}1.027 & K^{-1}2.053 & 1/3 & 1 \end{array}\right].$$

3.3 Comparison with other methods.

In this subsection we compare our approach with two well-know PCM completion methods, namely, Van Uden's rule (van Uden, 2002) and Harker's method (Harker, 1987).

Let A be an incomplete reciprocal $n \times n$ matrix ($n > 2$). If only one entry a_{ik} above the diagonal is missing, van Uden proposes the following equality for calculating the missing element

$$a_{ik} = \sqrt[n-2]{X/Y}, \qquad X = \prod_{j \neq k} a_{ij}, \; Y = \prod_{j \neq i} a_{kj}. \tag{20}$$

The intuitive idea for this proposal is the following: if we consider only the fixed indices i, k, and a third index j (varying in $\{1, \ldots, n\} \setminus \{i, k\}$), we get an incomplete 3×3 submatrix and to achieve the consistency of this submatrix, we should set $a_{ik} = a_{ij}a_{jk} = a_{ij}/a_{kj}$. Since index j can take $n-2$ possible values, then we have $n-2$ possible values of a_{ik}. It is natural to consider the geometric mean of these values. We shall see that our Theorem 5 includes van Uden's rule. We introduce the notation $\mathscr{R}(\cdot)$ for indicating the range space of a matrix.

Rearranging the indices, we can assume that the missing entries are a_{12} and a_{21}. Observe that the associate graph G_A is connected, and thus, the solution of Problem 1 is unique (Corollary 2). To find this solution, in view of Theorem 5 and (18), we must study the system $Y^T L(G_A)\boldsymbol{\theta} = Y^T Q(G_A)\boldsymbol{\rho}$, where Y is an $n \times (n-1)$ matrix whose columns form an orthogonal basis of $\mathrm{span}\{\mathbb{1}_n\}^\perp$, $\boldsymbol{\theta} \in \mathbb{R}^n$,

$$\boldsymbol{\rho} = [\log a_{13} \; \cdots \; \log a_{1n} \; \log a_{23} \; \cdots \; \log a_{2n} \; l_1 \; \cdots \; l_r]^T,$$

and any l_m is of the form $\log a_{i_m j_m}$ with $3 \leq i_m < j_m$. In view of Lemma 1, the equation $Y^T L(G_A)\boldsymbol{\theta} = Y^T Q(G_A)\boldsymbol{\rho}$ is equivalent to $L(G_A)\boldsymbol{\theta} = Q(G_A)\boldsymbol{\rho}$. It is evident, by the definition of the Laplacian matrix of the graph G_A, that

$$L(G_A) = \begin{bmatrix} (n-2)I_{n-2} & -U_{2,n-2} \\ -U_{n-2,2} & nI_{n-2} - U_{n-2} \end{bmatrix}.$$

Since G_A is the complete graph of order n without the edge connecting vertices 1 and 2, if we denote by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the standard basis of $\mathbb{R}^n$, then we can write

$$Q(G_A) = [\ \mathbf{e}_1 - \mathbf{e}_3 \mid \cdots \mid \mathbf{e}_1 - \mathbf{e}_n \mid \mathbf{e}_2 - \mathbf{e}_3 \mid \cdots \mid \mathbf{e}_2 - \mathbf{e}_n \mid \mathbf{f}_1 \mid \cdots \mid \mathbf{f}_r\ ],$$

where the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_r$ have the form $\mathbf{e}_i - \mathbf{e}_j$, where $3 \le i < j$, since in the graph G_A, if $i, j \in \{3, \ldots, n\}$, then i and j are connected. If we define $s_1 = \sum_{j=3}^n \log a_{1j}$ and $s_2 = \sum_{j=3}^n \log a_{2j}$, then

$$\begin{aligned} Q(G_A)\boldsymbol{\rho} &= \sum_{j=3}^n \log a_{1j}(\mathbf{e}_1 - \mathbf{e}_j) + \sum_{j=3}^n \log a_{2j}(\mathbf{e}_2 - \mathbf{e}_j) + \sum_{j=1}^r l_j \mathbf{f}_j \\ &= s_1\mathbf{e}_1 + s_2\mathbf{e}_2 - \sum_{j=3}^n (\log a_{1j} + \log a_{2j})\, \mathbf{e}_j + \sum_{j=1}^r l_j \mathbf{f}_j. \end{aligned}$$

Observe that $\mathbf{f}_1, \ldots, \mathbf{f}_r \in \text{span}\{\mathbf{e}_3, \ldots, \mathbf{e}_n\}$. Thus, exists $\mathbf{v} \in \mathbb{R}^{n-2}$ such that

$$Q(G_A)\boldsymbol{\rho} = \begin{bmatrix} s_1 \\ s_2 \\ \mathbf{v} \end{bmatrix}.$$

Since $Q(G_A)\boldsymbol{\rho} \in \mathscr{R}[Q(G_A)] = \mathscr{R}[Q(G_A)Q(G_A)^T] = \mathscr{R}[L(G_A)]$, there exists $\boldsymbol{\theta} \in \mathbb{R}^n$ such that $L(G_A)\boldsymbol{\theta} = Q(G_A)\boldsymbol{\rho}$. Hence, denoting $\mathbf{s} = [s_1\ s_2]^T$ and decomposing $\boldsymbol{\theta}^T = [\boldsymbol{\theta}_1^T\ \boldsymbol{\theta}_2^T]^T$, $\boldsymbol{\theta}_1 \in \mathbb{R}^2$ and $\boldsymbol{\theta}_2 \in \mathbb{R}^{n-2}$, we have

$$\begin{bmatrix} (n-2)I_2 & -U_{2,n-2} \\ -U_{n-2,2} & nI_{n-2} - U_{n-2} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{v} \end{bmatrix}.$$

Therefore, $(n-2)\boldsymbol{\theta}_1 - U_{2,n-2}\boldsymbol{\theta}_2 = \mathbf{s}$. If $\boldsymbol{\theta}_1 = [\xi_1, \xi_2]^T$ and $\boldsymbol{\theta}_2 = [\xi_3, \ldots, \xi_n]^T$, then

$$(n-2)\xi_1 - (\xi_3 + \cdots + \xi_n) = s_1 \qquad \text{and} \qquad (n-2)\xi_2 - (\xi_3 + \cdots + \xi_n) = s_2.$$

By subtracting these two equalities, $(n-2)(\xi_1 - \xi_2) = s_1 - s_2$. Now, since $s_1 - s_2 = \sum_{j=3}^n (\log a_{1j} - \log a2j) = \log(\prod_{j=3}^n a_{1j}/a_{2j})$, we get

$$a_{12} = \mathrm{e}^{\xi_1 - \xi_2} = \mathrm{e}^{(s_1 - s_2)/(n-2)} = \sqrt[n-2]{\mathrm{e}^{s_1 - s_2}} = \sqrt[n-2]{\prod_{j=3}^n a_{1j}/a_{2j}},$$

which is van Uden's rule (20) for $i = 1$ and $k = 2$.

There are other methods to deal with an incomplete reciprocal matrix when only one entry above the main diagonal is missing. We can cite the one proposed by Shiraishi, Obata, and Daigo (1998) and the heuristic approach given by Harker (1987). The foundation of the method proposed by Shiraishi, Obata, and Daigo (1998) is based on the following theorem: let A be a reciprocal $n \times n$ matrix ($n > 2$), if $p_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + c_n$, then $c_1 = -n$, $c_2 = 0$, and $c_3 \leq 0$. Furthermore, $c_3 = 0$ if and only if A is consistent. So, it is natural to maximize c_3 in this kind of problems. As one can see in section 3 in (Shiraishi, Obata, & Daigo, 1998), the van Uden's rule follows a different approach.

To better show the performance and validity of the method we propose, we finally compare, in an empirical way, Harker's method and ours. Let A be the following reciprocal matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1/2 & 1 & 4 & 2 \\ 1/3 & 1/4 & 1 & 1/2 \\ 1 & 1/2 & 2 & 1 \end{bmatrix},$$

with priorities given by the eigenvector $(0.361, 0.318, 0.097, 0.224)^T$. By using Theorem 3 in (Benítez, Izquierdo, Pérez-García, & Ramos-Martínez, 2014), we get that the consistent matrix closest to A is

$$X_A \simeq \begin{bmatrix} 1 & 1.107 & 3.464 & 1.565 \\ 0.9036 & 1 & 3.130 & 1.414 \\ 0.2887 & 0.3194 & 1 & 0.4518 \\ 0.6389 & 0.7071 & 2.213 & 1 \end{bmatrix}.$$

Let us proceed to delete some entries of A obtaining, as an example,

$$\widehat{A} = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 4 & \star \\ 1/3 & 1/4 & 1 & 1/2 \\ \star & \star & 2 & 1 \end{bmatrix}.$$

Note that the rank of the 3×3 upper left block of $\widehat{A}$ is 3; hence, the rank of $\widehat{A}$ is greater than 1 and, as a result, $\widehat{A}$ cannot be completed to be a consistent matrix.

We will estimate the missing data by Harker's rule. To this end, we build the derived reciprocal matrix:

$$\widetilde{A} = \begin{bmatrix} 2 & 2 & 3 & 0 \\ 1/2 & 2 & 4 & 0 \\ 1/3 & 1/4 & 1 & 1/2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

By using Octave, we find that the largest normalised eigenvalue is $\lambda_{\max} \simeq 4.083$, with associated eigenvector $\mathbf{v} \simeq (0.4243, 0.2927, 0.09939, 0.1836)^T$, the priority vector found by Harker's method. With this vector one can get the matrix $H = (H_{ij})$, where $H_{ij} = \mathbf{v}_i/\mathbf{v}_j$. In this example,

$$H \simeq \begin{bmatrix} 1 & 1.449 & 4.269 & 2.311 \\ 0.6900 & 1 & 2.945 & 1.595 \\ 0.2343 & 0.3400 & 1 & 0.5414 \\ 0.4327 & 0.6272 & 1.847 & 1 \end{bmatrix},$$

which, obviously, is not a completion of $\widehat{A}$.

Let us now use the method proposed in this paper (we omit the complete set of details). First of all, since the associated graph is connected, the optimal completion is unique. Let

$$X(a,b) = \begin{bmatrix} 1 & 2 & 3 & a \\ 1/2 & 1 & 4 & b \\ 1/3 & 1/4 & 1 & 1/2 \\ 1/a & 1/b & 2 & 1 \end{bmatrix}$$

be this solution. By Theorem 2, the entries $(1,4)$ and $(2,4)$ of $X(a,b)$ (and their respective symmetrical entries) coincide with the corresponding entries of Z, where Z is the consistent matrix such that $\mathrm{d}(X(a,b),Z) = \mathrm{d}(X(a,b),\mathscr{C}_4)$, and $\mathscr{C}_4$ is the set of 4×4 consistent matrices (recall that $\mathrm{d}(\cdot,\cdot)$ is the distance defined as $\mathrm{d}(M,N) = \|\,\mathrm{LOG}(M) - \mathrm{LOG}(N)\|_F$). By the previous consideration of Theorem 2, one has $Z = E(\phi_4(\boldsymbol{\theta}))$. This vector $\boldsymbol{\theta}$ can be obtained by equalities (18) and (19) getting $\boldsymbol{\theta} \simeq (0.6310, 0.2648, -0.7945, -0.1014)^T$, and thus,

$$Z = E(\phi_4(\boldsymbol{\theta})) \simeq \begin{bmatrix} 1 & 1.443 & 4.160 & 2.080 \\ 0.6933 & 1 & 2.884 & 1.443 \\ 0.2404 & 0.3467 & 1 & 0.5000 \\ 0.4808 & 0.6933 & 2.000 & 1 \end{bmatrix}.$$

Accordingly, the optimal completion of $\widehat{A}$ is

$$X(Z_{14}, Z_{24}) = \begin{bmatrix} 1 & 2 & 3 & Z_{14} \\ 1/2 & 1 & 4 & Z_{24} \\ 1/3 & 1/4 & 1 & 1/2 \\ Z_{41} & Z_{42} & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2.080 \\ 1/2 & 1 & 4 & 1.443 \\ 1/3 & 1/4 & 1 & 1/2 \\ 0.4808 & 0.6933 & 2 & 1 \end{bmatrix}.$$

We can see that matrices Z and H are similar. We can also check that $\mathrm{d}(X_A, Z) = 0.6355 < 0.7988 = \mathrm{d}(X_A, H)$, which shows that, in this example, the matrix Z obtained by our method is closer to X_A than the matrix H obtained by Harker's rule.

Observe that our method gives the optimal completion of matrix $\widehat{A}$ (evidently, $X(Z_{14}, Z_{24})$ is a completion of $\widehat{A}$), while Harker's rule only gives a priority vector $\mathbf{v}$, and the matrix H such that $H_{ij} = \mathbf{v}_i/\mathbf{v}_j$, is not, in general, a completion of $\widehat{A}$.

Additionally, it can be checked (by using Octave, for example) that the largest eigenvalue of $X(Z_{14}, Z_{24})$ is $\lambda_{\max} \simeq 4.081$, from which we easily find the consistency index of $X(Z_{14}, Z_{24})$, which equals $CI = (\lambda_{\max} - 4)/(4-1) \simeq 0.02714$. Finally, since $CR = CI/RI = 0.03050 < 0.1 = 10\%$, according to Saaty's criterion, the consistency of $X(Z_{14}, Z_{24})$ is acceptable and a priority vector is the normalised eigenvector of $X(Z_{14}, Z_{24})$ associated to $\lambda_{\max}$, which is $\mathbf{w} \simeq (0.4164, 0.2890, 0.1001, 0.1949)^T$.

This example shows that given an incomplete matrix which cannot be completed to be consistent, we can get a completion whose consistency is acceptable.

4 Case Study.

The present case study refers to an industrial layout reorganisation problem involving materials handling – specifically the reorganisation of storage space in a factory. This reorganisation concerns the best arrangement (using various criteria) for shelving to store pallets of finished products and cardboards. Moreover, a path for the transit of people and forklifts (*i.e.,* lines to transport the goods) must be defined by considering the available space inside the storage facility. The AHP technique is applied to select the best option from a set of three layout proposals ($\mathrm{LP}_1, \mathrm{LP}_2, \mathrm{LP}_3$), evaluated on the basis of five criteria ($\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_5$). The considered and mutually independent criteria are: safety & security; cost; innovation; transport; and placement.

The first criterion considers the aspect of safety and security at the workplace for the stakeholders of the storage facility. The second criterion refers to the cost of implementing a specific layout. The third criterion regards the innovative character of each alternative in terms of broad flexibility for enhancing the storage conditions (for example, by creating spaces for the employees to communicate and so better integrate operations). The fourth criterion is related to the movement of goods in the storage area on forklifts and managing the pedestrian areas crossed by employees and visitors inside the facility. The fifth criterion considers how a specific layout alternative may facilitate the placement of materials on shelves with the aim of distributing pallets of finished products and cardboard in different sectors of the shelves on the basis of their uses (and thus avoiding mixing materials).

The hierarchical structure of the problem is shown in Figure 3.

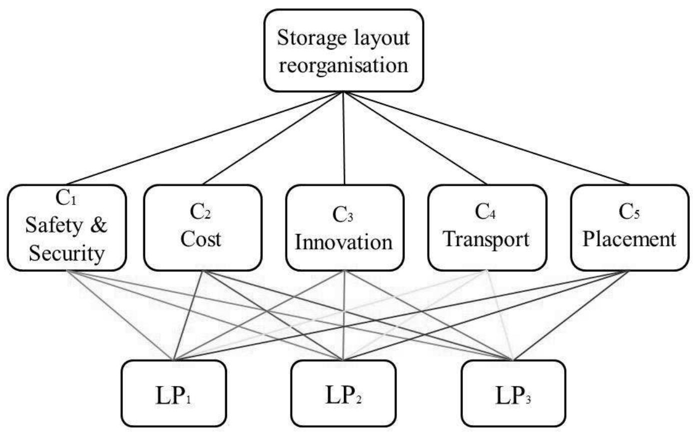


Figure 3: Hierarchical structure

Figure 4 shows the (feasible) schemes of the three layout proposals. The shelves to be arranged are highlighted as grey blocks numbered from one to five. Others blocks represent fixed elements in the facility. The topmost parts of the plants are the production areas of the firm that communicate with the storage and so more than two shelves cannot be allocated in this area (*e.g.* shelves 1 and 5 in LP2 in Figure 4). Observe that shelf 2 may be divided into two halves.

Table 2 shows the relative evaluations of the alternatives with respect to the criteria. In each table, the last two columns give, respectively, the normalised local priorities (the Perron vectors of the matrices calculated via the power method), and the consistency indices (CR).

Note that all the relative judgments are consistent because none of the CR indices exceed

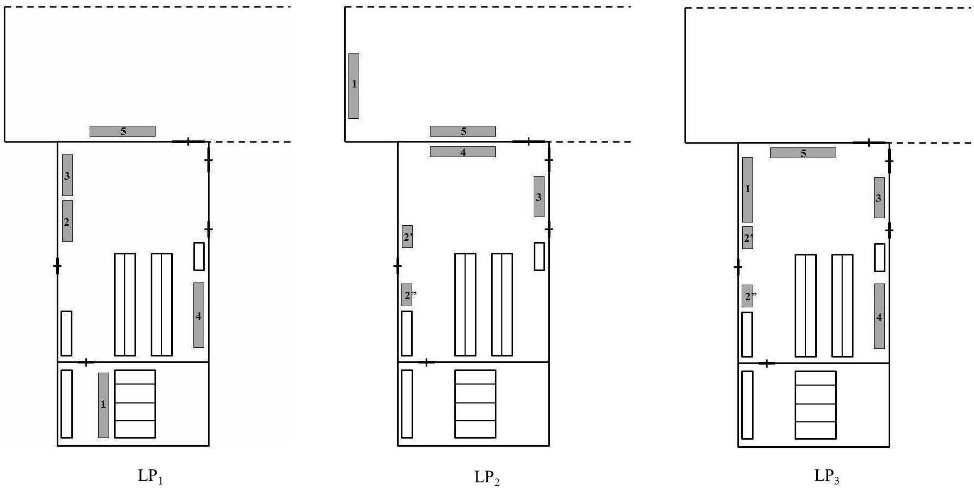


Figure 4: Layout proposals LP1, LP2, LP3

the value of 0.05 (the threshold for matrices of size 3×3, (Saaty, 1977)).

In addition to the calculation of the local priorities of alternatives, it is necessary to evaluate the vector of criteria weights. A decision group composed of three experts (D_1, D_2, D_3) was involved to this purpose. We will assume that the experts have the same weight in the decision process. Their roles are the following: consultant; chief of health and safety, and an employee representative. These decision-makers are involved in the management of the storage area from different – but complementary – perspectives. However, in formulating the judgements, the experts prefer not to express some evaluations. Since the presence of missing information often affects these kind of practical problems, the main difficulty of consistent completion regards the achievement of reliable values reflecting experts' opinions and preferences. Specifically, the experts were unwilling to give their judgements about the following pairwise comparison: C_2/C_5. In other terms, they preferred not to express any opinion comparing cost and placement. Moreover, experts D_2 and D_3 did not give their judgements about another pairwise comparison, C_2/C_3. In fact, they did not wish to express a judgement comparing cost and the pursuance of innovation. With relation to this last missing comparison, although the decision maker D_1 expressed his opinion by assigning a numerical value, he could not be totally exhaustive for evaluating the mentioned comparison. Indeed opinions of each single decision maker need to be balanced with the others and, to such an aim, the relative missing judgments must be calculated. Table 3 shows the incomplete pairwise comparisons judgments.

It is simple to check that the graphs corresponding to these matrices have only one connected component. According to Corollary 2, the completions of these matrices are unique in the sense of Theorem 1.

Van Uden's rule can be used for the first matrix, since only one upper-diagonal entry is unknown. The completion obtained is

$$a_{25} = \sqrt[3]{\frac{a_{21}a_{23}a_{24}}{a_{51}a_{53}a_{54}}}.$$

Table 2: Evaluation of alternatives with respect to criteria, local priorities and CR value

C_1	LP_1	LP_2	LP_3	Local Priorities	CR
LP_1	1	4	4	0.667	
LP_2	1/4	1	1	0.167	0
LP_3	1/4	1	1	0.167	

C_2	LP_1	LP_2	LP_3	Local Priorities	CR
LP_1	1	1/2	1/5	0.122	
LP_2	2	1	3	0.23	0.0035
LP_3	5	3	1	0.648	

C_3	LP_1	LP_2	LP_3	Local Priorities	CR
LP_1	1	6	6	0.75	
LP_2	1/6	1	3	0.125	0
LP_3	1/6	1	1	0.125	

C_4	LP_1	LP_2	LP_3	Local Priorities	CR
LP_1	1	1/2	1/4	0.136	
LP_2	2	1	1/3	0.238	0.0176
LP_3	4	3	1	0.625	

C_5	LP_1	LP_2	LP_3	Local Priorities	CR
LP_1	1	2	5	0.582	
LP_2	1/2	1	3	0.309	0.0036
LP_3	1/5	1/3	1	0.109	

The value of $\boldsymbol{\theta}$ for the second matrix is $\boldsymbol{\theta} = [0.900, -0.297, -0.099, -0.578, 0.074]^T$. This vector gives the best completion of the second matrix: $a_{23} = \exp(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_3) = 0.82019$ and $a_{25} = \exp(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_5) = 0.68980$. For the third matrix we get $\boldsymbol{\theta} = [0.461, -1.014, -0.194, 0.220, 0.528]^T$, $a_{23} = \exp(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_3) = 0.44068$ and $a_{25} = \exp(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_5) = 0.21394$.

By using these values, we can build the respective completions with the calculated entries in bold (shown in Table 4). The completed matrices were then shared with the team of decision makers, who did not show reasons to disagree with the assigned values, confirming the coherence of the found results.

To build a blend of these matrices we use the aggregation of individual judgments (AIJ) technique in which the individual comparison matrices are merged into one, so that the group normally becomes a 'new individual', in contrast to the aggregation of individual priorities (AIP) technique (in which individuals act with different value systems – producing alternative individual priorities (Forman & Peniwati, 1998) that are eventually merged into one priority vector). This approach agrees with (Guitouni & Martel, 1998), since the experts in our

Table 3: Criteria evaluation matrices provided by the experts

D_1	C_1	C_2	C_3	C_4	C_5
C_1	1	7	1	4	5
C_2	1/7	1	1/3	1/3	*
C_3	1	3	1	4	3
C_4	1/4	3	1/4	1	2
C_5	1/5	*	1/3	1/2	1

D_2	C_1	C_2	C_3	C_4	C_5
C_1	1	5	3	3	2
C_2	1/5	1	*	2	*
C_3	1/3	*	1	3	1/2
C_4	1/3	1/2	1/3	1	1
C_5	1/2	*	2	1	1

D_3	C_1	C_2	C_3	C_4	C_5
C_1	1	5	1	2	1
C_2	1/5	1	*	1/3	*
C_3	1	*	1	1/2	1/3
C_4	1/2	3	2	1	1
C_5	1	*	3	1	1

case study act together in a complementary manner and so combining individual judgments into a group judgment is recommended. To aggregate the individual priorities into group priorities, the geometric mean method (GMM) is used. Following these observations, the blended comparison matrix of criteria is shown in Table 5, in which the last column shows the priority vector, calculated via the power method.

Once the priority vectors for criteria and alternatives have been built, we aggregate the results through the distributive method and the final ranking of layout proposals is obtained (see Table 6).

The layout proposal LP_1 was recognised to be the best trade-off among all considered criteria, and the involved decision group, having previously agreed concerning completed matrices, eventually backed the selection as well. In particular, the application of the graph theory supports the goodness of the solution, this method being particularly advantageous in the manufacturing field (Rao Venkata, 2013). By adopting this solution, four of the five shelves (1 to 4) are arranged into the storage area, and the fifth shelf is placed in the production area. This solution permits a safe management of the available spaces and is well-balanced between the two departments. In fact, this arrangement enables an optimisation of the placement of pallets of finished products and cardboards according to the logistic strategies adopted by the organisation. At the same time, transport can be improved by establishing dedicated paths for people (employees and visitors) and forklifts (materials transport) inside the storage

Table 4: Completed matrices

D_1	C_1	C_2	C_3	C_4	C_5
C_1	1	7	1	4	5
C_2	1/7	1	1/3	1/3	**0.78090**
C_3	1	3	1	4	3
C_4	1/4	3	1/4	1	2
C_5	1/5	**1.28058**	1/3	1/2	1

D_2	C_1	C_2	C_3	C_4	C_5
C_1	1	5	3	3	2
C_2	1/5	1	**0.82019**	2	**0.68980**
C_3	1/3	**1.21922**	1	3	1/2
C_4	1/3	1/2	1/3	1	1
C_5	1/2	**1.44991**	2	1	1

D_3	C_1	C_2	C_3	C_4	C_5
C_1	1	5	1	2	1
C_2	1/5	1	**0.44068**	1/3	**0.21394**
C_3	1	**2.26923**	1	1/2	1/3
C_4	1/2	3	2	1	1
C_5	1	**4.67609**	3	1	1

department. Lastly, the selected layout proposal creates a special area (box) between the two doors in the upper right side of the storage area. This box can be used for employee meetings aimed at integrating the workforce and enhancing the level of communication inside the organisation.

5 Conclusions

Decision-making processes are connected with multiple aspects of human life and involve many levels and kinds of business activities. The multi-criteria decision-making method AHP is considered to be a particularly helpful tool in supporting decision-making, as well as in situations characterised by uncertainty in formulating opinions. When experts are asked to formulate pairwise comparison judgments, they may not be totally sure about one or more factors and may prefer not to express a preference. In this situation, the AHP is characterised by incomplete matrices of pairwise comparison judgments. With the aim of consistently building a complete PCM, which could enter the trade-off process between synthetic consistency and the judgments from the experts involved, this paper highlights that graph theory may be used to deal with the treatment of incomplete comparison matrices – and thanks to the described approach all the cases can be characterised and classified in terms

Table 5: Aggregated matrix and criteria weights

	C_1	C_2	C_3	C_4	C_5	**Weights**
C_1	1	5.593	1.442	2.884	2.154	**38.4** %
C_2	0.179	1	0.494	0.606	0.487	**8.43** %
C_3	0.693	2.025	1	1.817	0.794	**20.61** %
C_4	0.347	1.651	0.550	1	1.260	**14.82** %
C_5	0.464	2.055	1.260	0.794	1	**17.73** %

Table 6: Ranking of layout alternatives

Position	**Alternative**	**Score**
1^{st}	LP_1	**0.5442**
2^{nd}	LP_3	**0.2564**
3^{rd}	LP_2	**0.1993**

of matrix graph connectedness. Moreover, completion solutions are developed for all those cases and the solution for a quite synthetic general case with a two-component associated graph is produced.

The proposed approach is applied to a case study that refers to the storage layout reorganisation in a factory. In this case, three experts are involved and they decide not to express judgments about several pairs of criteria. Three incomplete matrices capture their opinions. One matrix presents just one unknown upper-diagonal entry, and the other two matrices have two gaps. In the first case, van Uden's rule may be used and provides the same result; whereas in the second case, the approach provided in this paper enables a consistent completion of the matrices and the production of a final ranking of alternatives.

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