UNIVERSITAT
POLITÈCNICA
DE VALÈNCIA

Ergodic properties of operators on spaces of functions

Author
Alberto Rodríguez Arenas

Advisers
José Antonio Bonet Solves
Enrique Jordá Mora

Valencia, February 2020

Por supuesto, esta tesis no habría sido posible sin Pepe y sin Enrique. Durante el máster, Pepe, me metiste en este tema, o según tus palabras, “me sacaste de la ignorancia.” Después, Enrique, me descubriste lo que era investigar. Los dos habéis tenido más confianza en mí de la que yo tengo. Gracias por todo.

The stays in Trier provided my best mathematical experiences (so far) although also the worst coffee. I'm thankful to everybody there, but specially Jochen and Leonhard. You've always been helpful and patient, thank you.

La primera vez que fui al IUMPA me perdí y al llegar seguía perdido. Poco a poco se convirtió en el sitio más cómodo para trabajar. Claro, no por el sitio, sino por la compañía. Ha pasado mucha gente y todos estaban ahí para echar una mano. Os mando gracias, que espero que salten el Muro.

Gracias a mis compañeros, con los que he compartido el día a día, tantas comidas y cenas olvidando la tesis y contando historias.

Muchas amistades han ido y venido, pero el spam siempre ha estado ahí, para todo, dando las mayores alegrías. Así que, gracias al spam.

Y, como no, mis padres. Por estar presentes a pesar de la distancia, por las llamadas en el peor momento y por no dejarme tranquilo. Gracias.

Contents

Summary

The aim of this thesis is to study the ergodic properties (power boundedness, mean ergodicity and uniform mean ergodicity) of some operators defined on several spaces of functions. In a locally convex Hausdorff space E, an operator $T \in \mathcal{L}(E)$ is called power bounded if the set of its iterates is equicontinuous. The Cesàro means of T are

$$T_{[n]} = \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}.$$

The operator T is called mean ergodic if the sequence $(T_{[n]})_n$ converges pointwise and it is called uniformly mean ergodic if the sequence converges uniformly on bounded sets.

In Chapter 1, the multiplication operator is studied when defined on weighted spaces of continuous functions and their inductive and projective limits. We work with a Hausdorff, normal, locally compact topological space X. Given a continuous function φ (a symbol), the multiplication operator is $M_\varphi : f \mapsto \varphi f$.

A continuous function v is a weight if it is strictly positive. The weighted (Banach) spaces of continuous functions are

$$C_v := C_v(X) = \{f \in C(X) \,:\, \|f\|_v := \sup_{x \in X} v(x)|f(x)| < \infty\},$$

$$C_v^0 := C_v^0(X) = \{f \in C(X) \,:\, vf \text{ vanishes at infinity}\},$$

with the norm $\|\cdot\|_v$.

We show that power boundedness of the multiplication operator on these spaces is equivalent to the multiplier being bounded by 1. In Section 1.2 we see that this is however only a necessary condition for mean ergodicity. In C_v^0 adding the condition of the pre-image of 1 by the symbol to be an open set suffices to obtain mean ergodicity. This is not enough for C_v, but including the condition of the symbol to be bounded away from 1 gives mean ergodicity and even uniform mean ergodicity on both spaces.

The Sections 1.3 and 1.4 are devoted to inductive and projective limits of the spaces in Section 1.2. If $V = (v_n)_n$ is a decreasing family of weights, the weighted

inductive limits of continuous functions are $VC = \text{ind}_n C_{v_n}$ and $V_0C = \text{ind}_n C^0_{v_n}$. If $A = (a_n)_n$ is an increasing family of weights, the weighted projective limits of continuous functions are $CA = \text{proj}_n C_{a_n}$ and $CA_0 = \text{proj}_n C^0_{a_n}$. The ergodic properties are characterized in terms of the family of weights and of the so-called Nachbin family $\overline{V}$ associated to V or A. The behaviour is different for the limits of the C_{v_n} (resp. C_{a_n}) and the limits of the $C^0_{v_n}$ (resp. $C^0_{a_n}$).

In Section 1.5 the spectrum and the Waelbroeck spectrum of the multiplication operator are completely determined. In the final Section 1.6 the topology of the set of multipliers between projective limits is compared with the one induced by the operator topology of uniform convergence on bounded sets.

The work of Chapter 2 is devoted to weighted sequence spaces and their inductive and projective limits. A sequence $v = (v(i))_i \in \mathbb{C}^{\mathbb{N}}$ is called a *weight* if it is strictly positive. The weighted Banach spaces of sequences considered are $\ell_p(v)$, $1 \leq p \leq \infty$ and $c_0(v)$.

Given $A = (a_n)_n$, a Köthe matrix (i.e. a_n is a weight and $a_n(i) \leq a_{n+1}(i)$ for all $i, n \in \mathbb{N}$), the echelon space of order $1 \leq p \leq \infty$ is defined by

$$\lambda_p(A) := \operatorname*{proj}_{n\in\mathbb{N}} \ell_p(a_n) \quad \text{and} \quad \lambda_0(A) = \operatorname*{proj}_{n\in\mathbb{N}} c_0(a_n).$$

The co-echelon space of order $1 \leq p \leq \infty$ is defined, for a decreasing family of weights $V = (v_n)_n$, by

$$\kappa_p(A) := \operatorname*{ind}_{n\in\mathbb{N}} \ell_p(v_n) \quad \text{and} \quad \kappa_0(A) := \operatorname*{ind}_{n\in\mathbb{N}} c_0(v_n).$$

In the Sections 2.2 and 2.3 ergodic and spectral properties of the multiplication operator are studied. The cases $p = \infty$ and $p = 0$ are particular cases of C_v and C^0_v and their limits and the results are deduced from Chapter 1. In the case $1 \leq p < \infty$ the results coincide with those of $p = 0$.

In Section 2.4 it is characterized when the multiplication operator is bounded or compact, in similar terms than continuity. In Section 2.5, as in Section 1.6, the topology of the set of multipliers between echelon spaces is compared with the one induced by the operator topology of uniform convergence on bounded sets. Also the topology of the set of bounded multiplication operators is studied. In the final Section 2.6, the results of the previous sections are applied to the power series spaces, as particular cases of echelon spaces.

Chapter 3 deals with the composition operator given by a holomorphic self-map of the complex open unit disc, when considered between different Banach spaces of holomorphic functions. If $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic, the composition operator is $C_\varphi : f \mapsto f \circ \varphi$.

In Section 3.2 necessary and sufficient conditions are given for ergodic properties of a composition operator defined on a general Banach space of holomorphic functions under the assumption of one or many of given properties.

The results of Section 3.2 are applied in Section 3.3 to classical spaces of holomorphic functions, particularly, Bergman spaces of infinite type H_v and H_v^0, Bloch spaces $\mathcal{B}_p$ and $\mathcal{B}_p^0$, weighted Bergman spaces A^p and Hardy spaces H^p. First, it is shown that C_φ is power bounded precisely when φ has one fixed point in the unit disc. Also, if φ is a periodic automorphism then C_φ is uniformly mean ergodic and if it is a non-periodic automorphism then C_φ is mean ergodic and not uniformly mean ergodic. On H_v^0, $\mathcal{B}_p^0$, A^p and H^p mean ergodicity is equivalent to φ having a fixed point and if φ is not an automorphism, then mean ergodicity is even equivalent to the weak convergence of the iterates of the operator. On H_v and $\mathcal{B}_p$ mean ergodicity is equivalent to uniform mean ergodicity. If φ is not an automorphism and has a fixed point $z_0 \in \mathbb{D}$, then, on all the spaces, C_φ is uniformly mean ergodic if, and only if, the operator is quasicompact if, and only if, the iterates converge in norm to the evaluation at the fixed point.

Resumen

El objetivo de esta tesis es estudiar las propiedades ergódicas (acotación en potencias, ergodicidad media y ergodicidad media uniforme) de operadores definidos en varios espacios de funciones. En un espacio Hausdorff localmente convexo E, un operador $T \in \mathcal{L}(E)$ es llamado acotado en potencias si el conjunto de sus iteradas es equicontinuo. Las medias de Cesàro de T son

$$T_{[n]} = \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}.$$

El operador T se dice ergódico en media si la sucesión $(T_{[n]})_n$ converge puntualmente y se dice uniformemente ergódico en media si la sucesión converge uniformemente en conjuntos acotados.

En el Capítulo 1 se estudia el operador de multiplicación cuando está definido sobre espacios ponderados de funciones continuas y sobre sus límites inductivos y proyectivos. Trabajamos sobre un espacio topológico Hausdorff, normal y localmente compacto X. Dada una función continua φ, el operador de multiplicacion se define como $M_\varphi : f \mapsto \varphi f$.

Una función continua v se llama peso si es estrictamente positiva. Los espacios (de Banach) ponderados de funciones continuas son

$$C_v := C_v(X) = \{f \in C(X) \,:\, \|f\|_v := \sup_{x \in X} v(x)|f(x)| < \infty\},$$

$$C_v^0 := C_v^0(X) = \{f \in C(X) \,:\, vf \text{ se anula en infinito}\},$$

con la norma $\|\cdot\|_v$.

Mostramos que la acotación en potencias del operador de multiplicación en estos espacios es equivalente a que φ esté acotado por 1. En la Sección 1.2 vemos que esto es, sin embargo, sólo una condición necesaria para la ergodicidad media. Para C_v^0, añadir la condición de que la antiimagen de 1 por el símbolo sea un abierto es suficiente para tener ergodicidad media. Esto no es suficiente para C_v, pero incluir la condición de que el símbolo esté alejado de 1 da ergodicidad media e incluso da ergodicidad en media uniforme en los dos espacios.

En las Secciones 1.3 y 1.4 se centra la atención en límites indutivos y proyectivos de los espacios de la Sección 1.2. Si $V = (v_n)_n$ es una familia decreciente de pesos, entonces los limites inductivos ponderados de funciones continuas son $VC = \mathrm{ind}_n C_{v_n}$ y $V_0C = \mathrm{ind}_n C^0_{v_n}$. Si $A = (a_n)_n$ es una familia creciente de pesos, los límites proyectivos ponderados de funciones continuas son $CA = \mathrm{proj}_n C_{a_n}$ y $CA_0 = \mathrm{proj}_n C^0_{a_n}$. Las propiedades ergódicas son caracterizadas en función de la familia de pesos y de la llamada familia de Nachbin $\overline{V}$ asociada a V o a A. El comportamiento es diferente para los límites de los C_{v_n} (resp. C_{a_n}) del de los límites de los $C^0_{v_n}$ (resp. $C^0_{a_n}$).

En la Sección 1.5 se determinan completamente el espectro y el espectro de Waelbroeck del operador de multiplicación. En la última Sección 1.6 se compara la topología del conjunto de multiplicadores entre límites proyectivos con la inducida por la topología de operadores de convergencia uniforme en acotados.

El Capítulo 2 se centra en estudiar espacios ponderados de sucesiones y sus límites inductivos y proyectivos. Una sucesión $v = (v(i))_i \in \mathbb{C}^{\mathbb{N}}$ se llama peso si es estrictamente positiva. Los espacios de Banach ponderados de sucesiones considerados son $\ell_p(v)$, $1 \leq p \leq \infty$ y $c_0(v)$.

Dada una matriz de Köthe $A = (a_n)_n$ (i.e. a_n es un peso y $a_n(i) \leq a_{n+1}(i)$ para todo $i, n \in \mathbb{N}$), el espacio escalonado de orden $1 \leq p \leq \infty$ se define como

$$\lambda_p(A) := \operatorname*{proj}_{n\in\mathbb{N}} \ell_p(a_n) \quad \text{y} \quad \lambda_0(A) = \operatorname*{proj}_{n\in\mathbb{N}} c_0(a_n).$$

El espacio co-escalonado de orden $1 \leq p \leq \infty$ se define, para una familia decreciente de pesos $V = (v_n)_n$, como

$$\kappa_p(A) := \operatorname*{ind}_{n\in\mathbb{N}} \ell_p(v_n) \quad \text{y} \quad \kappa_0(A) := \operatorname*{ind}_{n\in\mathbb{N}} c_0(v_n).$$

En las Secciones 2.2 y 2.3 se estudian las propiedades ergódicas y espectrales del operador de multiplicación. Los casos $p = \infty$ y $p = 0$ son casos particulares de C_v y C^0_v y de sus límites y los resultados se deducen del Capítulo 1. En el caso $1 \leq p < \infty$, los resultados coinciden con los de $p = 0$.

En la Sección 2.4 se caracteriza cuándo el operador de multiplicación es acotado o compacto, de manera similar a la continuidad. En la Sección 2.5, como en la Sección 1.6, la topología del conjunto de multiplicadores entre espacios escalonados se compara con la inducida por la topología de operadores de convergencia uniforme en acotados. También se estudia la topología del conjunto de operadores acotados. En la última Sección 2.6, los resultados de las secciones anteriores se aplican a los espacios de series de potencias, como casos particulares de los espacios escalonados.

El Capítulo 3 trata el operador de composición dado por una aplicación holomorfa del disco unidad abierto complejo en sí mismo, considerado entre diferentes espacios de Banach de funciones holomorfas. Si $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ es holomorfa, el operador de composición es $C_\varphi : f \mapsto f \circ \varphi$.

En la Sección 3.2 se dan condiciones necesarias y suficientes para las propiedades ergódicas del operador de composición definido en un espacio de Banach de funciones holomorfas general asumiendo una o varias propiedades dadas.

Los resultados de la Sección 3.2 se aplican en la Sección 3.3 a espacios clásicos de funciones holomorfas, en particular, espacios ponderados de Bergman de tipo infinito H_v y H_v^0, espacios de Bloch $\mathcal{B}_p$ y $\mathcal{B}_p^0$, espacios de Bergman A^p y espacios de Hardy H^p. Primero se prueba que C_φ es acotado en potencias precisamente cuando φ tiene un punto fijo en el disco unidad. También se muestra que si φ es un automorfismo periódico del disco, entonces C_φ es uniformemente ergódico en media y que si es un automorfismo no periódico, entonces C_φ es ergódico en media pero no uniformemente ergódico en media. En H_v^0, $\mathcal{B}_p^0$, A^p y H^p la ergodicidad media es equivalente a que φ tenga un punto fijo y si además φ no es un automorfismo, entonces la ergodicidad media es equivalente a que las iteradas del operador converjan débilmente. En H_v y $\mathcal{B}_p$ la ergodicidad media es equivalente a la ergodicidad en media uniforme. Si φ no es un automorfismo y tiene un punto fijo $z_0 \in \mathbb{D}$, entonces, en todos los espacios, C_φ es uniformemente ergódico si, y sólo si, el operador es quasicompacto si, y sólo si, las iteradas convergen en norma a la evaluación en el punto fijo.

Resum

L'objectiu d'aquesta tesi és estudiar les propietats ergòdiques (fitació en potències, ergodicitat mitjana i ergodicitat mitjana uniforme) d'operadors definits en diversos espais de funcions. En un espai Hausdorff localment convex E, un operador $T \in \mathcal{L}(E)$ s'anomena fitat en potències si el conjunt de les seues iterades és equicontinu. Les mitjanes de Cesàro de T són

$$T_{[n]} = \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}.$$

L'operador T és ergòdic en mitjana si la successió $(T_{[n]})_n$ convergeix puntualment i és uniformement ergòdic en mitjana si la successió convergeix uniformement en conjunts fitats.

Al Capítol 1 s'estudia l'operador de multiplicació quan està definit sobre espais ponderats de funcions contínues i sobre els seus límits inductius i projectius. Treballem sobre un espai topològic Hausdorff, normal i localment compacte X. Donada una funció contínua φ, l'operador de multiplicació es defineix com a $M_\varphi : f \mapsto \varphi f$.

Una funció contínua v s'anomena pes si és estrictament positiva. Els espais (de Banach) ponderats de funcions contínues són

$$C_v := C_v(X) = \{f \in C(X) \ : \ \|f\|_v := \sup_{x \in X} v(x)|f(x)| < \infty\},$$

$$C_v^0 := C_v^0(X) = \{f \in C(X) \ : \ vf \text{ s'anul·la a l'infinit}\},$$

amb la norma $\| \cdot \|_v$.

Provem que la fitació en potències de l'operador de multiplicació en aquestos espais és equivalent a provar que φ estiga fitada per 1. No obstant això, a la Secció 1.2 veiem que és només una condició necessària per a l'ergodicitat mitjana. Per a C_v^0, afegir la condició que l'antiimatge de 1 pel símbol siga un conjunt obert és suficient per obtindre ergodicitat mitjana. Aquesta no és suficient per a C_v, però incloure la condició que el símbol es trobe lluny de 1 dóna l'ergodicitat mitjana i, inclús, dóna l'ergodicitat en mitjana uniforme en els dos espais.

A les Seccions 1.3 i 1.4 es para atenció als límits inductius i projectius dels espais de la Secció 1.2. Si $V = (v_n)_n$ és una família decreixent de pesos, aleshores els límits inductius ponderats de funcions contínues són $VC = \mathrm{ind}_n C_{v_n}$ i $V_0C = \mathrm{ind}_n C^0_{v_n}$. Si $A = (a_n)_n$ és una família creixent de pesos, aleshores els límits projectius ponderats de funcions contínues són $CA = \mathrm{proj}_n C_{a_n}$ i $CA_0 = \mathrm{proj}_n C^0_{a_n}$. Les propietats ergòdiques estan caracteritzades en funció de la família de pesos i de l'anomenada família de Nachbin $\overline{V}$ associada a V o a A. El comportament és diferent per als límits dels C_{v_n} (resp. C_{a_n}) del dels límits dels $C^0_{v_n}$ (resp. $C^0_{a_n}$).

A la Secció 1.5 es determinen completament l'espectre i l'espectre de Waelbroeck de l'operador de multiplicació. A la darrera Secció 1.6 es compara la topologia del conjunt de multiplicadors entre límits projectius amb la induïda per la topologia d'operadors de convergència uniforme en fitats.

Al Capítol 2 es dedica l'estudi d'espais ponderats de successions i els seus límits inductius i projectius. Una successió $v = (v(i))_i \in \mathbb{C}^{\mathbb{N}}$ s'anomena pes si és estrictament positiva. Els espais de Banach ponderats de successions considerats són $\ell_p(v)$, $1 \le p \le \infty$ i $c_0(v)$.

Donada una matriu de Köthe $A = (a_n)_n$ (i.e. a_n és un pes i $a_n(i) \le a_{n+1}(i)$ per a tot $i, n \in \mathbb{N}$), l'espai esglaonat d'ordre $1 \le p \le \infty$ es defineix com a

$$\lambda_p(A) := \operatorname*{proj}_{n\in\mathbb{N}} \ell_p(a_n) \quad \text{i} \quad \lambda_0(A) = \operatorname*{proj}_{n\in\mathbb{N}} c_0(a_n).$$

L'espai co-esglaonat d'ordre $1 \le p \le \infty$ es defineix, per a una família decreixent de pesos $V = (v_n)_n$, com a

$$\kappa_p(A) := \operatorname*{ind}_{n\in\mathbb{N}} \ell_p(v_n) \quad \text{i} \quad \kappa_0(A) := \operatorname*{ind}_{n\in\mathbb{N}} c_0(v_n).$$

A les Seccions 2.2 i 2.3 s'estudien les propietats ergòdiques i espectrals de l'operador de multiplicació. Els casos $p = \infty$ i $p = 0$ són casos particulars de C_v i C^0_v i dels seus límits i els resultats es dedueixen del Capítol 1. En el cas $1 \le p < \infty$, els resultats coincideixen amb els de $p = 0$.

A la Secció 2.4 es caracteritza quan l'operador de multiplicació és fitat o compacte, d'un mode similar a la continuïtat. A la Secció 2.5, com a la Secció 1.6, la topologia del conjunt de multiplicadors entre espais esglaonats es compara amb la induïda per la topologia d'operadors de convergència uniforme en fitats. També s'estudia la topologia del conjunt d'operadors fitats. A la darrera Secció 2.6, els resultats de les seccions anteriors s'apliquen als espais de sèries de potències, com casos particulars dels espais esglaonats.

El Capítol 3 estudia l'operador de composició donat per una aplicació holomorfa del disc unitat obert complex en sí mateix, considerat entre diferents espais de Banach de funcions holomorfes. Si $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ és holomorfa, aleshores l'operador de composició és $C_\varphi : f \mapsto f \circ \varphi$.

A la Secció 3.2 es donen condicions necessàries i suficients per a les propietats ergòdiques de l'operador de composició definit en un espai de Banach de funcions holomorfes general assumint una o més propietats donades.

Els resultats de la Secció 3.2 s'apliquen a la Secció 3.3 per a espais clàssics de funcions holomorfes, en particular, espais ponderats de Bergman de tipus infinit H_v i H_v^0, espais de Bloch $\mathcal{B}_p$ i $\mathcal{B}_p^0$, espais de Bergman A^p i espais de Hardy H^p. Primer es prova que C_φ és fitat en potències precisament quan φ té un punt fix al disc unitat. També es prova que si φ és un automorfisme periòdic del disc, aleshores C_φ és uniformement ergòdic en mitjana i que si és un automorfisme no periòdic, aleshores C_φ és ergòdic en mitjana però no uniformement ergòdic en mitjana. En H_v^0, $\mathcal{B}_p^0$, A_p i H^p la ergodicitat mitjana és equivalent que φ tinga un punt fix i si a més a més φ no és un automorfisme, aleshores l'ergodicitat mitjana és equivalent que les iterades de l'operador convergisquen feblement. En H_v i $\mathcal{B}_p$ l'ergodicitat mitjana és equivalent a l'ergodicitat en mitjana uniforme. Si φ no és un automorfisme i té un punt fix $z_0 \in \mathbb{D}$, aleshores, a tots els espais, C_φ és uniformement ergòdic si i només si l'operador és quasi-compacte si i només si les iterades convergeixen en norma a l'avaluació al punt fixe.

UNIVERSITAT
POLITÈCNICA
DE VALÈNCIA

Ergodic properties of operators on spaces of functions

Author
Alberto Rodríguez Arenas

Advisers
José Antonio Bonet Solves
Enrique Jordá Mora

Valencia, February 2020

Introduction

The aim of this thesis is to study different dynamical properties of some (linear and continuous) operators defined on several spaces of functions. The dynamical properties we consider are mainly power boundedness, mean ergodicity and uniform mean ergodicity. In a locally convex Hausdorff space E, an operator $T \in \mathcal{L}(E)$ is called power bounded if the set of its iterates is equicontinuous. The Cesàro means of T are

$$T_{[n]} = \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}.$$

The operator T is called mean ergodic if the sequence $(T_{[n]})_n$ converges pointwise and it is called uniformly mean ergodic if the sequence converges uniformly on bounded sets.

Many authors have worked in this topic along the last century, starting with the seminal work of von Neumann in 1931 [66], where he proved that an unitary operator on a Hilbert space is always mean ergodic. Yosida in 1938 [89] showed that, under some conditions, the weak convergence of the averages of the iterates of an operator on a Banach space is a sufficient condition for pointwise convergence, i.e. for mean ergodicity. Later in 1941 Kakutani and Yosida [91] proved that a power bounded operator on a Banach spaces is uniformly mean ergodic precisely when there exists an iterate which is “close” to a compact operator. Dunford (1943) [42] and Lin (1974) [53] related the uniform mean ergodicity of operators on Banach spaces with the behaviour of 1 in their spectrum. Lotz stated in 1985 [56] that mean ergodicity and uniform mean ergodicity are equivalent for power bounded operators on a Banach Grothendieck space with the Dunford-Pettis property. The book of Krengel of 1985 [52, Chapter 2] contains an excellent compilation of some of these results. In 2009 Albanese, Bonet and Ricker [2] extended some of the results by Yosida and Lin to operators on Fréchet spaces. In Section A.2 we state these results, since they are of great importance in our work.

The relationship between power boundedness and (uniform) mean ergodicity has also been of interest. In 1938 Riesz [72] proved that on all L_p spaces with $1 < p < \infty$ power boundedness implies mean ergodicity. Lorch [55] extended this result to any reflexive Banach space in 1939, although this result is a direct consequence of the work by Yosida. Much more recently in 2001 Fonf, Lin and Wojtaszczyk [43] proved the converse of this result for Banach spaces with a Schauder basis. They found an operator on each non-reflexive Banach space with a Schauder basis which is power bounded and not mean ergodic and another which is power bounded and mean ergodic but not uniformly mean ergodic.

In this thesis we study the ergodic properties of different operators on various spaces. In Chapter 1 we consider the multiplication operator defined on weighted spaces of continuous functions and on inductive and projective limits of them. In Chapter 2 we also study the behaviour of multiplication operators (diagonal operators) on weighted spaces of sequences and their projective and inductive limits (Köthe echelon and co-echelon spaces). The work of Chapter 3 is significantly different, we study the composition operator on plenty of classical spaces of holomorphic functions on the complex open unit disc. There is no original work in Chapter A. It contains definitions and a compilation of classical results used in the rest of chapters. Its purpose is to ease the reading of the thesis.

The results in Sections 1.2 and 1.3 are contained in the paper [29] by the author and the advisers. The parts of Sections 2.2, 2.3 and 2.5 dealing with Köthe echelon spaces are included in the paper [73] by the author. The entirety of Chapter 3 is in the article [48] by the adviser Jordá and the author.

In Chapter 1 we study the multiplication operator defined on weighted spaces of continuous functions and their inductive and projective limits. We work with a Hausdorff, normal, locally compact topological space X. Given a continuous function φ, the multiplication operator is $M_\varphi : f \mapsto \varphi f$.

A continuous function v is a weight if it is strictly positive. The (Banach) weighted spaces of continuous functions are

$$C_v := C_v(X) = \{f \in C(X) \,:\, \|f\|_v := \sup_{x\in X} v(x)|f(x)| < \infty\},$$

$$C_v^0 := C_v^0(X) = \{f \in C(X) \,:\, vf \text{ vanishes at infinity}\},$$

with the norm $\|\cdot\|_v$. The space C_1 is precisely the space $CB(X)$ of continuous and bounded functions of X with the norm $\|\cdot\|_\infty$ and C_1^0 is the space of functions vanishing at infinity.

If $V = (v_n)_n$ is a decreasing family of weights, the weighted inductive limits of continuous functions are $VC = \operatorname{ind}_n C_{v_n}$ and $V_0C = \operatorname{ind}_n C_{v_n}^0$. If $A = (a_n)_n$ is an increasing family of weights, the weighted projective limits of continuous functions are $CA = \operatorname{proj}_n C_{a_n}$ and $CA_0 = \operatorname{proj}_n C_{a_n}^0$. The ergodic properties are characterized in terms of the family of weights and of the so-called Nachbin family, defined as

$$\overline{V} := \{\overline{v} : X \longrightarrow (0,\infty) \ : \overline{v} \text{ is upper semicontinuous and for each } k \in \mathbb{N}, \\ \frac{\overline{v}}{v_k} \text{ is bounded on } X\}.$$

Multiplication operators on weighted spaces of (vector valued) continuous functions have been investigated by Manhas [59], [60], Singh and Manhas [61], [77], [79], Oubbi [67], [68] and Klilou and Oubbi [50] among others. See also the book by Singh and Manhas [78]. Weighted inductive limits of spaces of continuous functions have been thoroughly investigated since the seminal work of Bierstedt, Meise and Summers [23] and [24]; see [15], [16], [18], [20], [19] and the references therein.

We show that power boundedness of the multiplication operator on these spaces is equivalent to $\|\varphi\|_\infty \leq 1$. In Section 1.2 we see that this is, in general, only a necessary condition for mean ergodicity. In C_v^0 adding the condition of $\varphi^{-1}(1)$ to be an open set suffices to have mean ergodicity. This is not enough for C_v, but including the condition $\inf\{|1-\varphi(x)| \ : \ x \notin \varphi^{-1}(1)\} > 0$, gives mean ergodicity for C_v and even uniform mean ergodicity on both spaces. The main results are Theorem 1.2.7 and Theorem 1.2.8.

In Section 1.3 we study weighted inductive limits of spaces of continuous functions. We find characterizations for mean ergodicity and uniform mean ergodicity in terms of the multiplier and the so-called Nachbin family associated to the family of weights, which depend deeply on properties of the family of weights. Again the mean ergodicity of M_φ on V_0C is equivalent to $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ being open. The uniform mean ergodicity on V_0C follows from mean ergodicity adding the condition that for every $k \in \mathbb{N}$ and every $\overline{v} \in \overline{V}$,

$$\lim_{n\to\infty} \sup_{x\notin\varphi^{-1}(1)} \frac{1}{n}\frac{\overline{v}(x)}{v_k(x)}|\varphi(x)|\frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = 0.$$

For the case of VC, mean ergodicity and uniform mean ergodicity are equivalent. Assuming the property (D) on V (Theorem A.3.5), we show that uniform mean ergodicity for V_0C is equivalent to uniform mean ergodicity on VC. We find some refinements of these results, to avoid the Nachbin family $\overline{V}$. The main results are Proposition 1.3.5 and Theorem 1.3.7 with its corollaries.

In Section 1.4 we see similar results for weighted projective limits of spaces of continuous functions. In this section we assume that X is even a σ-compact topological space. The mean ergodicity of M_φ on CA_0 is equivalent to $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ being open. The operator M_φ is mean ergodic on CA if, and only if, it is uniformly mean ergodic on CA if, and only if, it is uniformly mean ergodic on CA_0 if, and only if, $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open and or every $k \in \mathbb{N}$ and every $\overline{v} \in \overline{V}$,

$$\lim_{n\to\infty} \sup_{x\notin\varphi^{-1}(1)} \frac{\overline{v}(x)a_k(x)|\varphi(x)|}{n} \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = 0.$$

The main result of the section is Theorem 1.4.6.

In Section 1.5 the point spectrum, the spectrum and the Waelbroeck spectrum of the multiplication operator are completely determined. In all the spaces of this chapter we have

$$\sigma_{pt}(M_\varphi) \subseteq \{\varphi(x) \,:\, x \in X\} \subseteq \sigma(M_\varphi).$$

In C_v and C_v^0 we find $\overline{\{\varphi(x) \,:\, x \in X\}} = \sigma(M_\varphi)$, while for the rest of spaces,

$$\overline{\{\varphi(x) \,:\, x \in X\}} = \overline{\sigma(M_\varphi)} = \sigma^*(M_\varphi),$$

where $\sigma^*(M_\varphi)$ is the Waelbroeck spectrum. In VC and V_0C, $\mu \notin \sigma(M_\varphi)$ if, and only if, for each k, there exists $l \geq k$ such that

$$\sup_{x\in X} \frac{v_l(x)}{v_k(x)} \cdot \frac{1}{|\varphi(x)-\mu|} < \infty.$$

The case of CA and CA_0 is similar: $\mu \notin \sigma(M_\varphi)$ if, and only if, for each k, there exists $l \geq k$ such that

$$\sup_{x\in X} \frac{a_k(x)}{a_l(x)} \cdot \frac{1}{|\varphi(x)-\mu|} < \infty.$$

In the final Section 1.6 we see that the (PLB)–topology of the set of multipliers between projective limits coincides with the one induced by the operator topology of uniform convergence on bounded sets.

The aim of Chapter 2 is to study diagonal operators defined on weighted sequence spaces and on their inductive and projective limits. Given a sequence $\varphi = (\varphi_i)_i \in \mathbb{C}^{\mathbb{N}}$, the multiplication (or diagonal) operator is

$$M_\varphi : x = (x_i)_i \mapsto (\varphi_i x_i).$$

A sequence $v = (v_i)_i \in \mathbb{C}^{\mathbb{N}}$ is a weight if it is strictly positive. The weighted sequence spaces are defined by

$$\ell_p(v) := \{x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} \,:\, p_v(x) := \|(v(i)x_i)_i\|_p < \infty\},\ 1 \leq p \leq \infty,$$

$$c_0(v) := \{x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} \,:\, \lim_{i\to\infty} v(i)x_i = 0\},$$

where $\|\cdot\|_p$ denotes the usual ℓ_p norm and $c_0(v)$ has the $\ell_\infty(v)$ norm.

Given $A = (a_n)_n$, a Köthe matrix (i.e. a_n is a weight and $a_n(i) \leq a_{n+1}(i)$ for all $i, n \in \mathbb{N}$), the echelon space of order $1 \leq p \leq \infty$ is defined by

$$\lambda_p(A) := \operatorname*{proj}_{n\in\mathbb{N}} \ell_p(a_n) \quad \text{and} \quad \lambda_0(A) = \operatorname*{proj}_{n\in\mathbb{N}} c_0(a_n).$$

The co-echelon space of order $1 \le p \le \infty$ is defined, for a decreasing family of weights $V = (v_n)_n = (1/a_n)_n$, by

$$\kappa_p(A) := \operatorname*{ind}_{n\in\mathbb{N}} \ell_p(v_n) \quad \text{and} \quad \kappa_0(A) := \operatorname*{ind}_{n\in\mathbb{N}} c_0(v_n).$$

The relevance of echelon spaces is that they characterize a large class of Fréchet spaces, those with an absolute basis (see [63, Chapter 27]). Spaces of sequences and their operators have been investigated by many authors. We mention here [5], [4], [13], [14] and [49]. In the context of Köthe echelon spaces, diagonal operators were investigated by Crofts [38].

Our results for $\ell_\infty(v)$ and $c_0(v)$ follow directly from the results of Chapter 1. As we see in Section 2.2, it happens that the $\ell_p(v)$ spaces, with $1 \le p < \infty$, behave just like $c_0(v)$. The same behaviour is observed for projective limits (echelon spaces) and inductive limits (co-echelon spaces).

We have that, on all the spaces, M_φ is power bounded if, and only if, $\|\varphi\|_\infty \le 1$. If $p \ne \infty$, M_φ is mean ergodic if, and only if, it is power bounded. If $p = \infty$, uniform mean ergodicity is equivalent to mean ergodicity, in all the spaces. If $p \ne \infty$, M_φ is uniformly mean ergodic on $\lambda_p(A)$ if, and only if, it is uniformly mean ergodic on $\kappa_p(A)$ if, and only if, $\|\varphi\|_\infty \le 1$ and for each $n \in \mathbb{N}$ and each $\overline{v} \in \overline{V}$,

$$\lim_{k\to\infty} \sup_{i\in\mathbb{N}\setminus J} \frac{a_n(i)\overline{v}(i)|\varphi_i|}{k} \frac{|1-\varphi_i^k|}{|1-\varphi_i|} = 0,$$

where $J = \{i \in \mathbb{N} \,:\, \varphi_i = 1\}$. These three assertions are also equivalent to the operator being uniformly mean ergodic on $\lambda_\infty(A)$. Under the assumption that V has property (D), they are also equivalent to the operator being uniformly mean ergodic on $\kappa_\infty(V)$.

The main results about the ergodic properties are Theorems 2.2.6 and 2.2.8 and Theorems 2.2.11 and 2.2.13.

The spectrum of diagonal operators is characterized in Section 2.3. In all the spaces we have

$$\sigma_{pt}(M_\varphi) = \{\varphi \,:\, i \in \mathbb{N}\}.$$

In $\ell_p(v)$ and $c_0(v)$ we have $\sigma(M_\varphi) = \overline{\{\varphi \,:\, i \in \mathbb{N}\}}$, while for $\lambda_p(A)$ and $\kappa_p(A)$ we have $\overline{\sigma(M_\varphi)} = \overline{\{\varphi \,:\, i \in \mathbb{N}\}} = \sigma^*(M_\varphi)$. Also for $\lambda_p(A)$ and $\kappa_p(A)$, we have that $\mu \notin \sigma(M_\varphi)$ if, and only if, for each $n \in \mathbb{N}$, there exists $m \ge n$ such that

$$\sup_{i\in\mathbb{N}} \frac{a_n(i)}{a_m(i)} \frac{1}{|\varphi_i - \mu|} < \infty.$$

In Section 2.4 we study the boundedness and the compactness of the diagonal operator defined between different spaces. If A and B are Köthe matrices, we find that $M_\varphi : \lambda_p(A) \longrightarrow \lambda_p(B)$ is continuous if, and only if, for each $n \in \mathbb{N}$

there exists $m \in \mathbb{N}$ such that $\sup_{i\in\mathbb{N}} \frac{a_n(i)|\varphi_i|}{a_m(i)} < \infty$. It is bounded if, and only if, there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $\sup_i \frac{b_n(i)|\varphi_i|}{a_m(i)} < \infty$. It is compact if, and only if, there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $\lim_{i\to\infty} \frac{b_n(i)|\varphi_i|}{a_m(i)} = 0$. The result for continuity is analogous for $M_\varphi : \kappa_p(A) \longrightarrow \kappa_p(B)$. The result for boundedness also holds for κ_p with $p \neq 0$, but it holds for $p = 0$ if B is regularly decreasing. The result for compactness holds for κ_p, if B is regularly decreasing. The main results are Proposition 2.4.8 for echelon spaces and Propositions 2.4.10 and 2.4.14 for co-echelon spaces.

We also study the topology of the set of multipliers of echelon spaces in Section 2.5, but we consider as well the topology of the set of bounded multiplication operators. These results are in Theorem 2.5.1 and Theorem 2.5.3. Lastly in Section 2.6 we apply the results of the previous sections to a particular case of echelon spaces called power series spaces. They are defined in terms of an increasing non-negative sequence $\alpha = (\alpha_j)_j$ with $\lim_{j\to\infty} \alpha_j = \infty$, $r \in \mathbb{R} \cup \{+\infty\}$ and $1 \leq p < \infty$, as

$$\Lambda_r^p(\alpha) = \left\{ x \in \mathbb{C}^{\mathbb{N}} \ : \ \|x\|_t^p := \sum_{j\in\mathbb{N}} |x_j|^p e^{pt\alpha_j} < \infty, \text{ for all } t < r \right\},$$

$$\Lambda_r^\infty(\alpha) = \left\{ x \in \mathbb{C}^{\mathbb{N}} \ : \ \|x\|_t := \sup_{j\in\mathbb{N}} |x_j| e^{t\alpha_j} < \infty, \text{ for all } t < r \right\}.$$

The relevance of the power series spaces appears in the characterization of nuclear locally convex spaces and nuclear Fréchet spaces (see [63, Chapter 29]).

In Chapter 3 we work with the composition operator given by a holomorphic self-map of the complex open unit disc, when considered between different Banach spaces of holomorphic functions. Given a holomorphic function $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$, the composition operator is defined by $C_\varphi : f \mapsto f \circ \varphi$.

In Section 3.2 we define some properties on a general Banach space of holomorphic functions. We deduce necessary and sufficient conditions for the ergodic properties of the composition operator on these spaces under the assumption of one or many of these properties. We apply these results in Section 3.3 to classical spaces of holomorphic functions.

The properties of the composition operator acting between various spaces of holomorphic functions have been thoroughly studied, especially relating the properties of C_φ to those of φ. See the books of Cowen and McCluer [37] and Shapiro [74]. We study power boundedness, mean ergodicity and uniform mean ergodicity for composition operators defined on different spaces of holomorphic functions on the disc. This research for composition operators was firstly studied by Bonet and Domański in [26] when they studied the operator acting on $H(U)$, with U a domain in a Stein manifold. Later in [84], Wolf considered the composition operator on the general weighted Bergman spaces of infinite order $H_v^\infty(\mathbb{D})$. Bonet and Ricker [32] characterized the power boundedness and mean ergodicity of the multiplication operator on the general weighted Bergman spaces of infinite type $H_v(\mathbb{D})$ and $H_v^0(\mathbb{D})$. More recently, Beltrán-Meneu, Gómez-Collado, Jordá and Jornet considered in [11] and [12] the ergodic properties of the composition operator on the algebra of the disc $A(\mathbb{D})$ and on the space of holomorphic and bounded functions on the disc $H^\infty(\mathbb{D})$ and the weighted composition operators in the space of holomorphic functions $H(\mathbb{D})$. Arendt, Chalendar, Kummar and Srivastava extended the results of [11] to the convergence of the sequence of powers of operators besides the Cesáro means, including $A(\mathbb{D})$, $H^\infty(\mathbb{D})$ and also the Wiener algebra $W(\mathbb{D})$ [6]. The article [7] by the same authors includes some results close to ours. In [45], Han and Zhou also give some results contained in this thesis, obtained independently from [7] and ourselves.

We work with weighted Bergman spaces of infinite type H_v and H_v^0, Bloch spaces $\mathcal{B}_p$ and $\mathcal{B}_p^0$, Bergman spaces A^p and Hardy spaces H^p. For $p > 0$, the space of Bloch functions or Bloch space of order p is

$$\mathcal{B}_p := \{f \in H(\mathbb{D}) \ : \ \|f\|_{\mathcal{B}_p} := \sup_{z\in\mathbb{D}} (1-|z|^2)^p |f'(z)| < \infty\}.$$

It is a Banach space when endowed with the norm $\|f\| := |f(0)| + \|f\|_{\mathcal{B}_p}$. The closed subspace

$$\mathcal{B}_p^0 := \{f \in H(\mathbb{D}) \ : \ \lim_{|z|\to 1} (1-|z|^2)^p |f'(z)| = 0\}$$

is called the little Bloch space of order p and is also a Banach space with the norm $\|\cdot\|$. The spaces $\mathcal{B}_1$ and $\mathcal{B}_1^0$ are nothing but the classical Bloch spaces. Our study in Bloch spaces is focused on the case $p \geq 1$.

A holomorphic function $v : \mathbb{D} \longrightarrow]0, \infty[$ is called a weight. Associated to a weight v the weighted Bergman spaces of infinite type are

$$H_v := \{f \in H(\mathbb{D}) \ : \ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\},$$

$$H_v^0 := \{f \in H(\mathbb{D}) \ : \ \lim_{|z|\to 1} v(z)|f(z)| = 0\}.$$

Both of these are Banach spaces with the norm $\|\cdot\|_v$. A weight v is called *typical* if v is radial (i.e. $v(z) = v(|z|)$) and decreasing (i.e. $v(z) \leq v(w)$ if $|z| \geq |w|$) and $\lim_{|z|\to 1} v(z) = 0$. The associated weight is defined by

$$\tilde{v}(z) := 1/\|\delta_z\|_{H_v^*}.$$

We consider typical weights satisfying the Lusky condition

$$\inf_{n\in\mathbb{N}} \frac{\tilde{v}(1-2^{-n})}{\tilde{v}(1-2^{-(n-1)})} > 0.$$

For $p \geq 1$, the Bergman space of order p is

$$A^p := \{f \in H(\mathbb{D}) \;:\; \|f\|_{A^p}^p := \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty\},$$

where dA is the normalized Lebesgue measure on $\mathbb{D}$. This is a Banach space with the norm $\|\cdot\|_{A^p}$.

For $p \geq 1$, the Hardy space of order p is

$$H^p := \{f \in H(\mathbb{D}) \;:\; \|f\|_{H^p}^p := \lim_{r \to 1^-} \int_{\mathbb{D}} |f(re^{i\theta})|^p d\theta < \infty\},$$

which is a Banach space with the norm $\|\cdot\|_{H^p}$.

In Section 3.2, we deduce necessary and sufficient conditions for the ergodic properties of the composition operator on general spaces under the assumption of one or many properties. We also study the concept of quasicompactness, introduced as far as we know by Yosida and Kakutani [91]. An operator T is quasicompact if there exist $n \in \mathbb{N}$ and a compact operator K such that $\|T^n - K\| < 1$. We show that quasicompactness is equivalent to $r_e(T) < 1$, where $r_e(T)$ is the essential spectral radius, i.e. the spectral radius of the projection of T to the Calkin algebra.

In Section 3.3 we apply these results in Section 3.2 to classical spaces of holomorphic functions. These results depend deeply on the Denjoy–Wolff point of φ and on quasicompactness. First we show that C_φ is power bounded precisely when φ has one fixed point in the disc. We deduce that if φ is a periodic automorphism then C_φ is uniformly mean ergodic and if it is a non-periodic automorphism then C_φ is mean ergodic and not uniformly mean ergodic. On H_v^0, $\mathcal{B}_p^0$, A^p and H^p mean ergodicity is equivalent to φ having a fixed point $z_0 \in \mathbb{D}$ and if φ is not an automorphism (i.e. z_0 is an interior Denjoy–Wolff point), then mean ergodicity is even equivalent to the weak convergence of the iterates of the operator, i.e. C_φ^n converges to C_{z_0} in the weak operator topology. On H_v and $\mathcal{B}_p$ mean ergodicity is equivalent to uniform mean ergodicity. If φ is not an automorphism and has a Denjoy–Wolff point $z_0 \in \mathbb{D}$, then, on all the spaces, C_φ is uniformly mean ergodic if, and only if, the operator is quasicompact if, and only if, the iterates converge in norm to the evaluation at the fixed point, i.e. $\|C_\varphi^n - C_{z_0}\| \to 0$. The main results of this chapter are Theorem 3.3.7 and Theorem 3.3.8.

Chapter 1

Weighted spaces of continuous functions

1.1 Introduction and notation

We fix the notation for weighted spaces of continuous functions. Throughout this chapter X denotes a Hausdorff, normal, locally compact topological space. Examples of such an X are open subsets of $\mathbb{R}^n$ or discrete spaces, i.e. index sets (this particular case is studied in Chapter 2). The space of continuous functions from X to $\mathbb{C}$ is denoted by $C(X)$ and the locally convex topology on $C(X)$ of pointwise convergence is denoted by τ_p. A function $v \in C(X)$ is called a *weight* if it is strictly positive. A function $f \in C(X)$ *vanishes at infinity* if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ if $x \in X \setminus K$. The weighted spaces of continuous functions are defined by

$$C_v := C_v(X) = \{f \in C(X) \,:\, \|f\|_v := \sup_{x \in X} v(x)|f(x)| < \infty\},$$

$$C_v^0 := C_v^0(X) = \{f \in C(X) \,:\, vf \text{ vanishes at infinity}\},$$

and are Banach spaces when endowed with the norm $\|\cdot\|_v$. Clearly $C_v^0 \subseteq C_v$ and every continuous function with compact support on X is contained in C_v^0. Note that for $v \equiv 1$, C_v coincides with the Banach space $CB(X)$ of bounded continuous functions endowed with the supremum norm and C_v^0 coincides with the space $C_0(X)$ of continuous functions vanishing at infinity, also endowed with the supremum norm. Given $\varphi \in C(X)$, the *multiplication operator* $M_\varphi \in \mathcal{L}(C(X))$ is defined by $M_\varphi : f \mapsto \varphi f$.

Multiplication operators on weighted spaces of (vector valued) continuous functions have been investigated by Manhas [59], [60], Singh and Manhas [61], [77], [79], Oubbi [67], [68] and Klilou and Oubbi [50] among others. See also the book by Singh and Manhas [78].

1.2 Weighted Banach spaces of continuous functions

The aim of this section is to characterise the ergodic properties of the multiplication operator on the weighted spaces C_v and C_v^0 defined above. To ease the reading, the multiplication operator is denoted differently for each space as follows:

$$T_\varphi := M_\varphi|_{C_v} : C_v(X) \longrightarrow C(X)$$

and

$$S_\varphi := M_\varphi|_{C_v^0} : C_v^0(X) \longrightarrow C(X).$$

First we see when the operators satisfy $T_\varphi \in \mathcal{L}(C_v)$ and $S_\varphi \in \mathcal{L}(C_v^0)$. More general results can be seen in [67, Theorem 4], [77, Theorem 2.1] and [79, Theorem 3.1] for example.

Proposition 1.2.1 *Let v, w be weights and $\varphi \in C(X)$. The following assertions hold:*

(i) The operator $M_\varphi : C_v \longrightarrow C_w$ is continuous if, and only if, $\frac{w}{v}\varphi$ is bounded if, and only if, $M_\varphi : C_v^0 \longrightarrow C_w^0$ is continuous. Moreover, in both cases,

$$\|M_\varphi\| = \sup_{x\in X} \frac{w(x)}{v(x)} |\varphi(x)|\,.$$

(ii) $T_\varphi \in \mathcal{L}(C_v)$ if, and only if, $S_\varphi \in \mathcal{L}(C_v^0)$ if, and only if, φ is bounded. Moreover,

$$\|T_\varphi\| = \|S_\varphi\| = \|\varphi\|_\infty.$$

The following lemma gives us the first necessary condition for both power boundedness and mean ergodicity.

Lemma 1.2.2 *Let E be a locally convex space containing the functions with compact support on X such that E is continuously included in $(C(X), \tau_p)$. Assume that $M_\varphi \in \mathcal{L}(E)$. If $\lim_n (M_\varphi^n f)(x)/n = 0$ for every $f \in E$, then $|\varphi(x)| \leq 1$. In particular if M_φ is either power bounded or mean ergodic, then $\|\varphi\|_\infty \leq 1$.*

Proof. If M_φ is power bounded, then $\lim_n (M_\varphi^n f)(x)/n = 0$ for every $f \in E$ and every $x \in X$. The functional equation $M_\varphi^n/n = (M_\varphi)_{[n]} - \frac{n-1}{n}(M_\varphi)_{[n-1]}$ gives us that if M_φ is mean ergodic, then also $\lim_n (M_\varphi^n f)(x)/n = 0$ for every $f \in E$ and every $x \in X$.

Given $x \in X$, select a function g with compact support with $g(x) = 1$. We have $0 = \lim_n \left|(M_\varphi^n g)(x)\right|/n = \lim_n |\varphi(x)|^n/n$. This yields $|\varphi(x)| \leq 1$.

□

Since compactly supported functions are included in $C_v^0 \subseteq C_v$, we get immediately the following result.

Corollary 1.2.3 *Let $\varphi \in CB(X)$. Then, $T_\varphi \in \mathcal{L}(C_v)$ is power bounded if, and only if, $S_\varphi \in \mathcal{L}(C_v^0)$ is power bounded if, and only if, $\|\varphi\|_\infty \leq 1$.*

Remark 1.2.4 Let E be a locally convex space containing the functions with compact support on X such that E is continuously included in $(C(X), \tau_p)$ and assume that $M_\varphi \in \mathcal{L}(E)$. From the very definition of mean ergodicity we get that for each $f \in E$ and $n \in \mathbb{N}$ we have

$$((M_\varphi)_{[n]} f)(x) = \frac{f(x)}{n} \sum_{m=1}^{n} (\varphi(x))^m, \quad x \in X, \tag{1.2.1}$$

and

$$((M_\varphi)_{[n]} f)(x) = \frac{f(x)\varphi(x)}{n} \cdot \frac{1-(\varphi(x))^n}{1-\varphi(x)}, \quad x \in X \setminus \varphi^{-1}(1). \tag{1.2.2}$$

In the case $\|\varphi\|_\infty \leq 1$, these formulas imply the convergence of $(M_\varphi)_{[n]} f$ uniformly on the compact subsets of X to the (not necessarily continuous) function h_f defined by

$$h_f(x) = \begin{cases} f(x) & \text{if } \varphi(x) = 1 \\ 0 & \text{if } \varphi(x) \neq 1. \end{cases} \tag{1.2.3}$$

Proposition 1.2.5 *Let E be a locally convex space containing the functions with compact support on X such that E is continuously included in $(C(X), \tau_p)$. If $M_\varphi \in \mathcal{L}(E)$ and it is mean ergodic, then $\varphi^{-1}(1)$ is open.*

Proof. By Lemma 1.2.2, $\|\varphi\|_\infty \leq 1$. Now suppose $\varphi^{-1}(1)$ is not open, then there exists $x \in X$ such that $\varphi(x) = 1$ but it is not in the interior of $\varphi^{-1}(1)$. Therefore, for each $U \subset X$ open with $x \in U$, there exists $x_U \in U$ with $\varphi(x_U) \neq 1$. These x_U form a net converging to x. Select a function g with compact support such that $g(x) = 1$. By assumption $g \in E$. Since M_φ is mean ergodic, the function h_g defined in Remark 1.2.4 is continuous. However, $h_g(x) = 1$ and $h_g(x_U) = 0$, for every U, which is a contradiction, and $\varphi^{-1}(1)$ is open. □

Remark 1.2.6 Let $\varphi \in CB(X)$ and consider $S_\varphi \in \mathcal{L}(C_v^0)$. Assume that $Z = \varphi^{-1}(1)$ is open. Since Z is also a closed set, then Z and $X \setminus Z$ are both open and closed sets and thus $C_v^0(X) = C_v^0(Z) \oplus C_v^0(X \setminus Z)$. The restriction of S_φ to the first space is the identity operator, therefore we can restrict the proofs to the case $Z = \emptyset$ in X. This also holds for C_v.

The next two theorems are the main results of this section. In them the mean ergodicity and the uniform mean ergodicity of T_φ and S_φ are fully characterised.

Theorem 1.2.7 *Let $\varphi \in CB(X)$. Then, $S_\varphi \in \mathcal{L}(C_v^0)$ is mean ergodic if, and only if, $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.*

Proof. If S_φ is mean ergodic, the conclusion follows from Lemma 1.2.2 and Proposition 1.2.5.

To prove the converse, assume that $\|\varphi\|_\infty \leq 1$ and that $\varphi^{-1}(1)$ is open. By Remark 1.2.6, we may assume $\varphi \neq 1$ in X. From the formula (1.2.2) it follows that $(v\,(S_\varphi)_{[n]}f)_n$ is pointwise convergent to 0 for each $f \in C_v^0$ (even for the compact topology). If $\widehat{X}$ is the Alexandroff compactification of X, the isometry $C_v^0(X) \hookrightarrow C(\widehat{X})$, $f \mapsto vf$, together with Riesz's representation theorem yield that actually $((S_\varphi)_{[n]}f)_n$ is weakly convergent to 0 for every $f \in C_v^0$. Now, since S_φ is power bounded, it follows from the classical mean ergodic theorem (see Corollary A.2.2.(i)) that S_φ is mean ergodic. □

Theorem 1.2.8 *Let $\varphi \in C(X)$ with $\|\varphi\|_\infty \leq 1$. Then the following assertions are equivalent:*

(1) $T_\varphi \in \mathcal{L}(C_v)$ is uniformly mean ergodic,

(2) $S_\varphi \in \mathcal{L}(C_v^0)$ is uniformly mean ergodic,

(3) $\inf\{|\varphi(x) - 1| \,:\, x \in X \setminus \varphi^{-1}(1)\} > 0$, and

(4) $T_\varphi \in \mathcal{L}(C_v)$ is mean ergodic.

Proof. Clearly (1) implies (2) and (4).

To see that (2) implies (3) suppose that S_φ is uniformly mean ergodic. Then, S_φ is mean ergodic and by Theorem 1.2.7, $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open. Thus, by Remark 1.2.6, we may assume $\varphi \neq 1$ in X. In that case it is easy to see that $I - S_\varphi$ is injective. Since S_φ is uniformly mean ergodic, we can apply a theorem by Lin (see Theorem A.2.3) to conclude that $I - S_\varphi$ is also surjective. The continuous operator $I - S_\varphi$ is bijective if, and only if, $\frac{1}{1-\varphi}$ is bounded. This implies $\inf\{|\varphi(x) - 1| \,:\, x \in X\} > 0$.

We show that (3) implies (1). The assumption implies that $\varphi^{-1}(1)$ is open by the continuity of φ, therefore it is enough to consider the case $\varphi \neq 1$ in X. The assumption (3) and the formula (1.2.2) imply that there is $C > 0$ such that for each $f \in C_v$ and each $n \in \mathbb{N}$ we have $\big\|(T_\varphi)_{[n]}f\big\|_v \leq C\|f\|_v/n$. Thus, $\big\|(T_\varphi)_{[n]}\big\| \to 0$ as $n \to \infty$, and T_φ is uniformly mean ergodic.

Finally we prove that (4) implies (3). If T_φ is mean ergodic, also S_φ is mean ergodic and then $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open. Again we can restrict ourselves to the case $\varphi \neq 1$ in X. Assume $\inf\{|\varphi(x) - 1| \, : \, x \in X\} = 0$ and consider the complex polynomials

$$P_n(z) = \frac{1}{n} \sum_{m=1}^{n} z^m, \quad z \in \mathbb{C}, \, n \in \mathbb{N}.$$

Clearly $\lim_{z \to 1} P_n(z) = 1$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $|P_n(\varphi(x_n)) - 1| \leq 1/2$. Thus,

$$|P_n(\varphi(x_n))| \geq 1 - |P_n(\varphi(x_n)) - 1| \geq \frac{1}{2}.$$

Let $f = 1/v \in C_v$. We have $h_f = 0$ in (1.2.3). Then, for $n \in \mathbb{N}$,

$$\|(T_\varphi)_{[n]} f - h_f\|_v = \sup_{x \in X} v(x)|f(x)||P_n(\varphi(x))| \geq |P_n(\varphi(x_n))| \geq \frac{1}{2}.$$

Therefore, T_φ is not mean ergodic, which is a contradiction. □

Remark 1.2.9 We provide here an alternative topological proof of (4) implies (3) in Theorem 1.2.8.

If T_φ is mean ergodic, also S_φ is mean ergodic and then $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open. Again we can restrict ourselves to the case $\varphi \neq 1$ in X.

Let βX be the Stone-Čech compactification of X and denote by $\widehat{g}$ the unique extension of a function $g \in CB(X)$ to βX. Consider the isomorphisms $C_v(X) \longrightarrow CB(X)$, $f \mapsto vf$ and $CB(X) \longrightarrow C(\beta X)$, $h \mapsto \widehat{h}$. Also note that, since βX is compact, $C(\beta X)$ coincides with the space of functions vanishing at infinity $C_0(\beta X)$. Thus we have an isomorphism between $C_v(X)$ and $C_0(\beta X)$. Apply Theorem 1.2.7 to $S_{\widehat{\varphi}} \in \mathcal{L}(C_0(\beta X))$ to find that $(\widehat{\varphi})^{-1}(1)$ is open in βX. Since we have assumed that $\varphi \neq 1$ in X, we must have $(\widehat{\varphi})^{-1}(1) \subset \beta X \setminus X$. But $\beta X \setminus X$ has empty interior and thus $(\widehat{\varphi})^{-1}(1) = \emptyset$. From this follows that $\inf_{x \in X} |1 - \varphi(x)| > 0$.

Remark 1.2.10 If X is discrete, then $C_v = l_\infty(v)$ is a Grothendieck Banach space with the Dunford–Pettis property. In this case a theorem of Lotz (see Theorem A.2.8) implies that T_φ is mean ergodic if, and only if, it is uniformly mean ergodic. This argument cannot be used to prove Theorem 1.2.8 in full generality, since not all $C_v(X)$ spaces have the GDP property.

Example 1.2.11 This example shows a multiplication operator which is mean ergodic but not uniformly mean ergodic: Define $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ as $\varphi(2n) = 1$, $\varphi(2n+1) = (1 - 1/n)$. Let v be a weight on $\mathbb{N}$ and denote $c_0(v) = C_v^0(\mathbb{N})$ and $l_\infty(v) = C_v(\mathbb{N})$. We have that $S_\varphi : c_0(v) \longrightarrow c_0(v)$, $(a_n) \mapsto (\varphi(n)a_n)_n$

is mean ergodic by Theorem 1.2.7, since $\|\varphi\|_\infty \leq 1$ (and $\varphi^{-1}(1)$ is always open in $\mathbb{N}$). However S_φ is not uniformly mean ergodic by Theorem 1.2.8 since $\inf\{|\varphi(2n+1)-1| \,:\, n \in \mathbb{N}\} = 0$. For the same reason, $T_\varphi : l_\infty(v) \to l_\infty(v)$, $(a_n) \mapsto (\varphi(n)a_n)$ is not even mean ergodic.

1.3 Weighted inductive limits of spaces of continuous functions

Weighted inductive limits of spaces of continuous and holomorphic functions have been thoroughly investigated since the seminal work of Bierstedt, Meise and Summers [23] and [24]; see [15], [16], [18], [20] and the references therein. In this section we characterize power bounded and (uniformly) mean ergodic multiplication operators defined on weighted inductive limits of spaces of continuous functions defined on a normal locally compact Hausdorff topological space X.

Throughout this section $V = (v_k)_k$ denotes a decreasing sequence of continuous strictly positive weights. The weighted inductive limits associated to V are $VC = VC(X) := \mathrm{ind}_k\, C_{v_k}$ and $V_0C = V_0C(X) := \mathrm{ind}_k\, C^0_{v_k}$. They are Hausdorff (LB)-spaces and $V_0C \subset VC$ is a topological subspace of VC by [23, Theorem 1.3]. In case X is discrete, these spaces are precisely Köthe co-echelon spaces of order infnity and zero, [24] (also Chapter 2).

The *Nachbin family* associated to V is

$$\overline{V} := \{\overline{v} : X \longrightarrow (0,\infty) \;: \overline{v} \text{ is upper semicontinuous and for each } k \in \mathbb{N}, \quad \frac{\overline{v}}{v_k} \text{ is bounded on } X\}.$$

The weighted spaces of continuous functions associated with VC and V_0C are defined as follows.

$$C\overline{V} = \{f \in C(X) \;:\; p_{\overline{v}}(f) := \sup_{x\in X} \overline{v}(x)\,|f(x)| < \infty, \forall \overline{v} \in \overline{V}\}.$$

$$C\overline{V}_0 = \{f \in C(X) \;:\; \overline{v}f \text{ vanishes at infinity}, \forall \overline{v} \in \overline{V}\}.$$

They are endowed with the locally convex topology generated by the seminorms $p_{\overline{v}}, \overline{v} \in \overline{V}$. It is well known that $VC = C\overline{V}$ algebraically and the topology in $C\overline{V}$ is in general coarser, but they share bounded sets; in fact every bounded subset of $C\overline{V}$ is contained and bounded in a step C_{v_n}. Moreover, $C\overline{V}_0$ is a closed subspace of $C\overline{V}$ and V_0C is a topological subspace of $C\overline{V}_0$, hence of VC. We refer the reader to [23] for these results. More relations between these spaces arise when extra properties are assumed on V. These properties can be seen in the references already mentioned and are summarized in Section A.3.

Proposition 1.3.1 *The following assertions are equivalent:*

(1) $M_\varphi : VC(X) \longrightarrow VC(X)$ *is continuous.*

(2) For every k*, there exists* l*, such that* $M_\varphi : C_{v_k} \longrightarrow C_{v_l}$ *is continuous.*

(3) For every k*, there exists* l*, such that* $\frac{v_l}{v_k}\varphi$ *is bounded on* X*,*

(4) $M_\varphi : V_0C(X) \longrightarrow V_0C(X)$ *is continuous,*

(5) for every k*, there exists* l*, such that* $M_\varphi : C^0_{v_k} \longrightarrow C^0_{v_l}$ *is continuous.*

Proof. The equivalence of (2), (3) and (5) follows from Proposition 1.2.1.

We show (1) $\iff$ (2) Let $i_k : C_{v_k} \longrightarrow VC$ be the natural continuous injections. We claim that $M_\varphi : VC \longrightarrow VC$ is continuous if, and only if, for all $k \in \mathbb{N}$, $M_\varphi \circ i_k : C_{v_k} \longrightarrow VC$ is continuous. Indeed, if M_φ is continuous, then $M_\varphi \circ i_k$ is continuous for all $k \in \mathbb{N}$. Conversely, assume $M_\varphi \circ i_k$ is continuous, for all $k \in \mathbb{N}$. Fix an absolutely convex neighbourhood $W \subseteq VC$ of 0. Then for each $k \in \mathbb{N}$ there exists an absolutely convex neighbourhood $U_k \subseteq C_{v_k}$ of 0 such that $T \circ i_k(U_k) = T(U_k) \subseteq W$. Let $U = \Gamma(\bigcup_{k\in\mathbb{N}} U_k)$ be the absolutely convex hull of the union of the U_k. Then U is an absolutely convex neighbourhood of 0 in VC and $T(U) \subseteq W$, therefore $T : VC \longrightarrow VC$ is continuous.

Now, for each $k \in \mathbb{N}$, $M_\varphi \circ i_k : C_{v_k} \longrightarrow VC$ is continuous if, and only if, there exists $l = l(k) \in \mathbb{N}$ such that $M_\varphi \circ i_k(C_{v_k}) \subseteq C_{v_l}$ and such that even $M_\varphi \circ i_k : C_{v_k} \longrightarrow C_{v_l}$ is continuous, by Grothendieck's factorization theorem.

The equivalence of (4) and (5) is similar. □

Proposition 1.3.2 $M_\varphi \in \mathcal{L}(VC)$ *is power bounded if, and only if, the operator* $M_\varphi \in \mathcal{L}(V_0C)$ *is power bounded, if, and only if,* $\|\varphi\|_\infty \leq 1$.

Proof. The necessity of $\|\varphi\|_\infty \leq 1$ for both VC and V_0C follows from Lemma 1.2.2. For the sufficiency assume $\|\varphi\|_\infty \leq 1$. Since both V_0C and VC are barrelled spaces, by the Banach–Steinhaus theorem it is enough to show that every orbit $((M_\varphi)^n f)_n$ is bounded for each f in the space. Fix $f \in V_0C$. There exist k and $\lambda > 0$ such that $f \in \lambda B_k$, where B_k is the closed unit ball of C_{v_k}. This implies $(M_\varphi)^n f \in \lambda B_k \cap V_0C$ for each n. Now it is enough to keep in mind that $(kB_k \cap V_0C)_k$ is a fundamental system of bounded sets of V_0C by [23, Theorem 1.3]. The case of VC is similar. □

Proposition 1.3.3 $M_\varphi \in \mathcal{L}(V_0C)$ *is mean ergodic if, and only if,* $\|\varphi\|_\infty \leq 1$ *and* $\varphi^{-1}(1)$ *is open.*

Proof. If $M_\varphi \in \mathcal{L}(V_0C)$ is mean ergodic, then by Lemma 1.2.2 and Proposition 1.2.5, we get the necessary conditions.

Conversely, if $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open, then $S_\varphi \in \mathcal{L}(C^0_{v_k})$ is mean ergodic for every $k \in \mathbb{N}$ by Theorem 1.2.7. By the properties of inductive limits, it follows that $M_\varphi \in \mathcal{L}(V_0C)$ is mean ergodic. □

The next technical lemma was proved in [19, Proposition 3] and it is very useful in our setting.

Lemma 1.3.4 *There is a fundamental system $\mathcal{U}$ of neighbourhoods of 0 for VC such that if $U \in \mathcal{U}$ and $f \in U$, then, for every $g \in VC$ with $|g| \leq |f|$, one gets $g \in U$.*

Proposition 1.3.5 *The following assertions are equivalent:*

(1) $M_\varphi \in \mathcal{L}(VC)$ is mean ergodic.

(2) $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic.

Furthermore, if these equivalent conditions hold, then $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open, h_f as in (1.2.3) belongs to VC for each $f \in VC$, and $(M_\varphi)_{[n]} \to P$ in $\mathcal{L}_b(VC)$, where $P \in \mathcal{L}(VC)$ with $P(f) = h_f$.

Proof. It suffices to show that M_φ is uniformly mean ergodic whenever it is mean ergodic. Let B_k denote the closed unit ball of C_{v_k}. By [23] every bounded set of VC is contained in a multiple of some B_k (see Definition A.3.3). Since M_φ is mean ergodic, h_g belongs to VC for each $g \in VC$. Fix k. We have to show that $(M_\varphi)_{[n]}g - h_g$ converges to 0 uniformly on $g \in B_k$. Set $f := 1/v_k$. If $U \in \mathcal{U}$, where $\mathcal{U}$ is the basis of neighbourhoods of VC in Lemma 1.3.4, there exists n_0 such that for $n \geq n_0$, $(M_\varphi)_{[n]}f - h_f \in U$, since M_φ is mean ergodic. Every $g \in B_k$ satisfies $|g| \leq f$, therefore $|(M_\varphi)_{[n]}g| \leq |(M_\varphi)_{[n]}f|$ and we have, for $x \notin \varphi^{-1}(1)$,

$$\begin{aligned}|((M_\varphi)_{[n]}g)(x) - h_g(x)| &= |((M_\varphi)_{[n]}g)(x)| \\ &\leq |((M_\varphi)_{[n]}f)(x)| = |((M_\varphi)_{[n]}f)(x) - h_f(x)|,\end{aligned}$$

and for $x \in \varphi^{-1}(1)$,

$$|((M_\varphi)_{[n]}g)(x) - h_g(x)| = |g(x) - g(x)| = 0 = |((M_\varphi)_{[n]}f)(x) - h_f(x)|.$$

Therefore $|(M_\varphi)_{[n]}g - h_g| \leq |(M_\varphi)_{[n]}f - h_f|$ and Lemma 1.3.4 yields $(M_\varphi)_{[n]}g - h_g \in U$, for $n \geq n_0$.

The consequences of conditions (1) and (2) in the statement now follow from arguments in Lemma 1.2.2, Remark 1.2.4 and Proposition 1.2.5. □

Although M_φ is mean ergodic if, and only if, it is uniformly mean ergodic on VC, these conditions do not necessarily imply

$$\inf\{|\varphi(x)-1| \,:\, x\in X\setminus\varphi^{-1}(1)\}>0.$$

Compare with Theorem 1.2.8. The first examples can be obtained assuming a condition on the sequence V. This condition is called (S) and characterizes $VC=V_0C$ (see Theorem A.3.1 and [23]).

Corollary 1.3.6 *Assume that the sequence V satisfies condition (S). The following conditions are equivalent for $M_\varphi\in\mathcal{L}(VC)$:*

(1) $M_\varphi\in\mathcal{L}(VC)$ is uniformly mean ergodic.

(2) $M_\varphi\in\mathcal{L}(VC)$ is mean ergodic.

(3) $\|\varphi\|_\infty\le 1$ and $\varphi^{-1}(1)$ is open.

Proof. It is a consequence of Proposition 1.3.3 and Proposition 1.3.5, considering Theorem A.3.1. □

Now we characterize when M_φ is uniformly mean ergodic on V_0C.

Theorem 1.3.7 *The following assertions are equivalent:*

(1) $M_\varphi\in\mathcal{L}(V_0C)$ is uniformly mean ergodic.

(2) $\|\varphi\|_\infty\le 1$, $\varphi^{-1}(1)$ is open in X and for every $k\in\mathbb{N}$ and every $\overline{v}\in\overline{V}$,

$$\lim_{n\to\infty}\sup_{x\in Y}\frac{1}{n}\frac{\overline{v}(x)}{v_k(x)}|\varphi(x)|\frac{|1-\varphi(x)^n|}{|1-\varphi(x)|}=0,$$

where $Y=X\setminus\varphi^{-1}(1)$.

Proof. It is enough to prove the result when $\varphi(x)\neq 1$ for each $x\in X$. Thus we assume $X=Y$. We set, for each $k,n\in\mathbb{N}$, $\overline{v}\in\overline{V}$ and $x\in X$,

$$R_n^{k,\overline{v}}(x):=\frac{1}{n}\frac{\overline{v}(x)}{v_k(x)}|\varphi(x)|\frac{|1-\varphi(x)^n|}{|1-\varphi(x)|}.$$

Assume first that condition (2) holds. Fix $k\in\mathbb{N}$. For each $f\in V_0C$ such that $|f|\le 1/v_k$ and each $\overline{v}\in\overline{V}$ we have

$$\overline{v}(x)\left|(M_\varphi)_{[n]}f(x)\right|\le R_n^{k,\overline{v}}(x),$$

for each $n\in\mathbb{N}$ and each $x\in X$. Given $\varepsilon>0$ we apply (2) to find $n(0)\in\mathbb{N}$ such that $R_n^{k,\overline{v}}(x)<\varepsilon$ for each $x\in X$ and each $n\ge n(0)$. Hence $\overline{v}(x)\left|(M_\varphi)_{[n]}f(x)\right|<\varepsilon$

for each $x \in X, n \geq n(0)$ and each $f \in V_0C$ such that $|f| \leq 1/v_k$. By [23, Theorem 1.3] V_0C is a topological subspace of $C\overline{V}_0$ and every bounded set of V_0C is contained in a multiple of $\{f \in V_0C \ : \ |f| \leq 1/v_k\}$ for some k. Therefore $((M_\varphi)_{[n]})_n$ converges to 0 in $\mathcal{L}_b(V_0C)$, and M_φ is uniformly mean ergodic.

To prove the converse, suppose that $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic. Fix $k \in \mathbb{N}$ and $\overline{v} \in \overline{V}$. Given $\varepsilon > 0$ there is $n(0) \in \mathbb{N}$ such that $\overline{v}(x)\left|(M_\varphi)_{[n]}f(x)\right| < \varepsilon$ for each $x \in X, n \geq n(0)$ and each $f \in V_0C$ such that $|f| \leq 1/v_k$. Now, for an arbitrary $z \in X$, there is a continuous function with compact support $h \in C(X)$ such that $0 \leq h \leq 1$ and $h(z) = 1$. Then $g := h/v_k \in V_0C$ and $|g| \leq 1/v_k$ on X. This implies, for $n \geq n(0)$,

$$R_n^{k,\overline{v}}(z) = \frac{\overline{v}(z)}{k}\left|(M_\varphi)_{[n]}g(z)\right| < \varepsilon.$$

Since $z \in X$ is arbitrary, we have shown that $\lim_{n\to\infty}\sup_{z\in X} R_n^{k,\overline{v}}(z) = 0$. The other statements in condition (2) follow from Proposition 1.3.3. □

In general, working with $\overline{V}$ to check the uniform mean ergodicity of specific examples might be complicated. Our aim for the rest of the section is to solve this problem, replacing the Nachbin family by the easier known family of weights V. This leads to the following characterization in terms of the family of weights, under the assumption that V is regularly decreasing (see Theorem A.3.4).

Corollary 1.3.8 *If $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ there exists $l \geq k$ such that*

$$\lim_{n\to\infty}\sup_{x\in Y}\frac{1}{n}\frac{v_l(x)}{v_k(x)}|\varphi(x)|\frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = 0,$$

then $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic. Furthermore, if V is regularly decreasing, the converse also holds.

Proof. This is a consequence of Theorem 1.3.7 since, for every $\overline{v} \in \overline{V}$ and each $l \in \mathbb{N}$, there exists $\alpha_l > 0$ such that $\overline{v} \leq \alpha_l v_l$.

For the converse, assume that V is regularly decreasing. As in Theorem 1.3.7, consider the bounded sets $A_k = \{f \in V_0C \ : \ |f| \leq 1/v_k\}$, $k \in \mathbb{N}$. Since V_0C is strongly boundedly rectractive by Theorem A.3.4 and for each k, A_k is bounded in C_{v_k} we have that, for each k, there exists $l \geq k$ such that uniform convergence in A_k for the topology of V_0C and for the topology of C_{v_l} is the same. Then, since M_φ is uniformly mean ergodic, for each k there exists $l \geq k$ such that for each $\varepsilon > 0$ there exists $n(0) \in \mathbb{N}$ such that $v_l(x)|(M_\varphi)_{[n]}f(x)| < \varepsilon$, for each $x \in X, n \geq n(0)$ and each $f \in A_k$. To conclude, use compactly supported functions in A_k as in the proof of Theorem 1.3.7. □

For further results we may consider the topological identity $VC = C\overline{V}$. This condition is characterized by the property (D) on V (see Theorem A.3.5).

Corollary 1.3.9 *Assume that $VC = C\overline{V}$ holds topologically. Then the operator $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic if, and only if, condition (2) of Theorem 1.3.7 holds.*

Proof. If $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic, then $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic, since V_0C is a topological subspace of VC. We can apply Theorem 1.3.7 to conclude that condition (2) in this theorem holds.

Conversely, assume $VC = C\overline{V}$ algebraically and topologically. The topology of $C\overline{V}$ is given by the family $\{p_{\overline{v}} : \overline{v} \in \overline{V}\}$, therefore M_φ is uniformly mean ergodic if for each bounded set $B \subseteq VC$ and each $\overline{v} \in \overline{V}$,

$$\lim_{n\to\infty} \sup_{f\in B} p_{\overline{v}}((M_\varphi)_{[n]}f - h_f) = 0.$$

Let $B \subseteq VC$ be a bounded set, then by the regularity of VC (Definition A.3.3), there exist $k \in \mathbb{N}$ and $M > 0$ such that

$$B \subseteq \{f \in C_{v_k} : v_k|f| \leq M\}.$$

For $n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$, we have

$$\begin{aligned}
\sup_{f\in B} p_{\overline{v}}((M_\varphi)_{[n]}f - h_f) &= \sup_{f\in B}\sup_{x\in X} \overline{v}(x)|((M_\varphi)_{[n]}f)(x) - h_f(x)| \\
&= \sup_{f\in B}\sup_{x\in Y} \overline{v}(x)|((M_\varphi)_{[n]}f)(x)| \\
&= \sup_{f\in B}\sup_{x\in Y} \frac{\overline{v}(x)}{n}|f(x)||\varphi(x)|\frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} \\
&\leq \sup_{x\in Y} \frac{\overline{v}(x)}{n}\frac{M}{v_k(x)}|\varphi(x)|\frac{|1-\varphi(x)^n|}{|1-\varphi(x)|}.
\end{aligned}$$

Then by condition (2) in Theorem 1.3.7, we deduce the uniform mean ergodicity of M_φ.

□

Corollary 1.3.10 *Assume that $\|\varphi\|_\infty \leq 1$ and $\varphi(x) \neq 1$ for each $x \in X$.*

(i) If for all k there is $l > k$ such that

$$\sup_{x\in X} \frac{v_l(x)}{v_k(x)} \frac{1}{|1-\varphi(x)|} < \infty,$$

then $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic.

(ii) If $\inf\{|1-\varphi(x)| : x \in X\} > 0$, then $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic.

Proof. The hypothesis of part (i) implies the assumption in Corollary 1.3.8. Part (ii) follows from (i). □

Examples 1.3.11 We mention two examples of sequences $V = (v_n)_n$ which do not satisfy condition (S) and uniformly mean ergodic multiplication operators $M_\varphi \in \mathcal{L}(V_0C)$ such that $\inf\{|1-\varphi(x)| \,:\, x \in X \setminus \varphi^{-1}(1)\} > 0$ is not satisfied. These examples show that the converse of Corollary 1.3.10 does not hold and they should be compared with Theorem 1.2.8 and Corollary 1.3.6.

(1) Let $X = \mathbb{D}$ be the complex unit disc. Consider $V = (v_k)_k$ with $v_k(z) = \min(1, |1-z|^k)$. The sequence V does not satisfy condition (S), since $v_k(-1+a) = 1$ for each $a \in]0,1[$. Take $\varphi(z) = z$. Clearly $\varphi(z) \neq 1$ for each $z \in \mathbb{D}$, but $\inf\{|1-\varphi(x)| \,:\, x \in X\} = 0$.

On the other hand, for any $n \in \mathbb{N}$ we have, for $|1-z| \leq 1$,

$$\frac{v_{k+1}(z)}{v_k(z)} \cdot \frac{1}{|1-\varphi(z)|} = \frac{|1-\varphi(z)|^{k+1}}{|1-\varphi(z)|^{k+1}} = 1,$$

and for $|1-z| \geq 1$,

$$\frac{v_{k+1}(z)}{v_k(z)} \cdot \frac{1}{|1-\varphi(z)|} = \frac{1}{|1-z|} \leq 1.$$

We can apply Corollary 1.3.10.(i) to conclude that M_φ is uniformly mean ergodic in V_0C.

(2) Let $X = \{z \in \mathbb{C} \;:\; \mathrm{Im} z > 0\}$ be the upper half plane in $\mathbb{C}$. Define $v_k(z) := \exp(-k\mathrm{Im} z), z \in X, k \in \mathbb{N}$, and $V = (v_k)_k$. It is easy to check that the sequence V does not satisfy condition (S). The function $\varphi(z) = 1 - e^{iz}$, $z \in X$, satisfies $\varphi(z) \neq 1$ for each $z \in X$ and $\inf\{|1-\varphi(x)| \,:\, x \in X\} = 0$.

We have

$$\frac{v_{k+1}(z)}{v_k(z)} \cdot \frac{1}{|1-\varphi(z)|} = 1.$$

Therefore M_φ is uniformly mean ergodic in VC by Corollary 1.3.10 (i).

(3) It is not hard to show that both sequences V in the examples above are regularly decreasing in the sense of [23, Definition 2.1] (see Theorem A.3.4). This implies that $VC = C\overline{V}$ holds topologically and the multiplication operators M_φ are also uniformly mean ergodic on VC.

1.4 Weighted projective limits of spaces of continuous functions

Throughout this section $A = (a_k)_k$ denotes an increasing family of continuous strictly positive weights on a normal locally compact Hausdorff topological space X and $V = (1/a_k)_k$ denotes the corresponding decreasing family. The weighted projective limits associated to A are $CA = CA(X) := \mathrm{proj}_k C_{a_k}$ and $CA_0 = CA_0(X) := \mathrm{proj}_k C^0_{a_k}$. They are Fréchet spaces, where the seminorms are $\|\cdot\|_k = \|\cdot\|_{a_k}$ and $CA_0 \subset CA$ is a topological subspace of CA.

The Nachbin family $\overline{V}$ associated to A is the one associated to V (Section 1.3).

As in Section 1.3, we start with the continuity of the multiplication operator M_φ, with $\varphi \in C(X)$ a continuous function.

Proposition 1.4.1 *The following assertions are equivalent:*

(1) $M_\varphi : CA(X) \longrightarrow CA(X)$ is continuous.

(2) For every k, there exists l such that $M_\varphi : C_{a_l} \longrightarrow C_{a_k}$ is continuous.

(3) For every k, there exists l such that $\frac{a_k}{a_l}\varphi$ is bounded on X.

(4) $M_\varphi : CA_0(X) \longrightarrow CA_0(X)$ is continuous.

(5) For every k, there exists l such that $M_\varphi : C^0_{a_l} \longrightarrow C^0_{a_k}$ is continuous.

Proof. The equivalence of (2), (3) and (5) is a consequence of Proposition 1.2.1.

Now assume (3) holds, we prove (1). Fix k and let l as in (3), then $a_k|\varphi| \leq M a_l$, for some $M > 0$. We have

$$\|M_\varphi f\|_k = \sup_{x\in X} a_k(x)|\varphi(x)||f(x)| \leq M \sup_{x\in X} a_l(x)|f(x)| = M\|f\|_l,$$

for every $f \in CA$. This shows (1).

Assume (1), we show (4). Denote by $H \subseteq CA(X)$ the set of compactly supported functions. Then $\overline{H} = CA_0$. Since $\varphi f \in H$ for every $f \in H$, we have $M_\varphi H \subseteq H$. From the continuity of M_φ on CA, we deduce

$$M_\varphi(CA_0) = M_\varphi(\overline{H}) \subseteq \overline{M_\varphi H} \subseteq \overline{H} = CA_0,$$

and the continuity of M_φ on CA_0.

We conclude assuming (4) and proving (5). We have that for every k, there exists l such that

$$\|M_\varphi f\|_k \leq M\|f\|_l,$$

for some $M > 0$ and for every $f \in CA_0$ with compact support. This implies $M_\varphi(C^0_{a_l}) \subseteq C^0_{a_k}$ since the set of compactly supported functions is dense in every $C^0_{a_l}$. This concludes the proof using the closed graph theorem. □

Proposition 1.4.2 *$M_\varphi \in \mathcal{L}(CA)$ is power bounded if, and only if, the operator $M_\varphi \in \mathcal{L}(CA_0)$ is power bounded, if, and only if, $\|\varphi\|_\infty \le 1$.*

Proof. The necessity of $\|\varphi\|_\infty \le 1$ for both CA and CA_0 follows from Lemma 1.2.2. For the sufficiency, assume $\|\varphi\|_\infty \le 1$, then $\|M_\varphi^n f\|_k \le \|f\|_k$ for every $n, k \in \mathbb{N}$, thus $\{M_\varphi^n\}_{n=1}^\infty$ is an equicontinuous set of $\mathcal{L}(X)$, and M_φ is power bounded. □

Proposition 1.4.3 *$M_\varphi \in \mathcal{L}(CA_0)$ is mean ergodic if, and only if, $\|\varphi\|_\infty \le 1$ and $\varphi^{-1}(1)$ is open.*

Proof. If $M_\varphi \in \mathcal{L}(CA_0)$ is mean ergodic, then by Lemma 1.2.2 and Proposition 1.2.5, we get the necessary conditions.

Conversely, if $\|\varphi\|_\infty \le 1$ and $\varphi^{-1}(1)$ is open, then $S_\varphi \in \mathcal{L}(C^0_{a_k})$ is mean ergodic for every $k \in \mathbb{N}$ by Theorem 1.2.7. Then, for each $f \in CA_0$, $\lim_n \|(M_\varphi)_{[n]} f - h_f\|_k = \lim_n \|(S_\varphi)_{[n]} f - h_f\|_{a_k} = 0$, for each k, where $h_f \in CA_0$ is as in (1.2.3). This implies the mean ergodicity of $M\varphi$. □

Next we introduce known results regarding the Nachbin family $\overline{V}$. They are used later in the characterizations of uniform mean ergodicity.

Lemma 1.4.4 *If X is also σ-compact, then for each $\overline{v} \in \overline{V}$ there exists $\tilde{v} \in C(X) \cap \overline{V}$ such that $\overline{v} \le \tilde{v}$.*

Proof. The proof can be found in [23], we include it here for completeness. Fix $\overline{v} \in \overline{V}$. By the very definition of $\overline{V}$, there is a sequence $(\alpha_n)_n \subset \mathbb{R}$ with $\alpha_n > 0$ and such that $\overline{v} \le \alpha_n v_n$, for $n = 1, 2, \dots$ Since X is σ-compact, there exists a family $\{K_m : m \in \mathbb{N}\}$ of compact sets of X satisfying $K_m \subset K^\circ_{m+1}$, where K°_{m+1} denotes the interior of K_{m+1}, and $X = \bigcup_{m\in\mathbb{N}} K_m$.

Inductively choose positive numbers $\beta_n \ge \alpha_n$ by $\beta_1 = \alpha_1$ and for every $n = 2, 3, \dots$,

$$\beta_n \inf\{v_n(x) : x \in K_n\} \ge \sup\{\beta_j v_j(x) : x \in K_{n-1},\ j = 1, \dots, n-1\}.$$

Now define $\tilde{v} := \inf_{n\in\mathbb{N}} \beta_n v_n$. Clearly $\tilde{v} \in \overline{V}$ and $\overline{v} \le \tilde{v}$. We prove it is even continuous. Fix $m \in \mathbb{N}$, then for $n = m+1, m+2, \dots$, we have

$$\begin{aligned} \inf_{x\in K_m} \beta_n v_n(x) &\ge \beta_n \inf_{x\in K_n} v_n(x) \\ &\ge \sup\{\beta_j v_j(x) : x \in K_{n-1},\ j = 1, \dots, n-1\} \\ &\ge \sup\{\beta_j v_j(x) : x \in K_m,\ j = 1, \dots, n-1\}. \end{aligned}$$

Therefore, for $x \in K_m$ we have $\beta_n v_n(x) \ge \beta_m v_m(x)$, for $n = m+1, m+2, \dots$, and thus $\tilde{v}|_{K_m} = \inf_{j=1,\dots,m} \beta_j v_j$. Since all the weights v_j are continuous, $\tilde{v}$ is continuous on K_m. As $m \in \mathbb{N}$ was arbitrary, we conclude the proof. □

Lemma 1.4.5 *A set $B \subset CA$ is bounded if, and only if, there exists $\overline{v}_0 \in \overline{V}$ such that $B \subset B_{\overline{v}_0} := \{f \in C(X) \,:\, |f| \leq \overline{v}_0\}$.*

Proof. Let I denote X but with the discrete topology (as a set of indices). Then $\lambda_\infty(I, A) := \bigcap_k \ell_\infty(a_k)(I)$ and CA is a topological subspace of $\lambda_\infty(I, A)$. By [24, Proposition 2.5] (see also Lemma 2.2.7), a set B is bounded in $\lambda_\infty(I, A)$ if, and only if, there exists

$$\overline{v}_1 \in \lambda_\infty(I, A)_+ := \{\overline{v} \in \lambda_\infty(I, A) \,:\, \overline{v} \geq 0\}$$

such that $B \subset B_{\overline{v}_1}$. Thus, assume $B \subset CA$ is bounded, then it is bounded in $\lambda_\infty(I, A)$ and therefore $B \subset B_{\overline{v}_1}$, for some $\overline{v}_1 \in \lambda_\infty(I, A)_+$. Hence, there are $\alpha_n > 0$ such that $\overline{v}_1 a_n < \alpha_n$, for $n \in \mathbb{N}$. Let $\overline{v}_0 = \inf_n \alpha_n / a_n$. Then $\overline{v}_0$ is in $\overline{V}$ as it is upper semicontinuous. Since $\overline{v}_1 \leq \overline{v}_0$, we deduce $B \subseteq B_{\overline{v}_1} \subseteq B_{\overline{v}_0}$.

For the other implication, fix $\overline{v}_0 \in \overline{V}$ and let $B \subset B_{\overline{v}_0}$. Let $k \in \mathbb{N}$, $x \in X$ and $b \in B$, then $a_k(x)|b(x)| \leq a_k(x)\overline{v}_0(x)$, which is bounded by the definition of $\overline{V}$. Then B is bounded in CA. □

The next result characterizes uniform mean ergodicity in the projective case and should be compared with Proposition 1.3.5, Theorem 1.3.7 and, more precisely, with Corollary 1.3.9.

Theorem 1.4.6 *If X is also σ-compact, then the following assertions are equivalent:*

(1) $M_\varphi \in \mathcal{L}(CA)$ is mean ergodic,

(2) $M_\varphi \in \mathcal{L}(CA)$ is uniformly mean ergodic,

(3) $M_\varphi \in \mathcal{L}(CA_0)$ is uniformly mean ergodic,

(4) $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for each $k \in \mathbb{N}$ and each $\overline{v} \in \overline{V}$,

$$\lim_{n\to\infty} \sup_{x\in Y} \frac{a_k(x)\overline{v}(x)|\varphi(x)|}{n} \cdot \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = 0,$$

where $Y = X \setminus \varphi^{-1}(1)$.

Proof. Clearly (2) implies (1) and (3). Assume (1), we show (4). Fix $k \in \mathbb{N}$ and $\overline{v} \in \overline{V}$. By Lemma 1.4.4, there exists $\tilde{v} \in C(X) \cap \overline{V}$ such that $\overline{v} \leq \tilde{v}$. Since $\tilde{v} \in CA$ and M_φ is mean ergodic in CA,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{x\in Y} \frac{a_k(x)\overline{v}(x)|\varphi(x)|}{n} \cdot \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} \\
&\leq \lim_{n\to\infty} \sup_{x\in Y} \frac{a_k(x)\tilde{v}(x)|\varphi(x)|}{n} \cdot \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = 0,
\end{aligned}$$

and we conclude.

Now, assume (4) to deduce (2). Let $B \subset CA$ be bounded and let $\overline{v}_0 \in \overline{V}$ be as in Lemma 1.4.5. Define $P \in \mathcal{L}(CA)$ by $Pf = h_f$, where h_f is as in (1.2.3). Fix k, n, then

$$\sup_{f \in B} \|(M_\varphi)_{[n]} f - Pf\|_k \le \sup_{f \in B} \sup_{x \in Y} \frac{a_k(x)|f(x)||\varphi(x)|}{n} \cdot \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|}$$
$$\le \sup_{x \in Y} \frac{a_k(x)\overline{v}_0(x)|\varphi(x)|}{n} \cdot \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|},$$

which converges to 0 by assumption and thus M_φ is uniformly mean ergodic.

It only remains to show that (3) implies (4). To do so fix k and $\overline{v} \in \overline{V}$. For each $x \in Y$ define a compactly supported function f_x with $f_x(x) = \overline{v}(x)$ and $0 \le f_x \le \overline{v}(x)$, then $f_x \in B_{\overline{v}}^0 := B_{\overline{v}} \cap CA_0$ for each $x \in Y$. By uniform mean ergodicity, for each $\varepsilon > 0$ there exists $n(0)$ such that for $n \ge n(0)$,

$$\sup_{f \in B_{\overline{v}}^0} \|(M_\varphi)_n f - Pf\|_k < \varepsilon.$$

We conclude considering that for each $x \in Y$ and $n \ge n(0)$,

$$\frac{a_k(x)\overline{v}(x)|\varphi(x)|}{n} \cdot \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = \|(M_\varphi)_n f_x - Pf_x\|_k < \varepsilon.$$

□

Corollary 1.4.7 *If X is also σ-compact, $\|\varphi\|_\infty \le 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ there exists $l \ge k$ such that*

$$\lim_{n\to\infty} \sup_{x \in Y} \frac{1}{n} \frac{a_k(x)}{a_l(x)} |\varphi(x)| \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = 0,$$

then $M_\varphi \in \mathcal{L}(CA_0)$ is uniformly mean ergodic.

Proof. This is a consequence of Theorem 1.4.6 since, for every $\overline{v} \in \overline{V}$ and each $l \in \mathbb{N}$, there exists $\alpha_l > 0$ such that $\overline{v} \le \alpha_l v_l = \frac{\alpha_l}{a_l}$. □

Corollary 1.4.8 *Assume that X is also σ-compact, $\|\varphi\|_\infty \le 1$ and $\varphi(x) \neq 1$ for each $x \in X$.*

(i) If for all k there is $l > k$ such that

$$\sup_{x \in X} \frac{a_k(x)}{a_l(x)} \frac{1}{|1-\varphi(x)|} < \infty,$$

then $M_\varphi \in \mathcal{L}(CA_0)$ is uniformly mean ergodic.

(ii) If $\inf\{|1-\varphi(x)| \; : \; x \in X\} > 0$, then $M_\varphi \in \mathcal{L}(CA_0)$ is uniformly mean ergodic.

Proof. The hypothesis of part (i) implies the assumption in Corollary 1.4.7. Part (ii) follows from (i). □

Example 1.4.9 Set $X = \mathbb{N}$, $a_n(i) = i^n$ and $A = (a_n)_n$. Then (see Chapter 2), $CA(\mathbb{N}) = \lambda_\infty(A) = \Lambda_\infty^\infty((\log i)_i)$ is the space of all rapidly decreasing functions. Let $\varphi(i) = 1 - 2^{-i}$. Then $M_\varphi \in \mathcal{L}(CA(\mathbb{N}))$. In [2, Example 2.17] the authors show that M_φ is uniformly mean ergodic. However a simple computation shows that the equation in Corollary 1.4.8.(i) does not hold. Indeed, for all $k \in \mathbb{N}$ and all $l \geq k$,

$$\sup_{i\in\mathbb{N}} i^{k-l} 2^i = \infty.$$

Therefore the converse of Corollary 1.4.8.(i) is not true. Since

$$\inf_{i\in\mathbb{N}} 2^{-i} = 0,$$

the converse of 1.4.8.(ii) does not hold either.

1.5 Spectrum of the multiplication operator

We devote this section to compute the pointwise spectrum, the spectrum and the Waelbroeck spectrum of the multiplication operator defined on the spaces considered in the previous sections.

Let $V = (v_k)_k$ be a decreasing family of weights and denote the corresponding increasing family by $A = (a_k)_k = (1/v_k)_k$. Along this section φ is a continuous function. The results of this section hold when M_φ is a continuous operator on the spaces considered.

Lemma 1.5.1 *Let E be either C_v, C_v^0, VC, V_0C, CA or CA_0, for a weight v. Then*

$$\sigma_{pt}(M_\varphi, E) \subset \{\varphi(x) \,:\, x \in X\} \subset \sigma(M_\varphi, E).$$

Proof. The first inclusion is clear since the equation $M_\varphi f(x) = \lambda f(x)$ holds, for $f \in E$ with $f \not\equiv 0$, only if there exists $x \in X$ such that $\varphi(x) = \lambda$ whenever $f(x) \neq 0$.

For the second one, fix $x \in X$ then the inverse of $M_\varphi - \varphi(x)I = M_{\varphi-\varphi(x)}$ (if it exists) is $M_{\frac{1}{\varphi-\varphi(x)}}$, but $\frac{1}{\varphi-\varphi(x)}$ is not continuous and thus $M_\varphi - \varphi(x)I$ is not invertible. □

Example 1.5.2 Regarding the first inclusion in Lemma 1.5.1, we see examples concerning the (non)existence of an f satisfying the equations.

(1) Let $X = \mathbb{R}$ and $v \equiv 1$, and thus, $C_v = CB(\mathbb{R})$ is the space of bounded and continuous functions. Define the multiplier by $\varphi(x) = x$. Then $\lambda \in \sigma_{pt}(M_\varphi)$ if $xf(x) = \varphi(x)f(x) = \lambda f(x)$. Then $\lambda = x$ for every $x \in X$ with $f(x) \neq 0$. That would imply that f is 0 everywhere except at λ, which is impossible by continuity. Then $\sigma_{pt}(M_\varphi) = \emptyset$.

(2) Let $X = [0, 1]$ and $v \equiv 1$. Define the multiplier by

$$\varphi(x) = \begin{cases} 0, & 0 \leq x \leq 1/4 \\ 2x - \frac{1}{2}, & 1/4 \leq x \leq 3/4 \\ 1, & 3/4 \leq x \leq 1. \end{cases}$$

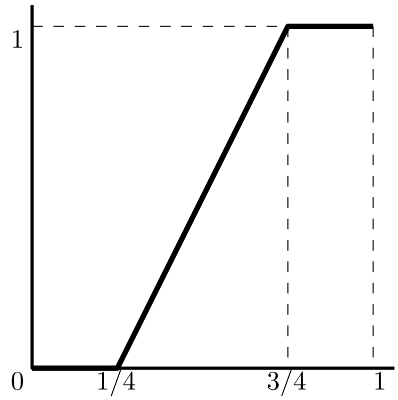

Figure: The multiplier φ.

Let f be a continuous function supported in $0 \leq x \leq 1/4$, then we have $M_\varphi f(x) = \varphi(x)f(x) = 0 = 0f(x)$, for every $x \in X$. Therefore, $0 \in \sigma_{pt}(M_\varphi)$. The same can be done in $3/4 \leq x \leq 1$ to obtain $1 \in \sigma_{pt}(M_\varphi)$. Similarly as in the first example, we can see that $\varphi(x) \notin \sigma_{pt}(M_\varphi)$ for any $1/4 < x < 3/4$. Hence, $\sigma_{pt}(M_\varphi) = \{0, 1\}$.

(3) Let $X = [0, 1]$ and $v \equiv 1$. Following the idea of the second example, we can deduce that $\lambda \in \sigma_{pt}(M_\varphi)$ if $\varphi \equiv \lambda$ in an open set of X. The extreme example of this being taking φ a constant. This idea can actually be extended to any Hausdorff, locally compact, normal topological space X, since the functions with compact support are contained in any $C_v(X)$.

(4) The case $X = \mathbb{N}$ gives us the equality $\varphi(\mathbb{N}) = \sigma_{pt}(M_\varphi)$ since every point in $\mathbb{N}$ is an open and compact set itself. More precisely, we have that $M_\varphi(e^j) = (\varphi(i)e_i^j)_i = \varphi(j)e^j$, where $e_j = (\delta_{ij})_i$ for every $j \in \mathbb{N}$, and then $\varphi(j) \in \sigma_{pt}(M_\varphi)$. See Section 2.3.

Proposition 1.5.3 *Let E be either C_v or C_v^0, for a weight v. Then*

$$\sigma(M_\varphi, E) = \overline{\{\varphi(x) \,:\, x \in X\}}.$$

Proof. Since E is a Banach space, $\sigma(M_\varphi, E)$ must be a compact set, and thus, using Lemma 1.5.1, $\overline{\{\varphi(x) : x \in X\}} \subset \sigma(M_\varphi, E)$.

For the other inclusion let $\mu \in \mathbb{C} \setminus \overline{\{\varphi(x) : x \in X\}}$. Then, there exists $\varepsilon > 0$ such that for every $x \in X$, $|\varphi(x) - \mu| \geq \varepsilon$. Define $\phi \in C(X)$ by $\phi(x) = \frac{1}{\varphi(x)-\mu}$, $x \in X$. Thus, $\phi \in CB$ is a bounded function and $M_\phi \in \mathcal{L}(E)$ by Lemma 1.2.1. Clearly M_ϕ is the continuous inverse of $M_\varphi - \mu I$, therefore, $\mu \notin \sigma(M_\varphi, E)$. □

Proposition 1.5.4 *Let E be either VC, V_0C, CA or CA_0. Then the following assertions are equivalent:*

(1) $\mu \in \rho(M_\varphi, E)$,

(2) for each k, there exists $l \geq k$ such that

$$\sup_{x\in X} \frac{v_l(x)}{v_k(x)} \cdot \frac{1}{|\varphi(x) - \mu|} = \sup_{x\in X} \frac{a_k(x)}{a_l(x)} \cdot \frac{1}{|\varphi(x) - \mu|} < \infty.$$

Proof. Set $\phi = \frac{1}{\varphi-\mu}$. Since M_ϕ is the inverse of $M_\varphi - \mu I$, whenever it exists, the equivalence of (1) and (2) is a consequence of Proposition 1.3.1 for VC and V_0C and of Proposition 1.4.1 for CA and CA_0. □

Corollary 1.5.5 *Let E be either VC, V_0C, CA or CA_0. Then,*

$$\overline{\sigma(M_\varphi, E)} = \overline{\{\varphi(x) : x \in X\}}.$$

Proof. By Lemma 1.5.1, we have

$$\overline{\{\varphi(x) : x \in X\}} \subset \overline{\sigma(M_\varphi, E)}.$$

For the other inclusion let $\mu \in \mathbb{C} \setminus \overline{\{\varphi(x) : x \in X\}}$. Then there is $\delta > 0$ such that $|\mu - \varphi(x)| > 2\delta$, for $x \in X$. Proposition 1.5.4 yields $\mu \in \rho(M_\varphi, E)$. Assume $\mu \in \overline{\sigma(M_\varphi, E)}$. Then we must actually have $\mu \in \partial\sigma(M_\varphi, E)$. Thus, there exists $\lambda \in \sigma(M_\varphi, E)$ such that $|\mu - \lambda| < \delta$ and therefore, for every $x \in X$, we have

$$|\lambda - \varphi(x)| \geq |\mu - \varphi(x)| - |\mu - \lambda| > 2\delta - \delta = \delta.$$

From this we deduce, again by Proposition 1.5.4, that $\lambda \in \rho(M_\varphi, E)$, which is a contradiction. Hence, $\mu \notin \overline{\sigma(M_\varphi, E)}$.

□

Example 1.5.6 Recall that in general the spectrum is not a closed set of $\mathbb{C}$. This example shows that in our case. Let $X = \mathbb{D}$ be the complex unit disc. Consider $V = (v_k)_k$ with $v_k(z) = \min(1, |1-z|^k)$. Take $\varphi(z) = z$. As we saw in Example 1.3.11,

$$\sup_{z\in\mathbb{D}} \frac{v_{k+1}(z)}{v_k(z)} \cdot \frac{1}{|1-\varphi(z)|} = 1,$$

for every k. Then, $1 \in \rho(M_\varphi)$, by Proposition 1.5.4. However, if $\lambda \in \partial\mathbb{D}$ with $\lambda \neq 1$, it can easily be checked that $\lambda \in \sigma(M_\varphi)$. Clearly $\mathbb{C} \setminus \overline{\mathbb{D}} \subset \rho(M_\varphi)$ and $\mathbb{D} = \varphi(\mathbb{D}) \subset \sigma(M_\varphi)$. Then, $\sigma(M_\varphi, VC) = \sigma(M_\varphi, V_0C) = \overline{\mathbb{D}} \setminus \{1\}$.

Proposition 1.5.7 *Let E be either VC, V_0C, CA or CA_0. Then*

$$\sigma^*(E, M_\varphi) = \overline{\sigma(E, M_\varphi)} = \overline{\{\varphi(x) \, : \, x \in X\}}.$$

Proof. By the definitions one deduces that $\sigma^*(E, M_\varphi)$ is closed and also that $\sigma(E, M_\varphi) \subset \sigma^*(E, M_\varphi)$. From this it follows that $\overline{\sigma(E, M_\varphi)} \subset \sigma^*(E, M_\varphi)$.

For the other inclusion, let $\lambda \in \mathbb{C} \setminus \overline{\sigma(E, M_\varphi)}$. We show that we also have $\lambda \in \rho^*(E, M_\varphi)$. There exists $\delta > 0$ such that if $|\mu - \lambda| \leq 2\delta$, then $\mu \in \rho(M_\varphi, X)$. We show that the set

$$\{(M_\varphi - \mu I)^{-1} \, : \, |\lambda - \mu| \leq \delta\}$$

is equicontinuous. By Lemma 1.5.1, we know that $\varphi(x) \in \sigma(E, M_\varphi)$, and thus $|\varphi(x) - \lambda| > 2\delta$, for all $x \in X$. Therefore we have, for $x \in X$,

$$|\varphi(x) - \mu| \geq |\varphi(x) - \lambda| - |\mu - \lambda| > 2\delta - \delta = \delta.$$

Now let $f \in E$. We have, for each $\mu \in \mathbb{C}$ with $|\lambda - \mu| \leq \delta$,

$$|((M_\varphi - \mu I)^{-1} f)(x)| = \left| \frac{f(x)}{\varphi(x) - \mu} \right| \leq \frac{1}{\delta} |f(x)|. \tag{1.5.1}$$

Now, if E is either CA or CA_0, we deduce that $\|(M_\varphi - \mu I)^{-1} f\|_k \leq \frac{1}{\delta} \|f\|_k$, for each $k \in \mathbb{N}$, and the set is equicontinuous.

Lastly, consider the case in which E is either VC or V_0C. From the existence of a fundamental system of solid neighbourhoods given in Lemma 1.3.4, we deduce the equicontinuity. Indeed, if $\mathcal{U}$ is as in Lemma 1.3.4, $U \in \mathcal{U}$ and $f \in U$, we get $((M_\varphi - \mu I)^{-1} f) \in (1/\delta) U$, for every $\mu \in \mathbb{C}$ with $|\lambda - \mu| \leq \delta$.

□

To conclude this section we show some relation between the spectral properties of M_φ and its ergodic properties. The following result must be compared with the results by Dunford and Lin stated in Theorem A.2.5.

Corollary 1.5.8 *Let $\varphi \in C(X)$ satisfy $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1) = \emptyset$. Then the following assertions hold:*

(i) The operator M_φ is uniformly mean ergodic on C_v and C_v^0 if, and only if, $1 \in \rho(M_\varphi)$.

(ii) If $1 \in \rho(M_\varphi)$, then M_φ is uniformly mean ergodic on V_0C. If V satisfies the property (D), then also M_φ is uniformly mean ergodic on VC.

(iii) If $1 \in \rho(M_\varphi)$ and X is σ-compact, then M_φ is uniformly mean ergodic on CA and CA_0.

Proof. (i) Since $\varphi(x) \neq 1$ for all $x \in X$, we have that $1 \in \rho(M_\varphi)$ if, and only if, $\inf_{x\in X} |1 - \varphi(x)| > 0$, according to Proposition 1.5.3. But this is precisely the condition for uniform mean ergodicity given in Theorem 1.2.8.

(ii) If $1 \in \rho(M_\varphi)$, then for each k, there exists $l \geq k$ such that

$$\sup_{x\in X} \frac{v_l(x)}{v_k(x)} \cdot \frac{1}{|\varphi(x) - 1|} < \infty,$$

by Proposition 1.5.4. The assertion holds by Corollary 1.3.10 for V_0C. If (D) holds for V, then the assertion is also satisfied by VC, due to Corollary 1.3.9.

(iii) If $1 \in \rho(M_\varphi)$, then for each k, there exists $l \geq k$ such that

$$\sup_{x\in X} \frac{a_k(x)}{a_l(x)} \cdot \frac{1}{|\varphi(x) - 1|} < \infty,$$

by Proposition 1.5.4. We deduce the uniform mean ergodicity on CA_0 by Corollary 1.4.8. From this we see it also holds for CA using Corollary 1.4.6.

□

1.6 Topology of the set of multipliers

Let $\mathcal{A} = (a_m)_m$ and $\mathcal{B} = (b_n)_n$ be increasing families of weights on a Hausdorff, normal locally compact topological space X. Define

$$C\mathcal{A} = \operatorname*{proj}_{m\in\mathbb{N}} C_{a_m}(X), \quad C\mathcal{B} = \operatorname*{proj}_{n\in\mathbb{N}} C_{b_n}(X),$$

and

$$C\mathcal{A}_0 = \operatorname*{proj}_{m\in\mathbb{N}} C^0_{a_m}(X), \quad C\mathcal{B}_0 = \operatorname*{proj}_{n\in\mathbb{N}} C^0_{b_n}(X).$$

In this subsection E and F denote either the pair $C\mathcal{A}$ and $C\mathcal{B}$ or the pair $C\mathcal{A}_0$ and $C\mathcal{B}_0$.

By Proposition 1.4.1, M_φ is continuous from E to F, that is φ is a multiplier, if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$\sup_{x \in X} \frac{b_n(x)|\varphi(x)|}{a_m(x)} < \infty.$$

This is actually equivalent to

$$\varphi \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} C_{\frac{b_n}{a_m}}(X) =: E_{\mathcal{AB}}.$$

Endow the set $E_{\mathcal{AB}}$ of *multipliers* from E to F with the topology

$$E_{\mathcal{AB}} = \operatorname*{proj}_n \operatorname*{ind}_m C_{\frac{b_n}{a_m}}(X).$$

Then $E_{\mathcal{AB}}$ is a countable projective limit of countable inductive limits of Banach spaces. These spaces are called (PLB)-spaces. We refer the reader to Wengenroth lecture notes [82] for more information about these spaces.

In fact, we can write $E_{\mathcal{AB}}$ as

$$E_{\mathcal{AB}} = \operatorname*{proj}_n \mathcal{V}_n C(X),$$

where $\mathcal{V}_n = \left(\frac{b_n}{a_m}\right)_m$ is a decreasing family of weights for each $n \in \mathbb{N}$. Now let $V = (1/a_m)_m$, then we have $\mathcal{V}_n C(X) \simeq VC(X)$, for each $n \in \mathbb{N}$. Furthermore, if we assume the property (D) holds for V (see Theorem A.3.5), then $\mathcal{V}_n C(X) \simeq C\overline{V}(X)$, for each $n \in \mathbb{N}$. This way, the topology of $\mathcal{V}_n C(X)$ is given by the seminorms

$$p_{\overline{v}}(f) := \sup_{x \in X} b_n(x)\overline{v}(x)|f(x)|, \quad f \in \mathcal{V}_n C(X), \overline{v} \in \overline{V}.$$

Then the fundamental system of seminorms of $E_{\mathcal{AB}}$ is

$$q_{n,\overline{v}}(f) = \sup_{x \in X} b_n(x)\overline{v}(x)|f(x)|, \quad f \in \mathcal{V}_n C(X), n \in \mathbb{N}, \overline{v} \in \overline{V}.$$

On the other hand, the topology of the space $\mathcal{L}_b(E, F)$ can be described as follows. Denote the seminorms of F by q_n, then the seminorms defining the topology of $\mathcal{L}_b(E, F)$ are

$$r_{n,B}(M) := \sup_{f \in B} q_n(Mf), \quad M \in \mathcal{L}_b(E, F), n \in \mathbb{N}, B \subset E \text{ bounded}.$$

The following theorem is the main result and ensures that the topology of the space $\mathcal{L}_b(E, F)$ induces on the space of multipliers $E_{\mathcal{AB}}$ from E into F the natural (PLB)-topology defined above.

Theorem 1.6.1 *Assume the property (D) holds for* $V = (1/a_m)_m$. *Then the map* $T : E_{\mathcal{AB}} \longrightarrow \mathcal{L}_b(E,F)$, $\varphi \mapsto M_\varphi$ *is a topological isomorphism into. Hence* $\mathcal{L}_b(E,F)$ *induces on the set of multipliers the (PLB)-topology.*
Proof. The injectivity is easy.

1. Continuity of T: Fix $n \in \mathbb{N}$ and let B be a bounded set in E. Apply Lemma 1.4.5 to find $\overline{v} \in \overline{V}$ such that $B \subset B_{\overline{v}}$. For $\varphi \in E_{\mathcal{AB}}$, we have

$$\begin{aligned} r_{n,B}(T(\varphi)) &= r_{n,B}(M_\varphi) \\ &= \sup_{f\in B}\sup_{x\in X} b_n(x)|f(x)||\varphi(x)| \\ &\le \sup_{x\in X} b_n(x)\overline{v}(x)|\varphi(x)| = q_{n,\overline{v}}(\varphi). \end{aligned}$$

2. Continuity of T^{-1}: Fix $n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$. For each $x \in X$ denote by f_x a function with compact support in E such that $0 \le f_x \le 1$ and $f_x(x) = 1$. Then, $f_x \cdot \overline{v} \in B_{\overline{v}}$. Fix $x \in X$, we have

$$\begin{aligned} r_{n,B_{\overline{v}}}(M_\varphi) &= \sup_{f\in B_{\overline{v}}} q_n(M_\varphi f) \\ &\ge q_n(M_\varphi(f_x\cdot\overline{v})) \\ &= q_n(\overline{v}\cdot\varphi\cdot f_x) \\ &\ge b_n(x)\overline{v}(x)|\varphi(x)|. \end{aligned}$$

Since $x \in X$ is arbitrary, we have $q_{n,\overline{v}}(\varphi) \le r_{n,B_{\overline{v}}}(M_\varphi)$. □

Chapter 2

Echelon and Co-Echelon spaces

2.1 Introduction and notation

Our notation for Köthe echelon and co-echelon sequence spaces is as in [24] and [63]. We recall here the terminology needed below. A complex sequence $v = (v(i))_i \in \mathbb{C}^{\mathbb{N}}$ is called a *weight* if it is strictly positive. The weighted Banach spaces of sequences are defined by

$$\ell_p(v) := \{x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} \, : \, p_v(x) := \|(v(i)x_i)_i\|_p < \infty\}, \; 1 \leq p \leq \infty,$$

$$c_0(v) := \{x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} \, : \, \lim_{i \to \infty} v(i)x_i = 0\},$$

where $\| \cdot \|_p$ denotes the usual ℓ_p norm. These spaces are Banach spaces with the corresponding norm p_v, and $c_0(v)$ is a Banach space with the norm of $\ell_\infty(v)$.

Now, given $A = (a_n)_n$, a Köthe matrix (i.e. each a_n is a weight and $a_n(i) \leq a_{n+1}(i)$ for all $i, n \in \mathbb{N}$), the echelon space of order $1 \leq p \leq \infty$ is defined by

$$\lambda_p(A) = \bigcap_{n \in \mathbb{N}} \ell_p(a_n) \quad \text{and} \quad \lambda_0(A) = \bigcap_{n \in \mathbb{N}} c_0(a_n),$$

endowed with the projective topologies

$$\lambda_p(A) := \operatorname*{proj}_{n \in \mathbb{N}} \ell_p(a_n) \quad \text{and} \quad \lambda_0(A) := \operatorname*{proj}_{n \in \mathbb{N}} c_0(a_n).$$

These spaces are Fréchet spaces with the topology defined by the corresponding seminorms $p_n := p_{a_n}$, with $n = 1, 2, \ldots$ Observe that we only consider here Köthe echelon spaces with a continuous norm. It is easy to extend our results to the general case, since every Köthe echelon space is the countable product of Köthe echelon spaces with a continuous norm; see [24] and [15].

From the same Köthe matrix $A = (a_n)_n$, define $V = (v_n)_n = ((1/a_n(i))_i)_n$. The co-echelon space of order $1 \leq p \leq \infty$ is defined by

$$\kappa_p(A) = \kappa_p(V) := \bigcup_{n\in\mathbb{N}} \ell_p(v_n) \quad \text{and} \quad \kappa_0(A) = \bigcap_{n\in\mathbb{N}} c_0(v_n),$$

endowed with the inductive topologies

$$\kappa_p(A) := \operatorname*{ind}_{n\in\mathbb{N}} \ell_p(v_n) \quad \text{and} \quad \kappa_0(A) := \operatorname*{proj}_{n\in\mathbb{N}} c_0(v_n).$$

The corresponding Nachbin family $\overline{V}$, defined in Section 1.3, is, in the discrete case, just $\overline{V} = \lambda_\infty(A)_+ := \{\overline{v} \in \lambda_\infty(A) \ : \ \overline{v} \geq 0\}$. We denote

$$K_p(A) := \operatorname*{proj}_{\overline{v}\in\overline{V}} \ell_p(\overline{v}), \quad K_0(A) := \operatorname*{proj}_{\overline{v}\in\overline{V}} c_0(\overline{v}),$$

for $1 \leq p \leq \infty$. The topology is the one given by the family of seminorms $\{p_{\overline{v}} \ : \ \overline{v} \in \overline{V}\}$. The relations between $\kappa_p(A)$ and $K_p(A)$ are well known and can be seen in Section A.4.

Along this chapter we work with diagonal (multiplication) operators on these echelon and co-echelon spaces. Given a sequence $\varphi = (\varphi_i)_i$, we define the multiplication (or diagonal) operator as $M_\varphi : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$, $(x_i)_i \mapsto (\varphi_i x_i)_i$. These operators have been investigated by many authors. We mention here [5] [13], [14] and [49]. In the context of Köthe echelon spaces, diagonal operators were investigated by Crofts [38].

Many of the properties we study for the multiplication operator are mere corollaries of the results in Chapter 1, since we have $CA(\mathbb{N}) = \lambda_\infty(A)$, $CA_0(\mathbb{N}) = \lambda_0(A)$, $VC(\mathbb{N}) = \kappa_\infty(A)$ and $V_0C(\mathbb{N}) = \kappa_0(A)$. Therefore we focus mainly in the cases $1 \leq p < \infty$. It turns out that those cases behave usually like the case $p = 0$.

2.2 Ergodic properties of diagonal operators

The aim of this section is to characterize the power boundedness, the mean ergodicity and the uniform mean ergodicity of the diagonal operators defined on Köthe echelon and co-echelon spaces.

First we characterize the multipliers acting on echelon and co-echelon spaces. The following result is well known and can be easily deduced from [38]. See also Propositions 1.2.1, 1.3.1 and 1.4.1.

Lemma 2.2.1 *Let $v, w \in \mathbb{C}^\mathbb{N}$ be weights and let $A = (a_m)_m$ and $B = (b_n)_n$ be Köthe matrices. The following assertions hold for $1 \leq p \leq \infty$ and $p = 0$,*

(i) $M_\varphi : \ell_p(v) \longrightarrow \ell_p(w)$ is continuous if, and only if,

$$\sup_{i\in\mathbb{N}} \frac{w(i)|\varphi_i|}{v(i)} < \infty.$$

(ii) $M_\varphi : \lambda_p(A) \longrightarrow \lambda_p(B)$ *is continuous if, and only if, for each* $n \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *such that*

$$\sup_{i\in\mathbb{N}} \frac{b_n(i)|\varphi_i|}{a_m(i)} < \infty.$$

(iii) $M_\varphi : \kappa_p(A) \longrightarrow \kappa_p(B)$ *is continuous if, and only if, for each* $m \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *such that*

$$\sup_{i\in\mathbb{N}} \frac{a_m(i)|\varphi_i|}{b_n(i)} < \infty.$$

(iv) If $A = B$, *then* $M_\varphi : \lambda_p(A) \longrightarrow \lambda_p(A)$ *is continuous if, and only if,* $M_\varphi : \kappa_p(A) \longrightarrow \kappa_p(A)$ *is continuous if, and only if, for each* $n \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *such that*

$$\sup_{i\in\mathbb{N}} \frac{a_n(i)|\varphi_i|}{a_m(i)} < \infty.$$

Proof. (i) First note that $M_\Phi : \ell_p \longrightarrow \ell_p$ is continuous if, and only if, Φ is bounded. The operators $M_v : \ell_p(v) \longrightarrow \ell_p$ and $M_w : \ell_p(w) \longrightarrow \ell_p$ are isomorphisms. Therefore $M_\varphi : \ell_p(v) \longrightarrow \ell_p(w)$ is continuous if, and only if, M_Φ is continuous, where $\Phi = \frac{w(i)\varphi_i}{v(i)}$, since $M_\Phi = M_w M_\varphi M_v^{-1}$.

(ii) This proof mimics the one of Proposition 1.4.1, using that M_φ is continuous if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $p_{b_n}(M_\varphi x) \leq p_{a_n}(x)$, for all $x \in \lambda_p(A)$.

(iii) According to Grothendieck's factorization theorem [63, Theorem 24.33], M_φ is continuous if, and only if, for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_\varphi : \ell_p((1/a_m(i))_i) \longrightarrow \ell_p((1/b_n(i))_i)$ is continuous. The conclusion follows from assertion (i).

(iv) This is a particular case of (ii) and (iii). □

Lemma 2.2.2 *Let* X *be* $\lambda_p(A)$ *or* $\kappa_p(A)$ *for* $1 \leq p \leq \infty$ *or* $p = 0$ *or be* ℓ_p *for* $1 \leq p \leq \infty$ *or* c_0. *If* $T := M_\varphi \in \mathcal{L}(X)$ *satisfies that* $\frac{T^k x}{k} \to 0$ *as* $k \to \infty$ *for each* $x \in X$, *then* $\|\varphi\|_\infty \leq 1$. *This holds in particular if* T *is power bounded or mean ergodic.*

Proof. This is a particular case of Lemma 1.2.2 for the cases $p = 0, \infty$ and c_0. We include it here in the general case for the sake of completeness.

For each $j \in \mathbb{N}$, let $e^j = (\delta_{ij})_i \in X$. Then

$$\lim_k \frac{|\varphi_j^k|}{k} = \lim_k \frac{|(T^k e^j)_j|}{k} = 0,$$

with $(T^k e^j)_j$ the j-th coordinate of $T^k e^j$. This implies $|\varphi_j| \leq 1$.

If T is power bounded, then $\left(T^k/k\right)_k$ clearly converges to 0 in $\mathcal{L}_s(X)$. On the other hand, if T is mean ergodic, then $\frac{T^k x}{k} = T_{[k]}x - \frac{k-1}{k}T_{[k-1]}x$ converges to 0 as $k \to \infty$ for each $x \in X$. □

Remark 2.2.3 Let v be a weight and let A be the Köthe matrix $(v)_n$. Then, for $1 \leq p \leq \infty$, $\lambda_p(A) = \ell_p(v)$ and $\lambda_0(A) = c_0(v)$ algebraically and topologically. Therefore the ergodic properties of the Banach sequence spaces are not explicitly proved, as they are particular cases of the echelon spaces.

2.2.1 Echelon Spaces

Throughout this section $A = (a_n)_n$ is a Köthe matrix and the corresponding decreasing family is $V = (v_n)_n = (1/a_n)_n$.

Proposition 2.2.4 *For $1 \leq p \leq \infty$ and $p = 0$, $M_\varphi \in \mathcal{L}(\lambda_p(A))$ is power bounded if, and only if, $\|\varphi\|_\infty \leq 1$.*

Proof. Necessity follows from Lemma 2.2.2. Conversely, if $\|\varphi\|_\infty \leq 1$, then $p_n(M_\varphi^k x) \leq p_n(x)$ for every $k, n \in \mathbb{N}$ and $x \in \lambda_p(A)$. Thus M_φ is power bounded. □

The following lemma follows from a more general argument, we give however a direct proof.

Lemma 2.2.5 *Let $(y^k)_k \subset \lambda_0(A)$ be bounded with $\lim_k y_i^k = 0$ for each $i \in \mathbb{N}$. Then $(y^k)_k$ converges to 0 for the weak topology $\sigma(\lambda_0(A), (\lambda_0(A))')$.*

Proof. Let $u = (u_i)_i \in \lambda_0(A)'$. There is $m \in \mathbb{N}$ such that $(u_i/a_m(i))_i \in \ell_1$. Since $(y^k)_k$ is bounded, there is $M > 0$ such that $a_m(i)|y_i^k| \leq M$ for each $i, k \in \mathbb{N}$. Given $\varepsilon > 0$, there is $i(0) \in \mathbb{N}$ such that

$$\sum_{i=i(0)+1}^{\infty} \frac{|u_i|}{a_m(i)} < \frac{\varepsilon}{2M}.$$

Now, there is $k_0 \in \mathbb{N}$ such that for all $k \geq k(0)$ and $i = 1, ..., i(0)$, we have $|y_i^k| < \varepsilon/(2i(0)(|u_i| + 1))$. We have, for $k \geq k(0)$,

$$|\langle y^k, u\rangle| = \left|\sum_{i=1}^{\infty} y_i^k u_i\right| \leq \sum_{i=1}^{i(0)} |y_i^k u_i| + \sum_{i=i(0)+1}^{\infty} |y_i^k| a_m(i) \left|\frac{u_i}{a_m(i)}\right| < \varepsilon.$$

Therefore $(y^k)_k$ converges to 0 for the weak topology $\sigma(\lambda_0(A), (\lambda_0(A))')$. □

Theorem 2.2.6 *For $1 \leq p < \infty$ or $p = 0$, $M_\varphi \in \mathcal{L}(\lambda_p(A))$ is mean ergodic if, and only if, $\|\varphi\|_\infty \leq 1$.*

Proof. If M_φ is mean ergodic, by Lemma 2.2.2, $\|\varphi\|_\infty \leq 1$.

Now we assume that $\|\varphi\|_\infty \leq 1$ and distinguish three cases.

1. If $1 < p < \infty$, the space $\lambda_p(A)$ is reflexive. By Corollary A.2.2.(ii), every power bounded operator in $\lambda_p(A), 1 < p < \infty$ is mean ergodic. The conclusion follows from Proposition 2.2.4.

For the rest of the proof, we may assume without loss of generality that $\varphi_i \neq 1$ for all $i \in \mathbb{N}$. Otherwise we split the space into two sectional subspaces and observe that in the subspace in which $\varphi_i = 1$, the diagonal operator acts as the identity.

Under this assumption, the following expression of the i-th coordinate of the means $(M_\varphi)_{[k]}x$ of the iterates evaluated at $x \in \lambda_p(A)$ is useful.

$$((M_\varphi)_{[k]}x)_i = \frac{\varphi_i + \cdots + \varphi_i^k}{k} x_i = \frac{\varphi_i}{k} \frac{1 - \varphi_i^k}{1 - \varphi_i} x_i.$$

2. For the case $p = 1$. Fix $x \in \lambda_1(A)$. We want to show that $(M_\varphi)_{[k]}x$ converges to 0 in $\lambda_1(A)$. Let $\varepsilon > 0$ and fix $n \in \mathbb{N}$. Since $x \in \lambda_1(A)$, there exists $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0+1}^{\infty} a_n(i)|x_i| < \frac{\varepsilon}{2}.$$

For each $i \in \mathbb{N}$ we have

$$\lim_{k\to\infty} |((M_\varphi)_{[k]}x)_i| = \lim_{k\to\infty} \frac{|\varphi_i|}{k} \frac{|1 - \varphi_i^k|}{|1 - \varphi_i|} |x_i| \leq \lim_{k\to\infty} \frac{2|x_i|}{k} \frac{1}{|1 - \varphi_i|} = 0.$$

Then, for $i = 1, \ldots, i_0$, select $k_i \in \mathbb{N}$ such that for $k \geq k_i$,

$$|((M_\varphi)_{[k]}x)_i| < \frac{\varepsilon}{2a_n(i)i_0}.$$

If $k \geq \max\{k_i \,:\, i = 1, \ldots, i_0\}$, then

$$\begin{aligned} p_n((M_\varphi)_{[k]}x) &= \sum_{i=1}^{\infty} a_n(i) \frac{|\varphi_i|}{k} \frac{|1 - \varphi_i^k|}{|1 - \varphi_i|} |x_i| \\ &\leq \sum_{i=1}^{i_0} a_n(i) \frac{|\varphi_i|}{k} \frac{|1 - \varphi_i^k|}{|1 - \varphi_i|} |x_i| + \sum_{i=i_0+1}^{\infty} a_n(i) \left| \frac{\varphi_i + \cdots + \varphi_i^k}{k} \right| |x_i| \\ &\leq \sum_{i=1}^{i_0} a_n(i) \frac{\varepsilon}{2a_n(i)i_0} + \sum_{i=i_0+1}^{\infty} a_n(i)|x_i| < \varepsilon. \end{aligned}$$

3. We now consider the case $p = 0$. Fix $x \in \lambda_0(A)$ and set $y^k := (M_\varphi)_{[k]}x$. Then

$$y_i^k = \frac{\varphi_i x_i}{k} \frac{1-\varphi_i^k}{1-\varphi_i}.$$

By Proposition 2.2.4, M_φ is power bounded and, in particular, $(y^k)_k$ is bounded. Clearly, for each $i \in \mathbb{N}$, $\lim_k y_i^k = 0$. Apply Lemma 2.2.5 to deduce that (y^k) converges to 0 for the weak topology $\sigma(\lambda_0(A), (\lambda_0(A))')$. By Yosida's mean ergodic theorem (see Corollary A.2.2.(i)) we conclude that M_φ is mean ergodic. □

The following useful description of the bounded sets in a Köthe echelon space is due to Bierstedt, Meise and Summers [23].

Lemma 2.2.7 *Let $1 \le p \le \infty$ or $p = 0$, then $B \subset \lambda_p(A)$ is bounded if, and only if, there exists $\overline{v}_0 \in \overline{V}$, with $\overline{v}_0 > 0$, such that*

$$B \subset B_{\overline{v}_0} := \left\{ (x_i)_i \in \mathbb{C}^{\mathbb{N}} \ : \ \left\| \left(\frac{x_i}{\overline{v}_0(i)} \right)_i \right\|_p \le 1 \right\}.$$

In the following result the equivalence of (1) – (3) and (5) is a particular case of Theorem 1.4.6. However we give a direct proof.

Theorem 2.2.8 *The following assertions are equivalent:*

(1) $M_\varphi \in \mathcal{L}(\lambda_\infty(A))$ is mean ergodic,

(2) $M_\varphi \in \mathcal{L}(\lambda_\infty(A))$ is uniformly mean ergodic,

(3) $M_\varphi \in \mathcal{L}(\lambda_0(A))$ is uniformly mean ergodic,

(4) for $1 \le p < \infty$, $M_\varphi \in \mathcal{L}(\lambda_p(A))$ is uniformly mean ergodic,

(5) $\|\varphi\|_\infty \le 1$ and for each $n \in \mathbb{N}$ and each $\overline{v} \in \overline{V}$,

$$\lim_{k\to\infty} \sup_{i \in \mathbb{N}\setminus J} \frac{a_n(i)\overline{v}(i)|\varphi_i|}{k} \frac{|1-\varphi_i^k|}{|1-\varphi_i|} = 0,$$

where $J = \{i \in \mathbb{N} \ : \ \varphi_i = 1\}$.

Proof. Clearly (2) implies (1) and (3).

We prove that (1) implies (5). Clearly $\|\varphi\|_\infty \le 1$, by Lemma 2.2.2. For the rest assume that $\varphi_i \ne 1$ for all $i \in \mathbb{N}$, i.e. $J = \emptyset$, and let $n \in \mathbb{N}$ and $\overline{v} \in \overline{V} \subset \lambda_\infty(A)$. Then, by mean ergodicity,

$$0 = \lim_{k\to\infty} p_n((M_\varphi)_{[k]}\overline{v}) = \lim_{k\to\infty} \sup_{i\in\mathbb{N}} a_n(i) \frac{|\varphi_i|}{k} \left| \frac{1-\varphi_i^k}{1-\varphi_i} \right| \overline{v}(i),$$

and we conclude.

We assume (5) and show (2) and (4) simultaneously. Assume that $\varphi_i \neq 1$ for all $i \in \mathbb{N}$. Fix $1 \leq p \leq \infty$, let $B \subset \lambda_p(A)$ be bounded and let $\overline{v}_0 \in \overline{V}$ be as in Lemma 2.2.7. Fix $n, k \in \mathbb{N}$. Then we have

$$\begin{aligned}\sup_{x \in B} p_n((M_\varphi)_{[k]}x) &= \sup_{x \in B} \left\| \left(a_n(i) \frac{\overline{v}_0(i)|\varphi_i|}{k} \frac{|1-\varphi_i^k|}{|1-\varphi_i|} \frac{|x_i|}{\overline{v}_0(i)} \right)_i \right\|_p \\ &\leq \sup_{i \in \mathbb{N}} \frac{a_n(i)\overline{v}_0(i)|\varphi_i|}{k} \frac{|1-\varphi_i^k|}{|1-\varphi_i|} \sup_{x \in B} \left\| \left(\frac{x_i}{\overline{v}_0(i)} \right)_i \right\|_p \\ &\leq \sup_{i \in \mathbb{N}} \frac{a_n(i)\overline{v}_0(i)|\varphi_i|}{k} \frac{|1-\varphi_i^k|}{|1-\varphi_i|},\end{aligned}$$

which converges to 0 by assumption.

Now we show that both (3) and (4) imply (5). Assume that $\varphi_i \neq 1$ for all $i \in \mathbb{N}$. Let $1 \leq p < \infty$ or $p = 0$ and fix $n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$. Let $B = \{x \in \lambda_p(A) : |x_i| \leq \overline{v}(i), \forall i \in \mathbb{N}\}$ and $\varepsilon > 0$. By uniform mean ergodicity, there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, $p_n((M_\varphi)_{[k]}x) < \varepsilon$ for every $x \in B$. For each $s \in \mathbb{N}$, let $x^s = (\delta_{is}\overline{v}(s))_i$. Note that $x^s \in B$ for every $s \in \mathbb{N}$. We have, for $s \in \mathbb{N}$ and $k \geq k_0$,

$$\frac{a_n(s)\overline{v}(s)|\varphi_s|}{k} \frac{|1-\varphi_s^k|}{|1-\varphi_s|} = p_n((M_\varphi)_{[k]}x^s) < \varepsilon.$$

□

The following result is a consequence of [2, Propositon 2.9].

Proposition 2.2.9 *Let $1 \leq p < \infty$ or $p = 0$ and let $A = (a_n)_n$ be a Köthe matrix. The space $\lambda_p(A)$ is Montel if, and only if, every mean ergodic diagonal operator M_φ on $\lambda_p(A)$ is uniformly mean ergodic.*

Proof. Every power bounded operator on a Fréchet Montel space is uniformly mean ergodic by Corollary A.2.2.(iii). Also M_φ is mean ergodic if, and only if, it is power bounded.

If $\lambda_p(A)$ is not Montel, then it contains a sectional subspace which is diagonally isomorphic to ℓ_p (or to c_0 if $p = 0$) by [30, proof of Proposition 2.5]. The conclusion follows, since there are mean ergodic not uniformly mean ergodic diagonal operators on ℓ_p, by [88, Example 2.4].

□

2.2.2 Co-Echelon Spaces

Throughout this section $A = (a_n)_n$ is a Köthe matrix and the corresponding decreasing family is $V = (v_n)_n = (1/a_n)_n$.

Lemma 2.2.10 *For $1 \leq p \leq \infty$ and $p = 0$, $M_\varphi \in \mathcal{L}(\kappa_p(A))$ is power bounded if, and only if, $\|\varphi\|_\infty \leq 1$.*

Proof. The cases $p = 0, \infty$ are particular cases of Proposition 1.3.2. We prove the case $1 \leq p < \infty$.

Necessity of $\|\varphi\|_\infty \leq 1$ follows directly from Lemma 1.2.2. Assume now $\|\varphi\|_\infty \leq 1$, then for each $\overline{v} \in \overline{V}$, $p_{\overline{v}}(M_\varphi^k x) \leq p_{\overline{v}}(x)$, for every $k \in \mathbb{N}$ and $x \in \kappa_p$. By Theorem A.4.2, the topology of κ_p is actually given by the seminorms $p_{\overline{v}}$. Thus, we conclude. □

Theorem 2.2.11 *For $1 \leq p < \infty$ and $p = 0$, $M_\varphi \in \mathcal{L}(\kappa_p)$ is mean ergodic if, and only if, $\|\varphi\|_\infty \leq 1$.*

Proof. The case $p = 0$ follows from Lemma 1.3.3. The case $1 < p < \infty$ is clear using reflexivity of κ_p, as in Theorem 2.2.6.

For the case $p = 1$, necessity is seen as usual. For the converse, assume $\|\varphi\|_\infty \leq 1$. This implies mean ergodicity of M_φ on any $\ell_1(v_n)$ (as a consequence of Theorem 2.2.6 and Remark 2.2.3). Then, for any $x \in \kappa_1$, there exists $n \in \mathbb{N}$ such that $x \in \ell_1(v_n)$ and $(M_\varphi)_{[k]}x$ converges there. By continuity of $\ell_1(v_n) \hookrightarrow \kappa_p(V)$, we conclude. □

Theorem 2.2.12 *$M_\varphi \in \mathcal{L}(\kappa_\infty)$ is uniformly mean ergodic if, and only if, it is mean ergodic.*

Proof. It is a particular case of Proposition 1.3.5. □

Theorem 2.2.13 *The following assertions are equivalent:*

(1) $M_\varphi \in \mathcal{L}(\kappa_0)$ is uniformly mean ergodic,

(2) for $1 \leq p < \infty$, $M_\varphi \in \mathcal{L}(\kappa_p)$ is uniformly mean ergodic,

(3) for each $n \in \mathbb{N}$ and each $\overline{v} \in \overline{V}$,

$$\lim_{k\to\infty} \sup_{i\in\mathbb{N}\setminus J} \frac{\overline{v}(i)|\varphi_i|}{v_n(i)k} \frac{|1-\varphi_i^k|}{|1-\varphi_i|} = 0,$$

where $J = \{i \in \mathbb{N} : \varphi_i = 1\}$.

Proof. (1) $\iff$ (3) follows from Theorem 1.3.7.

For (3) $\Rightarrow$ (2) let $B \subset \kappa_p$ be bounded. By regularity (Definition A.3.3, Theorem A.4.2), there exist $n \in \mathbb{N}$ and $M > 0$ such that $B \subset MB_n$, where B_n is the closed unit ball of $\ell_p(v_n)$. We want to show that

$$\lim_{k\to\infty} \sup_{x\in B} p_{\overline{v}}((M_\varphi)_{[k]}x - Px) = 0,$$

for each $\overline{v} \in \overline{V}$, where P is as usual. Fix $\overline{v} \in \overline{V}$ and $k \in \mathbb{N}$, we have,

$$\begin{aligned}\sup_{x\in B} p_{\overline{v}}((M_\varphi)_{[k]}x - Px) &= \sup_{x\in B}\left(\sum_{i\in\mathbb{N}\setminus J}\left(\overline{v}(i)|x_i||\varphi_i|\frac{|1-\varphi_i^k|}{|1-\varphi_i|}\right)^p\right)^{1/p}\\ &\leq \sup_{i\in\mathbb{N}\setminus J}\frac{\overline{v}(i)|\varphi_i|}{v_n(i)k}\frac{|1-\varphi_i^k|}{|1-\varphi_i|}\sup_{x\in B}\left(\sum_{i\in\mathbb{N}\setminus J}(v_n(i)|x_i|)^p\right)^{1/p}\\ &\leq M\sup_{i\in\mathbb{N}\setminus J}\frac{\overline{v}(i)|\varphi_i|}{v_n(i)k}\frac{|1-\varphi_i^k|}{|1-\varphi_i|}.\end{aligned}$$

For (2) $\Rightarrow$ (3) assume M_φ is uniformly mean ergodic and fix $k, n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$. For each $i \in \mathbb{N}$, $x^i = (\delta_{ij}\frac{1}{v_n(i)})_j$. Then $(x^i) \subset B_n$, which is a bounded set. We have

$$\begin{aligned}\sup_{i\in\mathbb{N}\setminus J}\frac{\overline{v}(i)|\varphi_i|}{v_n(i)k}\frac{|1-\varphi_i^k|}{|1-\varphi_i|} &= \sup_{i\in\mathbb{N}\setminus J}\frac{\overline{v}(i)|\varphi_i|}{k}\frac{|1-\varphi_i^k|}{|1-\varphi_i|}x_i^i\\ &\leq \sup_{x\in B_n}\sup_{i\in\mathbb{N}\setminus J}\frac{\overline{v}(i)|\varphi_i|}{k}\frac{|1-\varphi_i^k|}{|1-\varphi_i|}|x_i|\\ &\leq \sup_{x\in B_n} p_{\overline{v}}((M_\varphi)_{[k]}x - Px),\end{aligned}$$

which converges to 0 by assumption. □

We would like to connect the results of Theorem 2.2.11 with those of Theorem 2.2.13, as we did in Section 1.3. To do so, we consider the topological identity $\kappa_\infty = K_\infty$. See Theorem A.4.3.

Theorem 2.2.14 *Assume $\kappa_\infty = K_\infty$. Then $M_\varphi \in \mathcal{L}(\kappa_\infty)$ is uniformly mean ergodic if, and only if, condition (3) of Theorem 2.2.13 holds.*
Proof. It is a particular case of Corollary 1.3.9. □

2.3 Spectrum of diagonal operators

We start with the point spectrum. This result should be compared with Lemma 1.5.1 and the examples after.

Lemma 2.3.1 *Let E be $\lambda_p(A)$ or $\kappa_p(A)$ for $1 \leq p \leq \infty$ or $p = 0$ or be ℓ_p for $1 \leq p \leq \infty$ or c_0. Then,*

$$\sigma_{pt}(M_\varphi, E) = \{\varphi_i \, : \, i \in \mathbb{N}\}.$$

Proof. If $\lambda \in \sigma_{pt}(M_\varphi, E)$, then there exists $0 \neq x \in E$ such that $\varphi_i x_i = \lambda x_i$, for all $i \in \mathbb{N}$. Since $x \neq 0$, there is $i \in \mathbb{N}$ such that $\varphi_i = \lambda$, and therefore $\sigma_{pt}(M_\varphi, E) \subseteq \{\varphi_i \, : \, i \in \mathbb{N}\}$.

For the other inclusion, fix $j \in \mathbb{N}$ and let $e^j = (\delta_{ij})_i$. We have

$$M_\varphi(e^j) = (\varphi_i e^j_i)_i = \varphi_j e^j,$$

thus, $\varphi_j \in \sigma_{pt}(M_\varphi, E)$ and $\{\varphi_i \, : \, i \in \mathbb{N}\} \subseteq \sigma_{pt}(M_\varphi, E)$. □

Proposition 2.3.2 *Let v be a weight and let E be $\ell_p(v)$ for $1 \leq p \leq \infty$ or $c_0(v)$. Then,*

$$\sigma(M_\varphi, E) = \overline{\{\varphi_i \, : \, i \in \mathbb{N}\}}.$$

Proof. The proof mimics the one of Proposition 1.5.3 using Lemma 2.2.1. Indeed, since the spectrum is a closed set of $\mathbb{C}$ and $\{\varphi_i \, : \, i \in \mathbb{N}\} \subseteq \sigma(M_\varphi, E)$ by Lemma 2.3.1, we deduce $\overline{\{\varphi_i \, : \, i \in \mathbb{N}\}} \subseteq \sigma(M_\varphi, E)$.

For the other inclusion let $\mu \in \mathbb{C} \setminus \overline{\{\varphi_i \, : \, i \in \mathbb{N}\}}$. Then, there exists $\varepsilon > 0$ such that for every $i \in \mathbb{N}$, $|\varphi_i - \mu| \geq \varepsilon$. Define $\phi = (\phi_i)_i \in \mathbb{C}^{\mathbb{N}}$ by $\phi_i = \frac{1}{\varphi_i - \mu}$. Thus, ϕ is a bounded sequence and $M_\phi \in \mathcal{L}(E)$ by Lemma 2.2.1.(i). Clearly M_ϕ is the continuous inverse of $M_\varphi - \mu I$, therefore, $\mu \notin \sigma(M_\varphi, E)$. □

Proposition 2.3.3 *Let E be $\lambda_p(A)$ or $\kappa_p(A)$ for $1 \leq p \leq \infty$ or $p = 0$. Then the following assertions are equivalent:*

(1) $\mu \in \rho(M_\varphi, E)$,

(2) for each $n \in \mathbb{N}$, there exists $m \geq n$ such that

$$\sup_{i \in \mathbb{N}} \frac{a_n(i)}{a_m(i)} \frac{1}{|\varphi_i - \mu|} < \infty.$$

Proof. The equivalence follows from Lemma 2.2.1. □

Example 2.3.4 Recall Example 1.4.9. Set $A = (a_n)_n$, with $a_n(i) = i^n$ and $\varphi(i) = 1 - 2^{-i}$. Condition (2) in Proposition 2.3.3 does not hold for $\mu = 1$, since for all $k \in \mathbb{N}$ and all $l \geq k$,

$$\sup_{i \in \mathbb{N}} i^{k-l} 2^i = \infty.$$

Therefore, $1 \in \sigma(\lambda_p(A))$. Since $1 - 2^{-i} \in \sigma(\lambda_p(A))$ for all $i \in \mathbb{N}$ as well, then 1 is an accumulation point of the spectrum. In [2, Example 2.17] the authors show that M_φ is uniformly mean ergodic. These facts show that the Theorem A.2.5 by Dunford and Lin does not necessarily hold for spaces which are not Banach.

Lemma 2.3.5 *Let E be $\lambda_p(A)$ or $\kappa_p(A)$ for $1 \leq p \leq \infty$ or $p = 0$. Then*

$$\sigma^*(M_\varphi, E) = \overline{\sigma(M_\varphi, E)} = \overline{\{\varphi_i \,:\, i \in \mathbb{N}\}}.$$

Proof. The second equality can be proved in similar manner than that of Corollary 1.5.5.

The cases $p = 0, \infty$ are a particular case of Proposition 1.5.7. The proof of the cases with $1 \leq p < \infty$ is analogous to Proposition 1.5.7. We include it here with the notation for sequence spaces. Let $1 \leq p < \infty$. By the very definition, we have $\overline{\sigma(M_\varphi, E)} \subset \sigma^*(M_\varphi, E)$.

It remains to prove the other inclusion. Let $\lambda \in \mathbb{C} \setminus \overline{\{\varphi_i \,:\, i \in \mathbb{N}\}}$. There exists $\delta > 0$ such that if $|\mu - \lambda| \leq 2\delta$, then $\mu \in \rho(M_\varphi, E)$. We show that the set

$$\{(M_\varphi - \mu I)^{-1} \,:\, |\lambda - \mu| \leq \delta\}$$

is equicontinuous. By Lemma 2.3.1, we know that $\varphi_i \in \sigma(M_\varphi, E)$, and thus $|\varphi_i - \lambda| > 2\delta$. Therefore, if $\mu \in \mathbb{C}$ satisfies $|\lambda - \mu| \leq \delta$, then for every $i \in \mathbb{N}$ we have

$$|\varphi_i - \mu| \geq |\varphi_i - \lambda| - |\lambda - \mu| > 2\delta - \delta = \delta.$$

Now let $x \in X$ and let $y^\mu = (M_\varphi - \mu I)^{-1} x = M_{\left(\frac{1}{\varphi_i - \mu}\right)_i} x$, for each $\mu \in \mathbb{C}$ with $|\lambda - \mu| \leq \delta$. Then for every $i \in \mathbb{N}$ we have

$$|y_i^\mu| = \left| \frac{x_i}{\varphi_i - \mu} \right| \leq \frac{1}{\delta} |x_i|.$$

From this we deduce for $E = \lambda_p(A)$ that

$$p_n((M_\varphi - \mu I)^{-1} x) \leq \frac{1}{\delta} p_n(x),$$

for all $n \in \mathbb{N}$ and for all μ with $|\lambda - \mu| \leq \delta$. Therefore the set $\{(M_\varphi - \mu I)^{-1} \,:\, |\lambda - \mu| \leq \delta\}$ is equicontinuous, hence $\lambda \in \rho^*(M_\varphi, \lambda_p(A))$.

Now if $E = \kappa_p(A)$, by Theorem A.4.2, the topology of κ_p is given by the seminorms $p_{\overline{v}}$ and

$$p_{\overline{v}}((M_\varphi - \mu I)^{-1}x) \leq \frac{1}{\delta}\, p_{\overline{v}}(x),$$

for all $n \in \mathbb{N}$ and for all μ with $|\lambda - \mu| \leq \delta$. We conclude as well that the set $\{(M_\varphi - \mu I)^{-1} \,:\, |\lambda - \mu| \leq \delta\}$ is equicontinuous. □

2.4 Compactness and boundedness of diagonal operators

The first result of this section is well-known, we include a proof.

Lemma 2.4.1 *Let X be a Banach space with a Schauder basis $\{e_m : m \in \mathbb{N}\}$ and let P_n be the projection into the subspace generated by $\{e_1, \dots, e_n\}$, for $n \in \mathbb{N}$. Then a bounded set $K \subset X$ is relatively compact if, and only if, $(I - P_n)_n$ converges to 0 uniformly in K.*

Proof. Assume first that K is a compact set. By the very definition of a basis, $(I - P_n)_n$ converges to 0 pointwise in X. Therefore the set $\{I - P_n \,:\, n \in \mathbb{N}\}$ is equicontinuous. The compact-open topology coincides with the one of $\mathcal{L}_s(X)$ for equicontinuous sets. Therefore, $(I - P_n)_n$ converges to 0 uniformly in K.

For the converse fix $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for each $x \in K$ and for each $n \geq n_0$, we have

$$\|(I - P_n)x\| < \frac{\varepsilon}{2}.$$

Now, each $x \in K$ can be written as $x = P_n x + (I - P_n)x$. Therefore, if we fix $n \geq n_0$ we have

$$K \subseteq P_n K + B(0, \frac{\varepsilon}{2}).$$

Since $P_n K$ is a bounded set with finite dimension, it is already precompact (which is equivalent to relatively compact in metric complete spaces) by Heine-Borel theorem. Thus, there are $a_1, \dots, a_m \in X$ such that

$$P_n K \subseteq \bigcup_{j=1}^{m} B(a_j, \frac{\varepsilon}{2}).$$

From this we conclude, since

$$K \subseteq \bigcup_{j=1}^{m} B(a_j, \varepsilon)$$

implies that K is precompact. □

The following result is well known and can be found as an exercise in [63, Chapter 15].

Lemma 2.4.2 *Let $1 \leq p < \infty$, then the following assertions hold:*

(i) A bounded set $K \subset \ell_p$ is relatively compact if, and only if, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $x \in K$,

$$\sum_{j=N+1}^{\infty} |x_j|^p < \varepsilon^p, \quad \textit{and}$$

(ii) A bounded set $K \subset c_0$ is relatively compact if, and only if, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $x \in K$,

$$\sup_{j \geq N+1} |x_j| < \varepsilon.$$

Proof. Apply Lemma 2.4.1 computing the $\|(I - P_n)x\|$ with the corresponding norm. □

Lemma 2.4.3 *The multiplication operator $M_\phi : c_0 \longrightarrow c_0$ is compact if, and only if, $\phi \in c_0$.*

Proof. Assume M_ϕ is compact, then, if B_0 is the closed unit ball of c_0, $M_\phi B_0$ is relatively compact in c_0. Let $\varepsilon > 0$, then by Lemma 2.4.2, there exists $N \in \mathbb{N}$ such that for every $x \in B_0$ and every $j > N$, $|\phi_j x_j| < \varepsilon$. This holds in particular for $x = e_j \in B_0$, and thus for every $j > N$, $|\phi_j| < \varepsilon$ and $\phi \in c_0$.

Now assume $\phi \in c_0$. First note that, since $\phi \in \ell_\infty$, $M_\phi B_0 \in c_0$. It remains to show that it is relatively compact. To do so, let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $j > N$, $|\phi_j| < \varepsilon$. We have for $j > N$ and $x \in B_0$, $|x_j \phi_j| < \varepsilon |x_j| < \varepsilon$. And thus,

$$\sup_{z \in M_\phi B_0} \sup_{j \geq N+1} |z_j| = \sup_{x \in B_0} \sup_{j \geq N+1} |\phi_j x_j| < \varepsilon,$$

and we conclude using Lemma 2.4.2. □

Lemma 2.4.4 *If $1 \leq p < \infty$, then the multiplication operator $M_\phi : \ell_p \longrightarrow \ell_p$ is compact if, and only if, $\phi \in c_0$.*

Proof. Assume M_ϕ is compact, then, if B_p is the closed unit ball of ℓ_p, $M_\phi B_p$ is relatively compact in ℓ_p. Let $\varepsilon > 0$, then by Lemma 2.4.2, there exists $N \in \mathbb{N}$ such that for every $x \in B_0$,

$$\sum_{j=N+1}^{\infty} |\phi_j x_j|^p < \varepsilon^p.$$

This holds in particular for $x = e_j \in B_p$, and thus for every $j > N$, $|\phi_j|^p < \varepsilon^p$ and $\phi \in c_0$.

Now assume $\phi \in c_0$. First note that, since $\phi \in \ell_\infty$, $M_\phi B_p \in \ell_p$. It remains to show that it is relatively compact. To do so, let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $j > N$, $|\phi_j| < \varepsilon$. We have, for $x \in B_p$

$$\sum_{j=N+1}^{\infty} |\phi_j x_j|^p < \varepsilon^p \sum_{j=N+1}^{\infty} |x_j|^p < \varepsilon^p.$$

Conclusion follows by Lemma 2.4.2. □

Lemma 2.4.5 *Let $v, w : \mathbb{N} \longrightarrow \mathbb{C}$, with $v, w > 0$ and $1 \le p < \infty$. Then, the following assertions are equivalent:*

(1) The multiplication operator $M_\varphi : c_0(v) \longrightarrow c_0(w)$ is compact.

(2) The multiplication operator $M_\varphi : \ell_p(v) \longrightarrow \ell_p(w)$ is compact.

(3) $\lim_{i\to\infty} \frac{w_i|\varphi_i|}{v_i} = 0$.

Proof. We show (1) $\iff$ (3). Consider the isometries $M_v : c_o(v) \longrightarrow c_0$, $M_v(x) = (v_i x_i)_i$ and $M_w : c_o(w) \longrightarrow c_0$, $M_w(x) = (w_i x_i)_i$. Let $\phi = \left(\frac{w_i \varphi_i}{v_i}\right)$, then $M_\phi = M_w \circ M_\varphi \circ M_v^{-1}$. Thus, M_φ is compact if, and only if, M_ϕ is compact. Apply Lemma 2.4.3.

For (2) $\iff$ (3) follow the same reasoning, with Lemma 2.4.4. □

2.4.1 Projective limits

Let $E_m \supset E_{m+1}$ and $F_n \supset F_{n+1}$ be continuous injections of Banach spaces. Let $E = \bigcap_m E_m$, $F = \bigcap_n F_n$ and endow them with the projective topologies $E = \operatorname{proj}_m E_m$ and $F = \operatorname{proj}_n F_n$. Let $\pi_m : E \longrightarrow E_m$ and $\pi_n : F \longrightarrow F_n$ be the natural continuous injections.

Lemma 2.4.6 *A set $K \subset E$ is relatively compact in E if, and only if, $\pi_m(K)$ is relatively compact in E_m for every $m \in \mathbb{N}$.*

Proof. The direct implication follows since π_m is continuous, and thus $\pi_m(K)$ is relatively compact.

Now assume $\pi_m(K)$ is relatively compact for every $m \in \mathbb{N}$. Then by Tijonov's theorem, $\prod_m \pi_m(K)$ is relatively compact in $\prod_m E_m$. But $K \subset E$, then $K \subset E \cap \prod_m \pi_m(K)$ is relatively compact in $E \cap \prod_m E_m = E$. □

This lemma is a well-known result and can be seen, for example, in [5, Lemma 25].

Lemma 2.4.7 *Assume E is dense in E_m for all $m \in \mathbb{N}$. Let $T : E \longrightarrow F$ be a continuous linear operator. Then*

(i) T is bounded if, and only if, there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, T has a unique continuous linear extension $T_{m,n} : E_m \longrightarrow F_n$.

(ii) T is compact if, and only if, there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, T has a unique compact linear extension $T_{m,n} : E_m \longrightarrow F_n$.

Proof. Only the proof for the compact case is shown, since both are quite similar. Denote by B_m the unit ball of E_m. Assume there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, T has a unique compact linear extension $T_{m,n} : E_m \longrightarrow F_n$. We have $T(B_m \cap E) \subset T_{m,n}(B_m) \subset F_n$, and thus $T(B_m \cap E)$ is relatively compact in F_n for each $n \in \mathbb{N}$. This fact together with Lemma 2.4.6 and noting that $B_m \cap E$ is a neighbourhood of 0 in E gives us that T is compact.

Assume now that T is compact, then there exists a neighbourhood of 0 U such that $T(U)$ is relatively compact in F. There exist $m \in \mathbb{N}$ and $\alpha > 0$ such that $B_m \cap E \subset \alpha U$. Thus $T(B_m \cap E)$ is relatively compact in F, and therefore, in each F_n. If p_m is the norm of E_m, then (E, p_m) is a normed space and the operator $T : (E, p_m) \longrightarrow F_n$ is compact for each $n \in \mathbb{N}$, since $B_m \cap E$ is the unit ball of (E, p_m). Now, since (E, p_m) is dense in E_m, there exists a unique linear continuous extension $T_{m,n} : E_m \longrightarrow F_n$, which is also compact, for each $n \in \mathbb{N}$. □

Proposition 2.4.8 *For $1 \leq p < \infty$ or $p = 0$ and $A = (a_m)_m$ and $B = (b_n)_n$ Köthe matrices, then*

(i) $M_\varphi : \lambda_p(A) \longrightarrow \lambda_p(B)$ is bounded if, and only if, there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $\sup_i \frac{b_n(i)|\varphi_i|}{a_m(i)} < \infty$.

(ii) $M_\varphi : \lambda_p(A) \longrightarrow \lambda_p(B)$ is compact if, and only if, there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $\lim_{i\to\infty} \frac{b_n(i)|\varphi_i|}{a_m(i)} = 0$.

Proof. For both cases apply Lemma 2.4.7, since the dense space of null sequences is inside of $\lambda_p(A)$. For compactness use it with Lemma 2.4.5.(1) or Lemma 2.4.5.(2), for each case and with $v = a_m$ and $w = b_n$. □

2.4.2 Inductive limits

Lemma 2.4.9 *Let $E = \operatorname{ind}_m E_m$ and $F = \operatorname{ind}_n F_n$ be inductive limits of Banach spaces. Assume F is regular. Then the continuous and linear operator $T : E \longrightarrow F$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the restriction $T : E_m \longrightarrow F_n$ is continuous.*

Proof. Denote by B_m and C_n the unit balls of E_m and F_n respectively. By regularity the condition of T being bounded is equivalent to the condition

$$\text{there are a 0-neighbourhood } U, n \text{ and } \lambda > 0 \text{ such that } T(U) \subset \lambda C_n. \quad (*)$$

Thus we prove that $(*)$ is equivalent to the restrictions being continuous. First assume $(*)$ and let $m \in \mathbb{N}$. Since B_m is bounded, there exists $\lambda_m > 0$ such that $B_m \subset \lambda_m U$. Thus $T(B_m) \subset \lambda T(U) \subset \lambda\lambda_m T(C_n)$, and therefore $T : E_m \longrightarrow F_n$ is continuous.

Now we assume that there exists $n \in \mathbb{N}$ such that $T : E_m \longrightarrow F_n$ is continuous for each $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ there exists a $\lambda_m > 0$ such that $T(B_m) \subset \lambda_m C_n$. Define $U = \Gamma(\bigcup_{m=1}^{\infty} \lambda_m^{-1} B_m)$, where Γ denotes the absolutely convex hull of the union. Clearly U is a 0-neighbourhood. Let $x \in U$, then

$$x = \sum_{j=1}^{s} \mu_j \lambda_j^{-1} b_j,$$

for some $s \in \mathbb{N}$, $b_j \in B_j$ and with $\sum |\mu_j| \leq 1$. We have

$$Tx = \sum_{j=1}^{s} \mu_j \lambda_j^{-1} T b_j.$$

Since $\lambda_j^{-1} T b_j \in C_n$ and C_n is absolutely convex, we have $Tx \in C_n$ and $TU \subset C_n$ and thus, $(*)$. □

Proposition 2.4.10 *Let $1 \leq p < \infty$ and $A = (a_m)_m$ and $B = (b_n)_n$ Köthe matrices, then $M_\varphi : \kappa_p(A) \longrightarrow \kappa_p(B)$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $\sup_i \frac{b_n(i)|\varphi_i|}{a_m(i)} < \infty$.*

The result also holds for $p = 0$ if the Köthe matrix B is regularly decreasing.

Proof. Apply Lemma 2.4.9. The regularity of $\kappa_p(B)$, for $1 \leq p < \infty$ and also for $p = 0$, follows from Theorem A.4.2. □

Let $E = \operatorname{ind}_m E_m$. Recall that E is *precompactly retractive* if for each compact $K \subset E$, there exists m such that $K \subset E_m$ and it is compact there. Furthermore, E is called *sequentially retractive* if for each null sequence in E there exists m

such that the sequence is contained in E_m and is null there. It is well-known that these two concepts are equivalent and they are even equivalent to E being strongly boundedly retractive (Definition A.3.3) [35, Theorem 16] or [80]. For our case we have the following lemma, which can be found in [15, Theorem 4.11].

Lemma 2.4.11 *Let $1 \leq p \leq \infty$ or $p = 0$, then $\kappa_p(A)$ is precompactly retractive if, and only if, A is regularly decreasing.*

Lemma 2.4.12 *If $(K_n)_n$ is a sequence of precompact sets in a Fréchet space X, then there exists a sequence $(\mu_n)_n$ of strictly positive numbers such that $\bigcup_n \mu_n K_n$ is precompact.*

Proof. Let (U_n) be a decreasing basis of neighbourhoods of 0. Then for each $n \in \mathbb{N}$, there exists $\lambda_n > 0$ such that $K_n \subset \lambda_n U_n$. Let $\mu_n = \lambda_n^{-1}$ and $K = \bigcup_n \mu_n K_n$. Fix a 0-neighbourhood U, then there is $s \in \mathbb{N}$ with $U_s \subset U$. Then, for $n \geq s$, $U_n \subset U_s$ and

$$K^1 = \bigcup_{n=s}^{\infty} \mu K_n \subset \bigcup_{n=s}^{\infty} U_n \subset U_s \subset U.$$

On the other hand we have that the set

$$K^2 = \bigcup_{n=1}^{s-1} \mu_n K_n$$

is a finite union of precompact sets and thus precompact itself, then there exist $x_1, \ldots, x_t \in X$ such that

$$K^2 \subset \bigcup_{n=1}^{t} (x_i + U).$$

We finally get

$$K = K^1 \cup K^2 \subset \bigcup_{n=1}^{t} (x_i + U) \cup U,$$

and K is precompact. □

Note that in our inductive limits, since they are complete, precompactness and relative compactness are the same.

Lemma 2.4.13 *Let $E = \text{ind}_m E_m$ and $F = \text{ind}_n F_n$ be inductive limits of Banach spaces. Assume F is precompactly retractive. Then the continuous and linear operator $T : E \longrightarrow F$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the restriction $T : E_m \longrightarrow F_n$ is compact.*

Proof. Assume T is compact, then there exists a 0-neighbourhood U such that $T(U)$ is relatively compact in F. Since F is precompactly retractive, there exists $n \in \mathbb{N}$ such that $T(U) \subset F_n$ and it is relatively compact. For each $m \in \mathbb{N}$ there exists λ_m such that the unit ball B_m is inside of $\lambda_m U$, and thus $T(B_m)$ is relatively compact in F_n and $T : E_m \longrightarrow F_n$ is compact.

Now assume the other assertion, then $T(B_m)$ is relatively compact in F_n for every $m \in \mathbb{N}$. By Lemma 2.4.12, there exists a sequence $(\mu_m)_m$ such that $\bigcup_m \mu_m T(B_m)$ is relatively compact. Let $U = \Gamma(\bigcup_m \mu_m B_m)$ be the absolutely convex hull of that union, which is a 0-neighbourhood in E. Now,

$$T(U) \subset \Gamma(\bigcup_m \mu_m T(B_m)),$$

which is the absolutely convex hull of a relatively compact set, and thus relatively compact and $T : E \longrightarrow F$ is compact. □

Proposition 2.4.14 *Let $1 \leq p \leq \infty$ or $p = 0$, let $A = (a_m)_m$ be a Köthe matrix and let $B = (b_n)_n$ be a regularly decreasing Köthe matrix. Then $M_\varphi : \kappa_p(A) \longrightarrow \kappa_p(B)$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $\lim_i \frac{b_n(i)|\varphi_i|}{a_m(i)} = 0$.*

Proof. Apply Lemma 2.4.13, Lemma 2.4.11 and Lemma 2.4.5. □

2.5 About the topology of sets of symbols

2.5.1 Topology of the set of multipliers

The results given in Section 1.6 relating to the topology of the set of multipliers between projective limits of Banach spaces of continuous functions can be stated also for Köthe echelon spaces. The notation used in this section is that of Section 1.6, we recall it here.

Let $\mathcal{A} = (a_m)_m$ and $\mathcal{B} = (b_n)_n$ be Köthe matrices, let $1 \leq p \leq \infty$ or $p = 0$ and let $\varphi \in \mathbb{C}^{\mathbb{N}}$.

By Lemma 2.2.1, M_φ is continuous from $\lambda_p(\mathcal{A})$ to $\lambda_p(\mathcal{A})$, that is φ is a multiplier, if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$\sup_{i \in \mathbb{N}} \frac{b_n(i)|\varphi_i|}{a_m(i)} < \infty.$$

This is actually equivalent to

$$\varphi \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \ell_p\left(\frac{b_n}{a_m}\right) =: E_{\mathcal{AB}}.$$

Endow the set $E_{\mathcal{AB}}$ of *multipliers* from E to F with the topology

$$E_{\mathcal{AB}} = \operatorname*{proj}_n \operatorname*{ind}_m \ell_p \left(\frac{b_n}{a_m} \right),$$

In fact, we can write $E_{\mathcal{AB}}$ as

$$E_{\mathcal{AB}} = \operatorname*{proj}_n \kappa_p(\mathcal{V}_n),$$

where $\mathcal{V}_n = \left(\frac{b_n}{a_m}\right)_m$ is a decreasing family of weights for each $n \in \mathbb{N}$. Now let $V = (1/a_m)_m$, then we have $\kappa_p(\mathcal{V}_n) \simeq \kappa_p(V)$, for each $n \in \mathbb{N}$.

Now assume that $\kappa_p(V) = K_p(V)$. This condition is characterized in Section A.4. Then $\kappa_p(\mathcal{V}_n) \simeq K_p(V)$, for each $n \in \mathbb{N}$. This way, the topology of $\kappa_p(\mathcal{V}_n)$ is given by the seminorms

$$p_{\overline{v}}(x) := b_n(i)\overline{v}(i)|x_i|, \quad x \in \kappa_p(\mathcal{V}_n), \overline{v} \in \overline{V}.$$

Then the fundamental system of seminorms of $E_{\mathcal{AB}}$ is

$$q_{n,\overline{v}}(x) = \sup_{i \in \mathbb{N}} b_n(i)\overline{v}(i)|x_i|, \quad x \in \kappa_p(\mathcal{V}_n), n \in \mathbb{N}, \overline{v} \in \overline{V}.$$

On the other hand, the topology of the space $\mathcal{L}_b(\lambda_p(\mathcal{A}), \lambda_p(\mathcal{B}))$ can be described as follows. Denote the seminorms of $\lambda_p(\mathcal{B})$ by q_n, where

$$q_n(x) = \|(a_n(i)x_i)_i\|_p, \quad x \in \lambda_p(\mathcal{B}), n \in \mathbb{N}.$$

Then the seminorms defining the topology of $\mathcal{L}_b(E, F)$ are

$$r_{n,B}(M) := \sup_{x \in B} q_n(Mx),$$

where $M \in \mathcal{L}_b(\lambda_p(\mathcal{A}), \lambda_p(\mathcal{B})), n \in \mathbb{N}, B \subset \lambda_p(\mathcal{A})$ bounded.

The condition $\kappa_p(V) = K_p(V)$ holds directly for $1 \leq p < \infty$. The condition $\kappa_0(V) = K_0(V)$ is equivalent to V being regularly decreasing. Finally, $\kappa_\infty(V) = K_\infty(V)$ holds if, and only if, V has the property (D) if, and only if, $\lambda_1(V)$ is distinguished.

Theorem 2.5.1 *Let* $1 \leq p \leq \infty$ *or* $p = 0$. *Assume* $\kappa_p(V) = K_p(V)$. *Then the map* $T : E_{\mathcal{AB}} \longrightarrow \mathcal{L}_b(\lambda_p(\mathcal{A}), \lambda_p(\mathcal{B}))$, $\varphi \mapsto M_\varphi$, *is a topological isomorphism into.*
Proof. The injectivity of T is easy.

1. Continuity of T: Fix $n \in \mathbb{N}$ and let $B \subset \lambda_p(\mathcal{A})$ be bounded. Apply Lemma 2.2.7 to find $\overline{v} \in \overline{V}$ such that $B \subset B_{\overline{v}}$. For $\varphi \in E_{\mathcal{AB}}$ we have

$$\begin{aligned} r_{n,B}(T(\varphi)) &= r_{n,B}(M_\varphi) \\ &= \sup_{x\in B} \left\| \left(b_n(i)\overline{v}(i)\varphi_i \frac{x_i}{\overline{v}(i)} \right)_i \right\|_p \\ &\le \sup_{i\in\mathbb{N}} b_n(i)\overline{v}(i)|\varphi_i| \sup_{x\in B} \left\| \left(\frac{x_i}{\overline{v}(i)} \right)_i \right\|_p \\ &\le \sup_{i\in\mathbb{N}} b_n(i)\overline{v}(i)|\varphi_i| = q_{n,\overline{v}}(\varphi). \end{aligned}$$

2. Continuity of T^{-1}: Fix $n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$. For $j \in \mathbb{N}$ let $e^j = (\delta_{ij})_i$. Thus, $e^j\overline{v}(j) \in B_{\overline{v}}$. For $j \in \mathbb{N}$ we have

$$\begin{aligned} r_{n,B_{\overline{v}}}(T(\varphi)) &= r_{n,B_{\overline{v}}}(M_\varphi) \\ &= \sup_{x\in B_{\overline{v}}} q_n(M_\varphi x) \\ &\ge q_n(M_\varphi(e^j\overline{v}(j))) \\ &= q_n(\overline{v}(j)\varphi_j e^j) = b_n(j)\overline{v}(j)|\varphi(j)| \end{aligned}$$

Since $j \in \mathbb{N}$ is arbitrary, we have $q_{n,\overline{v}}(\varphi) \le r_{n,B_{\overline{v}}}(M_\varphi)$. □

The following result about the locally convex properties of the space $E_{\mathcal{AB}}$ of multipliers is a direct consequence of [1]. We refer the reader to this article and to [82] for the relevance of the technical conditions (Q) and (wQ).

Proposition 2.5.2 *Consider the family* $(\mathcal{V}_N)_N = \left(\left(\frac{b_N}{a_m}\right)_m\right)_N$.

(i) Assume $(\mathcal{V}_N)_N$ *satisfies condition* (Q), *i.e.*

$$\forall N\ \exists M \ge N\ \exists n\ \forall K \ge M\ \forall m, \varepsilon > 0\ \exists S\ \exists k\ \forall x\ :$$

$$\frac{a_m(x)}{b_M(x)} \le \max\left(\varepsilon\frac{a_n(x)}{b_N(x)}, S\frac{a_k(x)}{b_K(x)}\right). \qquad \text{(Q)}$$

Then $E_{\mathcal{AB}}$ *is barrelled.*

(ii) If $E_{\mathcal{AB}}$ *is barrelled, then* $(\mathcal{V}_N)_N$ *satisfies condition* (wQ), *i.e.*

$$\forall N\ \exists M \ge N\ \exists n\ \forall K \ge M\ \forall m\ \exists k\ \exists S \ge 0\ \forall x\ :$$

$$\frac{a_m(x)}{b_M(x)} \le S\max\left(\frac{a_n(x)}{b_N(x)}, \frac{a_k(x)}{b_K(x)}\right). \qquad \text{(wQ)}$$

2.5.2 Topology of the set of bounded multiplication operators

Let $A = (a_m)_m$ and $B = (b_n)_n$ be Köthe matrices. Let $1 \leq p < \infty$ or $p = 0$ and $E = \lambda_p(A)$ and $F = \lambda_p(B)$. By Proposition 2.4.8, the operator M_φ is bounded from E to F if, and only if, there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$\sup_{i \in \mathbb{N}} \frac{b_n(i)|\varphi_i|}{a_m(i)} < \infty.$$

This is equivalent to

$$\varphi \in \bigcup_m \bigcap_n \ell_\infty \left(\frac{b_n}{a_m} \right) =: F_{\mathcal{AB}}.$$

We can endow $F_{\mathcal{AB}}$ with the topology

$$F_{\mathcal{AB}} = \operatorname*{ind}_m \operatorname*{proj}_n \left(\frac{b_n}{a_m} \right),$$

and we get an (LF)-space. Clearly $F_{\mathcal{AB}} \subset E_{\mathcal{AB}}$, the next result gives the equality (even topological).

Theorem 2.5.3 *The following assertions hold for $F_{\mathcal{AB}}$ and $E_{\mathcal{AB}}$:*

(i) $F_{\mathcal{AB}} = E_{\mathcal{AB}}$ algebraically if, and only if, condition (B) holds, i.e.

$$\forall (n(\nu))_{\nu \in \mathbb{N}} \subset \mathbb{N}\ \exists n\ \forall N\ \exists M,\ c_N > 0\ :\ \frac{b_N}{a_n} \leq c_N \max_{\nu = 1, \ldots, M} \frac{b_\nu}{a_{n(\nu)}}. \tag{B}$$

(ii) If $F_{\mathcal{AB}} = E_{\mathcal{AB}}$ algebraically and A satisfies condition (Q), then the equality holds also topologically, where

$$\forall N\ \exists M \geq N\ \exists n\ \forall K \geq M\ \forall m, \varepsilon > 0\ \exists S\ \exists k\ \forall x\ :$$

$$\frac{a_m(x)}{b_M(x)} \leq \max \left(\varepsilon \frac{a_n(x)}{b_N(x)}, S \frac{a_k(x)}{b_K(x)} \right). \tag{Q}$$

(iii) If $F_{\mathcal{AB}} = E_{\mathcal{AB}}$ algebraically and topologically, then A satisfies condition (wQ), i.e.

$$\forall N\ \exists M \geq N\ \exists n\ \forall K \geq M\ \forall m\ \exists k\ \exists S \geq 0\ \forall x\ :$$

$$\frac{a_m(x)}{b_M(x)} \leq S \max \left(\frac{a_n(x)}{b_N(x)}, \frac{a_k(x)}{b_K(x)} \right). \tag{wQ}$$

Proof. Use [1, Theorem 3.10 and Corollary 3.11] considering $a_{N,n} = \frac{b_N}{a_n}$. □

2.6 Power series spaces

In this section we consider a particular case of Köthe echelon spaces, the power series spaces, and state the results obtained for the general case for them. Given $\alpha = (\alpha_j)_j$ an increasing non-negative sequence such that $\lim_j \alpha_j = \infty$, $r \in \mathbb{R} \cup \{+\infty\}$ and $1 \leq p < \infty$, we define

$$\Lambda_r^p(\alpha) = \left\{ x \in \mathbb{C}^{\mathbb{N}} \ : \ \|x\|_t^p := \sum_{j \in \mathbb{N}} |x_j|^p e^{pt\alpha_j} < \infty, \text{ for all } t < r \right\},$$

$$\Lambda_r^\infty(\alpha) = \left\{ x \in \mathbb{C}^{\mathbb{N}} \ : \ \|x\|_t := \sup_{j \in \mathbb{N}} |x_j| e^{t\alpha_j} < \infty, \text{ for all } t < r \right\}.$$

These spaces are the so-called *power series spaces.* General results on these spaces can be extracted from chapter 29 in [63] and will be stated without proof. The relevance of the power series spaces appears in the characterization of nuclear locally convex spaces and nuclear Fréchet spaces.

For a fixed $1 \leq p \leq \infty$ we have $\Lambda_r^p(\alpha) = \Lambda_0^p(\alpha)$, for each $r < \infty$. Furthermore, a definition like the one above could be done to define $\Lambda_0^0(\alpha)$ and $\Lambda_\infty^0(\alpha)$, but those spaces coincide, even topologically, with $\Lambda_0^\infty(\alpha)$ and $\Lambda_\infty^\infty(\alpha)$. Hence, the only spaces of our interest are $\Lambda_0^p(\alpha)$ and $\Lambda_\infty^p(\alpha)$. Both of these spaces are Montel spaces.

This first result shows that the power series spaces are, indeed, special cases of echelon spaces.

Lemma 2.6.1 *Let $1 \leq p \leq \infty$. If $\alpha = (\alpha_j)_j$ is an increasing non-negative sequence with $\lim_{j\to\infty} \alpha_j = \infty$, then*

(i) $A = (a_n)_n = ((e^{-\frac{\alpha_j}{n}})_j)_n$ is a Köthe matrix and

$$\lambda_p(A) = \Lambda_0^p(\alpha) \quad \textit{topologically},$$

(ii) $A = (a_n)_n = ((e^{n\alpha_j})_j)_n$ is a Köthe matrix and

$$\lambda_p(A) = \Lambda_\infty^p(\alpha) \quad \textit{topologically}.$$

Proposition 2.6.2 *Let $1 \leq p \leq \infty$. Then $M_\varphi : \Lambda_0^p(\alpha) \longrightarrow \Lambda_0^p(\alpha)$ is continuous if, and only if,*

$$\lim_{i\to\infty} \frac{\log^+ |\varphi_i|}{\alpha_i} = 0,$$

where $\log^+(a) = \max\{0, \log(a)\}$.

Proof. First let us assume that M_φ is continuous, then, by Lemma 2.2.1, for each $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $M > 1$ such that for all $i \in \mathbb{N}$

$$\frac{a_n(i)}{a_m(i)}|\varphi_i| \leq M.$$

From this, it follows

$$\frac{e^{-\frac{\alpha_i}{n}}}{e^{-\frac{\alpha_i}{m}}}|\varphi_i| \leq M,$$
$$\log|\varphi_i| \leq \log M + \left(\frac{1}{n} - \frac{1}{m}\right)\alpha_i,$$
$$\frac{\log|\varphi_i|}{\alpha_i} \leq \frac{\log M}{\alpha_i} + \frac{1}{n} - \frac{1}{m} < \frac{\log M}{\alpha_i} + \frac{1}{n}.$$

Since the right-hand side is positive we have

$$0 \leq \frac{\log^+|\varphi_i|}{\alpha_i} \leq \frac{\log M}{\alpha_i} + \frac{1}{n}.$$

Now, let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $n > 2/\varepsilon$. Then there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$, $\frac{\log M}{\alpha_i} < \varepsilon/2$. Then, for $i \geq i_0$,

$$\frac{\log^+|\varphi_i|}{\alpha_i} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, we have $\lim_{i\to\infty} \frac{\log^+|\varphi_i|}{\alpha_i} = 0$.

For the other implication let $n \in \mathbb{N}$ and let $m = n + 1$. Then there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$,

$$\frac{\log|\varphi_i|}{\alpha_i} \leq \frac{\log^+|\varphi_i|}{\alpha_i} < \frac{1}{n} - \frac{1}{n+1}.$$

From there we deduce that for $i \geq i_0$,

$$\frac{a_n(i)}{a_m(i)}|\varphi_i| \leq 1.$$

We conclude by Lemma 2.2.1. □

Proposition 2.6.3 *Let $1 \leq p \leq \infty$. Then, $M_\varphi : \Lambda^p_\infty(\alpha) \longrightarrow \Lambda^p_\infty(\alpha)$ is continuous if, and only if,*

$$\sup_{i\in\mathbb{N}} \frac{\log|\varphi_i|}{\alpha_i} < \infty.$$

Proof. Assume that M_φ is continuous, then, by Lemma 2.2.1, for each $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $M > 1$ such that for all $i \in \mathbb{N}$

$$\frac{\log|\varphi_i|}{\alpha_i} \leq \frac{\log M}{\alpha_i} + (m-n).$$

This holds in particular for $n = 1$, which gives us

$$\frac{\log|\varphi_i|}{\alpha_i} \leq \frac{\log M}{\alpha_1} + (m-1),$$

where M and m are in this case fixed, thus we finish.

For the converse, we know there exists $K > 0$ such that

$$\frac{\log|\varphi_i|}{\alpha_i} \leq K.$$

Now for each $n \in \mathbb{N}$ let $m \geq n + K$, and we get

$$\frac{e^{n\alpha_i}}{e^{m\alpha_i}}|\varphi_i| = e^{(n-m)\alpha_i + \log|\varphi_i|} \leq e^{\log|\varphi_i| - K\alpha_i} \leq 1 =: M.$$

We conclude using Lemma 2.2.1. □

Proposition 2.6.4 *Let $1 \leq p \leq \infty$. Then, $M_\varphi : \Lambda_0^p(\alpha) \longrightarrow \Lambda_0^p(\alpha)$ is compact if, and only if, it is bounded if, and only if, there exists $n \in \mathbb{N}$ such that $\varphi = (\varphi_i)_i \in \Lambda_{1/n}^\infty(\alpha)$.*

Proof. The equivalence between compactness and boundedness is straightforward since the space is Montel.

We have $\varphi \in \Lambda_{1/n}^\infty(\alpha)$ if, and only if, for each $m \in \mathbb{N}$,

$$\sup_i e^{(\frac{1}{n} - \frac{1}{m})\alpha_i}|\varphi_i| < \infty.$$

Apply Proposition 2.4.8 with $A = (e^{-\alpha_i/n})$. □

Proposition 2.6.5 *Let $1 \leq p \leq \infty$. Then, $M_\varphi : \Lambda_\infty^p(\alpha) \longrightarrow \Lambda_\infty^p(\alpha)$ is compact if, and only if, it is bounded if, and only if,*

$$\lim_i \frac{\log|\varphi_i|}{\alpha_i} = -\infty.$$

Proof. The equivalence between compactness and boundedness is straightforward since the space is Montel.

Assume $\lim_i \frac{\log|\varphi_i|}{\alpha_i} = -\infty$. Fix $n = 1$, let $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$. Then there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$, $\alpha_i > 1$ and

$$\frac{\log|\varphi_i|}{\alpha_i} < n - m + \log\varepsilon < n - m + \frac{\log\varepsilon}{\alpha_i}.$$

From here we deduce that for $n = 1$ and for every $m \in \mathbb{N}$,

$$\lim_{i\to\infty} \frac{a_m(i)}{a_n(i)}|\varphi_i| = 0,$$

with $a_n(i) = e^{n\alpha_i}$. Apply Proposition 2.4.8 to deduce the operator is compact (and thus, bounded).

Assume M_φ is bounded. Then there exists $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ there exists $M_m > 1$ with $e^{(m-n)\alpha_i}|\varphi_i| < M_m$. Thus,

$$\frac{\log|\varphi_i|}{\alpha_i} \leq \frac{\log M_m}{\alpha_i} + n - m.$$

Let $M > 0$, take $m > M + n + 1$, then

$$\frac{\log|\varphi_i|}{\alpha_i} \leq \frac{\log M_m}{\alpha_i} - M - 1.$$

Then, for i great enough $\frac{\log M_m}{\alpha_i} < 1$ and thus in that case,

$$\frac{\log|\varphi_i|}{\alpha_i} < -M.$$

□

Due to Proposition 2.2.9, the ergodic properties of these operators are direct.

Theorem 2.6.6 *Let $1 \leq p \leq \infty$. The following assertions are equivalent:*

(1) $M_\varphi \in \mathcal{L}(\Lambda_0^p(\alpha))$ is mean ergodic,

(2) $M_\varphi \in \mathcal{L}(\Lambda_0^p(\alpha))$ is uniformly mean ergodic,

(3) $M_\varphi \in \mathcal{L}(\Lambda_0^p(\alpha))$ is power bounded,

(4) $\|\varphi\|_\infty \leq 1$.

Proof. Since the space is Montel, the operator is mean ergodic if, and only if, it is uniformly mean ergodic, by Proposition 2.2.9. The equivalence with (3) and (4) follows from Proposition 2.2.4 and Theorem 2.2.6. □

Theorem 2.6.7 *Let $1 \leq p \leq \infty$. The following assertions are equivalent:*

(1) $M_\varphi \in \mathcal{L}(\Lambda_\infty^p(\alpha))$ *is mean ergodic,*

(2) $M_\varphi \in \mathcal{L}(\Lambda_\infty^p(\alpha))$ *is uniformly mean ergodic,*

(3) $M_\varphi \in \mathcal{L}(\Lambda_\infty^p(\alpha))$ *is power bounded,*

(4) $\|\varphi\|_\infty \leq 1$.

The spectral properties of diagonal operators on power series spaces follow directly from those of the Köthe echelon spaces in Section 2.3. In particular the following result gives a characterization of the spectrum.

Proposition 2.6.8 *Let $1 \leq p \leq \infty$. Then*

(i) $\mu \in \rho(M_\varphi, \Lambda_0^p(\alpha))$ *if, and only if,*

$$\lim_{i\to\infty} \frac{\log^+ |\varphi_i - \mu|}{\alpha_i} = 0,$$

(ii) $\mu \in \rho(M_\varphi, \Lambda_\infty^p(\alpha))$ *if, and only if,*

$$\sup_{i\in\mathbb{N}} \frac{-\log |\varphi_i - \mu|}{\alpha_i} < \infty.$$

Proof. This is a consequence of Proposition 2.3.3 and the continuity formulas in Propositions 2.6.2 and 2.6.3. □

Chapter 3

Banach spaces of analytic functions

3.1 Introduction and notation

3.1.1 Introduction

The properties of the composition operator acting between various spaces of holomorphic functions have been thoroughly studied, especially relating the properties of C_φ to those of φ. See the books of Cowen and McCluer [37] and Shapiro [74]. We study power boundedness, mean ergodicity and uniform mean ergodicity for composition operators defined on different spaces of holomorphic functions on the disc. This research for composition operators was firstly studied by Bonet and Domański in [26] when they studied the operator acting on $H(U)$, with U a domain in a Stein manifold. Later in [84], Wolf considered the composition operator on the general weighted Bergman spaces of infinite order $H_v^\infty(\mathbb{D})$. Bonet and Ricker [32] characterized the power boundedness and mean ergodicity of the multiplication operator on the general weighted Bergman spaces of infinite type $H_v(\mathbb{D})$ and $H_v^0(\mathbb{D})$. More recently, Beltrán-Meneu, Gómez-Collado, Jordá and Jornet considered in [11] and [12] the ergodic properties of the composition operator on the algebra of the disc $A(\mathbb{D})$ and on the space of holomorphic and bounded functions on the disc $H^\infty(\mathbb{D})$ and the weighted composition operators in the space of holomorphic functions $H(\mathbb{D})$. Arendt, Chalendar, Kummar and Srivastava extended the results of [11] to the convergence of the sequence of powers of operators besides the Cesáro means, including $A(\mathbb{D})$, $H^\infty(\mathbb{D})$ and also the Wiener algebra $W(\mathbb{D})$ [6]. The article [7] by the same authors includes some results close to ours, see Propositions 3.3.12 and 3.3.13. Independently from [7] and ourselves, Han and Zhou [45] characterized (uniform) mean ergodicity of composition operators on the Hardy spaces H^p, $1 \leq p < \infty$.

Some of the spaces we focus on here share analogies with $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$. In this chapter we consider composition operators on $\mathcal{B}_p$, $\mathcal{B}_p^0$ (Bloch spaces of order $p \geq 1$), H_v, H_v^0 (weighted Bergman space of infinite order, for appropriate weights v), A^p (Bergman space of order $p \geq 1$) and H^p (Hardy space of order $p \geq 1$). We prove (Theorem 3.3.7) that on these spaces power bounded composition operators (whenever they are well defined) are precisely those defined by a symbol with an interior fixed point. In the case of $\mathcal{B}_p^0$, H_v^0, A^p and H^p, power boundedness is even equivalent to mean ergodicity. The study of uniform mean ergodicity on all these spaces depends deeply on the concept of *quasicompactness* (Definition 3.1.3). We show a full characterization in Theorem 3.3.8.

3.1.2 Notation and preliminaries

Our interest falls on composition operators defined on Banach spaces of analytic functions on the disc. Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$ with the topology τ_c of uniform convergence on compact sets. Given a holomorphic function $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$, the *composition operator* $C_\varphi : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ is defined by $C_\varphi f = f \circ \varphi$. Let $X \hookrightarrow H(\mathbb{D})$ be a Banach space with continuous inclusion. A holomorphic self-map $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is said to be *a symbol for* X if $C_\varphi(X) \subset X$. As a consequence of the Closed Graph Theorem this is equivalent to $C_\varphi \in \mathcal{L}(X)$.

Since clearly $C_\varphi^n = C_{\varphi_n}$, where $\varphi_n := \varphi \circ \overset{(n)}{\cdots} \circ \varphi$, the study of the iterates of the operator shifts to the study of the iterates of the symbol φ. The behaviour of φ and its iterates has been deeply investigated and it has great importance in our study (see for example [37] or [74]). The self-map φ is called *elliptic* if it has a fixed point p. The holomorphic automorphism of the disc $\phi_p(z) := (p-z)/(1-\overline{p}z)$ interchanges the fixed point p of φ with 0 and $\Phi = \phi_p \circ \varphi \circ \phi_p$ defines a holomorphic function with fixed point 0 satisfying that $C_\Phi = C_{\phi_p} C_\varphi C_{\phi_p}$ and C_φ have the dynamical properties. We say φ is *equivalent* to Φ.

If φ is elliptic, then by the Schwarz Lemma, $|\varphi'(p)| \leq 1$. Furthermore, if $|\varphi'(p)| = 1$, then φ is an automorphism. When φ is an elliptic automorphism, then $\Phi(z) = \lambda z$. In this case Φ is called a *rational rotation* if there exists $n \in \mathbb{N}$ such that $\lambda^n = 1$ and it is called an *irrational rotation* otherwise.

The results shown in this chapter depend deeply on the well known Denjoy-Wolff Theorem [39, 86, 85, 87] (see [37, Theorem 2.51]).

Theorem 3.1.1 (Denjoy–Wolff) *If $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic and not an elliptic automorphism, then there exists a unique point $p \in \overline{\mathbb{D}}$ (called the Denjoy-Wolff point of φ) such that the sequence $(\varphi_n)_n$ of the iterates converges to p uniformly on compact sets of $\mathbb{D}$.*

If the Denjoy-Wolff point of φ is in $\mathbb{D}$, it is clearly also a fixed point and as stated before, we may always consider it is 0. If it is on the boundary of $\mathbb{D}$, then, using a rotation, we may consider it is 1 for simplicity, keeping the dynamical properties of C_φ as in the elliptic case.

The next classical result is due to Koenigs [51] (see also [75]) and related to the Schroeder equation. We state a particular case of this theorem, whose general case completely characterizes the point spectrum of C_φ from a functional analytic point of view.

Theorem 3.1.2 (Koenigs) *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic with an interior Denjoy-Wolff point. The composition operator $C_\varphi : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ satisfies $\sigma_p(C_\varphi) \cap \partial\mathbb{D} = \{1\}$.*

We recall here the concept of *spectral radius* of an operator T on a Banach space X, which is defined as $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$. Furthermore, if $\mathcal{K}(X) \subset \mathcal{L}(X)$ denotes the space of compact operators on X, then the *essential norm* $\|T\|_e := \inf\{\|T - K\| : K \in \mathcal{K}(X)\}$ defines indeed a norm on the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Denote by $r_e(T)$ the *essential spectral radius*, i.e. the spectral radius of the projection of the operator T to $\mathcal{L}(X)/\mathcal{K}(X)$. We write $r_e(T, X)$ if we need to stress the space X.

Following the work of Yosida and Kakutani in [91] we give the following definition. See also [47, Chapter 51, Exercise 4].

Definition 3.1.3 *We say an operator $T \in \mathcal{L}(X)$ is quasicompact if there exists $n \in \mathbb{N}$ such that $\|T^n\|_e < 1$.*

Notice that $r_e(T) < 1$ holds precisely when T is quasicompact. Indeed, if T is quasicompact, there is $n_0 \in \mathbb{N}$ such that $\|T^{n_0}\|_e < \rho < 1$. Then $\|T^{n_0 k}\|_e^{\frac{1}{n_0 k}} \leq \rho^{\frac{1}{n_0}} < 1$ for each $k \in \mathbb{N}$. Since the limit $r_e(T) = \lim_n \|T^n\|_e^{1/n}$ exists, we get $r_e(T) < 1$. The other implication is trivial.

The following result permits us to show that condition (c) in [6, Proposition 3.1] can be substituted by power boundedness.

Proposition 3.1.4 *The following conditions are equivalent for $T \in \mathcal{L}(X)$:*

(1) $(T^n)_n$ converges in $\mathcal{L}_b(X)$ to a finite rank projection P.

(2) T is power bounded, quasicompact and $\sigma_p(T) \cap \partial\mathbb{D} \subseteq \{1\}$.

(3) $r_e(T) < 1$, $\sigma_p(T) \cap \partial\mathbb{D} \subseteq \{1\}$ and if 1 is in the spectrum then it is a pole of order 1.

Proof. The equivalence between (1) and (3) is [6, Proposition 3.1] . If (1) is satisfied then T is power bounded and quasicompact. Hence the equivalence between (1) and (2) is due to [91, Theorem 3.4, Corollary (ii), (iii)]. □

Considering the spaces $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$, summarizing the results of [11] and [6] we get the following characterization for uniformly mean ergodic composition operators. In $H^\infty(\mathbb{D})$ uniform mean ergodicity is equivalent to mean ergodicity for power bounded operators because it is a Grothendieck space with the Dunford–Pettis property and composition operators on it have norm 1 (see Theorem A.2.8).

Theorem A *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic (resp. and also uniformly continuous). Then $C_\varphi : H^\infty(\mathbb{D}) \longrightarrow H^\infty(\mathbb{D})$ (resp. $C_\varphi : A(\mathbb{D}) \longrightarrow A(\mathbb{D})$) is uniformly mean ergodic if, and only if, either φ is equivalent to a rational rotation or one of the following equivalent conditions is satisfied:*

(1) There exists $z_0 \in \mathbb{D}$ such that $\lim \varphi_n(z) = z_0$ uniformly on $\mathbb{D}$.

(2) C_φ is quasicompact.

(3) There exists $n_0 \in \mathbb{N}$ such that $C_\varphi^{n_0}$ is a compact operator.

(4) There exists $z_0 \in \mathbb{D}$ such that $\lim_n \|C_\varphi^n - C_{z_0}\| = 0$, with $C_{z_0}(f) = f(z_0)$.

In our work on Bloch spaces, weighted Bergman spaces of infinite type, Bergman spaces and Hardy spaces (in these last two cases restricting to univalent symbols), we prove (Theorem 3.3.8) that uniform mean ergodicity of composition operators whose symbol not equivalent to a rational rotation can be characterized by having an interior Denjoy–Wolff point together with condition (2), which is actually equivalent to (4) in Theorem A as a consequence of [6, Theorem 3.4]. We provide examples showing that conditions (1) an (3) are sufficient but not necessary for uniform mean ergodicity. Briefly, the converse of the Yosida–Kakutani Mean Ergodic Theorem (Theorem A.2.4) is also true in the spaces we consider, but here the condition of T^n being compact for some $n \in \mathbb{N}$ is only sufficient.

The classical spaces of holomorphic functions on the unit disc we consider in this chapter are defined next.

For $p > 0$, the space of Bloch functions or *Bloch space of order p* is

$$\mathcal{B}_p := \{f \in H(\mathbb{D}) \, : \, \|f\|_{\mathcal{B}_p} := \sup_{z\in\mathbb{D}} (1-|z|^2)^p |f'(z)| < \infty\}.$$

It is a Banach space when endowed with the norm $\|f\| := |f(0)| + \|f\|_{\mathcal{B}_p}$. The closed subspace

$$\mathcal{B}_p^0 := \{f \in H(\mathbb{D}) \, : \, \lim_{|z|\to 1} (1-|z|^2)^p |f'(z)| = 0\}$$

is called the *little Bloch space of order p* and is also a Banach space with the same norm $\|\cdot\|$. The spaces $\mathcal{B}_1$ and $\mathcal{B}_1^0$ are nothing but the classical Bloch spaces.

All these spaces are continuously included in $H(\mathbb{D})$. We have $C_\varphi \in \mathcal{L}(\mathcal{B}_p)$ for every $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ analytic, with $\|C_\varphi f\|_{\mathcal{B}_p} \leq \|f\|_{\mathcal{B}_p}$ as a consequence of the

Schwarz-Pick Lemma. On the other hand $C_\varphi^0 := C_\varphi|_{\mathcal{B}_p^0} \in \mathcal{L}(\mathcal{B}_p^0)$ if, and only if, $\varphi \in \mathcal{B}_p^0$.

We restrict our study to Bloch spaces of order $p \geq 1$ as each $\mathcal{B}_p$ with $0 < p < 1$ is a space of Lipschitz functions and hence included in $A(\mathbb{D})$, see [92].

A continuous function $v : \mathbb{D} \longrightarrow]0, \infty[$ is called a *weight.* Associated to a weight v the *weighted Bergman spaces of infinite type* are defined as follows

$$H_v := \{f \in H(\mathbb{D}) \ : \ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\},$$

$$H_v^0 := \{f \in H(\mathbb{D}) \ : \ \lim_{|z|\to 1} v(z)|f(z)| = 0\}.$$

Both of these are Banach spaces with the norm $\|\cdot\|_v$. A weight v is called *typical* if v is radial (i.e. $v(z) = v(|z|)$), decreasing (i.e. $v(z) \leq v(w)$ if $|z| \geq |w|$) and satisfies $\lim_{|z|\to 1} v(z) = 0$.

A composition operator C_φ is well-defined and continuous on each of H_v and H_v^0 whenever v is a typical weight satisfying the Lusky condition, that is, that

$$\inf_{n\in\mathbb{N}} \frac{\hat{v}(1-2^{-n})}{\hat{v}(1-2^{-(n-1)})} > 0,$$

where $\hat{v}(z) := \frac{1}{\|\delta_z\|_{H_v^*}}$ $(z \in \mathbb{D})$ is the associated weight, see [27, Theorem 2.3].

For $p \geq 1$, the *Bergman space of order* p is

$$A^p := \{f \in H(\mathbb{D}) \ : \ \|f\|_{A^p}^p := \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty\},$$

where dA is the normalized Lebesgue measure on $\mathbb{D}$. This is a Banach space with the norm $\|\cdot\|_{A^p}$.

For $p \geq 1$, the *Hardy space of order* p is

$$H^p := \{f \in H(\mathbb{D}) \ : \ \|f\|_{H^p}^p := \frac{1}{2\pi} \lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty\},$$

which is a Banach space with the norm $\|\cdot\|_{H^p}$.

For every $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ analytic the operator C_φ is in $\mathcal{L}(A^p)$ and $\mathcal{L}(H^p)$, see [37].

3.2 Ergodic theorems on general Banach spaces of analytic functions

We consider the following conditions on a Banach space X continuously embedded in $H(\mathbb{D})$.

(PB1) Automorphisms are symbols for X.
(PB2) If $\alpha \in \partial\mathbb{D}$ then there is $f \in X$ such that $\lim_{z\to\alpha} \mathrm{Re}\, f(z) = +\infty$.
(PB3) If ψ is a symbol with $\psi(0) = 0$, then C_ψ is power bounded.

(ME) If $(f_n)_n$ is a bounded sequence in X, which is pointwise convergent to $f \in X$, then $(f_n)_n$ is weakly convergent to f.

(UME) If ψ is a symbol with Denjoy-Wolff point 0, then $\overline{B(0, r_e(C_\psi, X))} \subseteq \sigma(C_\psi, X)$.

In this section we give results about power boundedness, (uniform) mean ergodicity and asymptotic behaviour of the powers of composition operators on spaces satisfying (some of) these conditions. Examples of such spaces are considered in Section 3.3.

The relevance of these conditions can be seen in the results of this section. However some explanation is given here. The condition (PB1) is not difficult to satisfy, and allows us to shift the Denjoy–Wolff point of the symbol to either 0 or 1. The condition (PB2) shows the difference in the dynamics given by an interior or a boundary Denjoy–Wolff point. It can actually be relaxed to an easier condition (PB2') in some cases.

(PB2') If $\alpha \in \partial\mathbb{D}$ then there is $f \in X$ such that $\lim_{z\to\alpha} |f(z)| = +\infty$.

The condition (PB3), in combination with (PB1) simplifies the study of quasi-compactness, in order to use Proposition 3.1.4.

The aim of condition (ME) is bringing pointwise convergence to weak convergence and from there, to mean ergodicity, using Theorem A.2.2. We remark here that since $X \hookrightarrow H(\mathbb{D})$ and this space is Montel (even nuclear), the topology of pointwise convergence and the compact–open topology coincide on bounded sets. Hence condition (ME) is in fact equivalent to requiring any bounded sequence (f_n) in X to be weakly convergent whenever it converges in the compact–open topology.

Finally the condition (UME) is used with Theorem A.2.5 to deduce uniform mean ergodicity.

Along this section we consider also other conditions on the space X, regarding the set $\mathcal{P}(\mathbb{D})$ of all polynomials. These conditions are, however, explicitly assumed in each result.

Proposition 3.2.1 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ and containing the constants. Let φ be a symbol for X. Then:*

(i) If (PB1) and (PB3) hold for X and φ is elliptic, then C_φ is power bounded.

(ii) If (PB2') holds for X and C_φ is power bounded, then φ is elliptic.

Proof. (i) Assume that φ has an interior fixed point $z_0 \in \mathbb{D}$. Then there are holomorphic self-maps $\phi, \psi : \mathbb{D} \longrightarrow \mathbb{D}$ with ϕ automorphism and with $\phi(z_0) = 0$ and $\psi(0) = 0$ such that $\varphi = \phi^{-1} \circ \psi \circ \phi$. By (PB1), ϕ and ψ are symbols for X. Since $\|C_\varphi^n\| = \|C_{\phi^{-1}} C_\psi^n C_\phi\| \leq \|C_{\phi^{-1}}\| \|C_\psi^n\| \|C_\phi\|$, we deduce the power boundedness from (PB3).

(ii) If φ is not elliptic, then it has a Denjoy–Wolff α on the boundary. Take f as in (PB2'). Define $\delta_0 \in X'$ by $\delta_0(f) = f(0)$. We have that $(\delta_0(C_\varphi^n(f)))_n$ is not bounded since

$$\lim_{n\to\infty} |C_{\varphi_n}(f)(0)| = +\infty.$$

Thus, $(C_\varphi^n(f))_n$ is not bounded for the weak topology of X and therefore $(C_\varphi^n(f))_n$ is not bounded in X, since the bounded sets of both topologies coincide. □

Now we consider the mean ergodicity of C_φ giving a characterization in the case when the symbol is an elliptic automorphism.

Proposition 3.2.2 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ and containing the constants. Further assume (PB1) and (PB3) hold for X. Let φ be an automorphism with fixed point $z_0 \in \mathbb{D}$. Then,*

(i) if φ is equivalent to a rational rotation, then C_φ is uniformly mean ergodic with
$(C_\varphi)_{[n]} \longrightarrow \frac{1}{k}(C_\varphi + \cdots + C_\varphi^k)$ for some $k \in \mathbb{N}$,

(ii) if φ is equivalent to an irrational rotation and $\mathcal{P}(\mathbb{D}) \subseteq X$, then C_φ is not uniformly mean ergodic,

(iii) if φ is equivalent to an irrational rotation and $\overline{\mathcal{P}(\mathbb{D})}^X = X$, then C_φ is mean ergodic, with $(C_\varphi)_{[n]} \longrightarrow C_{z_0}$, where $C_{z_0}(f) = f(z_0)$.

Proof. We may restrict ourselves to the case when $\varphi(z) = \lambda z$ is a rotation.

(i) If it is a rational rotation, i.e., there exists $k \in \mathbb{N}$ such that $\lambda^k = 1$, then C_φ is periodic with period k (take the smallest k). Then, (see [11, Theorem 2.2] and [84, Proposition 18])

$$\lim_{n\to\infty} \left\| (C_\varphi)_{[n]} - \frac{1}{k} \sum_{j=1}^{k} C_\varphi^j \right\| = 0.$$

(ii) In case $\varphi(z) = \lambda z$ and $\lambda^n \neq 1$ for every $n \in \mathbb{N}$, we see that $\lambda^n \in \sigma(C_\varphi)$ for all $n \in \mathbb{N}$. This is clear since

$$C_\varphi(f_n) = \lambda^n f_n,$$

where $f_n(z) := z^n$, with $n = 1, 2, \ldots$.

By Kronecker's Theorem (see [71, Theorem 2.2.4] or [46, Section 23.1]), $\{\lambda^n : n \in \mathbb{N}\}$ is dense in $\partial\mathbb{D}$, therefore 1 is an accumulation point of the spectrum and by Theorem A.2.5, C_φ is not uniformly mean ergodic.

(iii) Again, if $\varphi(z) = \lambda z$ and $\lambda^n \neq 1$ for every $n \in \mathbb{N}$, we proceed as in [11, Theorem 2.2 (ii)] to see that the operator is mean ergodic. Let $f_k(z) := z^k$, with $k = 1, 2, \ldots$, then

$$\|((C_\varphi)_{[n]} f_k)(z)\| = \left\| \frac{1}{n} \sum_{j=1}^{n} \lambda^{jk} z^k \right\| \leq \frac{2}{n|1-\lambda^k|} \|f_k\|,$$

and therefore, $\lim_n (C_\varphi)_{[n]} = C_0$ on the monomials. For $k = 0$, we have

$$(C_\varphi)_{[n]}(1) = I(1) = 1.$$

Since we are assuming (PB1) and (PB3), the sequence $(C_\varphi)_{[n]}$ is equicontinuous. By density, we deduce the mean ergodicity of C_φ. □

Proposition 3.2.3 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ and containing the constants. Assume (PB1) and (PB3) hold for X. Let φ be a symbol for X. Then we have:*

(i) If (PB2) holds for X and C_φ is mean ergodic, then φ is elliptic.

(ii) If φ is elliptic, X satisfies (ME) and $\overline{\mathcal{P}(\mathbb{D})}^X = X$, then C_φ is mean ergodic. In case φ has a Denjoy-Wolff point $z_0 \in \mathbb{D}$, then we even have that $(C_\varphi^n)_n$ converges to C_{z_0} in the weak operator topology.

Proof. (i) If φ were not elliptic, it would have a boundary Denjoy-Wolff point α. Take f as in (PB2). We have

$$\lim_{n\to\infty} \mathrm{Re}\, C_{\varphi_n}(f)(0) = +\infty.$$

From this it also follows

$$\lim_{n\to\infty} \mathrm{Re}\, ((C_\varphi)_{[n]} f)(0) = +\infty,$$

and hence $((C_\varphi)_{[n]}(f))_n$ is not a bounded sequence in X since $(\delta_0((C_\varphi)_{[n]}(f)))_n$ is not bounded. Therefore C_φ is not mean ergodic. This shows (i).

(ii) Now assume that (ME) holds and that φ is elliptic. If φ is a rotation the mean ergodicity follows from Proposition 3.2.2. If z_0 is the Denjoy-Wolff point of φ then $C_\varphi^n(f)$ converges pointwise to $f(z_0)$ for every $f \in X$. Furthermore, (ME) ensures that $C_\varphi^n(f)$ converges weakly to $f(z_0)$ for every $f \in X$. Apply the Mean Ergodic Theorem (Corollary A.2.2) [70, Theorem 1.3] to obtain (*ii*). □

Corollary 3.2.4 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ with $\overline{\mathcal{P}(\mathbb{D})}^X = X$. Assume (PB1)–(PB3) and (ME) hold for X. Let φ be a symbol for X. Then, the operator C_φ is power bounded if, and only if, it is mean ergodic if, and only if, φ is elliptic.*

Proof. It is a direct consequence of Theorem 3.2.1, Proposition 3.2.2 and Theorem 3.2.3. □

The following result is helpful for the characterization of uniform mean ergodicity. It should be compared with Theorem A.2.4.

Proposition 3.2.5 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ with $\mathcal{P}(X) \subset X$. Assume (PB1) and (PB3) hold for X. Let φ be a symbol for X with Denjoy–Wolff point $z_0 \in \mathbb{D}$. Then the following assertions are equivalent:*

(1) $\lim_n \|C_{\varphi_n} - C_{z_0}\| = 0$, *and*

(2) C_φ *is quasicompact.*

Proof. Since φ is elliptic, C_φ is power bounded by Proposition 3.2.1. Recall that by Theorem 3.1.2, $\sigma_p(C_\varphi, X) \cap \partial\mathbb{D} \subseteq \{1\}$. Therefore the equivalence between (1) and (2) follows directly from Proposition 3.1.4, since the only possible limit for $(C_\varphi^n)_n$ is C_{z_0}. □

Theorem 3.2.6 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ with $\mathcal{P}(X) \subset X$. Assume (PB1)–(PB3) and (UME) hold for X. Let φ be a symbol for X. Then C_φ is uniformly mean ergodic if, and only if, either*

(i) φ *is equivalent to a rational rotation, or*

(ii) the Denjoy–Wolff point of φ is $z_0 \in \mathbb{D}$ and C_φ is quasicompact, or, equivalently, $\lim_n \|C_{\varphi_n} - C_{z_0}\| = 0$.

Proof. The equivalence in (ii) is Proposition 3.2.5.

If (i) holds, then C_φ is uniformly mean ergodic by Proposition 3.2.2.(i).

If (ii) holds, the uniform convergence of the iterates implies the uniform convergence of the means and C_φ is uniformly mean ergodic.

Assume C_φ is uniformly mean ergodic, then by Proposition 3.2.3.(i), φ has a fixed point. If φ is equivalent to a rotation, it must be a rational rotation because of Proposition 3.2.2.(ii).

Assume C_φ is uniformly mean ergodic and suppose that φ is not equivalent to a rotation. Then φ has a Denjoy–Wolff point $z_0 \in \mathbb{D}$. Without loss of generality, we assume $z_0 = 0$.

If C_φ is not quasicompact, then $\|C_\varphi^n\|_e \geq 1$ for all $n \in \mathbb{N}$. Furthermore we have by (PB3) that $\|C_\varphi^n\|_e \leq \|C_\varphi^n\| \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$. Then $r_{e,X}(C_\varphi) = \lim_{n\to\infty} \|C_\varphi^n\|_e^{1/n} = 1$. Hence (UME) implies that 1 is an accumulation point in $\sigma(C_\varphi, X)$. Therefore C_φ is not uniformly mean ergodic by Theorem A.2.5, which is a contradiction. □

Remark 3.2.7 In the proof of Theorem 3.2.6, the condition (PB2) is only used to ensure the symbol to be elliptic when the operator is uniformly mean ergodic.

3.3 Ergodic theorems on concrete spaces of analytic functions

In this section we apply the results of Section 3.2 to some classical spaces of holomorphic functions on the unit disc $\mathbb{D}$. We first study whether the spaces satisfy the properties defined in Section 3.2. These results are summarized below.

	H_v	H_v^0	$\mathcal{B}_p$	$\mathcal{B}_p^0$, $p>1$	$\mathcal{B}_1^0$	A^p, $p>1$	A^1	H^p, $p>1$	H^1
(PB1)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(PB2)	$*$	$*$	✓	✓	✓	✓	✓	✓	✓
(PB3)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(ME)	✗	✓	✗	✓	✓	✓	✗	✓	✗
(UME)	✓	✓	✓	✓	✓	$**$	$**$	$**$	$**$

$*$: (PB2) holds if v is convenient (Definition 3.3.2).
$**$: (UME) holds if ψ is univalent.

Lemma 3.3.1 *The properties (PB1) and (PB3) hold for the following spaces:*

(1) H_v and H_v^0, for a typical weight v satisfying the Lusky condition,

(2) $\mathcal{B}_p$ and $\mathcal{B}_p^0$, for $p \geq 1$,

(3) A^p and H^p, for $p \geq 1$.

Proof. The fact that $\|C_\psi\| \leq 1$ if $\psi(0) = 0$ follows for H_v and H_v^0 from the formulas for C_ψ for typical weights (see [27]). For the Bloch spaces it follows from the Schwarz-Pick lemma (see [58]). The case of A^p and H^p is a consequence of Littlewood subordination principle [54, 37]. □

We introduce a subset of typical weights in order to state our results for a wide class of these spaces.

Definition 3.3.2 *A typical weight v is said to be convenient if it satisfies the Lusky condition and there exists $f \in H_v^0$ such that* $\lim_{z\to 1} \mathrm{Re}\,(f) = +\infty$.

We remark that all the standard weights $v_p(z) = (1-|z|)^p$ satisfy that $f(z) = \log(1-z) \in H^0_{v_p}$, and also the Lusky condition, hence they are convenient. Also if v is a typical weight satisfying the Lusky condition with $v = O(v_p)$ as $|z| \to 1$ then v is convenient.

Lemma 3.3.3 *The property (PB2) holds for the following spaces:*

(1) H_v and H_v^0, for a convenient weight v,

(2) $\mathcal{B}_p$, for $p \geq 1$,

(3) $\mathcal{B}_p^0$, for $p \geq 1$,

(4) A^p and H^p, for $p \geq 1$.

Proof. The case of (1) is clear by the definition of convenient weight. For the rest of cases note that $f(z) = \log(1-z)$ is in all of the spaces considered except for $\mathcal{B}_1^0$. For the case of $\mathcal{B}_1^0$ fix $\alpha \in \partial\mathbb{D}$ and define

$$\Omega = \mathbb{D} \cup \{z \in \mathbb{C} \,:\, \mathrm{Re}\, z > 0,\, 0 < \mathrm{Im}\, z < e^{-\mathrm{Re}\, z}\}.$$

There exists a Riemann map $f : \mathbb{D} \longrightarrow \Omega$ with $f(0) = 0$ and $f(\alpha) = \infty$. Therefore (see [44, Theorem A]) we have $f \in \mathcal{B}_1^0$, since

$$\lim_{\omega\in\Omega,\, |\omega|\to\infty} d(\omega, \partial\Omega) = 0,$$

where $d(\cdot,\cdot)$ denotes the distance in $\mathbb{C}$. Since $\lim_{z\to\alpha} \mathrm{Re}\, f(z) + \infty$, the space $\mathcal{B}_1^0$ satisfies the property (PB2). □

Lemma 3.3.4 *The property (ME) holds for the following spaces:*

(1) H_v^0, for a typical weight v satisfying the Lusky condition,

(2) $\mathcal{B}_p^0$, for $p \geq 1$,

(3) A^p and H^p, for $p > 1$.

Proof. (1) By means of $g \mapsto vg$, H_v^0 is isometric to a subspace of $C(\widehat{\mathbb{D}})$, where $\widehat{\mathbb{D}}$ is the Alexandroff compactification of $\mathbb{D}$. We denote by H this subspace, which is formed of functions vanishing at infinity. Therefore, if $(f_n)_n \subset H_v^0$ is pointwise convergent to f in $\mathbb{D}$ then $(vf_n)_n$ is a bounded sequence in $C(\widehat{\mathbb{D}})$ which is pointwise convergent in $\widehat{\mathbb{D}}$ to $vf \in H$. For every functional $u \in C(\widehat{\mathbb{D}})^*$ there is a finite Radon measure μ on $\widehat{\mathbb{D}}$ such that $u(g) = \int_{\widehat{\mathbb{D}}} g d\mu$ for every $g \in C(\widehat{\mathbb{D}})$. Lebesgue's dominated convergence theorem implies that $(vf_n)_n$ is weakly convergent to vf in $C(\widehat{\mathbb{D}})$, and hence $(f_n)_n$ is weakly convergent to f in H_v^0.

(2) Let us now consider the case of $\mathcal{B}_p^0$. Assume $(f_n)_n$ is pointwise convergent to f in $\mathbb{D}$. Since $\mathcal{B}_p^0 \hookrightarrow (H(\mathbb{D}), \tau_c)$, we get that $(f_n)_n$ is relatively compact in $H(\mathbb{D})$ and hence $(f_n)_n$ is actually convergent to f in $(H(\mathbb{D}), \tau_c)$. We have the isometric identification $\mathcal{B}_p^0 = H_{v_p}^0 \oplus_{l_1} \mathbb{C}$, $g \mapsto (g', g(0))$. Since $(f_n)_n$ is τ_c convergent to f and the differentiation operator $g \mapsto g'$ is continuous on $(H(\mathbb{D}), \tau_c)$ we get that $(f_n')_n$ is a bounded sequence in $H_{v_p}^0$ which is pointwise convergent to f'. Thus we conclude by the previous case that $(f_n')_n$ is weakly convergent to f'. Since $(f_n(0))_n$ converges to $f(0)$ by hypothesis, the isometric identification gives us that $(f_n)_n$ is weakly convergent to f in $\mathcal{B}_p^0$.

(3) When $p > 1$ the spaces considered are reflexive and therefore (ME) holds since the pointwise topology in $\mathbb{D}$ is Hausdorff and bounded sets are relatively weakly compact in reflexive spaces. □

Proposition 3.3.5 *The property (ME) does* ***not*** *hold for the following spaces:*

(1) A^1, H^1,

(2) $\mathcal{B}_p$, *for* $p \geq 1$,

(3) H_v, *for any typical weight* v.

Proof. (1) For $v(z) := (1 - |z|)$ it follows from [76] that $(H_v^0)^* = A^1$ and $(A^1)^* = H_v$. Via the dual pairing defined in [76] we have that, for $w \in \mathbb{D}$, the evaluations δ_w can be identified with $K_w(z) = \frac{2}{(1-wz)^2}$, and K_w is in H_v^0. The topology of pointwise convergence is then a Hausdorff topology weaker than the weak* topology. Hence these topologies agree on bounded sets of A^1. From this we conclude that the bounded sequences (f_n) in A^1 which are pointwise convergent to some $f \in A^1$ are precisely those which are weak*–convergent. This set of sequences is strictly contained in that of (bounded) sequences which are weakly convergent because H_v^0 is not a Grothendieck space, since it is separable and not reflexive.

The same argument holds for H^1, since VMOA* $= H^1$ and $(H^1)^* =$ BMOA, where BMOA is the space of functions in H^1 with bounded mean oscillation on $\partial\mathbb{D}$ and VMOA is the closure of the polynomials in BMOA.

(2) and (3) The case for H_v and $\mathcal{B}_p$ is similar, and both are analogous to that of A^1. We only give the proof for H_v. In [25] it is shown that the subspace X of H_v^* formed by functionals which are continuous on the unit ball B_v of H_v for the compact–open topology satisfies $X^* = H_v$. X contains the span of the evaluations $H := \{\delta_z : z \in \mathbb{D}\}$ as a separable subspace which is separating in H_v, i.e. H is dense in X. Hence the bounded sequences in H_v which are pointwise ($\sigma(H_v, H)$-)convergent are precisely those which are weak*–convergent, and X is not a Grothendieck space since it is separable and not reflexive. □

Lemma 3.3.6 *The property (UME) holds for the following spaces:*

(1) H_v *and* H_v^0*, for a typical weight* v *satisfying the Lusky condition,*

(2) $\mathcal{B}_p$ *and* $\mathcal{B}_p^0$*, for* $p \geq 1$*,*

(3) A^p *and* H^p*, for* $p \geq 1$*, whenever* ψ *is univalent.*

Proof. It is a consequence of the spectral radii formulas given in [9, 28, 36, 62], which, summarized, are

$$\sigma(C_\psi, X) = \overline{B(0, r_e(C_\psi))} \cup \{\psi'(0)^n \ : \ n \in \mathbb{N} \cup \{0\}\},$$

where X is any of the above spaces and ψ any symbol of X with Denjoy–Wolff point 0. □

Theorem 3.3.7 *Let* v *be a convenient weight and* $p \geq 1$*. Let* X^0 *stand for* $H_v^0, \mathcal{B}_p^0, A^p$ *or* H^p *and let* X *stand for* H_v *or* $\mathcal{B}_p$*. Let* $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ *be holomorphic. Then the following assertions hold:*

(i) $C_\varphi \in \mathcal{L}(X)$ *is power bounded if, and only if,* φ *is elliptic.*

(ii) $C_\varphi \in \mathcal{L}(X^0)$ *is power bounded if, and only if, it is mean ergodic if, and only if,* φ *is elliptic.*

(iii) If $C_\varphi \in \mathcal{L}(X^0)$ *and* φ *has a Denjoy-Wolff point* $z_0 \in \mathbb{D}$*, then* $(C_\varphi^n)_n$ *converges weakly to* C_{z_0} *on* X^0*, and consequently* C_φ *is mean ergodic.*

Proof. (i) By Lemmata 3.3.1 and 3.3.3, the properties (PB1), (PB2) and (PB3) hold for X. The assertion follows by Proposition 3.2.1.

(ii) and (iii) By Lemma 3.3.1, X^0 satisfies the properties (PB1), (PB2) and (PB3). Also, property (ME) holds for X^0, if $X^0 \neq A^1, H^1$, by Lemma 3.3.4 .

• If $X^0 \neq A^1, H^1$, then Corollary 3.2.4 implies (ii) and we can deduce (iii) from Proposition 3.2.3.(ii).

• If X^0 is A^1 or H^1, then C_φ is power bounded if, and only if, φ is elliptic, by Proposition 3.2.1. If C_φ is mean ergodic, then φ is elliptic, by Proposition 3.2.1.

If φ is a similar to a rotation, then C_φ is mean ergodic, by Proposition 3.2.2. If φ is not similar to a rotation, then it has a Denjoy–Wolff point $z_0 \in \mathbb{D}$, and C_φ is mean ergodic, by assertion (iii). We show assertion (iii).

If φ has a Denjoy–Wolff point $z_0 \in \mathbb{D}$, then assertion (iii) holds for A^2 and H^2. The continuous inclusions $A^2 \hookrightarrow A^1$ and $H^2 \hookrightarrow H^1$ have dense range, therefore $(C_\varphi^n)_n$ is an equicontinuous sequence, which is convergent to C_{z_0} in the weak operator topology on a dense subspace. Thus $(C_\varphi^n)_n$ also converges in the weak operator topology to C_{z_0} in $\mathcal{L}(A^1)$ and $\mathcal{L}(H^1)$. □

Theorem 3.3.8 *Let X^0, X and φ be as in Theorem 3.3.7. Further assume φ to be univalent if $X^0 = H^p$ or $X^0 = A^p$. Then the following assertions hold:*

(i) $C_\varphi \in \mathcal{L}(X^0)$ is uniformly mean ergodic if, and only if, either

- *(a) φ is equivalent to a rational rotation, or*
- *(b) φ has a Denjoy-Wolff point $z_0 \in \mathbb{D}$ and C_φ is quasicompact, or, equivalently, $\lim_n \|C_\varphi^n - C_{z_0}\| = 0$.*

(ii) $C_\varphi \in \mathcal{L}(X)$ is uniformly mean ergodic if, and only if, it is mean ergodic if, and only if, either

- *(a) φ is equivalent to a rational rotation, or*
- *(b) φ has a Denjoy-Wolff point $z_0 \in \mathbb{D}$ and C_φ is quasicompact, or, equivalently, $\lim_n \|C_\varphi^n - C_{z_0}\| = 0$.*

Proof. For (ii), we use Lotz's Theorem A.2.8 to deduce the equivalence of mean ergodicity and uniform mean ergodicity, considering that the spaces H_v and $\mathcal{B}_p$ are isomorphic to either l_∞ or H^∞ by [57, Theorem 1.1], and are therefore Grothendieck spaces with the Dunford–Pettis property.

The properties (PB1), (PB2), (PB3) and (UME) hold for X and X^0 by Lemmata 3.3.1 and 3.3.6. Then, for X and X^0, the assertions hold by Theorem 3.2.6. □

Corollary 3.3.9 *Let v be a convenient weight and $p \geq 1$. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic. The following assertions hold:*

(i) C_φ is mean ergodic on H_v if, and only if, C_φ is uniformly mean ergodic on H_v if, and only if, C_φ is uniformly mean ergodic on H_v^0.

(ii) If $\varphi \in \mathcal{B}_p^0$, then C_φ is mean ergodic on $\mathcal{B}_p$ if, and only if, C_φ is uniformly mean ergodic on $\mathcal{B}_p$ if, and only if, C_φ is uniformly mean ergodic on $\mathcal{B}_p^0$.

We provide here an extension of [6, Theorem 4.10, Theorem 4.12], where the authors prove that if φ has an interior Denjoy-Wolff point then the sequence of iterates $(C_\varphi^n)_n$ is convergent in $\mathcal{L}_b(H^p)$ if, and only if, φ is not inner for $1 \leq p < \infty$. We see below that for the Hardy case we can drop the hypothesis of φ being univalent in Theorem 3.3.8.(i). Also this Theorem extends [45, Theorem 8].

Theorem 3.3.10 *Let $1 \leq p < \infty$. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic and not an elliptic automorphism. Then $C_\varphi \in \mathcal{L}(H^p)$ is uniformly mean ergodic if, and only if, φ has a Denjoy–Wolff point $z_0 \in \mathbb{D}$, and it is not inner if, and only if, $\|C_\varphi^n - C_{z_0}\| \longrightarrow 0$.*

Proof. By Proposition 3.2.3 (i) we restrict ourselves to the case when φ has an interior Denjoy Wolff point, which we can as usual assume to be 0. If φ is not inner the sequence of iterates is norm convergent by [33, Theorem 3.1] .

Let us assume that φ is inner. If φ is univalent, since $\varphi(0) = 0$, then φ is an elliptic automorphism ([37, Corollary 3.28]). If φ is not univalent, then $\sigma(C_\varphi) = \sigma_e(C_\varphi) = \overline{\mathbb{D}}$ (see [37, Theorem 7.43]), thus C_φ is not uniformly mean ergodic by Theorem A.2.5. □

Example 3.3.11 Next we show examples of composition operators on H_{v_p} and $\mathcal{B}_p$ which are uniformly mean ergodic but such that none of its iterates is compact. Compare this with condition (iii) in Theorem A to see the difference with $H^\infty(\mathbb{D})$ and $A(\mathbb{D})$.

Let $v_p(z) = (1 - |z|)^p$ and let us consider the operator $C_\varphi : H_{v_p} \longrightarrow H_{v_p}$ with $\varphi(z) = az + (1-a)z^2$ for $0 < a < 1$. From [27], since 0 is a fixed point, it follows $\|C_\varphi\| = 1$. From [65, Theorem 2.1] we get

$$\|C_{\varphi_n}\|_e = \limsup_{z \to 1} \frac{v_p(z)}{v_p(\varphi_n(z))} = \limsup_{z \to 1} \left(\frac{1 - |z|}{1 - |\varphi_n(z)|} \right)^p.$$

By the Julia–Carathéodory Theorem ([37, Theorem 2.44], [74, Section 4.2]), we have

$$\liminf_{z \to 1} \frac{1 - |\varphi_n(z)|}{1 - |z|} = \varphi_n'(1).$$

Further, since 1 is a fixed point of φ, we have

$$\varphi_n'(1) = \varphi_{n-1}'(\varphi(1)) \cdot \varphi'(1) = \cdots = (\varphi'(1))^n = (2-a)^n.$$

Therefore, $\|C_{\varphi_n}\|_e = (2-a)^{-np}$.

This yields that C_φ is quasicompact, since $\|C_{\varphi_n}\|_e < 1$, but C_{φ_n} is not compact for any $n \in \mathbb{N}$, since $\|C_{\varphi_n}\|_e > 0$.

Corollary 10 in [9] gives that $C_\varphi : \mathcal{B}_p \longrightarrow \mathcal{B}_p$ for $\varphi(z) = sz/(1 - sz)$ also satisfies $0 < \|C_{\varphi_n}\|_e < 1$ for each $n \in \mathbb{N}$.

In [7], Arendt, Chalendar, Kumar and Srivastava obtained some results, which are similar to those presented by us. They show that in $H^0_{v_p}$, H_{v_p} for $p > 0$, $\mathcal{B}_p$, for $p > 1$ and A^p, for $p \geq 1$, mean ergodicity is equivalent to uniform convergence of the iterates of the operator. One of the tools they use is Proposition 3.3.12 below. Using this proposition and Theorem 3.3.8 we can also deduce the same result.

Proposition 3.3.12 *Let X be either $H^0_{v_p}$, H_{v_p} for $p > 0$, $\mathcal{B}_p$, $\mathcal{B}^0_p$, for $p > 1$ or A^p, for $p \geq 1$. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic with $\varphi(0) = 0$. Assume $C_\varphi \in \mathcal{L}(X)$. If φ is not a rotation, then*

$$r_e(C_\varphi) < 1.$$

Proof. The proof of the cases of $H^0_{v_p}$, H_{v_p}, $\mathcal{B}^0_p$ and $\mathcal{B}_p$ can be found in [7, Theorems 4.10, 5.1].

For A^p we have, by [62, Theorem 5], that

$$\big(r_e(C_\varphi, A^p)\big)^p = \big(r_e(C_\varphi, A^2)\big)^2.$$

By [34, Proposition 2.1], if C_φ is considered in A^2, then $\|C_\varphi\|_e < 1$ and therefore,

$$\big(r_e(C_\varphi, A^2)\big)^2 < 1.$$

□

Proposition 3.3.13 *Let X be either $H^0_{v_p}$, H_{v_p} for $p > 0$, $\mathcal{B}_p$, $\mathcal{B}^0_p$, for $p > 1$ or A^p, for $p \geq 1$. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic and not an automorphism. Assume $C_\varphi \in \mathcal{L}(X)$. Then, C_φ is mean ergodic if, and only if, the Denjoy–Wolff of φ is $z_0 \in \mathbb{D}$ and $\|C^n_\varphi - C_{z_0}\| \longrightarrow 0$.*

Proof. If C_φ is mean ergodic, then φ is elliptic, and therefore by Proposition 3.3.12, C_φ is quasicompact. Apply Theorem 3.3.8. □

We see below that the situation of Proposition 3.3.13 differs for H^0_v for a typical weight v, and also for Bloch spaces (with $p = 1$). We show operators that are mean ergodic but whose iterates do not converge in norm.

Proposition 3.3.14 *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic with $\varphi(0) = 0$. Assume φ is not a rotation and $\varphi([0,1)) \subseteq [0,1)$. Further assume the restriction $g(x) := \varphi|_{[0,1)}(x)$ admits a continuous extension to 1, with $g(1) = 1$, which is left differentiable at 1 and such that g' is left continuous at 1. The following assertions hold:*

(i) If $v(z) := (1 - \log(1 - |z|))^{-t}$, *with* $t > 0$, *then* $r_e(C_\varphi, H_v^0) = 1$.

(ii) If, in addition, $\varphi \in \mathcal{B}_1^0$, *then* $r_e(C_\varphi, \mathcal{B}_1^0) = 1$.

Proof. We show $\|C_\varphi^n\|_e = 1$, for $n \in \mathbb{N}$ and thus $r_e(C_\varphi) = \lim_n \|C_\varphi^n\|_e^{1/n} = 1$. If φ satisfies the assumptions, so do its iterates φ_n. Hence, we only write the computation for $\|C_\varphi\|_e = 1$.

First we prove (i). Consider $C_\varphi : H_v^0 \longrightarrow H_v^0$. By [65, Theorem 2.1],

$$\|C_\varphi\|_e = \limsup_{|z|\to 1^-} \frac{v(z)}{\tilde{v}(\varphi(z))},$$

where $\tilde{v}$ is the so-called *associated weight*. However by [21, Corollary 1.6], we have $v = \tilde{v}$. Therefore, using L'Hôpital's rule

$$\begin{aligned} 1 \geq \|C_\varphi(z)\|_e &= \limsup_{|z|\to 1^-} \frac{v(z)}{v(\varphi(z))} \\ &\geq \lim_{x\to 1^-} \left(\frac{1 - \log(1-x)}{1 - \log(1 - g(x))}\right)^{-t} \\ &= \left(\lim_{x\to 1^-} \frac{1 - g(x)}{g'(x)(1-x)}\right)^{-t} \\ &= \left(\lim_{x\to 1^-} \frac{1 - g(x)}{(1-x)}\right)^{-t} \left(\lim_{x\to 1^-} \frac{1}{g'(x)}\right)^{-t} = 1. \end{aligned}$$

In order to prove (ii), we use [64, Prop. 2.2],

$$1 \geq \|C_\varphi(z)\|_e = \limsup_{|\varphi(z)|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \geq \lim_{x\to 1} \frac{1-x^2}{1-g(x)^2} g'(x) = 1.$$

□

Corollary 3.3.15 *Set the weight* $v(z) := (1 - \log(1 - |z|))^{-t}$ *and let the symbol be* $\varphi(z) := az + (1-a)z^n$, $0 \leq a < 1$. *Then* C_φ^n *converges weakly to* C_0 *(hence is mean ergodic) on* H_v^0 *and on* $\mathcal{B}_1^0$, *but* C_φ *is not uniformly mean ergodic on these spaces. Therefore it is not mean ergodic on* H_v *or* $\mathcal{B}_1$.

Corollary 3.3.15 above contradicts [84, Theorem 10], since φ has an attracting fixed point, but C_φ is not uniformly mean ergodic.

The next example shows that we can easily get mean ergodic operators on $\mathcal{B}_1^0$ whose sequence of iterates converges weakly but not pointwise.

Example 3.3.16 Let $e_n(z) := z^n$. Then

$$\|e_n\|_{\mathcal{B}} = \sup_{0<x<1} nx^{n-1}(1-x^2) = \left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}} \frac{2n}{n+1}.$$

This implies that, for $\varphi(z) := e_2(z) = z^2$, by Proposition 3.3.14.(ii), we have

$$\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = 1,$$

but $C_\varphi : \mathcal{B}_1^0 \longrightarrow \mathcal{B}_1^0$ is not an isometry since

$$1 = \|e_1\|_{\mathcal{B}} \neq \|C_\varphi(e_1)\|_{\mathcal{B}} = \|e_2\|_{\mathcal{B}} = \frac{4}{3\sqrt{3}}.$$

Then neither $C_\varphi : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ is an isometry. This shows that [8, Theorem 1.1] is not correct.

The results in [7, Section 3] derived from this wrong assertion are unfortunately also not correct, particularly [7, Theorem 3.11], which states that in $\mathcal{B}_1^0$ weak convergence of the iterates of C_φ implies uniform convergence. Here by Theorem 3.3.7, if $\varphi(z) = z^2$, then C_φ^n converges weakly to C_0. However the operator C_φ is not uniformly mean ergodic by Theorem 3.3.8, since $r_e(C_\varphi) = 1$ by Proposition 3.3.14. Furthermore C_φ^n does not even converge pointwise, since $\lim_{n\to\infty} C_\varphi^n(e_1) = C_0(e_1) = 0$ weakly but

$$\lim_{n\to\infty} \|C_\varphi^n(e_1)\|_{\mathcal{B}_1} = \lim_{n\to\infty} \|\varphi_n\|_{\mathcal{B}_1} = \lim_{n\to\infty} \|e_{2^n}\|_{\mathcal{B}_1} = \frac{2}{e}.$$

Chapter A

Appendix

A.1 Notation and definitions

We devote this section to fix the notation for sets and define the properties used in the thesis. For undefined notation about set theory we refer the reader to [83]. We denote by $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of natural numbers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{C}$ the set of complex numbers and by $\mathbb{D}$ the open unit disc of $\mathbb{C}$.

The notation for functional analysis and locally convex spaces is standard (see [69], [63] and [90]). If E is a Banach space, a Fréchet space, or in general a locally convex Hausdorff space, we denote by E' its topological dual space. The weak topology of E is denoted by $\sigma(E, E')$.

The set of all linear and continuous operators between two locally convex Hausdorff spaces E and F is denoted by $\mathcal{L}(E, F)$ and if $E = F$, we denote it by $\mathcal{L}(E)$. If $\mathcal{L}(E, F)$ has the topology of pointwise convergence (strong operator topology) then it is denoted by $\mathcal{L}_s(E, F)$. If the space is endowed with the topology of uniform convergence on bounded sets of E, then we denote it by $\mathcal{L}_b(E, F)$. We denote by $I \in \mathcal{L}(E)$ the identity operator and by $T' \in \mathcal{L}(F', E')$ the adjoint operator of $T \in \mathcal{L}(E, F)$.

Let $T \in \mathcal{L}(E)$ be an operator on a locally convex Hausdorff space. We use the notation

$$T^n = T \circ \overset{(n)}{\cdots} \circ T$$

for the n-th iterate of the operator, with $n \in \mathbb{N}$. We say T is *power bounded* if the set $\{T^n : n \in \mathbb{N}\}$ is equicontinuous. Particularly, if E is a Banach space, then T is power bounded if the set $\{\|T^n\| : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$.

Along this thesis we use the notation

$$T_{[n]} = \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N},$$

for the averages of the iterates or *Cesàro means* of $T \in \mathcal{L}(E)$. We say $T \in \mathcal{L}(E)$ is *mean ergodic* if the sequence $(T_{[n]})_n$ converges in $\mathcal{L}_s(E)$. We say the operator is *uniformly mean ergodic* if the sequence converges in $\mathcal{L}_b(E)$. The standard text for mean ergodic operators is [52]. See also [90].

We define the *pointwise spectrum* of an operator $T \in \mathcal{L}(E)$ as the set $\sigma_{pt}(T, E)$ of its eigenvalues, i.e. the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The *spectrum* is defined as the set $\sigma(T, E)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible (does not have a continuous inverse). The *resolvent* set $\rho(T, E) = \mathbb{C} \setminus \sigma(T, E)$ is the complement of the spectrum in $\mathbb{C}$. We drop the E in the notation when the space is clear to $\sigma_{pt}(T)$, $\sigma(T)$ and $\rho(T)$.

Whenever E is a Banach space, the spectrum is a compact set, hence the resolvent is an open set of $\mathbb{C}$. This is not, in general, the case for Fréchet spaces and leads to the definition of the set $\rho^*(T, E)$. A point $\lambda \in \mathbb{C}$ is in $\rho^*(T, E)$ if there exists $\delta > 0$ such that if $|\lambda - \mu| < \delta$, then $\mu \in \rho(T, E)$ and such that the set $\{(T - \mu I)^{-1} \ : \ |\lambda - \mu| < \delta\}$ is equicontinuous. The set $\rho^*(T, E)$ is open and its complement in $\mathbb{C}$ is the *spectrum of Waelbroeck* $\sigma^*(T, E)$, which coincides with $\sigma(T, E)$ when E is a Banach space. See [81, Chapter III, Sect.3].

We refer the reader to [15] and [16] for the theory of inductive and projective limits. Let $(E_n)_n$ be a sequence of Banach spaces satisfying $E_n \subsetneq E_{n+1}$ with continuous inclusion. We define the *inductive limit* of $(E_n)_n$ as

$$E = \operatorname*{ind}_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} E_n,$$

endowed with the finest locally convex topology making all the natural injections $E_n \hookrightarrow E$ continuous. In what follows we assume that this topology is always Hausdorff.

We include here a famous result by Grothendieck [63, Theorem 24.33] regarding operators and inductive limits.

Grothendieck's Factorization Theorem *Let F be a Fréchet space and $E = \operatorname{ind}_{n \in \mathbb{N}} E_n$ be an inductive limit of Banach spaces. If $T \in \mathcal{L}(F, E)$, then there exists $n_0 \in \mathbb{N}$ such that $T(F) \subseteq E_{n_0}$ and $T \in \mathcal{L}(F, E_{n_0})$.*

Let $(E_n)_n$ be a sequence of Banach spaces satisfying $E_{n+1} \subsetneq E_n$ with continuous inclusion. We define the *projective limit* of $(E_n)_n$ as

$$E = \operatorname*{proj}_{n \in \mathbb{N}} E_n = \bigcap_{n \in \mathbb{N}} E_n,$$

endowed with the coarsest locally convex topology making all the natural injections $E \hookrightarrow E_n$ continuous.

A.2 Ergodic properties of operators

In this section we state classical results of the theory of mean ergodic operators on locally convex spaces and, in particular, on Banach and Fréchet spaces.

The following result is an extension given by Albanese, Bonet and Ricker [2] of a classical theorem by Yosida for Banach spaces [89] (see also page 72 in [52]).

Theorem A.2.1 *Let X be a barrelled locally convex Hausdorff space and let $T \in \mathcal{L}(X)$. Then T is mean ergodic if, and only if,* $\lim_n T^n x/n = 0$*, for all $x \in X$ and the set*

$$\left\{ T_{[n]}x := \frac{1}{n}\sum_{j=1}^{n} T^j(x) \ : \ n \in \mathbb{N} \right\}$$

is relatively sequentially $\sigma(X, X')$-compact, for all $x \in X$. Furthermore, if T is mean ergodic and $P = \lim_{n\to\infty} T_{[n]}$ (in $\mathcal{L}_s(X)$), then $P(X) = \ker(I - T)$ and $\ker P = \overline{(I - T)X}$, with

$$X = \overline{(I - T)X} \oplus \ker(I - T).$$

Some consequences of this theorem, which are corollaries in the same paper, are stated next.

Corollary A.2.2

(i) If X is a barrelled locally convex Hausdorff space and $T \in \mathcal{L}(X)$ satisfies that the sequence $(T_{[n]}x)_n$ converges for the weak topology $\sigma(X, X')$, for all $x \in X$, then T is mean ergodic.

(ii) If X is a reflexive locally convex Hausdorff space, which is a (DF)- or a (LF)-space, then every power bounded operator on X is mean ergodic.

(iii) If X is a (DF)- or a (LF)-space, which is Montel, then every power bounded operator on X is uniformly mean ergodic.

The next result is due to Lin [53] (see also [52, Theorem 2.1]) and characterizes uniform mean ergodicity on Banach spaces.

Theorem A.2.3 *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then T is uniformly mean ergodic if, and only if,* $\lim_n \|T^n\|/n = 0$ *and $(I - T)X$ is closed. Consequently, if T is uniformly mean ergodic, then*

$$X = (I - T)X \oplus \ker(I - T).$$

Another way to ensure uniform mean ergodicity in terms of *quasicompact* operators (see Chapter 3) was given by Yosida and Kakutani in [91].

Theorem A.2.4 *An operator T on a Banach space X is uniformly mean ergodic if it is power bounded and there exist $n \in \mathbb{N}$ and a compact operator K such that $\|T^n - K\| < 1$.*

The necessity of the condition in next theorem is due to Dunford [42, Theorem 3.16] and the sufficiency goes back to Lin [53]. The result connects spectral properties with the uniform mean ergodicity. In [52, Theorem 2.7] it is stated for power bounded operators.

Theorem A.2.5 *An operator T on a Banach space X is uniformly mean ergodic if and only if both $(\|T^n\|/n)_n$ converges to 0 and, either $1 \in \mathbb{C} \setminus \sigma(T)$ or 1 is a pole of order 1 of the resolvent $R_T : \mathbb{C} \setminus \sigma(T) \longrightarrow \mathcal{L}(X)$, where $R_T(\lambda) := (T - \lambda I)^{-1}$. Consequently if 1 is an accumulation point of $\sigma(T)$, then T is not uniformly mean ergodic.*

Next we state a result by Lotz [56], which deals with uniform mean ergodicity on Banach spaces which are Grothendieck spaces with the Dunford–Pettis property (GDP spaces). We only consider here Banach GDP spaces, even if the theory has developed to Fréchet [31] and locally convex Hausdorff spaces [3]. Known examples include L^∞, $H^\infty(\mathbb{D})$, injective Banach spaces (e.g. ℓ_∞) and certain $C(K)$ spaces. See [41, Section VI.5] and [40].

Definition A.2.6 *A Banach space X is a Grothendieck space if any sequence $(x'_j) \subset X'$ which is convergent to 0 for the weak* topology $\sigma(X', X)$ is also convergent to 0 for the weak topology $\sigma(X', X'')$.*

Definition A.2.7 *A Banach space X has the Dunford–Pettis property if for any sequence $(x_j) \subset X$ which is convergent to 0 for the weak topology $\sigma(X, X')$ and any sequence $(x'_j) \subset X'$ which is convergent to 0 for the weak topology $\sigma(X', X'')$ one gets*

$$\lim_{j\to\infty} \langle x_j, x'_j \rangle = 0.$$

Theorem A.2.8 *If X is a Banach GDP space and $T \in \mathcal{L}(X)$ is power bounded, then T is mean ergodic if, and only if, it is uniformly mean ergodic.*

A.3 Properties of weighted inductive limits of spaces of continuous functions

In this section further results of the ones given in Section 1.3 about weighted inductive limits of spaces of continuous functions are stated. They concern the implications of extra assumptions on the family of weights defining the inductive limit. Let us start with a normal locally compact Hausdorff topological space

X. Let $V = (v_k)_k$ be a family of strictly positive weights on X and let $VC = \mathrm{ind}_k\, C_{v_k}(X)$ and $V_0C = \mathrm{ind}_k\, C^0_{v_k}(X)$.

The *Nachbin family* associated to V is

$$\overline{V} := \{\overline{v} : X \longrightarrow (0,\infty) \ : \overline{v} \text{ is upper semicontinuous and for each } k \in \mathbb{N}, \frac{\overline{v}}{v_k} \text{ is bounded on } X\}.$$

The weighted spaces of continuous functions associated with VC and V_0C are defined as follows:

$$C\overline{V} = \{f \in C(X) \ : \ p_{\overline{v}}(f) := \sup_{x\in X} \overline{v}(x)\,|f(x)| < \infty, \forall \overline{v} \in \overline{V}\},$$

$$C\overline{V}_0 = \{f \in C(X) \ : \ \overline{v}f \text{ vanishes at infinity, } \forall \overline{v} \in \overline{V}\}.$$

They are endowed with the locally convex topology generated by the seminorms $p_{\overline{v}}, \overline{v} \in \overline{V}$. It is well known that $VC = C\overline{V}$ algebraically and the topology in $C\overline{V}$ is in general coarser, but they share bounded sets; in fact every bounded subset of $C\overline{V}$ is contained and bounded in a step C_{v_n}. Moreover, $C\overline{V}_0$ is a closed subspace of $C\overline{V}$ and V_0C is a topological subspace of $C\overline{V}_0$, hence of VC. We refer the reader to [23] and [16] for these results. More relations between these spaces arise when extra properties are assumed on V.

The following characterization was given by Bierstedt, Meise and Summers in [23, Theorem 1.3].

Theorem A.3.1 *The following assertions are equivalent:*

(1) V possesses property (S): for each $n \in \mathbb{N}$ there exists $m > n$ such that v_m/v_n vanishes at infinity on X, and

(2) $VC = V_0C = C\overline{V}_0 = C\overline{V}$ algebraically and topologically.

Example A.3.2 If $X = \mathbb{D}$, then the sequence of weights $V = (v_n)_n$, where $v_n(z) := (1-|z|)^n$, $n \in \mathbb{N}$, $z \in \mathbb{D}$, posseses property (S).

We refer the reader to [23] for the relevance and more examples of condition (S).

Definition A.3.3 *An inductive limit $(E,\tau) = \mathrm{ind}_n(E_n,\tau_n)_n$ is*

(i) ***regular*** *if for every bounded set B, there exists $n \in \mathbb{N}$ such that $B \subset E_n$ and B is bounded for τ_n.*

(ii) ***strongly boundedly retractive*** *if it is regular and for each $k \in \mathbb{N}$, there exists $l \geq k$ such that (E,τ) and (E_l,τ_l) induce the same topology on each bounded set of (E_k,τ_k).*

Note that VC is always a regular inductive limit. V_0C is also regular under the assumption that V is regularly decreasing. This assumption actually characterizes the property of being strongly boundedly retractive, as stated next. See Section 3 in [23], in particular Theorem 2.3 and Theorem 2.6 for these results.

Theorem A.3.4 *The following assertions are equivalent:*

(1) V is regularly decreasing, i.e. for each $n \in \mathbb{N}$, there exists $m \geq n$ such that, for every subset A of X,

$$\inf_{y \in A} \frac{v_m(y)}{v_n(y)} > 0 \text{ implies } \inf_{y \in A} \frac{v_k(y)}{v_n(y)} > 0, \text{ for all } k > m,$$

(2) VC is strongly boundedly retractive,

(3) V_0C is strongly boundedly retractive, and

(4) $V_0C = C\overline{V}_0$ algebraically and topologically.

The property of being regularly decreasing is less restrictive than property (S). Even less restrictive than regularly decreasing is property (D), stated next. This result was given by Bierstedt and Meise in [22, Theorem 6.9]. See also [18, Proposition 7].

Theorem A.3.5 *Assume V possesses property (D), i.e. there exists an increasing sequence $J = (X_m)_m$ of subsets of X such that,*

(N,J) for every $m \in \mathbb{N}$, there exists $n_m \geq m$ with

$$\inf_{x \in X_m} \frac{v_k(x)}{v_{n_m}(x)} > 0,$$

for all $k > n_m$, while

(M,J) for each $n \in \mathbb{N}$ and each subset $Y \subset X$ with $Y \cap (X \setminus X_m) \neq \emptyset$ for all $m \in \mathbb{N}$, there exists $n'(n, Y) > n$ such that

$$\inf_{y \in Y} \frac{v_{n'}(y)}{v_n(y)} = 0.$$

Then, $VC = C\overline{V}$ algebraically and topologically.

In [22, Appendix] Bierstedt and Meise gave a counterexample of the converse of this Theorem. Later Bastin in [10] found that the converse holds under the assumption that every weight in $\overline{V}$ is dominated by a continuous weight in $\overline{V} \cap C(X)$. This is always satisfied if the space X is also assumed to be σ-compact (see Lemma 1.4.4 in this thesis). See also [18, Theorem 11].

A.4 Properties of Echelon spaces and Co-Echelon spaces

We state in this section general known results for echelon and co-echelon spaces. They are mostly extracted from [24].

Definition A.4.1 *We say a Köthe matrix* $A = (a_n)_n$ *is regularly decreasing if* $V = (1/a_n)_n$ *is regularly decreasing.*

The assertions in the following theorem can be found in [24] (particularly in Theorem 2.3.(a), in the proof of Theorem 2.7 and in Theorem 3.7, respectively).

Theorem A.4.2 *For* $1 \leq p < \infty$, $\kappa_p(A) = K_p(A)$ *algebraically and topologically. Furthermore,* $\kappa_p(A)$ *is a regular inductive limit. If the Köthe matrix A is regularly decreasing, then also* $\kappa_0(A)$ *is regular.*

The equivalence of (1), (2) and (4) in the next theorem is in [24, Corollary 2.8.(f)]. The implication (3)⇒(2) is due to Bierstedt and Meise [22, Theorem 2.3]. The converse is due to Bierstedt and Bonet [17, Theorems 2.4 and 2.6].

Theorem A.4.3 *The following assertions are equivalent:*

(1) $\kappa_\infty(A) = (\lambda_1(A))'_b$,

(2) $\lambda_1(A)$ *is distinguished,*

(3) V possesses property (D) in Theorem A.3.5 and

(4) $\kappa_\infty = K_\infty$ *algebraically and topologically.*

Bibliography

[1] S. Agathen, K.D. Bierstedt, J. Bonet, Projective limits of weighted (LB)-spaces of continuous functions. *Arch. Math.* **92** (2009), 384–398.

[2] A.A. Albanese, J. Bonet, W.J. Ricker, Mean ergodic operators in Fréchet spaces. *Ann. Acad. Sci. Fenn. Math.* **34** (2009), no. 2, 401–436.

[3] A.A. Albanese, J. Bonet, W.J. Ricker, Grothendieck spaces with the Dunford-Pettis property. *Positivity* **14** (2010), no. 1, 145–164.

[4] A.A. Albanese, J. Bonet, W.J. Ricker, The Fréchet spaces $ces(p+)$, $1 < p < \infty$ *J. Math. Anal. Appl.* **458** (2018) 1314–1323.

[5] A.A. Albanese, J. Bonet, W.J. Ricker, Operators on the Fréchet sequence spaces $ces(p+)$, $1 < p < \infty$, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **113** (2019), 1533—1556.

[6] W. Arendt, I. Chalendar, M. Kumar, S. Srivastava, Asymptotic behaviour of the powers of composition operators on Banach spaces of holomorphic functions. *Indiana Univ. Math. J.* **67** (2018), no. 4, 1571–1595.

[7] W. Arendt, I. Chalendar, M. Kumar, S. Srivastava, Powers of composition operators: asymptotic behaviour on Bergman, Dirichlet and Bloch spaces. *J. Aust. Math. Soc.* to appear.

[8] R.F. Allen, F. Colonna. On the isometric composition operators on the Bloch space in $\mathbb{C}^n$, *J. Math. Anal. Appl.* **355** (2009) 675–688.

[9] R. Aron, M. Lindström, Spectra of weighted composition operators on weighted Banach spaces of analytic functions, *Israel J. Math.* **141** (2004), 263–276.

[10] F. Bastin, On bornological $C\overline{V}(X)$ spaces. *Arch. Math.* (Basel) **53** (1989), no. 4, 394-–398.

[11] M.J. Beltrán Meneu, M.C. Gómez-Collado, E. Jordá, D. Jornet, Mean ergodic composition operators on Banach spaces of holomorphic functions, *J. Funct. Anal.* **270** (2016), no. 12, 4369–4385.

[12] M.J. Beltrán Meneu, M.C. Gómez-Collado, E. Jordá, D. Jornet, Mean ergodicity of weighted composition operators on spaces of holomorphic functions. *J. Math. Anal. Appl.* **444** (2016), no. 2, 1640–1651.

[13] G. Bennett, Some elementary inequalities, *Quart. J. Math.* **38** (1987), 401–425.

[14] G. Bennett, Factoring the classical inequalities, *Mem. Amer. Math. Soc.* **120** (576) (1996)

[15] K.D. Bierstedt, An introduction to locally convex inductive limits, *Functional analysis and its applications (Nice, 1986)*, 35–133, ICPAM Lecture Notes, World Sci. Publishing, Singapore, 1988.

[16] K.D. Bierstedt, A survey of some results and open problems in weighted inductive limits and projective description for spaces of holomorphic functions, *Bull. Soc. Roy. Sci. Liège* **70** (2001), no. 4-6, 167–182.

[17] K.D. Bierstedt, J. Bonet, Stefan Heinrich's density condition for Fréchet spaces and the characterization of the distinguished Köthe echelon spaces. *Math. Nachr.* **135** (1988), 149—180.

[18] K.D. Bierstedt, J. Bonet, Some recent results on $VC(X)$, *Advances in the theory of Fréchet spaces (Istanbul, 1988)*, 181–194, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 287, Kluwer Acad. Publ., Dordrecht, 1989.

[19] K.D. Bierstedt, J. Bonet, Completeness of the (LB)-spaces VC(X), *Arch. Math. (Basel)* **56** (1991), no. 3, 281–285.

[20] K.D. Bierstedt, J. Bonet, Some aspects of the modern theory of Fréchet spaces, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **97** (2003), no. 2, 159–188.

[21] K.D. Bierstedt, J. Bonet, J. Taskinen, Associated weights and spaces of holomorphic functions. *Studia Math.* **127** (1998), no. 2, 137–168.

[22] K.D. Bierstedt, R. Meise, Distinguished echelon spaces and the projective description of weighted inductive limits of type $\mathcal{V}_d\mathcal{C}(X)$. *Aspects of mathematics and its applications*, 169—226, North-Holland Math. Library, 34, North-Holland, Amsterdam, 1986.

[23] K.D. Bierstedt, R. Meise, W.H. Summers, A projective description of weighted inductive limits, *Trans. Amer. Math. Soc.* **272** (1982), no. 1, 107–160.

[24] K.D. Bierstedt, R. Meise, W.H. Summers, Köthe sets and Köthe sequence spaces, *Functional analysis, holomorphy and approximation theory (Rio de Janeiro, 1980)*, 27–91.

[25] K.D. Bierstedt, W.H. Summers, Biduals of weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A* **54** (1993), no. 1, 70–79.

[26] J. Bonet, P. Domański, A note on mean ergodic composition operators on spaces of holomorphic functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **105** (2011), no. 2, 389–396.

[27] J. Bonet, P. Domański, M. Lindström, M.J. Taskinen, Composition operators between weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A* **64** (1998), no. 1, 101–118.

[28] J. Bonet, P. Galindo, M. Lindström, Spectra and essential spectral radii of composition operators on weighted Banach spaces of analytic functions. *J. Math. Anal. Appl.* 340 (2008) 884–891.

[29] J. Bonet, E. Jordá, A. Rodríguez-Arenas, Mean ergodic multiplication operators on weighted spaces of continuous functions, *Mediterr. J. Math* (2018) 15:108.

[30] J. Bonet, W.J. Ricker, The canonical spectral measure in Köthe echelon spaces, *Integral Equations Operator Theory* **53** (2005), 477-–496.

[31] J. Bonet, W.J. Ricker, Schauder decompositions and the Grothendieck and Dunford-Pettis properties in Köthe echelon spaces of infinite order. *Positivity* **11** (2007), no. 1, 77—93.

[32] J. Bonet, W.J. Ricker, Mean ergodicity of multiplication operators in weighted spaces of holomorphic functions, *Arch. Math.* **92** (2009), 428–437.

[33] P.S. Bourdon, V. Matache, J.H. Shapiro, On convergence to the Denjoy-Wolff point. *Illinois J. Math.* **49** (2005), no. 2, 405-–430.

[34] B.J. Carswell, C. Hammond, Composition operators with maximal norm on weighted Bergman spaces. *Proc. Amer. Math. Soc.* **134** (2006), no. 9, 2599-–2605.

[35] B. Cascales, J. Orihuela, On compactness in locally convex spaces, *Math. Z.*, **195** (1987), 365—381.

[36] M.D. Contreras, A.G. Hernandez-Diaz, Weighted composition operators in weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A* **69** (2000), no. 1, 41–60.

[37] C. Cowen, B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[38] G. Crofts, Concerning perfect Fréchet spaces and transformations, *Math. Ann.* **182** (1969), 67–76.

[39] A. Denjoy, Sur l'itération des fonctions analytiques, *C.R. Acad. Sci. Paris* **182** (1926) 255–257.

[40] J. Diestel, A survey of results related to the Dunford-Pettis property. *Proceedings of the Conference on Integration, Topology, and Geometry in Linear Spaces* (Univ. North Carolina, Chapel Hill, N.C., 1979), pp. 15—60, *Contemp. Math.*, **2**, Amer. Math. Soc., Providence, R.I., 1980.

[41] J. Diestel, J.J. Uhl, *Vector measures.* Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.

[42] N. Dunford, Spectral theory. I. Convergence to projections. *Trans. Amer. Math. Soc.* **54**, (1943). 185–217.

[43] V. Fonf, M. Lin, P. Wojtaszczyk, Ergodic characterizations of reflexivity of Banach spaces, *J. Funct. Anal.* **187**, (2001), no. 1, 146—162.

[44] P. Galanopoulos, D. Girela, R. Hernández, Univalent functions, VMOA and related spaces. *J. Geom. Anal.* **21** (2011), no. 3, 665-–682.

[45] S.A. Han, Z.H. Zhou, Mean ergodicity of composition operators on Hardy space. *Proc. Indian Acad. Sci. Math. Sci.* **129** (2019), no. 4, Art. 45, 10 pp.

[46] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Sixth Edition, Oxford University Press, Oxford, 2008.

[47] H.G. Heuser, *Functional analysis.* A Wiley–Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1982.

[48] E. Jordá, A. Rodríguez-Arenas, Ergodic properties of composition operators on Banach spaces of analytic functions, *J. Math. Anal. Appl.*, to appear.

[49] C.N. Kellogg, An extension of the Hausdorff-Young theorem, *Michigan Math. J.* **18** (1971), 121–127.

[50] M. Klilou, L. Oubbi, Multiplication operators on generalized weighted spaces of continuous functions, *Mediterr. J. Math.* **13** (2016), no. 5, 3265–3280.

[51] G. Koenigs, Recherches sur les intégrales de certaines équationes functionelles, *Ann. Sci. École Norm. Sup.* (3) **1** (1884), Supplément, 3—41.

[52] U. Krengel, *Ergodic Theorems*, de Gruyter, Berlin, 1985.

[53] M. Lin, On the uniform ergodic theorem, *Proc. Am. Math. Soc.*, Vol.**43**, 2, April 1974.

[54] J.E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.*, **23** (1925) 481–519.

[55] E.R. Lorch, Means of iterated transformations in reflexive vector spaces. *Bull. Amer. Math. Soc.* **45** (1939), 945-–947

[56] H.P. Lotz, Uniform convergence of operators on L^∞ and similar spaces, *Math. Z.*, **190** (1985), 207–220.

[57] W. Lusky, On the isomophism classes of weighted spaces of harmonic and holomorphic functions, *Studia Math.* **175** (2006), 19–45

[58] K. Madigan, A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347** (1995), no. 7, 2679—2687.

[59] J.S. Manhas, Compact multiplication operators on weighted spaces of vector-valued continuous functions, *Rocky Mountain J. Math.* **34** (2004), no. 3, 1047–1057.

[60] J.S. Manhas, Compact and weakly compact multiplication operators on weighted spaces of vector-valued continuous functions, *Acta Sci. Math. (Szeged)* **70** (2004), no. 1-2, 361–372.

[61] J.S. Manhas, R.K. Singh, Compact and weakly compact weighted composition operators on weighted spaces of continuous functions, *Integral Equations Operator Theory* **29** (1997), no. 1, 63–69.

[62] B. McCluer, K. Saxe, Spectra of composition operators on Bloch and Bergman spaces. *Israel J. Math* **128** (2002), 325–354.

[63] R. Meise, D. Vogt, *Introduction to Functional Analysis*, The Clarendon Press, Oxford University Press, New York, 1997.

[64] A. Montes-Rodríguez, The essential norm of a composition operator on Bloch spaces, *Pac. J. Math.*, **188** (1999), No. 2, 338–351.

[65] A. Montes-Rodríguez, Weighted composition operators on weighted Banach spaces of analytic functions, *J. London Math. Soc* (2) **61** (2000), 872–884.

[66] J. von Neumann, Proof of the quasi-ergodic hypothesis, *Proc. Natl. Acad. Sci. USA*, **18** (1932), 70–82.

[67] L. Oubbi, Multiplication operators on weighted spaces of continuous functions, *Port. Math. (N.S.)* **59** (2002), no. 1, 111–124.

[68] L. Oubbi, Weighted composition operators on non-locally convex weighted spaces, *Rocky Mountain J. Math.*, **35** (2005), no. 6, 2065–2087.

[69] P. Pérez Carreras, J. Bonet, *Barrelled locally convex spaces.* North-Holland Mathematics Studies, 131. Amsterdam, 1987.

[70] K. Petersen, *Ergodic theory.* Cambridge Studies in Advanced Mathematics, 2. Cambridge University Press, Cambridge, 1983.

[71] H. Queffélec, M. Queffélec, *Diophantine Approximation and Dirichlet series*, Hindustain Book Agency, New Delhi, 2013.

[72] F. Riesz, Some mean ergodic theorems, *J. Lond. Math. Soc.*, **13** (1938), 274–278.

[73] A. Rodríguez-Arenas, Some results about diagonal operators on Köthe echelon spaces, *Rev. Real Acad. Cienc. Exact. Fís. Natur.*, **113** (2019), 2959–2968.

[74] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, Berlin, 1993.

[75] J.H. Shapiro, Composition operators and Schröder's functional equation, *Contemp. Math.*, **213** (1998), Providence, RI, 213–228.

[76] A.L. Shields, D.L. Williams, Bonded projections, duality, and multipliers in spaces of analytic functions. *Trans. Amer. Math. Soc.* **162** (1971), 287–302.

[77] R.K. Singh, J.S. Manhas, Multiplication operators on weighted spaces of vector-valued continuous functions, *J. Austral. Math. Soc. Ser. A* **50** (1991), no. 1, 98–107.

[78] R.K. Singh, J.S. Manhas, *Composition operators on function spaces*, North-Holland Publishing Co., Amsterdam, 1993.

[79] R.K. Singh, J.S. Manhas, Operators and dynamical systems on weighted function spaces, *Math. Nachr.* **169** (1994), 279–285.

[80] M. Valdivia, *Topics in locally convex spaces*, North-Holland Mathematics Studies, 67. Notas de Matemática , Amsterdam-New York, 1982.

[81] F.H. Vasilescu, *Analytic Functional Calculus and Spectral Decompositions*, D. Reidel Publ. Co., Dordrecht, 1982.

[82] J. Wengenroth, *Derived functors in functional analysis*, Springer-Verlag, Berlin, 2003.

[83] A. Wilanski, *Topology for Analysis*, Ginn, Waltham, 1970.

[84] E. Wolf, Power bounded composition operators, *Comput. Methods Funct. Theory* **12** (1) (2012) 105–117.

[85] J. Wolff, Sur l'itération des fonctions bornées, *C.R. Acad. Sci. Paris* **182** (1926) 200–201.

[86] J. Wolff, Sur l'itération des fonctions holomorphes dans une région, et dont les valeurs appartiennent à cette région, *C.R. Acad. Sci. Paris* **182** (1926) 42–43.

[87] J. Wolff, Sur une généralisation d'un théorème de Schwarz, *C.R. Acad. Sci. Paris* **182** (1926) 918–920.

[88] M. Yahdi, Super-ergodic operators, *Proc. Amer. Math. Soc.* **134** (2006), 2613–2620.

[89] K. Yosida, Mean ergodic theorem in Banach spaces. *Proc. Imp. Acad.* **14** (1938), no. 8, 292–294.

[90] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[91] K. Yosida, S. Kakutani, Operator-theoretical treatment of Markoff's process and mean ergodic theorem, *Ann. Math.*, **42** (1941), No. 1.

[92] K.H. Zhu, Bloch type spaces of analytic functions. *Rocky Mountain J. Math.* **23** (1993), no. 3, 1143–1177.