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Research Article

Local Convergence Balls for Nonlinear Problems with Multiplicity and Their Extension to Eighth-Order Convergence

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The main contribution of this study is to present a new optimal eighth-order scheme for locating zeros with multiplicity $m \geq 1$. An extensive convergence analysis is presented with the main theorem in order to demonstrate the optimal eighth-order convergence of the proposed scheme. Moreover, a local convergence study for the optimal fourth-order method defined by the first two steps of the new method is presented, allowing us to obtain the radius of the local convergence ball. Finally, numerical tests on some real-life problems, such as a Van der Waals equation of state, a conversion chemical engineering problem, and two standard academic test problems, are presented, which confirm the theoretical results established in this paper and the efficiency of this proposed iterative method. We observed from the numerical experiments that our proposed iterative methods have good values for convergence radii. Further, they not only have faster convergence towards the desired zero of the involved function but also have both smaller residual error and a smaller difference between two consecutive iterations than current existing techniques.

1. Introduction

The construction of higher-order optimal multipoint iterative methods for locating multiple zeros with multiplicity $m \geq 1$ of the involved function f (where $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ is analytic in the enclosed region enclosing the required zero) is one of the toughest, most challenging, and most important tasks in the field of numerical analysis. Paramount importance of optimal multipoint iterative methods in the class of multipoint iteration functions is because they overcome theoretical limitations of one-point iterative methods regarding both computational efficiency and order of convergence (for details one can see some standard text books such as Traub [1] and Petkovíc et al. [2]).

No doubts with the advancement of digital computer, advanced computer arithmetic, and symbolic computation, the construction of higher-order multipoint methods becomes more vital and popular because the calculation of asymptotic error constant term or error equations of iterative methods for multiple zeros is quite easier than in the earlier times. Therefore, in the last decade, several scholars from worldwide like Li et al. [3] in 2009, Neta [4] and Li et al. [5] in 2010, Zhou et al. [6] in 2011, Sharifi et al. [7] in 2012, Soleymani and Babajee [8], Soleymani et al. [9], and Zhou et al. [10] in 2013, Hueso et al. [11] and Behl et al. [12] in 2015, and Behl et al. [13] in 2016 proposed fourth-order multipoint iterative methods. Only the iterative methods/schemes presented by Li et al. [5] (except only two methods) and Neta's [4] are nonoptimal and the rest of the above listed multipoint methods are optimal according to the classical Kung-Traub conjecture [14]. Most of them are the extension of modified Newton's method (also known as Rall's method [1]) or Newton-like method at the expense of additional functional evaluations or increase the substep of the original methods.

In recent years, many researchers have tried to construct optimal iterative schemes for multiple zeros of the involved function with multiplicity $m \geq 1$ that present order of convergence greater than four.

In 2015, Geum et al. [15] proposed a two-point sixth-order iterative scheme based on a bivariate weight function approach for multiple zeros, which is given as follows:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \quad m > 1, \\ x_{n+1} &= y_n - Q(u_n, s_n)\frac{f(y_n)}{f'(y_n)}, \end{aligned} \tag{1}$$

where $u_n = \sqrt[m]{f(y_n)/f(x_n)}$, $s_n = \sqrt[m-1]{f'(y_n)/f'(x_n)}$, and $Q : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a holomorphic function in the neighborhood of origin $(0, 0)$.

The above scheme (1) uses four functional evaluations in order to attain sixth-order convergence with the efficiency index $6^{1/4} = 1.5650$. According to the classical Kung-Traub's conjecture [14], the scheme is nonoptimal. In addition to this, scheme (1) does not work for simple zeros (i.e., m=1) either.

In 2017, Zafar et al. [16] proposed a new eighth-order scheme for a known multiplicity $m \geq 1$ of the desired multiple zero, which is given as follows:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - mu_nH(u_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - u_nv_n(A_2 + A_3u_n)P(v_n)G(w_n)\frac{f(y_n)}{f'(y_n)}, \end{aligned} \tag{2}$$

where $A_2, A_3 \in \mathbb{C}$ are free parameters and the weight functions $H : \mathbb{C} \longrightarrow \mathbb{C}$, $P : \mathbb{C} \longrightarrow \mathbb{C}$, $G : \mathbb{C} \longrightarrow \mathbb{C}$ are analytic function in the neighborhood of 0, with $u_n = (f(y_n)/f(x_n))^{1/m}$, $v_n = (f(z_n)/f(y_n))^{1/m}$, $w_n = (f(z_n)/f(x_n))^{1/m}$.

Recently, we find an interesting work due to Geun, Kim, and Neta, see [17], given by

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - mL_f(u_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - mH_f(u_n, v_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{3}$$

where $L_f : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic in a neighborhood of 0 and $K_f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is holomorphic in a neighborhood of $(0, 0)$ and u_n, v_n are appropriate analytic branches of the m-th root. In this paper the authors not only have constructed a new generic family of optimal eighth-order modified Newton-type multiple-zero finders but also have studied their dynamical behavior; also this kind of study was presented in [18].

But, none of these schemes have been studied from their local convergence treatment in Banach spaces. Studies of these type have a special interest from a mathematical point of view, since these studies allow us to get the local convergence balls, which are balls contained in the domain of the function with center at the solution and where any point of this ball can be taken as a starting point for getting the sequence of iterates that converges to the root.

Therefore, our objective is to introduce methods for multiple roots of high order of convergence and to carry out a study of the local convergence of these.

For this purpose, our aim is to extend for the case of multiple roots, the optimal method of eighth order for simple roots of Chun and Neta [19].

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)}\left[v_n^2 - \frac{1}{v_n - 1}\right], \\ x_{n+1} &= z_n \\ &\quad - \frac{f(x_n)}{f'(x_n)}\left[\phi_f(v_n) + \frac{f(z_n)}{f(y_n) - af(z_n)} + \frac{4f(z_n)}{f(x_n)}\right], \end{aligned} \tag{4}$$

where $v_n = f(y_n)/f(x_n)$, $\phi_f(v_n)$ is a real-valued weight function and a is a real parameter.

The paper is organized as follows. In Section 2, our aim is to perform a local convergence study of this extended method. Since the radius of local convergence for higher-order methods decreases with order, it is necessary to study its behavior when we present a new iterative method. In this paper, we introduce a new idea for establishing local convergence results of iterative methods for locating multiple zeros, under the assumption of a bounding condition for the $(m + 1)$th derivative of the function $f(x)$.

In Section 3, we present an optimal scheme with eighth-order convergence which will work for multiple zeros with multiplicity $m \geq 1$. The proposed scheme is the extension of Chun and Neta's scheme [19]. Moreover, their whole paper becomes the special case of our proposed scheme for $m = 1$. The proposed family requires four functional evaluations in order to obtain eighth-order convergence with the efficiency index $8^{1/4} = 1.6817$ which is higher than the efficiency index of any of the methods for multiple zeros in literature and of the families of Thukral [20] and Geum et al. [15, 21].

In Section 4, we present several numerical tests on some real-life problems which confirm the theoretical results established in this paper and show the validity, accuracy, and efficiency of our proposed iterative method. Finally, we present some concluding remarks in Section 5.

2. Local Convergence of an Optimal Fourth-Order Scheme

Using the first two steps of the family Chun and Neta (4) for simple zeros we can obtain an optimal fourth-order scheme for multiple zeros of $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ by implementing it in the following way:

$$y_n = x_n - m u_n,$$
$$x_{n+1} = x_n - m u_n \left[v_n^2 - \frac{1}{v_n - 1} \right], \tag{5}$$

where $u_n = f(x_n)/f'(x_n)$, $v_n = (f(y_n)/f(x_n))^{1/m}$.

Although (5) uses only real values, the value v_n can be extended to complex value by employing a principal branch whose detailed description is given in [14, 20]. In addition to the optimal convergence, the beauty of this scheme is that we can easily obtain Chun and Neta's [19] scheme as a special case of the above algorithm for $m = 1$.

In this section, our goal is to obtain a local convergence result for the optimal fourth-order two-step method of the new method described by (5).

We look for the radius of the local convergence ball, that is, a real positive number r such that the sequence x_n generated by this iterative method, starting from any point in the open ball $]\alpha - r, \alpha + r[$, remains in this ball and converges to α. For this kind of study, the larger value of r is the best; however, this will obviously depend on the conditions that the nonlinear function satisfies.

This kind of study has been performed for methods of second and third order. See [22] where the authors obtain an estimate of the convergence radius of the well-known modified Newton's method for multiple zeros when the involved function satisfies a Hölder and center-Hölder continuity condition. This result is improved in [23]. Third-order methods have been considered in [24, 25]. In these papers the authors establish the local convergence study by using different properties of divided differences. We do not follow this line of inquiry. We instead utilize Taylor's developments to introduce a new set of bounds to guarantee convergence. Notice that the authors in [25] also use Taylor's developments, but work in a different way from our technique.

Lemma 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in the open domain I, containing α, a zero with multiplicity m with $m > 1$, of the nonlinear equation $f(x) = 0$. Then, for all $x \in I$, the function $f(x)$ can be expressed as*

$$f(x) = (x - \alpha)^m g(x), \quad g(\alpha) \neq 0, \tag{6}$$

where

$$g(x) = \frac{f^{(m)}(\alpha)}{m!} + \frac{1}{(m-1)!} \cdot \int_0^1 \left[f^{(m)}(\alpha + \theta(x - \alpha)) - f^{(m)}(\alpha) \right] \cdot (1 - \theta)^{m-1} \, d\theta. \tag{7}$$

Proof. By approximating the function $f(x)$ with the Taylor development around the multiple zero α and performing some calculations, we obtain

$$\begin{aligned} f(x) &= \frac{f^{(m)}(\alpha)}{m!}(x - \alpha)^m + \int_\alpha^x \frac{f^{(m+1)}(t)}{m!}(x - t)^m \, dt \\ &= \frac{f^{(m)}(\alpha)}{m!}(x - \alpha)^m + \frac{1}{(m-1)!} \int_\alpha^x \left[f^{(m)}(t) - f^{(m)}(\alpha) \right](x - t)^{m-1} \, dt = \frac{f^{(m)}(\alpha)}{m!}(x - \alpha)^m \\ &+ \frac{1}{(m-1)!} \int_0^1 \left[f^{(m)}(\alpha + \theta(x - \alpha)) - f^{(m)}(\alpha) \right](x - \alpha)^m (1 - \theta)^{m-1} \, d\theta = \left[\frac{f^{(m)}(\alpha)}{m!} + \frac{1}{(m-1)!} \cdot \int_0^1 \left[f^{(m)}(\alpha + \theta(x - \alpha)) - f^{(m)}(\alpha) \right] \cdot (1 - \theta)^{m-1} \, d\theta \right](x - \alpha)^m. \end{aligned} \tag{8}$$

Therefore we deduce the expression of function $g(x)$ given in (7). □

Lemma 2. *Under the same conditions of Lemma 1, the function $g(x)$ satisfies*

$$g(\alpha) = \frac{f^{(m)}(\alpha)}{m!}$$
$$g'(x) = \frac{1}{(m-1)!} \int_0^1 f^{(m+1)}(\alpha + \theta(x - \alpha)) \theta (1 - \theta)^{m-1} \, d\theta. \tag{9}$$

Proof. The result holds by direct differentiation under the integral operator of (7). □

This expression suggests that we consider the following bound property for obtaining the local convergence study:

$$\left| f^{(m)}(\alpha)^{-1} f^{(m+1)}(x) \right| \le k_1, \quad \forall x \in I \tag{10}$$

where k_1 is a positive real number. We first obtain certain bounds that we will use for our local convergence study. We define $r_0 = m/k_1$; by taking a starting point $x_0 \in I_0 = [\alpha - r_0, \alpha + r_0[$ and denoting the local error by $|e_0| = |x_0 - \alpha| < r_0$ we have the following results.

Lemma 3. *Under the same conditions of Lemma 1 for all $x_0 \in I_0$ we obtain these bounds:*

$$\begin{aligned} &(B_1) \quad \left| g(\alpha)^{-1} g(x_0) \right| \le \frac{m + 1 + k_1 |e_0|}{m + 1} \\ &(B_2) \quad \left| g(\alpha)^{-1} g'(x_0) \right| \le \frac{k_1}{m + 1} \\ &(B_3) \quad \left| g(x_0)^{-1} g(\alpha) \right| \le \frac{m + 1}{m + 1 - k_1 |e_0|} \\ &(B_4) \quad \left| g(x_0)^{-1} g'(x_0) \right| \le \frac{k_1}{m + 1 - k_1 |e_0|} \end{aligned} \tag{11}$$

Proof. By using Lemma 2 and (10) and applying the Mean Value Theorem we have (B_1)

$$\begin{aligned}\left|g(\alpha)^{-1}g(x_0)\right| &= \left|1 + m\int_0^1 f^{(m)}(\alpha)^{-1}\right.\\ &\cdot\left[f^{(m)}(\alpha+\theta(x_0-\alpha)) - f^{(m)}(\alpha)\right]\\ &\left.\cdot(1-\theta)^{m-1}d\theta\right| \le \left|1 + m\int_0^1 f^{(m)}(\alpha)^{-1}\right.\\ &\left.\cdot f^{(m+1)}(\alpha_1)\theta\left|e_0\right|(1-\theta)^{m-1}d\theta\right|\\ &\le \frac{m+1+k_1\left|e_0\right|}{m+1},\end{aligned} \tag{12}$$

where in the last inequality we have used $\int_0^1 \theta(1-\theta)^{m-1}d\theta = 1/m(m+1)$.

Similar reasoning allows us to get (B_2)

$$\begin{aligned}\left|g(\alpha)^{-1}g'(x_0)\right| &= \left|m!f^{(m)}(\alpha)^{-1}\frac{1}{(m-1)!}\right.\\ &\left.\cdot\int_0^1 f^{(m+1)}(\alpha+\theta(x_0-\alpha))\theta(1-\theta)^{m-1}d\theta\right|\\ &\le \frac{k_1}{m+1}.\end{aligned} \tag{13}$$

We use the Mean Value Theorem to derive $(B3)$ where, for some φ between α and x_0,

$$\begin{aligned}\left|1-g(\alpha)^{-1}g(x_0)\right| &= \left|g(\alpha)^{-1}(g(\alpha)-g(x_0))\right|\\ &= \left|g(\alpha)^{-1}g'(\varphi)\left|e_0\right|\right| \le \frac{k_1\left|e_0\right|}{m+1}\\ &< 1,\end{aligned} \tag{14}$$

where in the last inequality we have used the previous bound (B_2); from this, first we obtain that $g(x_0) \neq 0$, so there exists $g(x_0)^{-1}$ and we can apply Banach Lemma so that

$$\left|g(x_0)^{-1}g(\alpha)\right| \le \frac{m+1}{m+1-k_1\left|e_0\right|}. \tag{15}$$

The following bound can be obviously achieved by using previous ones. □

2.1. Main Result. Now we can establish the main result for obtaining the local convergence radius of the new optimal order iterative method introduced in (5).

Theorem 4. *Let $I \subset \mathbb{R}$ be an open, convex, and nonempty set and $f : I \longrightarrow \mathbb{R}$ be in $\mathscr{C}^m(I)$ with α, a root of multiplicity m for the equation $f(x) = 0$. If boundary conditions (10) are satisfied, let $r_0 = m/k_1$. Then there exists $r \le r_0$ such that, for any initial point $x_0 \in]\alpha - r_0, \alpha + r_0[= I_0$, the sequence $\{x_n\}$, $n \ge 0$, generated by (5) is well-defined, remains in I_0, and converges to solution α, that is, the unique solution in $[\alpha - r_0, \alpha + r_0]$.*

Proof. As has been stated in Lemma 1 for the solution α of multiplicity m, we have for all $x_0 \in I$

$$\begin{aligned}f(x_0) &= g(x_0)(x_0-\alpha)^m = g(x_0)e_0^m,\\ f'(x_0) &= g'(x_0)e_0^m + mg(x_0)e_0^{m-1}.\end{aligned} \tag{16}$$

Now, we start an induction procedure where for $n = 0$ the fourth-order iteration of (5) is written as

$$\begin{aligned}y_0 &= x_0 - mu_0,\\ x_1 &= x_0 - mu_0\left[v_0^2 - \frac{1}{v_0-1}\right],\end{aligned} \tag{17}$$

and then by subtracting α to both sides of the first step and substituting (16), we get

$$\begin{aligned}\tilde{e}_1 = y_0 - \alpha &= \frac{g'(x_0)e_0}{g'(x_0)e_0 + mg(x_0)}e_0\\ &= \frac{g(\alpha)^{-1}g'(x_0)e_0/m}{g(\alpha)^{-1}g'(x_0)e_0/m + g(\alpha)^{-1}g(x_0)}e_0.\end{aligned} \tag{18}$$

In order to apply Banach lemma to the denominator, by applying the Mean Value Theorem, it follows that

$$\begin{aligned}&\left|1 - \frac{g(\alpha)^{-1}g'(x_0)e_0}{m} - g(\alpha)^{-1}g(x_0)\right|\\ &\quad\le \left|g(\alpha)^{-1}(g(\alpha)-g(x_0))\right| + \left|g(\alpha)^{-1}g'(x_0)\right|\left|\frac{e_0}{m}\right|\\ &\quad\le \frac{k_1\left|e_0\right|}{m} < 1,\end{aligned} \tag{19}$$

so we have $|(g(\alpha)^{-1}g'(x_0)e_0/m + g(\alpha)^{-1}g(x_0))^{-1}| < m/(m - k_1|e_0|)$ and therefore

$$\left|\tilde{e}_1\right| \le \frac{k_1\left|e_0\right|}{(m+1)(m-k_1\left|e_0\right|)}\left|e_0\right| = G_1(\left|e_0\right|)\left|e_0\right|, \tag{20}$$

where $G_1(t)$ is an increasing function defined as follows:

$$G_1(t) = \frac{k_1 t}{(m+1)(m-k_1 t)}, \tag{21}$$

so, by taking $h_1(t) = G_1(t) - 1$, it satisfies $h(0) = -1$ and $h(r_0^-) \longrightarrow +\infty$. We take the smallest positive real root of this function, $r_1 \in]0, r_0[$, and then $0 < G_1(t) < 1\ \forall t \in]0, r_1[$. Turning to (20) we get

$$\tilde{e}_1 \le G_1(\left|e_0\right|)\left|e_0\right| < \left|e_0\right|. \tag{22}$$

Now by subtracting α from both sides of the second step in (5), one has

$$e_1 = x_1 - \alpha = e_0 - \frac{v_0^3 - v_0^2 - 1}{v_0 - 1}(e_0 - \tilde{e}_1), \tag{23}$$

and, after applying (16) we have

$$v_0 = \left(\frac{f(y_0)}{f(x_0)}\right)^{1/m} = \left(\frac{g(y_0)}{g(x_0)}\right)^{1/m} \frac{\tilde{e}_1}{e_0} = \frac{g'(x_0)e_0}{g'(x_0)e_0 + mg(x_0)}\left(\frac{g(y_0)}{g(x_0)}\right)^{1/m}. \tag{24}$$

Substituting in (18), it follows that

$$e_1 = e_0 - \frac{mg(x_0)e_0}{mg(x_0) - g'(x_0)e_0\left((g(y_0)/g(x_0))^{1/m} - 1\right)} - \frac{mg'(x_0)^2 g(x_0)(g(y_0)/g(x_0))^{2/m} e_0^3}{(g'(x_0)e_0 + mg(x_0))^3}. \tag{25}$$

If we denote $C = ((g(y_0)/g(x_0))^{1/m} - 1)$ and assign $\widetilde{D}_1$ and $\widetilde{D}_2$ to be the denominators of the previous expression, we obtain $e_1 = \widetilde{N}/D$, where D is the common denominator and $\widetilde{N}$ the corresponding numerator:

$$\begin{aligned}
\widetilde{D}_1 &= mg(x_0) - g'(x_0)Ce_0,\\
\widetilde{D}_2 &= \left(g'(x_0)e_0 + mg(x_0)\right)^3,\\
D &= \widetilde{D}_1\widetilde{D}_2,\\
\widetilde{N} &= \widetilde{D}_1\widetilde{D}_2 e_0 - \widetilde{D}_2 mg(x_0)e_0\\
&\quad - \widetilde{D}_1(mg'(x_0)^2 g(x_0)\left(\frac{g(y_0)}{g(x_0)}\right)^{2/m} e_0^3.
\end{aligned} \tag{26}$$

By performing the calculation we have

$$\begin{aligned}
\widetilde{N} &= -Cg'(x_0)^4 e_0^5\\
&\quad + Cg'(x_0)^3 g(x_0)m\left(\frac{g(y_0)}{g(x_0)}\right)^{2/m} - 3)e_0^4\\
&\quad - g'(x_0)^2 g(x_0)^2 m^2\left(3C + \left(\frac{g(y_0)}{g(x_0)}\right)^{2/m}\right)e_0^3\\
&\quad - Cg'(x_0)g(x_0)^3 m^3 e_0^2.
\end{aligned} \tag{27}$$

Now, we express the error equation in the following terms, $e_1 = N/De_0^2$, with

$$\begin{aligned}
N = -\Bigg[&Cg'(x_0)^4 e_0^3\\
&+ Cg'(x_0)^3 g(x_0)m\left(3 - \left(\frac{g(y_0)}{g(x_0)}\right)^{2/m}\right)e_0^2\\
&+ m^2 g'(x_0)^2 g(x_0)^2\left(3C + \left(\frac{g(y_0)}{g(x_0)}\right)^{2/m}\right)e_0\\
&+ Cm^3 g'(x_0)g(x_0)^3\Bigg]
\end{aligned} \tag{28}$$

and the denominator

$$\begin{aligned}
D &= m^4 g(x_0)^4 + m^3(3 - C)g'(x_0)g(x_0)^3 e_0\\
&\quad + 3m^2(1 - C)g'(x_0)^2 g(x_0)^2 e_0^2\\
&\quad + mg'(x_0)^3 g(x_0)(1 - 3C)e_0^3 - Cg'(x_0)^4 e_0^4.
\end{aligned} \tag{29}$$

In order to apply the established bounds from previous lemmas, we multiply the numerator and denominator by $g(\alpha)^{-4}$ and consider from now on

$$e_1 = -\frac{g(\alpha)^{-4}N}{g(\alpha)^{-4}D}e_0^2 = -\frac{N_1 + N_2 + N_3 + N_4}{D_1 + D_2 + D_3 + D_4 + D_5}e_0^2, \tag{30}$$

where N_i, D_i $i = 1, 2, \ldots$, are

$$\begin{aligned}
N_1 &= m^3 C\left(g(\alpha)^{-1}g'(x_0)\right)\left(g(\alpha)^{-1}g(x_0)\right)^3,\\
N_2 &= m^2\left(g(\alpha)^{-1}g'(x_0)\right)^2\left(g(\alpha)^{-1}g(x_0)\right)^2\\
&\quad \cdot\left(3C + \left(\frac{g(y_0)}{g(x_0)}\right)^{2/m}\right)e_0,\\
N_3 &= Cm\left(g(\alpha)^{-1}g'(x_0)\right)^3\left(g(\alpha)^{-1}g(x_0)\right)\\
&\quad \cdot\left(3 - \left(\frac{g(y_0)}{g(x_0)}\right)^{2/m}\right)e_0^2,\\
N_4 &= C\left(g(\alpha)^{-1}g'(x_0)\right)^4 e_0^3,
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
D_1 &= m^4\left(g(\alpha)^{-1}g(x_0)\right)^4,\\
D_2 &= m^3(3 - C)\left(g(\alpha)^{-1}g'(x_0)\right)\left(g(\alpha)^{-1}g(x_0)\right)^3 e_0,\\
D_3 &= 3m^2(1 - C)\left(g(\alpha)^{-1}g'(x_0)\right)^2\left(g(\alpha)^{-1}g(x_0)\right)^2\\
&\quad \cdot e_0^2,\\
D_4 &= m(1 - 3C)\left(g(\alpha)^{-1}g'(x_0)\right)^3 g(\alpha)^{-1}g(x_0)e_0^3,\\
D_5 &= -Ce_0^4\left(g(\alpha)^{-1}g'(x_0)\right)^4 e_0^4.
\end{aligned} \tag{32}$$

Using the established bounds from Lemma 2 and applying the Mean Value Theorem, there exists ζ between x_0 and y_0 verifying

$$\begin{aligned} |C| &= \left|\left(\frac{g(y_0)}{g(x_0)}\right)^{1/m} - 1\right| \\ &\leq |g(x_0)|^{-1/m}\left|g(x_0)^{1/m} - g(y_0)^{1/m}\right| \\ &\leq \frac{1}{m}|g(x_0)|^{-1/m}\left|g'(\zeta)\, g(\zeta)^{(1-m)/m}(x_0 - y_0)\right|. \end{aligned} \tag{33}$$

Then, if $|g(x_0)| \leq |g(\zeta)|$ we have that $|g(x_0)|^{(1-m)/m} \geq |g(\zeta)|^{(1-m)/m}$ so

$$\begin{aligned} |C| &\leq \frac{1}{m}|g(x_0)|^{-1/m}|g'(\zeta)||g(\zeta)|^{(1-m)/m}|x_0 - y_0| \\ &\leq \frac{1}{m}|g(x_0)|^{-1/m}|g'(\zeta)||g(x_0)|^{(1-m)/m}|x_0 - y_0| \\ &\leq \frac{1}{m}|g(x_0)|^{-1}|g'(\zeta)||x_0 - y_0| \\ &\leq \frac{1}{m}\frac{k_1}{m+1-k_1|e_0|}(1 + G_1(|e_0|))|e_0|. \end{aligned} \tag{34}$$

Similar reasoning is applied if $|g(x_0)| \geq |g(\zeta)|$, where it then holds that $|g(x_0)|^{-1/m} \leq |g(\zeta)|^{-1/m}$ and so the same bound for C is obtained. By using this bound and those of Lemma 3, we have the following bounds for the numerator terms:

$$\begin{aligned} |N_1| &\leq m^3\frac{k_1(1+G_1(|e_0|))|e_0|}{m(m+1-k_1|e_0|)} \cdot \frac{k_1}{m+1}\left(\frac{m+1+k_1|e_0|}{m+1}\right)^3 = n_1(|e_0|), \\ |N_2| &\leq m^2\left(\frac{k_1}{m+1}\right)^2\left(\frac{m+1+k_1|e_0|}{m+1}\right)^2 \cdot \left(\frac{3}{m}\frac{k_1(1+G_1(|e_0|))|e_0|}{m+1-k_1|e_0|} + \left(\frac{m+1+k_1G_1(|e_0|)|e_0|}{m+1-k_1|e_0|}\right)^{2/m}\right)|e_0| = n_2(|e_0|), \\ |N_3| &\leq m\frac{k_1(1+G_1(|e_0|))}{m(m+1-k_1|e_0|)}\left(\frac{k_1}{m+1}\right)^3 \cdot \left(\frac{m+1+k_1|e_0|}{m+1}\right)\left(3 + \left(\frac{m+1+k_1G_1(|e_0|)|e_0|}{m+1-k_1|e_0|}\right)^{2/m}\right)|e_0|^2 = n_3(|e_0|), \\ |N_4| &\leq \left(\frac{k_1}{m+1}\right)^4\frac{k_1(1+G_1(|e_0|))|e_0|}{m(m+1-k_1|e_0|)}|e_0|^3 = n_4(|e_0|). \end{aligned} \tag{35}$$

Let us first consider the first term from the denominator, D_1. By recalling that $|a-b| \geq |a| - |b|$, we have

$$\begin{aligned} |D_1| &= \left|m^4 g(\alpha)^{-4} g(x_0)^4\right| \\ &= \left|m^4 - m^4 g(\alpha)^{-4}\left(g(\alpha)^4 - g(x_0)^4\right)\right| \\ &\geq \left|m^4\right| - \left|m^4 g(\alpha)^{-4}\left(g(\alpha)^4 - g(x_0)^4\right)\right|. \end{aligned} \tag{36}$$

If we apply the Mean Value Theorem to the function $g(x)^4$, it follows that there exists δ between α and x_0 with

$$g(\alpha)^4 - g(x_0)^4 = 4g(\delta)^3 g'(\delta)(\alpha - x_0), \tag{37}$$

so that by using bounds (B_1) and (B_2) it follows that

$$\begin{aligned} &\left|g(\alpha)^{-4}\left(g(\alpha)^4 - g(x_0)^4\right)\right| \\ &\quad = \left|4g(\alpha)^{-3} g(\delta)^3 g(\alpha)^{-1} g'(\delta)(\alpha - x_0)\right| \\ &\quad \leq 4\left(\frac{m+1+k_1|e_0|}{m+1}\right)^3\frac{k_1}{m+1}|e_0|. \end{aligned} \tag{38}$$

Then, D_1 satisfies

$$|D_1| \geq m^4 - 4m^4\left(\frac{m+1+k_1|e_0|}{m+1}\right)^3\frac{k_1}{m+1}|e_0|, \tag{39}$$

and the other terms in the denominator likewise satisfy

$$\begin{aligned} |D_2| &\leq m^3\left(3 + \frac{k_1(1+G_1(|e_0|))|e_0|}{m(m+1-k_1|e_0|)}\right) \cdot \left(\frac{m+1+k_1|e_0|}{m+1}\right)^3\frac{k_1}{m+1}|e_0| = d_2(|e_0|), \\ |D_3| &\leq 3m^2\left(1 + \frac{k_1(1+G_1(|e_0|))|e_0|}{m(m+1-k_1|e_0|)}\right)\left(\frac{k_1}{m+1}\right)^2 \cdot \left(\frac{m+1+k_1|e_0|}{m+1}\right)^2|e_0|^2 = d_3(|e_0|), \\ |D_4| &\leq m\left(1 + \frac{3k_1(1+G_1(|e_0|))|e_0|}{m(m+1-k_1|e_0|)}\right)\left(\frac{k_1}{m+1}\right)^3 \cdot \left(\frac{m+1+k_1|e_0|}{m+1}\right)|e_0|^3 = d_4(|e_0|), \\ |D_5| &\leq \left(\frac{k_1(1+G_1(|e_0|))|e_0|}{m(m+1-k_1|e_0|)}\right)\left(\frac{k_1}{m+1}\right)^4|e_0|^4 = d_5(|e_0|), \end{aligned} \tag{40}$$

so that we get

$$\begin{aligned} &\left|D_1 + D_2 + D_3 + D_4 + D_5\right| \\ &\quad \geq \left|D_1\right| - \left|D_2 + D_3 + D_4 + D_5\right| \\ &\quad \geq m^4 - 4m^4 \left(\frac{m+1+k_1\left|e_0\right|}{m+1}\right)^3 \frac{k_1}{m+1}\left|e_0\right| - \left|D_2\right| \\ &\qquad - \left|D_3\right| - \left|D_4\right| - D_5\left|.\right. \end{aligned} \quad (41)$$

Notice that the precedent bound can result in a negative value, so we define function $G_2(t)$ with $t \geq 0$ such that

$$\begin{aligned} G_2(t) = {} & m^4 - 4m^4 \left(\frac{m+1+k_1 t}{m+1}\right)^3 \frac{k_1}{m+1} t - d_2(t) \\ & - d_3(t) - d_4(t) - d_5(t), \end{aligned} \quad (42)$$

G_2 satisfies $G_2(0) > 0$ and $G_2(r_0^-) \longrightarrow -\infty$, so there exists r_2, the smallest positive root in $]0, r_0[$, such that $G_2(t) > 0 = G_2(r_2)$, $\forall t \in\,]0, r_0[$. If we take x_0 such that $|e_0| = |x_0 - \alpha| < r_2$, then

$$\left|D_1 + D_2 + D_3 + D_4 + D_5\right| \geq G_2\left(\left|e_0\right|\right) \geq G_2\left(r_2\right) = 0. \quad (43)$$

Then, if we take $\widetilde{r} = \min\{r_0, r_1, r_2\}$, we have for all $x_0 \in\,]\alpha - \widetilde{r}, \alpha + \widetilde{r}[$

$$\begin{aligned} \left|e_1\right| &\leq \frac{n_1\left(\left|e_0\right|\right) + n_2\left(\left|e_0\right|\right) + n_3\left(\left|e_0\right|\right) + n_4\left(\left|e_0\right|\right)}{G_2\left(\left|e_0\right|\right)}\left|e_0\right|^2 \\ &\leq G_3\left(\left|e_0\right|\right)\left|e_0\right|, \end{aligned} \quad (44)$$

where

$$G_3(t) = \frac{n_1(t) + n_2(t) + n_3(t) + n_4(t)}{G_2(t)} t. \quad (45)$$

Defining $h_3(t) = G_3(t) - 1$, which satisfies $h_3(0) = -1$ and $h_3(r_2^-) \longrightarrow +\infty$, thus, there exists r_3, the smallest positive root such that $G_3(t) < 1$, for all $t \in\,]0, r_3[$.

Finally we take $r = \min\{\widetilde{r}, r_3\}$ and we have for all $x_0 \in\,]\alpha - r, \alpha + r[$

$$\begin{aligned} \left|\widetilde{e}_1\right| &\leq G_1\left(\left|e_0\right|\right)\left|e_0\right| < \left|e_0\right|, \\ \left|e_1\right| &\leq G_3\left(\left|e_0\right|\right)\left|e_0\right| < \left|e_0\right|. \end{aligned} \quad (46)$$

$G_3(t)$ is an increasing function for n_i, $i = 1, \ldots, 4$ being increasing and $G_2(t)$ a decreasing function.

The same process holds starting from x_1 and getting x_2, and then by an inductive procedure one has $y_k, x_k \in\,]\alpha - r, \alpha + r[$ for all $k > 0$ by using the fact that G_1 and G_3 are increasing functions as follows:

$$\begin{aligned} &\left|\widetilde{e}_k\right| \leq G_1(r)\, G_3(r)^{k-1}\left|e_0\right| < \left|e_0\right| \\ &\left|e_k\right| \leq G_3(r)\left|e_{k-1}\right| \leq G_3(r)^2\left|e_{k-2}\right| \leq \cdots \leq G_3(r)^k\left|e_0\right| \\ &\quad < \left|e_0\right|, \end{aligned} \quad (47)$$

where $|\widetilde{e}_k| = |y_{k-1} - \alpha|$ and $|e_k| = |x_k - \alpha|$.

By taking limits in the last expressions and using $\lim_{k \longrightarrow +\infty} G_3(r)^k = 0$, we get that $\lim_{k \longrightarrow +\infty} x_k = \alpha$. Therefore we have obtained r, the radius of the local convergence ball. That is, for any starting guess $x_0 \in\,]\alpha - r, \alpha + r[$ the sequences obtained by the fourth-order iterative method y_k, and x_k remain in this interval and converge to the solution α.

To show uniqueness in the interval centered at the solution and radius r_0, we assume that there exists a second solution $\beta \in\,]\alpha - r, \alpha + r[$, and by Lemma 1 we have

$$f(\beta) = g(\beta)(\beta - \alpha)^m. \quad (48)$$

By using

$$\begin{aligned} \left|1 - g(\alpha)^{-1} g(\beta)\right| &= \left|g(\alpha)^{-1}\left(g(\alpha) - g(\beta)\right)\right| \\ &= \left|g(\alpha)^{-1} g'(\varphi)\left|\alpha - \beta\right|\right| \\ &\leq \frac{k_1}{m+1}\left|e_0\right| < \frac{m}{m+1} < 1, \end{aligned} \quad (49)$$

we deduce that $g(\beta) \neq 0$ and then by (48) we have that $\beta = \alpha$. □

3. Development of an Optimal Eighth-Order Scheme

In an analogous way, as we have done the extension of the Chun and Neta method from order four to multiple roots, introducing one more step, we have managed to extend this method to order eight.

$$\begin{aligned} y_n &= x_n - m u_n, \\ z_n &= x_n - m u_n\left[v_n^2 - \frac{1}{v_n - 1}\right], \\ x_{n+1} &= z_n - m t_n u_n\left[\phi_f\left(v_n\right) + \frac{t_n}{v_n - a t_n} + 4 t_n\right], \end{aligned} \quad (50)$$

where the weight function $\phi_f : \mathbb{R} \longrightarrow \mathbb{R}$ is an analytic function [26] in a neighborhood of (0) with $u_n = f(x_n)/f'(x_n)$, $v_n = (f(y_n)/f(x_n))^{1/m}$, $t_n = v_n(f(z_n)/f(y_n))^{1/m}$, and a free variable $a \in \mathbb{R}$. In addition to the optimal convergence, the beauty of this scheme is that we can easily obtain Chun and Neta's [19] scheme as a special case of the above algorithm for $m = 1$.

In Theorem 5, we demonstrate that the order of convergence of the proposed scheme will reach eight without using any additional functional evaluations. It is interesting to observe that ϕ_f contributes its role in the construction of the desired eighth-order convergence (for details please see Theorem 5).

Theorem 5. *Let $x = \alpha$ be a multiple zero with multiplicity $m \geq 1$ of an analytic function $f : \mathbb{R} \longrightarrow \mathbb{R}$ in a region enclosing* *the required zero of $f(x)$. Then, the scheme defined by (50) has an eighth-order convergence if the following expressions hold:*

$$\phi(0) = 1,$$
$$\phi'(0) = 2,$$
$$\phi''(0) = 4, \tag{51}$$
$$\phi'''(0) = -6.$$

Proof. Let us expand the functions $f(x_n)$ and $f'(x_n)$ about $x = \alpha$ with the help of Taylor's series expansion which produce

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^m\left(1 + c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O\left(e_n^9\right)\right), \tag{52}$$

and

$$f'(x_n) = \frac{f^m(\alpha)}{m!}e_n^{m-1}\left(m + (m+1)c_1e_n + (m+2)c_2e_n^2 + (m+3)c_3e_n^3 + (m+4)c_4e_n^4 + (m+5)c_5e_n^5 + (m+6)c_6e_n^6 + (m+7)c_7e_n^7 + (m+8)c_8e_n^8 + O\left(e_n^9\right)\right), \tag{53}$$

respectively, where $c_k = (m!/(m-1+k)!)(f^{m-1+k}(\alpha)/f^m(\alpha))$, $k = 2, 3, 4\ldots, 8$, and $e_n = x_n - \alpha$ is the error at nth iteration.

By using expressions (52) and (53), we have

$$u_n = \frac{f(x_n)}{f'(x_n)} = \frac{1}{m}e_n - \frac{c_1}{m^2}e_n^2 + \frac{(m+1)c_1^2 - 2mc_2}{m^3}e_n^3 + \sum_{i=0}^{4}G_ie_n^{i+4} + O\left(e_n^9\right), \tag{54}$$

where $G_i = G_i(m, c_1, c_2, \ldots, c_8)$ are given in terms of $m, c_2, c_3, \ldots, c_8$ with the first two coefficients explicitly written as $G_0 = (1/m^4)[m(3m+4)c_1c_2 - 3m^2c_3 - (m+1)^2c_1^3]$ and $G_1 = (1/m^5)[2m^2(2m+3)c_1c_3 - 2m(2m^2+5m+3)c_1^2c_2 + 2m^2((m+2)c_2^2 - 2mc_4) + (m+1)^3c_1^4]$.

Again using the Taylor Series expansion and expression (54), we further obtain

$$f(y_n) = f^{(m)}(\alpha)e_n^{2m}\left[\frac{(c_1/m)^m}{m!} + \frac{\left(2mc_2 - (m+1)c_1^2\right)(c_1/m)^m e_n}{m!c_1} + \left(\frac{c_1}{m}\right)^{1+m} \cdot \frac{1}{2m!c_1^3}\left\{\left(3 + 3m + 3m^2 + m^3\right)c_1^4 - 2m\left(2 + 3m + 2m^2\right)c_1^2c_2 + 4(m-1)m^2c_2^2 + 6m^2c_1c_3\right\}e_n^2 + \sum_{i=0}^{5}\overline{G}_ie_n^{i+3} + O\left(e_n^9\right)\right]. \tag{55}$$

With the help of expressions (52) and (55), we get

$$v_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{1/m} = \frac{c_1e_n}{m} + \frac{2mc_2 - (m+2)c_1^2}{m^2}e_n^2 + \frac{\left(2m^2 + 7m + 7\right)c_1^3 + 6m^2c_3 - 2m(3m+7)c_1c_2}{2m^3} \cdot e_n^3 + \theta_1e_n^4 + \theta_2e_n^5 + O\left(e_n^6\right). \tag{56}$$

where $\theta_1 = (1/6m^4)[12m^2(2m+5)c_1c_3 + 12m^2((m+3)c_2^2 - 2mc_4) - 6m(4m^2+16m+17)c_1^2c_2 + (6m^3+29m^2+51m+34)c_1^4]$ and $\theta_2 = (1/24m^5)[12m^2(10m^2+43m+49)c_1^2c_3 - 24m^3((5m+17)c_2c_3 - 5mc_5) + 12m^2((10m^2 + 47m + 53)c_2^2 - 2m(5m+13)c_4)c_1 - 4m(30m^3 + 163m^2 + 306m + 209)c_1^3c_2 + (24m^4 + 146m^3 + 355m^2 + 418m + 209)c_1^5]$.

Inserting expressions (52)–(56) into the second substep of scheme (50), we further obtain

$$z_n - \alpha = \frac{(m+7)c_1^3 - 2mc_1c_2}{2m^3}e_n^4 + \sum_{i=0}^{3}H_ie_n^{i+5} + O\left(e_n^9\right), \tag{57}$$

where $H_i = H_i(m, c_1, c_2, \ldots, c_8)$ are given in terms of $m, c_2, c_3, \ldots, c_8$ with the first two coefficients explicitly written as $H_0 = -(1/6m^4)[(7m^2+66m+89)c_1^4 + 12m^2c_1c_3 + 12m^2c_2^2 - 12m(2m+11)c_1^2c_2]$ and $H_1 = (1/24m^5)[156m^2(m+5)c_1^2c_3 - 168m^3c_2c_3 - 4m(53m^2 + 468m + 655)c_1^3c_2 + 12m^2((17m+91)c_2^2 - 6mc_4)c_1 + (46m^3 + 533m^2 + 1310m + 991)c_1^5]$.

Again, with the help of Taylor series expansion and expression (57), we further have

$$f(z_n) = f^{(m)}(\alpha) \cdot e_n^{4m}\left[\frac{2^{-m}\left(\left((m+7)c_1^3 - 2mc_1c_2\right)/m^3\right)^m}{m!} + \sum_{i=1}^{5}\overline{H}_ie_n^i + O\left(e_n^6\right)\right]. \tag{58}$$

With the help of expressions (55) and (58), we further obtain

$$t_n = v_n\left(\frac{f(z_n)}{f(y_n)}\right)^{1/m} = \frac{\left\{(m+7)c_1^3 - 2mc_2\right\}c_1}{2m^3}e_n^3 + \gamma_1e_n^4 + \gamma_2e_n^5 + O\left(e_n^6\right), \tag{59}$$

where $\gamma_1 = -(1/6m^4)[(7m^2 + 69m + 110)c_1^4 + 12m^2c_1c_3 + 12m^2c_2^2 - 6m(4m + 23)c_1^2c_2]$ and $\gamma_2 = (1/24m^5)[12m^2(13m + 69)c_1^2c_3 + 12m^2((17m + 97)c_2^2 - 6mc_4)c_1 - 168m^3c_2c_3 - 4m(53m^2 + 498m + 811)c_1^3c_2 + (46m^3 + 567m^2 + 1622m + 1389)c_1^5]$.

It is clear from expression (56) that v_n is of order e_n. Therefore, we can expand the weight function $\phi_f(v_n)$ in a neighborhood of the origin (0) by using Taylor series expansion and expand up to fourth-order terms as follows:

$$\phi_f(v_n) = \phi(0) + \phi'(0)v_n + \frac{1}{2!}\phi''(0)v_n^2 + \frac{1}{3!}\phi'''(0)v_n^3 + \frac{1}{4!}\phi''''(0)v_n^4. \tag{60}$$

By using expressions (52)–(60) in the last substep of proposed scheme (50), we obtain

$$e_{n+1} = -\frac{c_1(\phi(0)-1)\left((m+7)c_1^2 - 2mc_2\right)}{2m^3}e_n^4 + \sum_{i=1}^{4}L_ie_n^{i+4} + O\left(e_n^9\right), \tag{61}$$

where $L_i = L_i(m, a, \phi(0), \phi'(0), \phi''(0), \phi'''(0), c_1, c_2, \ldots, c_8)$.

It is noteworthy that we will obtain at least fifth-order convergence if we choose

$$\phi(0) = 1. \tag{62}$$

Insert the value of $\phi(0) = 1$ in $L_1 = 0$. Then, we obtain

$$\frac{\left(\phi'(0)-2\right)\left((7+m)c_1^2 - 2mc_2\right)c_1^2}{2m^4} = 0, \tag{63}$$

which further deduce to at least sixth-order convergence, when we use the following expression:

$$\phi'(0) = 2. \tag{64}$$

By substituting expressions (62) and (64) in $L_2 = 0$, we have

$$-\frac{\left(\phi''(0)-4\right)\left((7+m)c_1^2 - 2mc_2\right)c_1^3}{4m^5} = 0, \tag{65}$$

which further yields

$$\phi''(0) = 4. \tag{66}$$

We again use expressions (62), (64), and (66) in $L_3 = 0$. Then, we get

$$-\frac{\left(\phi'''(0)+6\right)\left((7+m)c_1^2 - 2mc_2\right)c_1^4}{12m^6} = 0, \tag{67}$$

which further gives

$$\phi'''(0) = -6. \tag{68}$$

Finally, substitute expressions (62), (64), (66), and (68) into expression (61) for obtaining the optimal asymptotic error constant term, which is given as follows:

$$\begin{aligned} e_{n+1} = {} & -\frac{c_1\left((m+7)c_1^2 - 2mc_2\right)}{48m^7}\Big[\Big(\phi''''(0) \\ & + 6a(m+7)^2 - 14m^2 - 192m - 730\Big)c_1^4 \\ & - 24m\left(a(m+7) - 2(m+8)\right)c_1^2c_2 + 24(a-1) \\ & \cdot m^2c_2^2 - 24m^2c_1c_3\Big]e_n^8 + O\left(e_n^9\right), \end{aligned} \tag{69}$$

where $\phi''''(0), a \in \mathbb{R}$. Expression (69) demonstrates that our proposed scheme (50) reaches eighth-order convergence by using only four functional evaluations (viz. $f(x_n)$ $f'(x_n)$ $f(y_n)$ and $f(z_n)$) per iteration. Therefore, it is an optimal scheme according to Kung-Traub conjecture, completing the proof. □

3.1. Special Cases of the Proposed Scheme. In this section, we will discuss some special cases of our proposed scheme (50) by assigning different weight functions ϕ_f. In this regard, please see the following cases, where we have mentioned some different kinds of members of the proposed scheme:

(1) Let us consider the following weight function directly from the proposed Theorem 5

$$\phi(v_n) = 1 + 2v_n + 2v_n^2 - v_n^3 + v_n^4\frac{\phi''''(0)}{24}, \tag{70}$$

which further has

$$\begin{aligned} y_n &= x_n - mu_n, \\ z_n &= x_n - mu_n\left[v_n^2 - \frac{1}{v_n - 1}\right], \\ x_{n+1} &= z_n - mt_nu_n\Bigg[1 + 2v_n + 2v_n^2 - v_n^3 + v_n^4\frac{\phi''''(0)}{24} \\ &\quad + \frac{t_n}{v_n - at_n} + 4t_n\Bigg], \end{aligned} \tag{71}$$

a new optimal scheme with eighth-order convergence.

(2) Let us choose another weight function which satisfies the conditions of Theorem 5. Then, we obtain

$$\phi(v_n) = \frac{16}{v_n + 2} + 6v_n - 7, \tag{72}$$

which further yields another following optimal eighth-order scheme:

$$\begin{aligned} y_n &= x_n - mu_n, \\ z_n &= x_n - mu_n\left[v_n^2 - \frac{1}{v_n - 1}\right], \end{aligned}$$

$$[5pt]x_{n+1} = z_n - mt_n u_n \left[\frac{16}{2+v_n} + 6v_n - 7 + \frac{t_n}{v_n - at_n} + 4t_n \right]. \quad (73)$$

It satisfies the following optimal asymptotic error constant term:

$$e_{n+1} = -\frac{c_1\left((m+7)c_1^2 - 2mc_2\right)}{24m^7} \left[\left(3a(m+7)^2 - 7m^2 - 96m - 359\right) c_1^4 + 12(a-1)m^2c_2^2 - 12m\left(a(m+7) - 2(m+8)\right)c_1^2c_2 - 12m^2c_1c_3 \right] e_n^8 + O\left(e_n^9\right). \quad (74)$$

(3) Consider one more weight function of the following form:

$$\phi(v_n) = \frac{1 - v_n^3}{1 - 2v_n + 2v_n^2}. \quad (75)$$

Then, we find another optimal eighth-order iteration function, which is given as follows:

$$\begin{aligned} y_n &= x_n - mu_n, \\ z_n &= x_n - mu_n \left[v_n^2 - \frac{1}{v_n - 1} \right], \\ x_{n+1} &= z_n - mt_n u_n \left[\frac{1 - v_n^3}{1 - 2v_n + 2v_n^2} + \frac{t_n}{v_n - at_n} + 4t_n \right], \end{aligned} \quad (76)$$

which has the following optimal asymptotic error constant term:

$$e_{n+1} = -\frac{c_1\left((m+7)c_1^2 - 2mc_2\right)}{24m^7} \left[\left(3a(m+7)^2 - 7m^2 - 96m - 437\right) c_1^4 + 12(a-1)m^2c_2^2 - 12m\left(a(m+7) - 2(m+8)\right)c_1^2c_2 - 12m^2c_1c_3 \right] e_n^8 + O\left(e_n^9\right). \quad (77)$$

(4) Again, we consider

$$\phi(v_n) = \frac{v_n + 1}{3v_n^3 - v_n + 1}, \quad (78)$$

With the above weight function, we will obtain another new optimal eighth-order iteration function, which is given below:

$$\begin{aligned} y_n &= x_n - mu_n, \\ z_n &= x_n - mu_n \left[v_n^2 - \frac{1}{v_n - 1} \right], \\ x_{n+1} &= z_n - mt_n u_n \left[\frac{v_n + 1}{3v_n^3 - v_n + 1} + \frac{t_n}{v_n - at_n} + 4t_n \right]. \end{aligned} \quad (79)$$

It satisfies the following optimal asymptotic error constant term:

$$e_{n+1} = -\frac{c_1\left((m+7)c_1^2 - 2mc_2\right)}{24m^7} \left[\left(3a(m+7)^2 - 7m^2 - 96m - 449\right) c_1^4 + 12(a-1)m^2c_2^2 - 12m\left(a(m+7) - 2(m+8)\right)c_1^2c_2 - 12m^2c_1c_3 \right] e_n^8 + O\left(e_n^9\right). \quad (80)$$

(5) With the help of the following weight function

$$\phi(v_n) = \frac{2v_n^2 + 3v_n + 2}{4v_n^3 - v_n + 2}. \quad (81)$$

we have another following optimal eighth-order iteration function:

$$\begin{aligned} y_n &= x_n - mu_n, \\ z_n &= x_n - mu_n \left[v_n^2 - \frac{1}{v_n - 1} \right], \\ x_{n+1} &= z_n - mt_n u_n \left[\frac{2v_n^2 + 3v_n + 2}{4v_n^3 - v_n + 2} + \frac{t_n}{v_n - at_n} + 4t_n \right], \end{aligned} \quad (82)$$

which satisfies the following optimal asymptotic error constant term:

$$e_{n+1} = -\frac{c_1\left((m+7)c_1^2 - 2mc_2\right)}{24m^7} \left[\left(3a(m+7)^2 - 7m^2 - 96m - 419\right) c_1^4 + 12(a-1)m^2c_2^2 - 12m\left(a(m+7) - 2(m+8)\right)c_1^2c_2 - 12m^2c_1c_3 \right] e_n^8 + O\left(e_n^9\right). \quad (83)$$

In a similar fashion arbitrary weight functions $\phi(v_n)$ are chosen provided the conditions of Theorem 5 should be satisfied. Then, we can obtain several new optimal methods of eighth-order for multiple zeros.

4. Numerical Experiments

Here, we will check the efficiency, effectiveness, and convergence behavior of our proposed scheme with the weight functions. Therefore, we choose some of expressions from our schemes, namely, expression (76) for ($a = 0, a = 2(m+8)/(m+7)$ & $a = (7m^2+96m+437)/3(m+7)^2$) and expression (79) for ($a = 0$ & $a = 2(m+8)/(m+7)$), denoted by $PM1$, $PM2$, $PM3$, $PM4$, and $PM5$, respectively. Therefore, we consider a total number of six test problems: first one is eigen value problem; second is Van der Waals equation of state; third one is again from real-life problem but for simple and complex zeros, and the last three are standard test problems, which can be found in examples (1)–(6).

Now, we want to compare our methods with other existing robust methods of the same order on the basis of difference between two consecutive iterations, computational order of convergence ρ, and residual errors in the function. We choose the following two schemes from eight-order iterative methods proposed by Zafar [16], expression (2) for ($H(u_n) = 6u_n^3 - u_n^2 + 2u_n + 1$, $P(v_n) = 1 + v_n$, & $G(w_n) = 2mw_n/A_2P_0 + m/A_2P_0$) and expression (2) for ($H(u_n) = (1 - 5u_n^2 + 8u_n^3)/(1 - 2u_n)$, $P(v_n) = 1 + v_n$, & $G(w_n) = (3m\omega_n + m)/A_2P_0(1 + \omega_n)$), denoted by $ZM1$ and $ZM2$, respectively, for comparison of what the highest-order is till date.

(1) Scheme ZM1

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - mu_n\left(6u_n^3 - u_n^2 + 2u_n + 1\right)\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= z_n - mu_nv_n\left(1 + 2u_n\right)\left(1 + v_n\right)\left(\frac{2\omega_n + 1}{A_2P_0}\right)\frac{f(x_n)}{f'(x_n)} \end{aligned} \tag{84}$$

(2) Scheme $ZM2$

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - mu_n\left(\frac{1 - 5u_n^2 + 8u_n^3}{1 - 2u_n}\right)\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= z_n - mu_nv_n\left(1 + 2u_n\right)\left(1 + v_n\right) \cdot\left(\frac{3\omega_n + 1}{A_2P_0\left(1 + \omega_n\right)}\right)\frac{f(x_n)}{f'(x_n)} \end{aligned} \tag{85}$$

where

$$\begin{aligned} u_n &= \left(\frac{f(y_n)}{f(x_n)}\right)^{1/m}, \\ v_n &= \left(\frac{f(z_n)}{f(y_n)}\right)^{1/m}, \\ \omega_n &= \left(\frac{f(z_n)}{f(x_n)}\right)^{1/m} \\ A_2 &= 1 \\ A_3 &= 2 \\ P_0 &= 1 \end{aligned} \tag{86}$$

Table 1: Computational cost.

Methods	Computational cost
$PM1$	$3l + \kappa + 10 + 6\gamma + 2\delta$
$PM2$	$3l + \kappa + 11 + 6\gamma + 2\delta$
$PM3$	$3l + \kappa + 11 + 6\gamma + 2\delta$
$PM4$	$3l + \kappa + 9 + 6\gamma + 2\delta$
$PM5$	$3l + \kappa + 10 + 6\gamma + 2\delta$
$ZM1$	$3l + \kappa + 13 + 4\gamma + 3\delta$
$ZM2$	$3l + \kappa + 15 + 6\gamma + 3\delta$

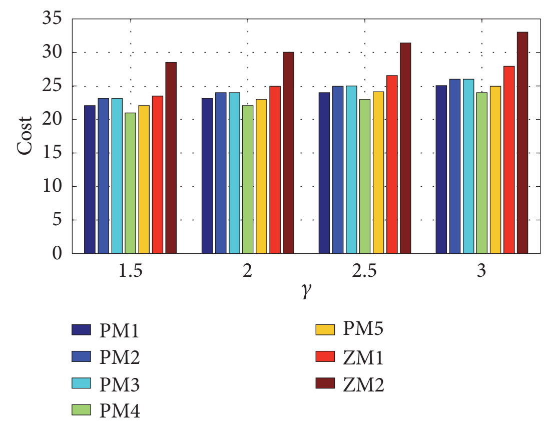

Figure 1: Computational costs for $\gamma = 1.5$ and $1.5 \le \delta \le 3$.

4.1. Computational Cost. We also make a comparison between the different methods studied above, calculating their computational cost in terms of products.

So, if γ is the ratio between the computational cost of a division and the computational cost of a product, δ is the ratio between the computational cost of a radical function and the computational cost of a product and l and κ are the ratios between functional evaluations of f and f' and products, respectively; we can express the total computational cost for each method in terms of products in Table 1.

For establishing the comparison, we have not considered the functional evaluations, since all the methods are optimal of order eight and, therefore, perform the same number of functional evaluations.

Figure 1 shows the different computational costs of the methods studied when it is assumed that a quotient is equal to 1.5 times a product and for different values of δ. In Figure 2 we find the computational cost of the methods, when we assign a quotient equal to 1.5 times a product, and γ takes different values between 1.5 and 2.5.

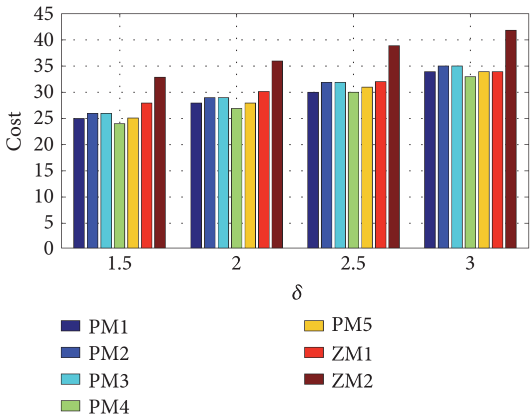

FIGURE 2: Computational costs for $\delta = 3$ and $1.5 \le \gamma \le 2.5$.

As we can see in Figure 1, the methods $PM2$ and $PM3$ have the same cost value for all values of γ; the $PM1$ and $PM4$ schemes have a low computational cost, the $PM4$ method being the most efficient between the two. On the other hand, the $ZM1$ scheme, which has been compared with ours, has a cost lower than the $ZM2$ (which always remains high), but without being less than the low cost of the $PM4$ method.

In Figure 2 we observe, as in the Figure 1, that the schemes $PM2$ and $PM3$ are always maintained with an equal cost value between the two in all cases. However, the rest of the methods behave differently in each case, with the method $PM4$ being most efficient for $\delta = 1.5, 2, 2.5$ and the most efficient $ZM1$ scheme for $\delta = 3$. From these observations we can affirm that the method $PM4$ is the most efficient of the schemes studied.

In Tables 2 and 3, we display the number of iteration indexes (n), error in the consecutive iterations $|x_{n+1} - x_n|$, computational order of convergence (ρ) (we used the formula given by Cordero and Torregrosa [27] in order to calculate ρ), and absolute residual error of the corresponding function ($|f(x_n)|$). We make our calculations with several numbers of significant digits (minimum 3000 significant digits) to minimize the round-off error.

As we mentioned in the above paragraph we calculate the values of all the constants and functional residuals up to several numbers of significant digits. However, due to the limited paper space, we display the value of errors in the consecutive iterations $|x_{n+1} - x_n|$ and absolute residual errors in the function $|f(x_n)|$ up to 2 significant digits with exponent power which are mentioned in Tables 2 and 3. Moreover, computational order of convergence is provided up to 5 significant digits. Finally, we displayed the values of approximated zeros up to 30 significant digits for each of the examples.

All computations have been performed using the programming package *Mathematica* 11 with multiple precision arithmetic. Further, the meaning of $a(\pm b)$ is shorthand for $a \times 10^{(\pm b)}$ in Tables 2 and 3.

Example 6 (eigenvalue problem). One of the main problems of linear algebra is concerned with calculating the eigenvalues of a square matrix. In addition, finding the roots of the characteristic equation of a square matrix greater than 4 is another big challenge. Therefore, we consider the following 9×9 matrix:

$$A = \frac{1}{8}\begin{bmatrix} -12 & 0 & 0 & 19 & -19 & 76 & -19 & 18 & 437 \\ -64 & 24 & 0 & -24 & 24 & 64 & -8 & 32 & 376 \\ -16 & 0 & 24 & 4 & -4 & 16 & -4 & 8 & 92 \\ -40 & 0 & 0 & -10 & 50 & 40 & 2 & 20 & 242 \\ -4 & 0 & 0 & -1 & 41 & 4 & 1 & 2 & 25 \\ -40 & 0 & 0 & 18 & -18 & 104 & -18 & 20 & 462 \\ -84 & 0 & 0 & -29 & 29 & 84 & 21 & 42 & 501 \\ 16 & 0 & 0 & -4 & 4 & -16 & 4 & 16 & -92 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{bmatrix}. \tag{87}$$

The corresponding characteristic polynomial of matrix (A) is given by

$$\begin{aligned} f_1(x) = {} & x^9 - 29x^8 + 349x^7 - 2261x^6 + 8455x^5 \\ & - 17663x^4 + 15927x^3 + 6993x^2 - 24732x \\ & + 12960. \end{aligned} \tag{88}$$

The above function has one multiple zero at $x = 3$ of multiplicity 4. For this $f_1(x)$, we considered the initial guess $x_0 = 3.1$.

Example 7 (Van der Waals equation of state).

$$\left(P + \frac{a_1 n^2}{V^2}\right)(V - na_2) = nRT \tag{89}$$

explains the behavior of a real gas by introducing in the ideal gas equations two parameters, a_1 and a_2, specific for each gas. The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V,

$$PV^3 - (na_2P + nRT)V^2 + a_1n^2V - a_1a_2n^2 = 0. \tag{90}$$

Given the constants a_1 and a_2 of a particular gas, one can find values for n, P, and T, such that this equation has three simple roots. By using the particular values, we obtain the following nonlinear function:

$$f_2(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675. \tag{91}$$

which has three zeros and out of them one is the multiple zero $\alpha = 1.75$ of multiplicity 2, and the other is the simple zero $\alpha = 1.72$. Our desired zero is $\alpha = 1.75$ and we chose initial approximation $x_0 = 1.8$.

Example 8 (chemical engineering problem). Let us consider a quartic equation from [28, 29], which describes the fraction of the nitrogen-hydrogen feed that gets converted to

Table 2: Comparison based on residual error (i.e., $|f(x_n)|$) of different iteration functions.

$f(x)$	n	$ZM1$	$ZM2$	$PM1$	$PM2$	$PM3$	$PM4$	$PM5$
$f_1(x)$	1	6.8(−5)	6.7(−5)	2.5(−9)	1.1(−6)	1.4(−6)	2.5(−9)	1.1(−6)
	2	1.2(−53)	1.5(−53)	8.8(−90)	8.7(−74)	2.4(−70)	9.4(−90)	2.3(−73)
	3	5.0(−443)	5.7(−442)	2.2(−733)	1.9(−610)	1.9(−142)	3.8(−733)	1.1(−606)
$f_2(x)$	1	4.6(−9)	5.1(−9)	3.4(−9)	7.3(−10)	1.8(−11)	3.5(−9)	7.5(−10)
	2	8.0(−35)	2.9(−34)	2.0(−36)	8.3(−43)	2.4(−60)	2.3(−36)	1.2(−42)
	3	1.1(−240)	4.3(−236)	3.9(−254)	2.5(−306)	2.7(−500)	1.3(−253)	6.3(−305)
$f_3(x)$	1	7.3(−3)	1.2(−2)	7.1(−3)	4.6(−3)	3.8(−3)	7.4(−3)	4.8(−3)
	2	6.8(−21)	4.2(−19)	2.6(−21)	2.7(−23)	7.1(−25)	3.5(−21)	4.3(−23)
	3	4.1(−165)	1.2(−150)	7.5(−169)	3.5(−185)	1.1(−198)	9.5(−168)	1.7(−183)
$f_4(x)$	1	1.5(−20)	2.6(−20)	3.9(−21)	6.3(−22)	1.0(−22)	4.2(−21)	7.5(−22)
	2	5.3(−158)	8.5(−156)	3.0(−163)	1.7(−170)	7.4(−178)	5.7(−163)	7.8(−170)
	3	1.7(−1257)	1.3(−1239)	3.9(−1300)	3.8(−1359)	4.9(−1419)	6.4(−1298)	1.2(−1353)
$f_5(x)$	1	1.0(−24)	2.0(−24)	1.9(−25)	1.1(−26)	7.9(−28)	2.0(−25)	1.3(−26)
	2	1.2(−193)	4.8(−191)	1.6(−200)	4.9(−212)	4.4(−223)	3.2(−200)	3.1(−211)
	3	3.4(−1545)	5.3(−1524)	3.6(−1601)	1.0(−1694)	4.4(−1785)	1.1(−1598)	4.0(−1688)
$f_6(x)$	1	2.4(−3)	2.2(−3)	1.1(−6)	3.4(−5)	4.2(−5)	1.1(−6)	3.4(−5)
	2	2.4(−29)	4.7(−29)	1.2(−55)	1.4(−45)	5.8(−46)	1.5(−55)	2.2(−45)
	3	2.9(−235)	2.4(−232)	4.1(−447)	2.9(−368)	1.8(−372)	2.3(−446)	1.6(−366)

* Method is not working for simple zero ($m = 1$).

TABLE 3: Difference between two consecutive iterations (i.e., $|x_{n+1} - x_n|$) of different iteration functions.

$f(x)$	n	$ZM1$	$ZM2$	$PM1$	$PM2$	$PM3$	$PM4$	$PM5$
$f_1(x)$	1	3.0(−2)	3.0(−2)	2.4(−3)	1.1(−2)	1.2(−2)	2.4(−3)	1.1(−2)
	2	2.0(−14)	2.1(−14)	1.8(−23)	1.8(−19)	1.3(−18)	1.9(−23)	2.3(−19)
	3	8.9(−112)	1.63(−111)	2.3(−184)	1.2(−153)	1.2(−36)	3.8(−184)	1.1(−152)
	ρ	7.9887	7.9884	7.9995	7.9975	1.1305	7.9995	7.9972
$f_2(x)$	1	3.9(−4)	4.1(−4)	3.4(−4)	1.6(−4)	2.4(−5)	3.4(−4)	1.6(−4)
	2	5.2(−17)	9.8(−17)	8.2(−18)	5.3(−21)	8.9(−30)	8.8(−18)	6.4(−21)
	3	5.9(−120)	1.2(−117)	1.1(−126)	9.2(−153)	9.5(−250)	2.1(−126)	4.6(−152)
	ρ	7.9945	7.9941	7.9963	7.9991	8.9998	7.9963	7.9990
$f_3(x)$	1	7.2(−4)	1.2(−3)	7.0(−4)	4.6(−4)	3.7(−4)	7.3(−4)	4.8(−4)
	2	6.7(−22)	4.2(−20)	2.5(−22)	2.7(−24)	7.0(−26)	3.5(−22)	4.3(−24)
	3	4.0(−166)	1.2(−151)	7.4(−170)	3.5(−186)	1.1(−199)	9.4(−169)	1.7(−184)
	ρ	8.0004	8.0007	8.0000	8.0000	8.0000	8.0000	8.0000
$f_4(x)$	1	1.2(−10)	1.6(−10)	6.1(−11)	2.5(−11)	1.0(−11)	6.5(−11)	2.7(−11)
	2	2.3(−79)	2.9(−78)	5.5(−82)	1.3(−85)	2.7(−89)	7.5(−82)	2.8(−85)
	3	4.1(−629)	3.6(−620)	2.0(−650)	6.2(−680)	7.0(−710)	2.5(−649)	3.4(−677)
	ρ	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
$f_5(x)$	1	8.0(−9)	1.0(−8)	4.5(−9)	1.7(−9)	7.3(−10)	4.7(−9)	1.9(−9)
	2	3.9(−65)	2.9(−64)	2.0(−67)	2.9(−71)	6.1(−75)	2.5(−67)	5.4(−71)
	3	1.2(−515)	1.4(−508)	2.6(−534)	1.7(−565)	1.3(−595)	1.8(−533)	2.7(−563)
	ρ	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
$f_6(x)$	1	2.6(−1)	2.6(−1)	4.1(−2)	9.4(−2)	9.9(−2)	4.1(−2)	9.4(−2)
	2	8.9(−8)	1.0(−7)	2.4(−14)	7.7(−12)	6.2(−12)	2.5(−14)	8.6(−12)
	3	2.9(−59)	1.6(−58)	3.2(−112)	1.6(−92)	1.5(−93)	4.9(−112)	4.5(−92)
	ρ	8.0000	8.0000	7.9983	7.9967	7.9989	7.9983	7.9963

**COC (ρ) cannot be calculated for this method.

Table 4: Radius of the convergence ball.

Example	α	m	$[a,b]$	k_1	r_H	r_0	r_1	r_2	$r_3 = r$
f_a	0	2	$[-\pi/2,\pi/2]$	1	1.2679	2	1.5	0.374785	0.370698
f_b	0	2	$\mathbb{R}$	12	0.1152	0.166667	0.125	0.031203	0.030895
f_c	$\pi/2$	3	$\mathbb{R}$	1	1.972	3	2.4	0.540385	0.537669

Table 5: Radius of the convergence ball.

Example	α	m	$[a,b]$	k_1	r_0	r_1	r_2	$r_3 = r$
f_1	3	4	$[2,4]$	21.5	0.184971	0.154142	0.032548	0.032450
f_2	1.75	2	$\mathbb{R}$	100	0.02	0.015	0.003744	0.003707
f_4	0	2	$]-\infty,1/2[$	8.06	0.248012	0.186008	0.046432	0.045973
f_5	1	3	$]1/2,3/2[$	11	0.272727	0.218181	0.049129	0.048882

ammonia (this fraction is called fractional conversion). By considering 250 atm and 500°C, the mentioned equation can be converted into the form

$$f_3(z) = z^4 - 7.79075z^3 + 14.7445z^2 + 2.511z - 1.674. \tag{92}$$

The above function has a total of four zeros, and out of them two are real and the other two are complex conjugate to each other. However, our desired zero is $\alpha = 3.9485424455620457727 + 0.3161235708970163733i$. The reason of considering the simple complex zeros problem is to confirm that our methods also work for simple zeros. In addition, it allows us to investigate the behavior of the algorithms on simple and complex zeros. Moreover, we considered the initial guess $x_0 = 3.8 + 0.32i$ in this case.

Example 9. Let us pick a standard nonlinear test function from Petkovíc et al. [2] which is the mixture of trigonometric, exponential, and polynomial functions given by

$$f_4(x) = x^2 \exp(x) - \sin(x) + x. \tag{93}$$

The above function has a multiple zero at $x = 0$ of multiplicity 2. We assumed the initial approximation $x_0 = 0.05$ for this function f_4.

Example 10. We choose another standard test problem from Petkovíc et al. [2], which is defined by

$$f_5(x) = x^5 - 8x^4 + 24x^3 - 34x^2 + 23x - 6. \tag{94}$$

This function f_5 has a multiple zero at $x = 1$ of multiplicity 3. For this function f_5, we considered initial guess $x_0 = 0.9$.

Example 11. Finally, we consider one more standard nonlinear test function, which is given as follows:

$$f_6(x) = \frac{\left(x - \sqrt{5}\right)^4}{(x-1)^2 + 1}. \tag{95}$$

The above function has a multiple zero at $x = \sqrt{5}$ of multiplicity 4. We have chosen the initial approximation $x_0 = 2.5$ for the function f_6.

Remark 12. It is worthwhile to note from Tables 2 and 3 that our proposed methods are efficient for determining multiple zeros of nonlinear functions and are far better than other well-recognized efficient sixth-order iterative methods, with our methods obtaining better results in all the considered problems analyzed. In addition, our methods also have the minimum residual errors corresponding to the considered test functions f. The minimum error between the consecutive iterations corresponding to the considered functions also belongs to our proposed iterative methods. We confirm that our methods converge faster towards the required zero of the corresponding function as compared to other existing methods.

Now, in order to perform the local convergence study we take the following examples $f_a(x) = \cos(x) - 1$, $f_b(x) = x^2(x^2-1)$, and $f_c(x) = x+\cos(x)-\pi/2$ from the literature, see [24], where the local convergence radius has been calculated for a third-order iterative method, so we compare the local convergence radius r_H of Halley's methods with that obtained in this paper for a fourth-order method, r. See the results in Table 4.

Finally, we take the examples of our numerical experience where the exact solution is known and the corresponding bound established in (10) can be obtained in $\mathbb{R}$.

Remark 13. In Tables 4 and 5 we show the values that restrict the local convergence radius. Notice that the value r_3 in all the examples is the final radius since we obtain a decreasing sequence of values $r_0 < r_1 < r_2 < r_3$. So, the last value is the radius of local convergence. This gave us an open interval where one can choose the initial guess for our iterative method, but more importantly, it proves that, despite being a higher-order method, the interval of local convergence still remains considerably good.

5. Conclusion

In this paper, we contributed further to the development of the theory of iteration processes and presented optimal eighth-order iterative methods for multiple zeros. The proposed scheme is the extension of an earlier work proposed by Chun and Neta [19] for simple zeros. The beauty of the proposed methods is that they have minimum error between the consecutive iterations and minimum residual errors corresponding to considered test functions f. Our methods also exhibit a stable computational order of convergence. We can even obtain the whole paper of Chun and Neta [19] as a special case of our paper for $m = 1$. The proposed scheme is optimal in the sense of the classical Kung-Traub conjecture. The computational efficiency index is defined as $E = p^{1/\theta}$, where p is the order of convergence and θ is the number of functional evaluations per iteration. Thus, the efficiency index of the proposed methods is $E = \sqrt[4]{8} \approx 1.682$ which is better than the classical Newton's method $E = \sqrt[2]{2} \approx 1.414$ and sixth-order methods proposed by Guem et al. [15, 21], $E = \sqrt[4]{6} \approx 1.565$. We can obtain several new optimal and interesting iterative methods of order eight by considering different types of weight functions. Moreover, we obtain a local convergence study for the intermediate optimal fourth-order method that defines the two first steps of our method, which is important because it allows us to obtain an open interval where one can choose the initial guess, but more important than this is the fact that it proves that, despite being of a higher order, the interval of local convergence still remains considerably good. Finally, on account of the results obtained, it can be concluded that our proposed methods are highly efficient and perform better than the existing methods.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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