Isomorphic copies of ℓ_1 for m-homogeneous non-analytic Bohnenblust–Hille polynomials

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Abstract

We employ a classical result by Toeplitz (1913) and the seminal work by Bohnenblust and Hille on Dirichlet series (1931) to show that the set of continuous m-homogeneous non-analytic polynomials on c_0 contains an isomorphic copy of ℓ_1. Moreover, we can have this copy of ℓ_1 in such a way that every non-zero element of it fails to be analytic at precisely the same point.

1 Introduction and statement of our main result

An m-homogeneous polynomial in n variables is a function of the form

$$P(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ \alpha_1+\cdots+\alpha_n=m}} c_\alpha z_1^{\alpha_1}\cdots z_n^{\alpha_n} = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=m}} c_\alpha z^\alpha \text{ for } z\in\mathbb{C}^n\,. \tag{1}$$

These are somehow the blocks with which the Taylor series expansion of a holomorphic function at 0 is built. When we go to infinitely many variables (say to c_0, the Banach space of null sequences), one is tempted to define an m-homogenous polynomial as a function given by an power series like

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in (1) (since we have infinitely many variables we have no longer a finite sum). This was actually one of the first attempts at the beginning of the 20th century to define a theory of holomorphic functions in infinitely many variables, assuming that such a power series would converge everywhere under certain conditions (basically that the suprema on the open unit ball of finite dimensional sections are uniformly bounded). It soon became clear that this was not the right approach: Toeplitz [11] provided an example of a 2-homogenous polynomial for which the power series expansion does not converge everywhere (see Section 2 for all needed definitions)

Proposition 1.1. *There exists $P = \sum_\alpha c_\alpha(P) z^\alpha \in \mathcal{P}(^2 c_0)$ such that for every $\varepsilon > 0$ there is $\tilde{z} \in \ell_{4+\varepsilon}$ with $\sum_\alpha |c_\alpha(P)\tilde{z}^\alpha| = \infty$.*

The existence of elements for which the convergence at a certain point fails is not a isolated phenomenon. Bohnenblust and Hille in [5] solved a long standing problem on Dirichlet series and, as one of the tools for the solution, extended this construction of Toeplitz to m-homogeneous polynomials (see Proposition 3.1).

Recently, the work by Bohnenblust and Hille [5] on Dirichlet series has resurfaced and attracted the attention of many authors who became interested in classical problems such as obtaining the optimal values for the constants in the Bohnenblust–Hille and Hardy–Littlewood inequalities, or in estimating the asymptotic value of the n-dimensional Bohr radius (see, e.g., [3, 6, 9]).

Bohnenblust and Hille's work, beyond inspiring the aforementioned line of research, also has implications in the study of the analyticity of continuous m-homogeneous polynomials. Nowadays the set of points on which the power series expansion of every m-homogeneous polynomial converges is known to be exactly the space $\ell_{\frac{m-1}{2m},\infty}$ (see [2]). Here we look at the problem from a different point of view: we are interested in the set of m-homogeneous polynomials for which there is a point at which the power series expansion does not converge. Our work here is a contribution to the ongoing search (see, e.g., [10, 4, 1]) of what are often large subspaces of mathematical objects enjoying "special" properties in a stronger way. Namely we show that:

Theorem 1.2. *For every $m \geq 2$, the set of continuous, m-homogeneous non-analytic polynomials on c_0 contains an isomorphic copy of ℓ_1.*

The main section in this paper, Section 3, is dedicated to the construction (in full detail) of a linear subspace of polynomials sharing a common vector for which the analyticity fails (and that is isomorphic to the Banach space ℓ_1). First we provide all the necessary notations and background to follow our construction.

2 Basic definitions

A mapping $P : c_0 \to \mathbb{C}$ is a (continuous) m-homogeneous polynomial if there exists a (continuous) m-linear mapping $L : c_0 \times \cdots \times c_0 \to \mathbb{C}$ such that $P(z) = L(z, \ldots, z)$ for every $z \in c_0$. It is a well known fact that the mapping L can be taken to be symmetric (see e.g. [8, Chapter 2]). We denote by $\mathcal{P}(^m c_0)$ the set of continuous m-homogeneous polynomials on c_0, which endowed with the norm $\|P\| = \sup_{z \in B_{c_0}} |P(z)|$ is a Banach space. Every polynomial is going to be assumed to be continuous.
We consider $\mathbb{N}_0^{(\mathbb{N})}$, the set of multi-indices that eventually become zero. In other words, if we make the identification $\mathbb{N}_0^n = \mathbb{N}_0^n \times \{0\}$, then $\mathbb{N}_0^{(\mathbb{N})} = \bigcup_{n=1}^{\infty} \mathbb{N}_0^n$. For each such index $\alpha = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots)$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n = m$ a polynomial P defines a coefficient through its associated symmetric m-linear form

$$c_\alpha(P) = \frac{m!}{\alpha_1! \cdots \alpha_n!} L(e_1, \overset{\alpha_1}{\ldots}, e_1, e_2, \overset{\alpha_2}{\ldots}, e_2, \ldots, e_n, \overset{\alpha_n}{\ldots}, e_n) .$$

In this way, each m-homogeneous polynomial defines a formal power series

$$P \sim \sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = m}} c_\alpha(P) z^\alpha .$$

We say that an m-homogeneous polynomial is analytic at $z_0 \in c_0$ if the corresponding power series converges absolutely, that is if $\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = m}} |c_\alpha(P) z_0^\alpha| < \infty$.
A polynomial is analytic if it is analytic at every $z \in c_0$.
Finally, we define the support of a multi-index $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ as $\operatorname{supp} \alpha := \{k \in \mathbb{N} : \alpha_k \neq 0\}$.

3 Main section

We start with the construction of a continuous m-homogenous polynomial that fails to be analytic. This whole construction was first presented in [5, Sections 3 and 4] and a detailed study can be found at [7, pages 70–84]. We sketch here the proof and point out the facts that will be needed later.

Proposition 3.1. *For each $m \geq 2$ there exists an m-homogeneous polynomial $P \in \mathcal{P}(^m c_0)$ that is not analytic.*

Let us note that for $m = 1$ we have that $\mathcal{P}(^1 c_0)$ is just ℓ_1, the dual space of c_0. In other words, in this case the coefficients are $c_i = P(e_i)$ for $i \in \mathbb{N}$.

Then, for each $x \in c_0$ we have $\sum_{i=1}^{\infty} \|c_i x_i\| \leq \|x\|_\infty \sum_{i=1}^{\infty} |c_i| < \infty$. So, every 1-homogeneous polynomial is analytic.

Proof. To begin with, we fix $m \in \mathbb{N}$ and a prime number p with $p > m$. Let us consider the $p \times p$ matrix $M_1 = (m_{rs})_{r,s} = \left(e^{2\pi i \frac{rs}{p}}\right)_{r,s}$. We now define $M_2 = M_1 \otimes M_1$ and, inductively, $M_n = M_1 \otimes M_{n-1}$ (where $\otimes$ denotes the Kronecker product of matrices), for $n \geq 2$. Then $M_n = (a_{rs})_{r,s}$ is a $p^n \times p^n$ matrix that satisfies

$$\sum_{t=1}^{p^n} a_{rt}\overline{a_{st}} = p^n \delta_{rs} \text{ and } |a_{rs}| = 1.$$

With this, for each $n \in \mathbb{N}$, we are going to define an m-homogeneous polynomial Q_n in p^n variables as

$$Q_n(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^{p^n} \\ |\alpha| = m}} c_\alpha(Q_n) z^\alpha,$$

in the following way. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_{p^n}) \in \mathbb{N}_0^{p^n}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_{p^n} = m$, we consider $(i_1, \dots, i_m) = (1, \overset{\alpha_1}{\dots}, 1, 2, \overset{\alpha_2}{\dots}, 2, \dots, p^n, \overset{\alpha_{p^n}}{\dots}, p^n)$. If we also denote by Σ_m the group of permutations of $\{1, \dots, m\}$, the coefficients of Q are defined to be

$$c_\alpha(Q_n) = \frac{1}{\alpha_1! \cdots \alpha_{p^n}!} \sum_{\sigma \in \Sigma_m} a_{i_{\sigma 1} i_{\sigma 2}} \cdots a_{i_{\sigma m-1} i_{\sigma m}}$$

The polynomials in the sequence $(Q_n)_n$ satisfy $\|Q_n\| \leq \left(p^n\right)^{\frac{m+1}{2}}$ and

$$\begin{aligned} 0 < \eta = \inf \left\{ |c_\alpha(Q_n)| \colon \alpha \in \mathbb{N}_0^{p^n},\, |\alpha| = m,\, n \in \mathbb{N} \right\} \\ \leq \sup \left\{ |c_\alpha(Q_n)| \colon \alpha \in \mathbb{N}_0^{p^n},\, |\alpha| = m,\, n \in \mathbb{N} \right\} \leq m! \end{aligned}$$

Now, let us consider the decomposition of c_0 as $c_0(\ell_\infty^{p^n})$ (that is, we decompose each sequence $z \in c_0$ into blocks of increasing length $p, p^2, p^3, \dots$, that we denote by $z^{(n)}$). Then, we can rewrite every $z \in c_0$ as

$$z = \Big(\underbrace{z_1^{(1)}, \overset{p}{\dots}, z_p^{(1)}}_{z^{(1)}}, \underbrace{z_1^{(2)}, \dots \overset{p^2}{\dots} \dots, z_{p^2}^{(2)}}_{z^{(2)}}, \underbrace{z_1^{(3)}, \dots \dots \overset{p^3}{\dots} \dots \dots, z_{p^3}^{(3)}}_{z^{(3)}}, \dots \dots \dots \Big) \tag{2}$$

and then define the m-homogenous polynomial $P \in \mathcal{P}(^m c_0)$ by

$$P(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} Q_n\left(z^{(n)}\right). \tag{3}$$

We affirm that P is an m-homogeneous polynomial that fails to be analytic: Let us define the point $\tilde{z}$ at which the monomial series expansion of P does not converge. Choosing $1/p < b < 1$, we define $\tilde{z}$ blockwise as

$$\tilde{z}_k^{(n)} = \left(\frac{b}{p}\right)^{n\frac{m-1}{4m}} \qquad \text{for } k = 1, \ldots, p^n, \text{ and } n \in \mathbb{N}. \tag{4}$$

Clearly $\tilde{z} \in c_0$ and, moreover, $\sum_{\alpha \in \mathbb{N}_0^{p^n},\, |\alpha|=m} c_\alpha(P)\, |\tilde{z}^\alpha| = \infty$. $\square$

We proceed showing that there exists certain m-homogeneous non analytic polynomials such that no linear combination of them is analytic.

Proposition 3.2. *There are non-analytic m-homogeneous polynomials P_1, $P_2 \in \mathcal{P}(^m c_0)$ such that $\lambda P_1 + P_2$ is not analytic for every $\lambda \in \mathbb{C}$.*

Proof. First, we are going to block again c_0 in a slightly different way from (2). Instead of taking blocks of length $p, p^2, p^3, \ldots$, we are going to take two consecutive blocks of each length

$$z = (\underbrace{z_{1,1}^{(1)}, \overset{p}{\ldots}, z_{1,p}^{(1)}}_{z_1^{(1)}}, \underbrace{z_{2,1}^{(1)}, \overset{p}{\ldots}, z_{2,p}^{(1)}}_{z_2^{(1)}} \underbrace{z_{1,1}^{(2)}, \overset{p^2}{\ldots\ldots}, z_{1,p^2}^{(2)}}_{z_1^{(2)}}, \underbrace{z_{2,1}^{(2)}, \overset{p^2}{\ldots\ldots}, z_{2,p^2}^{(2)}}_{z_2^{(2)}}, \ldots).$$

To be more precise, let us define

$$b_n = \frac{2p(p^{n-1}-1)}{p-1} + 1 \text{ and } c_n = b_n + p^n \quad \text{for every } n \in \mathbb{N}.$$

Then each $z \in c_0$ is decomposed as $z = z_1 + z_2$, where each of these two z_j's is defined blockwise and the n-th block is given by

$$z_1^{(n)} = \sum_{k=0}^{p^n-1} z_{b_n+k} e_{b_n+k} \text{ and } z_2^{(n)} = \sum_{k=0}^{p^n-1} z_{c_n+k} e_{c_n+k}, \quad \text{for every } n \in \mathbb{N}.$$

The polynomials are then defined as a modification of the original P given in (3):

$$P_1(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} Q_n\big(z_1^{(n)}\big) \text{ and } P_2(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} Q_n\big(z_2^{(n)}\big).$$

On the one hand, for each fixed $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \|Q_n\| \le \sum_{n=1}^{N} \frac{1}{n^2} p^{-n\frac{m+1}{2}} p^{n\frac{m+1}{2}} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \tag{5}$$

that shows, as in the case of P [7, page 82], that $P_1, P_2 \in \mathcal{P}(^m c_0)$. Moreover, we also have that

$$\max\{\|P_1\|, \|P_2\|\} \leq \frac{\pi^2}{6}.$$

On the other hand, we first have to show that none of these polynomials is analytic, so we need to find a point for each of them at which the monomial series expansion does not converge. We are going to produce this point by alternating two copies of the $\tilde{z}$ already defined in (4) by repeating twice each one of the blocks; that is, we define $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$, where $\tilde{w}_1$ and $\tilde{w}_2$ are defined blockwise from $\widetilde{z}$ in the following fashion:

$$\widetilde{w}_1 = (\underbrace{\widetilde{z}_1^{(1)} \overset{p}{\ldots}, \widetilde{z}_p^{(1)}}_{\widetilde{z}^{(1)}}, 0, \overset{p}{\ldots}, 0, \underbrace{\widetilde{z}_1^{(2)}, \overset{p^2}{\ldots\ldots}, \widetilde{z}_{p^2}^{(2)}}_{\widetilde{z}^{(2)}}, 0, \overset{p^2}{\ldots\ldots}, 0, \ldots\ldots\ldots)$$

and

$$\widetilde{w}_2 = (0, \overset{p}{\ldots}, 0, \underbrace{\widetilde{z}_1^{(1)}, \overset{p}{\ldots}, \widetilde{z}_p^{(1)}}_{\widetilde{z}^{(1)}} 0, \overset{p^2}{\ldots\ldots}, 0, \underbrace{\widetilde{z}_1^{(2)}, \overset{p^2}{\ldots\ldots}, \widetilde{z}_{p^2}^{(2)}}_{\widetilde{z}^{(2)}}, \ldots\ldots\ldots)$$

or, to be more precise,

$$\tilde{w}_1 = \sum_{n=1}^{\infty} \sum_{k=0}^{p^n-1} \left(\frac{b}{p}\right)^{n\frac{m-1}{4m}} e_{b_n+k} \text{ and } \tilde{w}_2 = \sum_{n=1}^{\infty} \sum_{k=0}^{p^n-1} \left(\frac{b}{p}\right)^{n\frac{m-1}{4m}} e_{c_n+k}\,.$$

Next, define $B_n = \{b_n, \ldots, b_n + p^n - 1\}$ and observe that, by the construction of the polynomial P_1, $c_\alpha(P_1) \neq 0$ only if $\operatorname{supp}\alpha \subseteq B_n$ for some n. Then we have

$$\begin{aligned}
\sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m}} |c_\alpha(P_1)\tilde{w}^\alpha| &= \sum_{n=1}^{\infty} \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m\\\operatorname{supp}\alpha\subseteq B_n}} |c_\alpha(P_1)\tilde{w}^\alpha| = \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m\\\operatorname{supp}\alpha\subseteq B_n}} |c_\alpha(Q_n)\tilde{w}^\alpha| \\
&\geq \eta \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \sum_{\substack{\alpha\in\mathbb{N}_0^{p^n}\\|\alpha|=m}} \left|\tilde{z}^{(n)}\right|^\alpha \geq \frac{\eta}{m!} \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \left(\sum_{k=1}^{p^n} \left(\frac{b}{p}\right)^{n\frac{m-1}{4m}}\right)^m \\
&= \frac{\eta}{m!} \sum_{n=1}^{\infty} \frac{1}{n^2} \left((bp)^{\frac{m-1}{4}}\right)^n
\end{aligned}$$

and the last series diverges to ∞ since $bp > 1$. With the same argument, using $C_n = \{c_n, \ldots, c_n + p^n - 1\}$ (observe that now $c_\alpha(P_2) \neq 0$ only if $\operatorname{supp}\alpha \subseteq C_n$

for some n) we conclude that

$$\sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m}} |c_\alpha(P_2)\tilde{w}^\alpha| = \sum_{n=1}^{\infty} \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m\\ \operatorname{supp}\alpha\subseteq C_n}} |c_\alpha(P_2)\tilde{w}^\alpha| = \infty$$

Let us note that $\mathbb{N} = \bigcup_{n=1}^{\infty}(B_n \cup C_n)$ and that $B_i \cap C_j = \varnothing$ for all i, j; this means that the coefficients of P_1 and of P_2 have mutually disjoint supports. Then for every linear combination we have

$$\begin{aligned}
\sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m}} |c_\alpha(\lambda P_1 + P_2)\tilde{w}^\alpha| &= \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m}} |\lambda c_\alpha(P_1) + c_\alpha(P_2)|\ |\tilde{w}^\alpha| \\
&= \sum_{n=1}^{\infty} \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m\\ \operatorname{supp}\alpha\subseteq B_n\cup C_n}} |\lambda c_\alpha(P_1) + c_\alpha(P_2)|\ |\tilde{w}^\alpha| \\
&= \sum_{n=1}^{\infty} \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m\\ \operatorname{supp}\alpha\subseteq B_n}} |\lambda|\ |c_\alpha(P_1)\tilde{w}^\alpha| + \sum_{n=1}^{\infty} \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=m\\ \operatorname{supp}\alpha\subseteq C_n}} |c_\alpha(P_2)\tilde{w}^\alpha|
\end{aligned}$$

and this shows that $\lambda P_1 + P_2$ is not analytic at $\tilde{w}$. □

Our aim now is to provide a sequence of non-analytic m-homogeneous polynomials for which any linear combination is non-analytic. The construction this time is in some sense a step forward in the philosophy for the case of two polynomials. We hope that having given the construction in the previous case helps the reader to understand the general case.

Proposition 3.3. *There exists a sequence $\{P_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}(^m c_0)$ of non-analytic polynomials such that every $\sum_{i=1}^{N} \lambda_i P_i$ with $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots, N$ and $\lambda_N \neq 0$ is not analytic.*

Proof. The idea is quite similar to that of Proposition 3.2: to block c_0 in such a way that we can produce infinitely many copies of P and a vector that contains infinitely many copies of $\tilde{z}$. What we do is to divide c_0 in blocks, first of length p, then p and p^2, then p, p^2, p^3, then p, p^2, p^3, p^4, and so on. . . . We will then decompose each $z \in c_0$ as

$$z = \left(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_3^{(1)}, z_2^{(2)}, z_1^{(3)}, z_4^{(1)}, z_3^{(2)}, z_2^{(3)}, z_1^{(4)}, \ldots\ldots\right),$$

where each $z_k^{(n)}$ is a block of length p^n. We locate the starting point of each of these blocks by defining $s_1^{(1)} = 1$,

$$s_k^{(1)} = \left(\sum_{i=1}^{k} \sum_{j=1}^{i} p^j \right) + 1$$

and

$$s_{k-i}^{(1+i)} = s_k^1 + \sum_{j=1}^{i} p^j \quad \text{for} \quad i = 1, \ldots, k-1.$$

Then the block $z_k^{(n)}$ is defined as follows

$$z_k^{(n)} = \sum_{i=0}^{p^n-1} z_{s_k^{(n)}+i} \, e_{s_k^{(n)}+i} \, .$$

Now for each $k \in \mathbb{N}$ we define the polynomial

$$P_k(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} Q_n\big(z_k^{(n)}\big) \, .$$

Just like in Proposition 3.2 we have that $P_k \in \mathcal{P}(^m c_0)$ and $\|P_k\| \leq \frac{\pi^2}{6}$ for every $k \in \mathbb{N}$.
Let us now define the following sets

$$M_k^{(n)} := \left\{ s_k^{(n)}, s_k^{(n)} + 1, \ldots, s_k^{(n)} + p^n - 1 \right\}$$

for every $k, n \in \mathbb{N}$. Each $M_k^{(n)}$ is the support of the block starting at $s_k^{(n)}$ and has length p^n. Note that all these sets are mutually disjoint. Then we are going to use $M_k^{(n)}$ as the support for the n-th block of the k-th copy of $\tilde{z}$; we define, for each $n \in \mathbb{N}$,

$$\tilde{v}_k = \sum_{n=1}^{\infty} \sum_{j \in M_k^{(n)}} \tilde{z}_j^{(n)} e_j = \sum_{n=1}^{\infty} \sum_{j \in M_k^{(n)}} \left(\frac{b}{p} \right)^{n\frac{m-1}{4m}} e_j.$$

Each one of these $\tilde{v}_k' s$ is a copy of $\tilde{z}$, all of them scattered in such a way that all of them have disjoint support,

$$\tilde{v}_1 = (\tilde{z}^{(1)}, 0, \overset{p}{\ldots}, 0, \tilde{z}^{(2)}, 0, \overset{p}{\ldots}, 0, 0, \overset{p^2}{\ldots}, 0, \tilde{z}^3, 0, \overset{p}{\ldots}, 0, 0, \overset{p^2}{\ldots}, 0, 0, \overset{p^3}{\ldots}, 0, \tilde{z}^{(4)} \ldots),$$
$$\tilde{v}_2 = (0, \overset{p}{\ldots}, 0, \tilde{z}^{(1)}, 0, \overset{p^2}{\ldots}, 0, 0, \overset{p}{\ldots}, 0, \tilde{z}^{(2)}, 0, \overset{p^3}{\ldots}, 0, 0, \overset{p}{\ldots}, 0, 0, \overset{p^2}{\ldots}, 0, \tilde{z}^{(3)}, 0, \overset{p^4}{\ldots}, 0, \ldots),$$
$$\tilde{v}_3 = (0, \overset{p}{\ldots}, 0, 0, \overset{p}{\ldots}, 0, 0, \overset{p^2}{\ldots}, 0, \tilde{z}^{(1)}, 0, \overset{p^2}{\ldots}, 0, 0, \overset{p^3}{\ldots}, 0, 0, \overset{p}{\ldots}, 0, \tilde{z}^{(2)}, 0, \overset{p^3}{\ldots}, 0, 0, \overset{p^4}{\ldots}, 0, \ldots),$$

and so on. The idea now is to paste all of these into one single vector $\tilde{v}$ on which each linear combination of the P_k's is not analytic. However, if we do that as it is now we would get infinitely many copies of the block $z^{(1)}$ and the resulting vector, although in ℓ_∞, would not be in c_0. We solve this by weighting the $\tilde{v}_k$'s

$$\tilde{v} = \sum_{k=1}^{\infty} \frac{1}{k^2} \tilde{v}_k.$$

As in Proposition 3.2, none of the polynomials P_k is analytic at $\tilde{v}$:

$$\begin{aligned}
\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha|=m}} |c_\alpha(P_k)\tilde{v}^\alpha| &= \sum_{n=1}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha|=m \\ \operatorname{supp}\alpha \subseteq M_k^{(n)}}} |c_\alpha(P_k)\tilde{v}^\alpha| \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha|=m \\ \operatorname{supp}\alpha \subseteq M_k^{(n)}}} |c_\alpha(Q_n)\tilde{v}^\alpha| \\
&\geq \frac{\eta}{k^2} \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \sum_{\substack{\alpha \in \mathbb{N}_0^{p^n} \\ |\alpha|=m}} \left|\tilde{z}^{(n)}\right|^\alpha \\
&\geq \frac{\eta}{m!\,k^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left((bp)^{\frac{m-1}{4}}\right)^n,
\end{aligned}$$

and the last series diverges to ∞. Again, the fact that all the sets $M_k^{(n)}$ are pairwise disjoint gives that for every $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ we have

$$
\begin{aligned}
\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = m}} \left| c_\alpha \big(\textstyle\sum_{k=1}^N \lambda_k P_k \big) \tilde{v}^\alpha \right| &= \sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = m}} \left| \sum_{k=1}^N \lambda_k c_\alpha(P_k) \right| |\tilde{v}^\alpha| \\
&= \sum_{n=1}^\infty \sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = m \\ \operatorname{supp} \alpha \subseteq \bigcup_{k=1}^N M_k^{(n)}}} \Big| \sum_{k=1}^N \lambda_k c_\alpha(P_k) \Big| |\tilde{v}^\alpha| \\
&= \sum_{k=1}^N \sum_{n=1}^\infty \sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = m \\ \operatorname{supp} \alpha \subseteq M_k^{(n)}}} |\lambda_k| \, |c_\alpha(P_k)| \, |\tilde{v}^\alpha| \qquad (6) \\
&= \sum_{k=1}^N |\lambda_k| \sum_{n=1}^\infty \sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = m \\ \operatorname{supp} \alpha \subseteq M_k^{(n)}}} |c_\alpha(P_k)| \, |\tilde{v}^\alpha| \\
&\geq \frac{\eta}{m!} \sum_{k=1}^N \frac{|\lambda_k|}{k^2} \sum_{n=1}^\infty \frac{1}{n^2} \Big((bp)^{\frac{m-1}{4}} \Big)^n = \infty \,,
\end{aligned}
$$

and this completes the proof. □

Now we have at hand all the ingredients we need to prove our main result.

Proof of Theorem 1.2. Consider the sequence of m-homogeneous polynomials $\{P_k\}_k$ given in Proposition 3.3. Just like in (5) we have that $\|P_k\| \leq \frac{\pi^2}{6}$ for every k and then for each $(\lambda_k)_k \in \ell_1$ we have that $\sum_{k=1}^\infty \|\lambda_k P_k\| \leq \frac{\pi^2}{6} \sum_{k=1}^\infty |\lambda_k| < \infty$. Then the series $\sum_{k=1}^\infty \lambda_k P_k$ converges absolutely, and hence it converges in the Banach space $\mathcal{P}(^m c_0)$. Therefore it defines an m-homogeneous polynomial on c_0. Then we consider

$$
X = \left\{ \sum_{k=1}^\infty \lambda_k P_k : \ (\lambda_k)_k \in \ell_1 \right\} \subseteq \mathcal{P}(^m c_0)
$$

that is clearly isomorphic to ℓ_1. Finally, proceeding as in (6) we have

$$
\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = m}} \left| c_\alpha \left(\textstyle\sum_{k=1}^\infty \lambda_k P_k \right) \tilde{v}^\alpha \right| \geq \frac{\eta}{m!} \sum_{k=1}^\infty \frac{|\lambda_k|}{k^2} \sum_{n=1}^\infty \frac{1}{n^2} \Big((bp)^{\frac{m-1}{4}} \Big)^n = \infty \,,
$$

which shows that none of the polynomials in X is analytic. □

Remark 3.4. We recall that in our construction in the proof of Proposition 3.3 none of the polynomials P_k is analytic at a particular vector $\tilde{v}$. A careful examination of this proof permits us to find a copy of ℓ_1 within the set of continuous m-homogeneous non-analytic polynomials for which the analyticity fails at an infinitely many vectors. The idea is the following:

Let V be the (infinite dimensional) linear space generated by the polynomials P_k defined in the proof of Proposition 3.3. Let us denote by

$$Z_V = \{z \in c_0 : P \text{ is not analytic at } z, \text{ for every } P \in V \setminus \{0\}\},$$

in other words, we consider the set of vectors $z \in c_0$ such that no linear combination of the P_k's is analytic at z. The proof of that result can be modified in order to obtain infinitely many vectors having disjoint supports (call them, for instance, $\tilde{w_k}$, $k \in \mathbb{N}$). This could be achieved by working with the original $\tilde{v}$ and modifying it in each case by inserting a suitable number of zero coordinates. Clearly, any vector $w \in \operatorname{span}\{w_k : k \in \mathbb{N}\} \setminus \{0\}$ belongs to Z_V, and then the set Z_V is lineable. The spirit of this construction would be the same as in the proof of Proposition 3.3, thus we spare the details of such a construction here.

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