THE CONTINUOUS SCHWARZ-CHRISTOFEL TRANSFORMS (CSCT)

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SUMMARY

This paper shows a method, developped by the authors, to transform series of curvilinear segments, which intersect at angles of non zero magnitude, into the real axis, as a complex function transform.

The method initiates from the Schwarz-Chritoffel transformation and generalizes it. Examples of how to carry out the computations are shown and some new results are presented.

1.- THE CSCT

Several methods exist to transform a whole curve into the real axis. As an example, if the parametric equations of a curve are

$$x = x(t); \quad y = y(t)$$

where the parameter must go from $-\infty$ to $+\infty$. The transformation

$$z = x(w) + i \cdot y(w)$$

maps the curve, in the **z** plane, into the real axis, in the **w** plane.

The Continuous Schwarz-Christoffel Transforms consists in relating a curvilinear segment in the **z**, **(x+iy)**, plane with a segment of the **u** axis in the **w**, **(u+iv)**, plane.

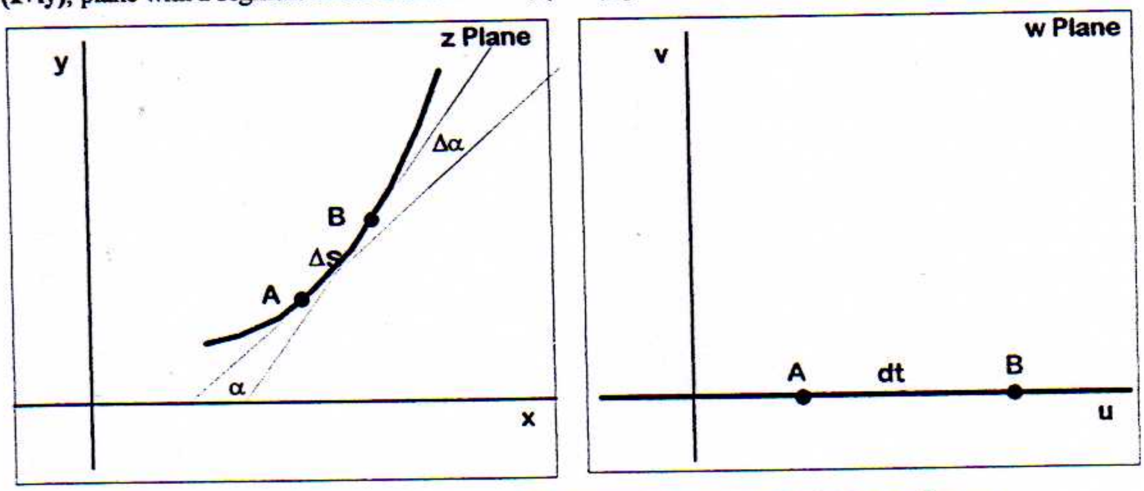


Figure 1.- The continuous Schwartz-Christoffel Transforms.

In figure 1, it is supposed that the curve has a continuous tangent. In the elementary length, Δs, the change in orientation is Δα, so the factor to be introduced in the Schwarz.Christoffel transforms, due to this point, is

$$(w - t)^{-\frac{\Delta\alpha}{\pi}} = e^{-\frac{\Delta\alpha}{\pi} \cdot Ln(w-t)}$$

and, along the complete curve segment, the product of all these factors has to be put

$$dz = dw \cdot \prod (w - t)^{-\frac{\Delta\alpha}{\pi}} = dw \cdot e^{-\sum \frac{\Delta\alpha}{\pi} \cdot Ln(w-t)}$$

and taking the limit when Δs tends to zero,

$$dz = dw \cdot e^{-\frac{1}{\pi}\int d\alpha \cdot Ln(w-t)}$$

When the point follows the curve, in the **z** plane, a piece of the axis , in the **w** plane, is followed. If the relation between both paths is known, $t = t(\alpha)$, or $\alpha = \alpha(t)$, the integral can be solved, yielding

$$dz = dw \cdot e^{-\frac{1}{\pi} \cdot F(w)}$$

where

$$F(w) = \int d\alpha \cdot Ln\{w - t(\alpha)\}$$

Evidently, the final result depends on how the **u** axis is run while going through the curve segment, that is, it depends on the function $t(\alpha)$. As an example, it will be seen what happens when a circumference is followed once or an infinity of times, while the **u** axis is run once.

When several pieces of curves intersect at angles, the corresponding factor have to be put in the formula, yielding

$$dz = dw \cdot e^{\frac{1}{\pi} F(w)} \cdot \prod_i \frac{1}{(w - a_i)^{\Delta\alpha_i / \pi}}$$

where a_i is the point where the transformed curve has a sudden angle change of $\Delta\alpha_i$.

If the sudden changes are represented by a delta function, $\Delta\alpha_i \cdot \delta(t - a_i)$, the transformation, including smooth curves as well as finite angle changes can be put in the following way

$$dz = dw \cdot e^{-\frac{1}{\pi} \int_{-\infty}^{\infty} d\alpha \cdot Ln(w-t)}$$

2.- FIRST EXAMPLES

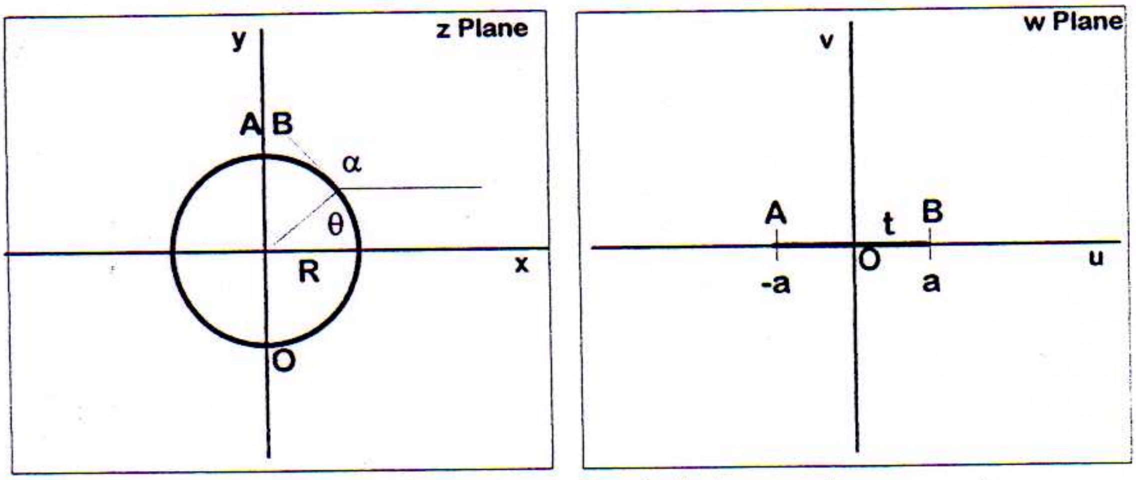


Figura 2.- The transformation of a circle in an axis segment.

When in the circle an angle of $2 \cdot \pi$ is followed, in the axis a segment of **2a** is followed, so it is

$$\alpha = \pi \cdot t / a; \quad t = a \cdot \alpha / \pi$$

so that the following steps are

$$F(w) = \int_{-\infty}^{\infty} d\alpha \cdot Ln(w - a \cdot \alpha / \pi)$$

$$F'(w) = \int_{-\infty}^{\infty} \frac{d\alpha}{w - a \cdot \alpha / \pi} = -\frac{\pi}{a} \lim_{\alpha \to \infty} Ln\left(\frac{w - a \cdot \alpha / \pi}{w + a \cdot \alpha / \pi}\right) = -\frac{\pi}{a} \cdot Ln(-1) = -\frac{\pi^2 \cdot i}{a}$$

$$F(w) = -\frac{\pi^2 \cdot i}{a} \cdot w + Ln(C)$$

$$dz = dw \cdot C \cdot e^{\frac{\pi w}{a} \cdot i}$$

$$z = -i \cdot R \cdot e^{\frac{\pi w}{a} \cdot i}$$

The complex functios components are

$$x = R \cdot e^{-\pi\psi/a} \cdot sen(\pi\phi / a)$$
$$y = -R \cdot e^{-\pi\psi/a} \cdot \cos(\pi\phi / a)$$

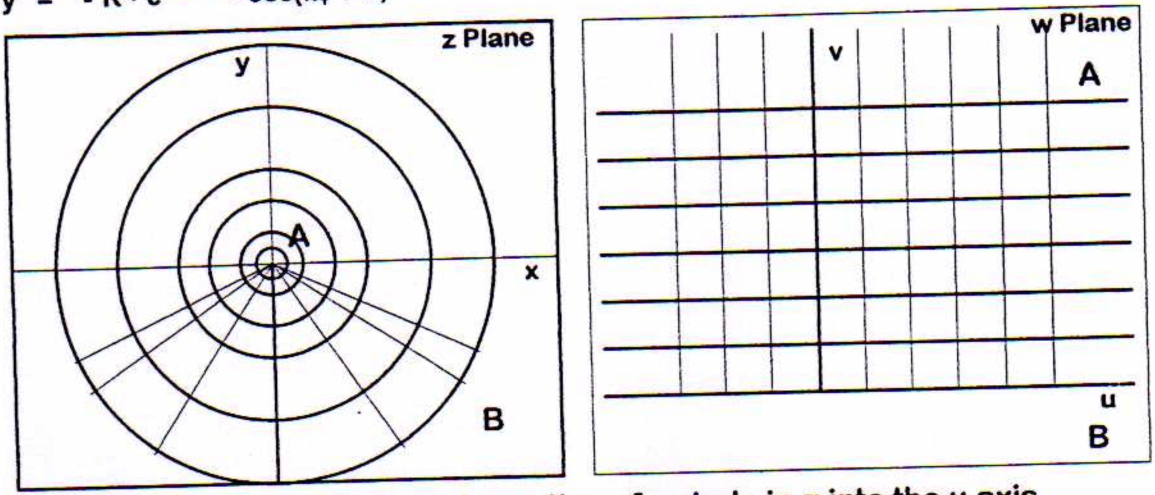


Figure 3.- Plot of the transformation of a circle in z into the u axis.

The straight lines, parallel to the **u** axis, become the circles

$$x^2 + y^2 = R^2 \cdot e^{-2\pi\phi/a}$$

and the straight lines, parallel to the **v** axis, become the straight lines

$$\frac{y}{x} = -\cot(\pi\phi / a)$$

Figure 3 is a plot of this transformation..

In the case that the circle is run only once while following the complete **u** axis, several **t-a** relationships can be used. As an example, the following transformation, shown in figure 4,

$$t = a \cdot \tan(\alpha / 2); \quad \alpha = 2 \cdot atn(t / a); \quad d\alpha = \frac{2 \cdot a \cdot dt}{a^2 + t^2}$$

from where, the following steps are obtained

$$F(w) = \int_{-\pi}^{\pi} d\alpha \cdot Ln(w - t) = \int_{-\infty}^{\infty} Ln(w - t) \cdot \frac{2 \cdot a \cdot dt}{a^2 + t^2}$$

$$F'(w) = \int_{-\infty}^{\infty} \frac{2 \cdot a \cdot dt}{(a^2 + t^2) \cdot (w - t)} = 2 \cdot \pi \cdot i \cdot 2 \cdot a \cdot \left\{ \frac{1}{2 \cdot a \cdot i \cdot (w - ai)} - \frac{1}{w^2 + a^2} \right\} = \frac{2 \cdot \pi}{w + ai}$$

$$F(w) = \int \frac{2 \cdot \pi}{w + a \cdot i} \cdot dw = 2 \cdot \pi \cdot Ln\{(w + a \cdot i) / C\}$$

$$dz = dw \cdot e^{-2 \cdot Ln\{(w + a \cdot i)/C\}} = \frac{C \cdot dw}{(w + a \cdot i)^2}$$

$$w = A + \frac{B}{z + a \cdot i} = R \cdot i - \frac{2 \cdot a \cdot R}{z + a \cdot i} = \frac{z \cdot R \cdot i - 3 \cdot a \cdot R}{z + a \cdot i}$$

where the integration constants have been adjusted to fit the shown points **A, B** and **O**. The transformed functions are

$$x = \frac{2 \cdot a \cdot R \cdot u}{u^2 + (v + a)^2}; \quad y = R - \frac{2 \cdot a \cdot R \cdot (v + a)}{u^2 + (v + a)^2}$$

and, putting $u = (v + a) \cdot \cot(\alpha / 2)$ it follows that

$$x = \frac{a \cdot R}{v + a} \cdot \cos\alpha; \quad y = R - \frac{a \cdot R}{v + a} - \frac{a \cdot R}{v + a} \cdot \sin\alpha$$

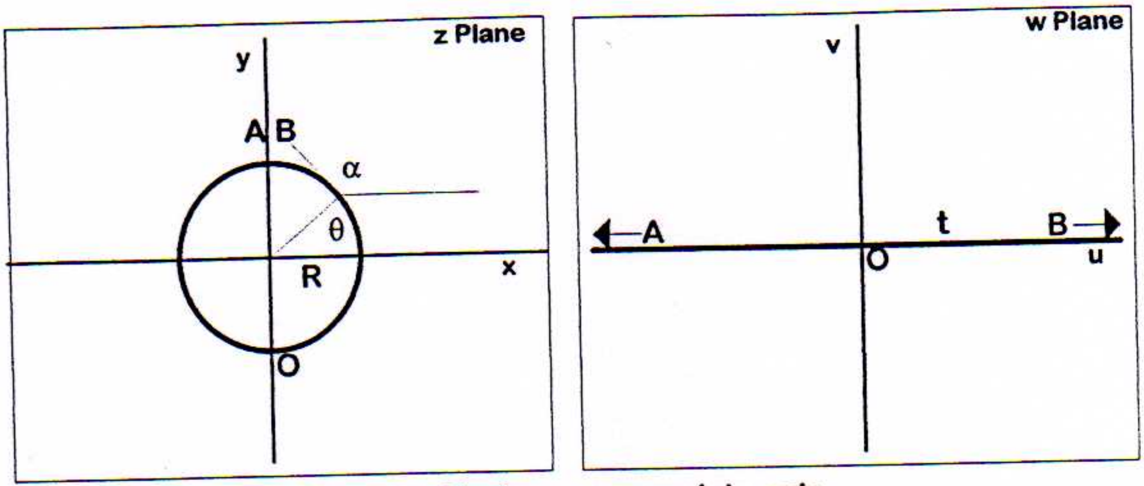


Figura 4.- Circle on a complete axis.

which are the parametric equations of a circle with center at **{0, R - ar / (y + a)}** and radius **ar / (y + a)**. All of them go through the point **(0, R)**. If it is **y + a = x · tan(α / 2)**, it follows that

$$x = \frac{a \cdot R}{\phi} \cdot (1 + \cos\alpha); \quad y = R - \frac{a \cdot R}{\phi} \cdot \mathrm{sen}\alpha$$

which are circles with center at **(ar / x, R)** and radius **ar / x**. All of them pass through the point **(0, R)**. Figure 5 shows this transformation where the straight lines parallel to the **u** and **v** axes become two sets of circles.

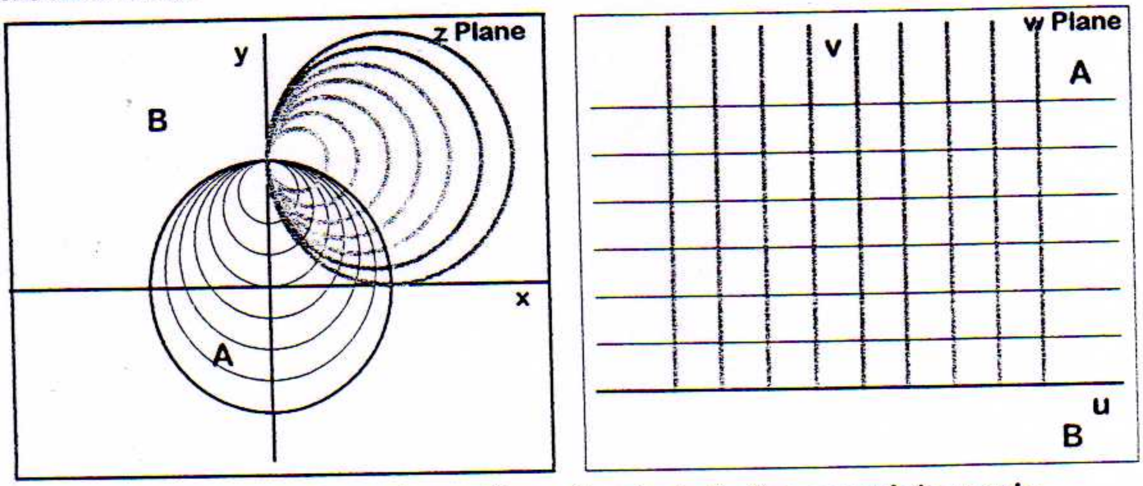


Figure 5.- The transformation of a circle in the complete u axis.

3.- MIXED COMBINATIONS

The solutions found in the previous paragraph can be obtained by other methods. Where the CSCT transforms is most suitable is combinating several curve segments which intersect at different angles. The following examples show some old and new transformations obtained using this new method,

The first example is the circular obstacle, shown in figure 6.

First, the case where the semicircle is run so that its projection on the **x** axis is proportional to the corresponding path in the **u** axis, is shown. The transform is

$$dw = C \cdot \frac{dz}{\sqrt{z^2 - a^2}} \cdot e^{-\frac{1}{x} \int d\alpha \cdot Ln(z-t)} = C \cdot \frac{dz}{\sqrt{z^2 - a^2}} \cdot e^{-\frac{1}{x} \cdot F(z)}$$

where the square root stands for the right angles at points **A** and **B**.

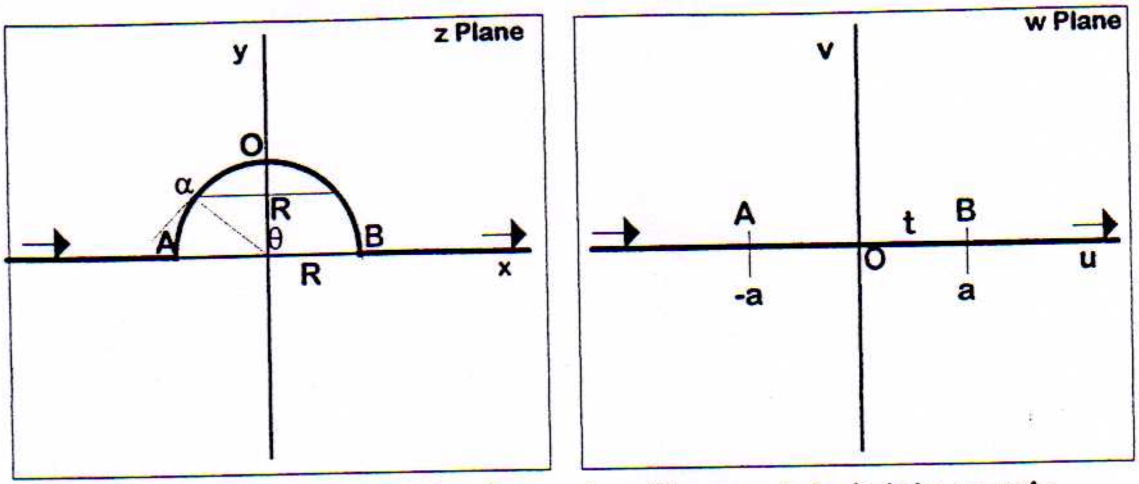


Figure 6.- Transformatión of an axis with a semicircle into an axis.

The following operations show what it is obtained

$$\frac{3\pi}{2} \to \alpha \to \frac{\pi}{2}; \quad -a \to t \to a; \quad \pi \to \theta \to 0$$

$$\theta = \alpha - \frac{\pi}{2}; \quad t = a \cdot \cos\theta; \quad d\alpha = d\theta$$

$$F(w) = \int_{\pi}^{0} d\theta \cdot Ln(w - a \cdot \cos\theta)$$

$$F'(w) = -\int_{0}^{\pi} \frac{d\theta}{w - a \cdot \cos\theta} = -\frac{\pi}{\sqrt{w^2 - a^2}}$$

$$F(w) = -\pi \cdot Ach(w / a) = -\pi \cdot Ln\left\{\frac{w + \sqrt{w^2 - a^2}}{a}\right\} = \pi \cdot Ln\left\{\frac{w - \sqrt{w^2 - a^2}}{a}\right\}$$

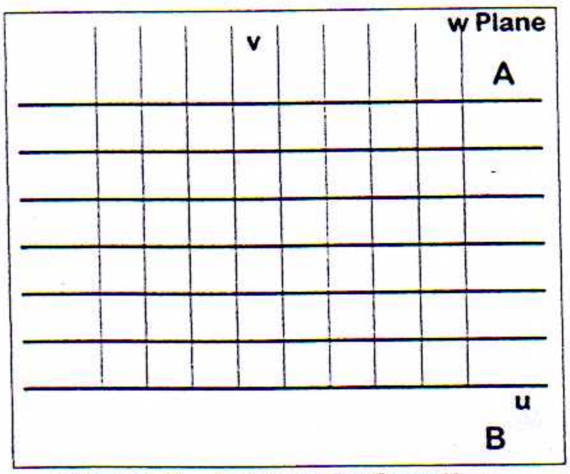
$$dz = C \cdot \frac{dw}{\sqrt{w^2 - a^2}} \cdot \left\{\frac{w + \sqrt{w^2 - a^2}}{a}\right\} = C \cdot \frac{dw}{a} \cdot \left\{1 + \frac{w}{\sqrt{w^2 - a^2}}\right\}$$

$$z = \frac{R}{a} \cdot \left(w + \sqrt{w^2 - a^2}\right)$$

and the inverse is the well known Joukovsky transformation.

$$\frac{w}{a} = \frac{1}{2} \cdot \left(\frac{z}{R} + \frac{R}{z}\right)$$

a plot of which is shown in figure 7.



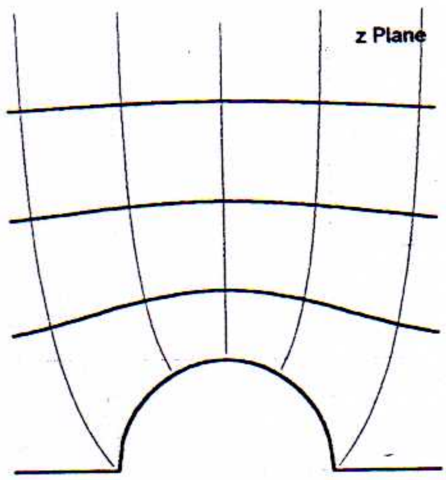


Figure 7.- Joukovsky function

If the circle and the axis are run at constant velocities, one fInds

$$\frac{3\pi}{2} \rightarrow \alpha \rightarrow \frac{\pi}{2}; \quad -a \rightarrow t \rightarrow a$$

$$\alpha = \pi \cdot \left(1 - \frac{t}{2a}\right); \quad d\alpha = -\frac{\pi}{2a} \cdot dt$$

$$F(w) = -\frac{\pi}{2a} \int_{-a}^{a} dt \cdot Ln(w - t)$$

$$F(w) = \pi + \frac{\pi}{2a} \cdot \{(w - a) \cdot Ln(w - a) - (w + a) \cdot Ln(w + a)\}$$

$$dz = C \cdot dw \cdot \frac{(w - a)^{(w-2a)/2a}}{(w + a)^{(w+2a)/2a}}$$

an integral, to solve which, numerical methods or series developments must be used.

4.- SOME NEW RESULTS

If the path in the z plane is the one shown in figure 7, consisting in the positive **u** axis from ∞, to **R**, a complete circle of radius **R** and the positive **u** axis from **R** to ∞, the following operatios are

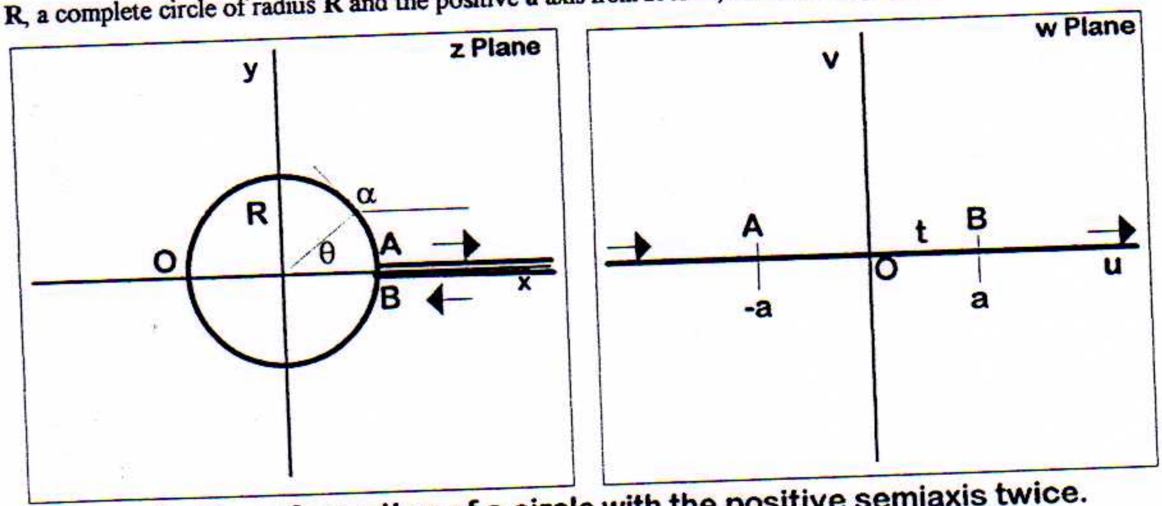


Figure 8.- Transformation of a circle with the positive semiaxis twice.

$$dz = C \cdot \frac{dw}{\sqrt{w^2 - a^2}} \cdot e^{-\frac{1}{\pi} \cdot \int d\alpha \cdot Ln(w-t)} = C \cdot \frac{dw}{\sqrt{w^2 - a^2}} \cdot e^{-\frac{1}{\pi} \cdot F(w)}$$

where the square root accounts for the de 90° angles in the points **A** y **B**. The following equations are obtained succesively:

$$\frac{\pi}{2} \rightarrow \alpha \rightarrow \frac{5\pi}{2}; \quad -a \rightarrow t \rightarrow a; \quad 0 \rightarrow \theta \rightarrow 2\pi$$

$$\theta = \alpha - \frac{\pi}{2}; \quad t = a \cdot \cos\theta; \quad d\alpha = d\theta$$

$$F(w) = 2\int_{0}^{\pi} d\theta \cdot Ln(w - a \cdot \cos\theta)$$

$$F'(w) = 2\int_{0}^{\pi} \frac{d\theta}{w - a \cdot \cos\theta} = \frac{2\pi}{\sqrt{w^2 - a^2}}$$

$$F(w) = 2 \cdot \pi \cdot Ach(w / a) = 2 \cdot \pi \cdot Ln\left\{\frac{w + \sqrt{w^2 - a^2}}{a}\right\} = -2 \cdot \pi \cdot Ln\left\{\frac{w - \sqrt{w^2 - a^2}}{a}\right\}$$

$$dz = C \cdot \frac{dw}{\sqrt{w^2 - a^2}} \cdot \left\{ \frac{w - \sqrt{w^2 - a^2}}{a} \right\}^2$$

$$\frac{z}{R} = \frac{1}{a^2} \cdot \left(w - \sqrt{w^2 - a^2} \right)^2$$

and the inverse transformation is

$$\frac{w}{a} = \frac{1}{2}\left(\sqrt{\frac{z}{R}} + \sqrt{\frac{R}{z}} \right)$$

The implicit equations of the equipotentials and flow lines are

$$\frac{u}{a} = \frac{1}{2}\left\{ \sqrt{\frac{1}{2 \cdot R}\left[\sqrt{x^2 + y^2} + x\right]} \cdot \left(1 + \frac{R}{\sqrt{x^2 + y^2}}\right) \right\}; \quad \frac{v}{a} = \frac{1}{2}\left\{ \sqrt{\frac{1}{2 \cdot R}\left[\sqrt{x^2 + y^2} - x\right]} \cdot \left(1 - \frac{R}{\sqrt{x^2 + y^2}}\right) \right\}$$

and the parametric equations are

$$\frac{x}{R} = \frac{(v + \beta)^2 - (u + \alpha)^2}{a^2}; \quad \frac{y}{R} = -\frac{2 \cdot (v + \beta) \cdot (u + \alpha)}{a^2}$$

where

$$2 \cdot \alpha^2 = \sqrt{(u^2 - u^2 - a^2)^2 + 4 \cdot u^2 \cdot v^2} + (u^2 - v^2 - a^2)$$

$$2 \cdot \beta^2 = \sqrt{(u^2 - v^2 - a^2)^2 + 4 \cdot u^2 \cdot v^2} - (u^2 - v^2 - a^2)$$

The two previous results suggest that the transformation of a quarter circle with two perpendicular semiaxes is

$$\frac{z}{R} = \frac{1}{a} \cdot \sqrt{w + \sqrt{w^2 - a^2}}$$

$$\frac{w}{a} = \frac{1}{2}\left(\frac{z^2}{R^2} + \frac{R^2}{z^2} \right)$$

The case of a semicircle with straight lines parallel to the negative axis is shown in figure 9.

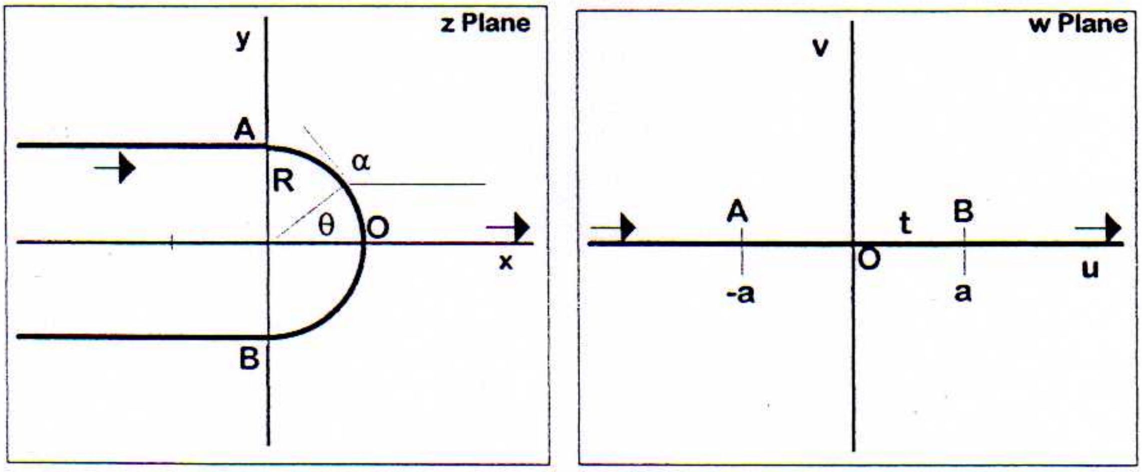


Figura 9.- Semicircle with two parallels to the negative axis.

If the semicircle is run so that its projection on the **y** axis is proportional to the path in the **u** axis, the method yields

$$\pi \to \alpha \to 0; \quad -a \to t \to a; \quad \pi/2 \to \theta \to -\pi/2$$

$$\theta = \alpha - \frac{\pi}{2}; \quad t = a \cdot \text{sen}\theta; \quad d\alpha = d\theta$$

$$F(w) = \int_{\pi/2}^{-\pi/2} d\theta \cdot Ln(w - a \cdot \text{sen}\theta); \quad F'(w) = -\int_{-\pi/2}^{\pi/2} \frac{d\theta}{w - a \cdot \text{sen}\theta} = \frac{-\pi}{\sqrt{w^2 - a^2}}$$

$$F(w) = -\pi \cdot Ach(w / a) = -\pi \cdot Ln\left\{\frac{w + \sqrt{w^2 - a^2}}{a}\right\} = \pi \cdot Ln\left\{\frac{w - \sqrt{w^2 - a^2}}{a}\right\}$$

$$dz = C \cdot dw \cdot \left\{\frac{w + \sqrt{w^2 - a^2}}{a}\right\} = C \cdot \frac{dw}{a} \cdot \left\{w + \sqrt{w^2 - a^2}\right\}$$

$$z = -\frac{2 \cdot R \cdot i}{\pi \cdot a^2} \cdot \left\{w \cdot \left(w + \sqrt{w^2 - a^2}\right) - a^2 \cdot Ln\left(\frac{w + \sqrt{w^2 - a^2}}{a}\right)\right\}$$

The obstacle with a semicircular form is shown in figure 10.

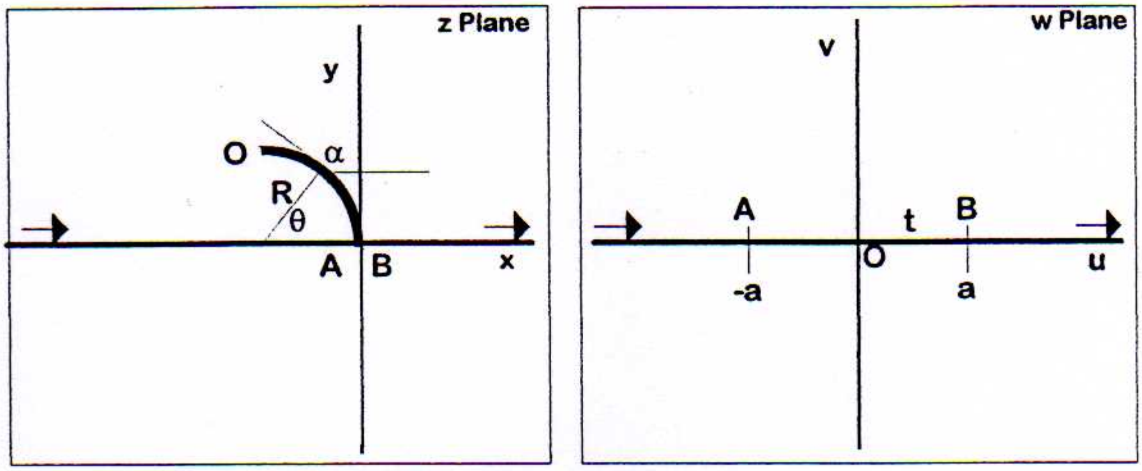


Figure 10.- Semicircular obstacle.

If the semicircle is run so that its projection on the **x** axis is proportional to the path in the **u** axis, the set of results is shown

$\pi / 2 \to \alpha_1 \to \pi;\ 0 \to \theta \to \pi / 2;\ -a \to t \to 0;\ \alpha_1 = \theta + \pi / 2;\ d\alpha_1 = d\theta;\ t = -a \cdot \cos\theta$

$\pi \to \alpha_2 \to \pi / 2;\ \pi / 2 \to \theta \to 0;\ 0 \to t \to a;\ \alpha_2 = \theta + \pi / 2;\ d\alpha_2 = d\theta;\ t = a \cdot \cos\theta$

$$F_1(w) = \int_0^{\pi/2} d\theta \cdot Ln(w + a \cdot \cos\theta); \quad F_2(w) = \int_{\pi/2}^{0} d\theta \cdot Ln(w - a \cdot \cos\theta)$$

$$F'(w) = F_1'(w) + F_2'(w) = \frac{-2}{\sqrt{w^2 - a^2}} \cdot atn\left(\frac{a}{\sqrt{w^2 - a^2}}\right)$$

This result proves that the method can lead to integrals which are difficult to solve in close form, so that numerical analysis must be brought up frequently.

5.- CONCLUSIONS

A transformation of the real axis into a piecewise curve yuxtaposition intersecting at non zero angles can be solved by the transformation defined by

$$dz = dw \cdot e^{-\frac{1}{\pi}\int_{-\infty}^{\infty} d\alpha \cdot Ln(w-t)}$$

where each piece has a relation $\alpha = \alpha(t)$ and each non zero angle is represented by a function $\Delta\alpha_i \cdot \delta(w - a_i)$

Few cases can be solved directly and numerical methods must be used in many cases.

No bibliography is given because the Schwarz-Christoffel transformation can be found in most books on complex variables.