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ARTICLE TYPE

Solving random boundary heat model using the finite difference method under mean square convergence

Abstract

This contribution is devoted to construct numerical approximations to the solution of the one-dimensional boundary-value problem for the heat model with uncertainty in the diffusion coefficient. Approximations are constructed via random numerical schemes. This approach permits discussing the effect of the random diffusion coefficient, which is assumed to be a random variable. We establish results about the consistency and stability of the random difference scheme using mean square convergence. Finally, an illustrative example is presented.

KEYWORDS:

Random difference scheme; random boundary heat model; mean square consistency; mean square stability

1 | INTRODUCTION

Several problems in many areas of science are formulated via partial differential equations. The coefficients of any partial differential equation can be deterministic values or random variables. Partial differential equations, whose coefficients are deterministic, have been discussed for a long time. Theory and methods for solving, both analytic and numeric, are well developed. However, in many problems, partial differential equations with random coefficients are better suited to describe the real behaviour of phenomenon than their deterministic counterpart. The randomness in the coefficients may arise because of errors involved in measurement data or uncertainties due to lack of knowledge. Partial differential equations with random coefficients or incorporating stochastic effects have been increasingly used in the last few decades to deal with errors and uncertainty and represent a growing field of great scientific interest.

We want to emphasize that there exist two main approaches to consider uncertainty within the context of partial differential equations each one of them leading to different ways of performing the corresponding numerical analysis. The most common approach is based upon SDEs where uncertainty is forced via the perturbation of model parameters by means of an irregular) stochastic process such as a Wiener process or Brownian motion (whose trajectories are nowhere differentiable). This kind of equations are typically written in terms of both Lebesgue and Itô stochastic integrals. The rigorous treatment of SDEs requires of a special calculus usually referred to as Itô calculus whose cornerstone result is termed the Itô's Lemma. A less known approach is based upon RDEs for which random effects are manifested directly in coefficients, initial/boundary conditions and/or source term which are assumed to behave regular (e.g., continuous) with respect to time and space. In recent literature [1] has been pointed out that there is a growing trend in the uncertainty quantification community to treat the terms SDE and RDE as synonymous when in fact they are distinctly different and they require completely different techniques for analysis and approximation. As rightly indicated in recent contributions [2], this confusion could be because RDEs seem to have had a shadow existence to SDEs, although they have been around for as long as if not longer and have many important applications. Throughout this manuscript we will only work in the context of partial RDEs. This decision has specific advantages with respect to the Itô approach. First, we do not require to apply Itô-Taylor type expansions, which usually involve technical hypotheses, to conduct the corresponding

[0] **Abbreviations:** IBVP, initial boundary value problem; PDE, partial differential equation; RDE, random differential equation; RFDS, random finite difference scheme; SDE, stochastic differential equation

numerical analysis for consistency since we are dealing with partial RDEs rather than partial SDEs. Second, there is a lack of results in the context of random PDEs that need further investigation since the majority of contributions have focused on partial SDEs. Third, model parameters can have a wider variety of probability distributions including the important Gaussian pattern. For instance, the diffusion parameter in the heat equation is positive so Gaussian distribution is not suitable to describe it and other probabilities distributions with positive support could be more appropriate like the Beta distribution. Nevertheless, allowing a wider kind of uncertainty in the context of RDEs is not straightforward from the properties of its deterministic counterpart since the corresponding operational rules must be carefully legitimated. An appropriate overview of such difficulties can be checked in recent contributions [3].

Heat is energy that flows from higher to lower temperature. The heat can be transferred by conduction or convection [4,5]. Heat equation arises from many fields, for example, the heat transfer, fluid dynamics, astrophysics, finance or other areas of applied mathematics and physics. The temperature, $u(x,t)$, in a random heat conducting insulated rod along the interval $[0,1]$ on x-axis at the time instant $t>0$ is modelled by means of the random partial differential heat equation in the form:

$$u_t(x,t) = \beta\, u_{xx}(x,t), \qquad t \geq 0, \quad 0 \leq x \leq 1, \tag{1}$$

where t is the time variable, x is the space coordinate and β is the conductivity coefficient. The initial condition is given by

$$u(x,0) = u_0(x), \tag{2}$$

and we take boundary conditions as

$$u(0,t) = 0, \quad u(1,t) = 0. \tag{3}$$

In recent years, mathematical models are described as PDEs in many areas of science and engineering, also in medicine and finance for example. The heat equation is a model of diffusive systems since the physical meaning may be imagined in which heat is considered to be a fluid inside matter, free to flow from one position to another [4,6]. The heat equation has a great deal of application in many branches of sciences, naturally in different models from chemistry, theoretical physics and others [4]. There are analytical and numerical processing for dealing with problem (1). In the case that the conductivity diffusive coefficient is a constant or a deterministic function, some authors study the problem by taken the initial condition to be a step function, others, using the green function under the existence and uniqueness of solution and others solve this problem by using the separation of variables when the spatial domain is bounded [7,8]. An algorithm [9] for solving the heat problem on unbounded domains has been developed. This algorithm depends on the evolution of the continuous spectrum of the solution. Fast solver [10] for heat equation in free space is proposed and some authors presented an exact artificial boundary condition to reduce the original initial-boundary-value problem heat equation on a finite computational domain [11,12]. The finite difference method is very useful to approximate partial differential equations in the deterministic scenario [13,14,15,16]. Some authors have studied the stability and consistency for the finite difference method in mean square but, for solving initial value boundary conditions problem for heat equation by taken the diffusion coefficient as constant value and also by adding a white noise term [17]. This approach is usually solved by means of the Itô calculus [18,19,20]. Also, some others discussed the mean square consistency and mean square stability for the finite difference method in order to solve the initial value boundary conditions problem by taken the diffusion coefficient as a random variable [21].

In this problem, we assume that $u_0(x)$ is a deterministic initial data function measuring the temperature along the whole spatial, while $\beta = \beta(\omega)$, $\omega \in \Omega$, is a positive random variable defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and it represents the random conductivity coefficient of the material used to make the rod. Since we assume β is a positive random variable, it implies that material properties are random variables and depend on the location in the rod. It means that we are considering an inhomogeneous material, that is, containing impurities. The physical meaning of the thermal diffusion coefficient is associated with the speed of the flux of heat into the material during changes of temperature over time. The propagation rate of heat is proportional to the thermal diffusivity [4,22]. As β is a random variable, the solution of the IVBP (1)–(3) is a stochastic process, namely, $u(x,t)(\omega)$. In order to avoid a cumbersome notation, throughout this paper has been omitted the notation of ω parameter, often referred to as the *hidden* parameter [23].

This paper is concerned with the application of the mean square consistency and the mean square stability for finite difference methods. Some authors recently developed random difference schemes to solve some Cauchy problems strongly related to this contribution [24,25].

In the following, we introduce some definitions and important results in the context that will be used and extended to the random context throughout this paper.

Definition 1. [14,26] A finite difference scheme $\mathrm{L}_k^n u_k^n = G_k^n$ ($\mathrm{L}_\mathrm{k}^\mathrm{n}$ is the discretization operator) approximating the partial differential equation $\mathrm{L}v = G$ (L denotes the differential operator) is said to be mean square consistent if the solution, v, of the PDE satifies:

$$V^{n+1} = Q\, V^n + (\Delta t) G^n + (\Delta t)\tau^n,$$

with $\|\tau^n\| \to 0$ as $\Delta x, \Delta t \to 0$, and being $\|\cdot\|$ an arbitrary norm. V^n denotes the vector whose k-th component is $v(k\Delta x, n\Delta t)$.

Definition 2. [14,26] A finite difference scheme $\mathrm{L}_k^n u_k^n = \mathrm{G}_\mathrm{k}^\mathrm{n}$ (L_k^n is the discretization operator) approximating the partial differential equation $\mathrm{L}v = \mathrm{G}$ (L denotes the differential operator) is said to be mean square stable, if there exist some positive constants ϵ, δ and non-negative constants η, ξ such that

$$\left\|u^{n+1}\right\| \le \eta \mathrm{e}^{\xi t} \left\|u^0\right\|,$$

being u^0 the initial data, for $t = (n+1)\Delta t$, $0 < \Delta x \le \epsilon$, $0 < \Delta t \le \delta$, and $\|\cdot\|$ denotes an arbitrary norm.

Definition 3. [14] A finite difference scheme $\mathrm{L}_k^n u_k^n = \mathrm{G}_\mathrm{k}^\mathrm{n}$ (L_k^n is the discretization operator) approximating the partial differential equation $\mathrm{L}v = \mathrm{G}$ (L denotes the differential operator) is said to be accurate of order (p, q) to a given PDE if:

$$\|\tau^\mathrm{n}\| = \mathcal{O}(\Delta x^p) + \mathcal{O}(\Delta t^q),$$

where τ^n is the truncation error, and $\|\cdot\|$ denotes an arbitrary norm.

To extend previous definitions to random scenario, throughout this paper, $\|\cdot\|$ in random context will denote the mean square norm defined as

$$\|\mathbf{x}\| = \left(\mathbb{E}\left[\left(\sup_k |x_k|\right)^2\right]\right)^{1/2}, \tag{4}$$

where $\mathbf{x} = \left(x_1, \dots, x_s\right)$ is a random vector and $\mathbb{E}[\cdot]$ denotes the expectation operator. Throughout this paper, we will work with bounded random variables. In particular this hypothesis is assumed for the single random input in our target problem (1), i.e., β. This is a realistic assumption since in most of the physical problems the involved input parameters take finite values.

This contribution is organized in the following form. In Section 2 a random difference scheme is constructed to solve problem (1)–(3). Next, sufficient conditions are established in order to guarantee consistency and stability in a mean square sense. In Section 3 theoretical results are illustrated by means of a numerical example. The obtained results are compared with other numerical methods. Finally, some conclusions are drawn in Section 4.

2 | RANDOM FINITE DIFFERENCE TECHNIQUE

This section is devoted to introduce the finite difference technique, that will be applied later, in order to find the solution stochastic process to the random IBVP (1)–(3). For the sake of clarity, we introduce some notation that will be useful throughout our subsequent analysis. Therefore, let us consider a uniform grid for space,

$$x_0 = 0 < x_1 < \cdots < x_{M-1} < x_M = 1,$$

where

$$\Delta x = x_k - x_{k-1}, \qquad 1 \le k \le M,$$

and also, a uniform grid for time

$$t_0 = 0 < t_1 < \cdots,$$

where

$$\Delta t = t_n - t_{n-1}, \quad n \ge 1,$$

which defines a two-dimensional time-space mesh grid, where the exact solution stochastic process to the random IBVP (1)–(3), $u(x,t)$, will be obtained numerically.

Let

$$u_k^n \approx u(k\Delta x, n\Delta t),$$

denotes the approximation solution for the random IBVP (1)–(3), $u(x,t)$, at the point $(x_k, t_n) = (k\Delta x, n\Delta t)$. To obtain the numerical method, the following expression is used to approximate the time derivative in (1)

$$\frac{\partial u}{\partial t} = \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} + \mathcal{O}(\Delta t)\,. \tag{5}$$

For the spatial derivative we use a centered second order approximation

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x-\Delta x,t) - 2u(x,t) + u(x+\Delta x,t)}{(\Delta x)^2} + \mathcal{O}\left((\Delta x)^2\right). \tag{6}$$

By replacing derivatives in (1) by the corresponding approximations, (5)–(6), in each interior point $(k\Delta x, n\Delta t)$, $1 \leq k \leq M-1$, $n \geq 1$, of the mesh, we obtain

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} = \beta \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{(\Delta x)^2}. \tag{7}$$

So, the numerical method associated to the interior points, for each $n \geq 1$, can be stated as

$$u_k^{n+1} = u_k^n + r(u_{k+1}^n - 2u_k^n + u_{k-1}^n), \qquad k = 1, 2, \ldots, M-1, \tag{8}$$

where

$$r = \frac{\beta \Delta t}{(\Delta x)^2}. \tag{9}$$

For the initial condition in (2) we have:

$$u_k^0 = u_0(k\,\Delta x), \qquad k = 0, 1, \ldots, M. \tag{10}$$

For the boundary conditions in (3) we have:

$$\begin{aligned} u_0^n &= u(0, n\,\Delta t) = 0, \qquad n \geq 1, \\ u_M^n &= u(M\,\Delta x, n\,\Delta t) = u(1, n\,\Delta t) = 0, \qquad n \geq 1. \end{aligned} \tag{11}$$

Notice that the initial and boundary solutions, given by (10) and (11), are established directly from (2) and (3), respectively. So, taking the vector of unknowns in a time t_n,

$$\mathbf{u}^n = \left(u_1^n, u_2^n, \ldots, u_{M-1}^n\right),$$

RDFS (8)–(11) can be rewritten as

$$\mathbf{u}^{n+1} = \begin{pmatrix} 1-2r & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1-2r & 1 & 0 & \cdots & 0 \\ & & & & & \\ & \ddots & \ddots & \ddots & & \\ & & & & & \\ 0 & \cdots & 0 & 1 & 1-2r & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1-2r \end{pmatrix} \mathbf{u}^n, \qquad n \geq 1, \tag{12}$$

then, we can advance one step to calculate the whole numerical solution at time instant t_1, $\mathbf{u}^1$, in an explicit way by means of RFDS (12), and the initial condition $\mathbf{u}^0$ defined by (10). The approximations of the solution at the time instant t_n are recursively obtained from the approximations calculated at t_{n-1}.

Consistency, stability and convergence are important topics in deterministic and stochastic theory for many numerical methods [14]. In the next two subsections, we extend these concepts to the numerical scheme (8) using the mean square approach [27].

2.1 | Mean square consistency of the random finite difference scheme

Definition 4. A random finite difference scheme $\mathrm{L}_k^n \mathrm{u}_k^n = \mathrm{G}_k^n$ that approximate the random partial differential equation $\mathrm{Lv} = \mathrm{G}$ is said to be mean square consistent if the solution of the random partial differential equation, v, satifies:

$$\mathrm{V}^{n+1} = Q\mathrm{V}^n + (\Delta t)\mathrm{G}^n + (\Delta t)\tau^n, \tag{13}$$

and

$$\mathbb{E}\left[\left(\sup_k |\tau_k^n|\right)^2\right] \to 0,$$

as $\Delta x, \Delta t \to 0$. V^n denotes the vector whose k-th component is $v(k\Delta x, n\Delta t)$.

Theorem 1. The RFDS (8) associated to problem (1) is mean square consistent.

Proof. Let us consider RFDS (8)–(9),

$$u_k^{n+1} = u_k^n + r(u_{k+1}^n - 2u_k^n + u_{k-1}^n), \qquad r = \frac{\beta\Delta t}{(\Delta x)^2},$$

and let $u(x,t)$ be a solution to random PDE (1). Using Taylor expansions at (x_k, t_n) we have

$$u_k^{n+1} = u_k^n + (u_t)_k^n(\Delta t) + \mathcal{O}\left((\Delta t)^2\right),$$

$$u_{k-1}^n = u_k^n - (u_x)_k^n(\Delta x) + (u_{xx})_k^n\frac{(\Delta x)^2}{2} - (u_{xxx})_k^n\frac{(\Delta x)^3}{6} + \mathcal{O}\left((\Delta x)^4\right),$$

$$u_{k+1}^n = u_k^n + (u_x)_k^n(\Delta x) + (u_{xx})_k^n\frac{(\Delta x)^2}{2} + (u_{xxx})_k^n\frac{(\Delta x)^3}{6} + \mathcal{O}\left((\Delta x)^4\right).$$

Now, we compute $(\Delta t)\tau_k^n$ using these Taylor expressions

$$\begin{aligned}
(\Delta t)\tau_k^n &= \mathrm{u}_k^{n+1} - \left\{\mathrm{u}_k^n + r\left[\mathrm{u}_{k+1}^n - 2\mathrm{u}_k^n + \mathrm{u}_{k-1}^n\right]\right\}\\
&= \mathrm{u}_k^n + (\mathrm{u_t})_k^n(\Delta t) + \mathcal{O}\left((\Delta t)^2\right) - \mathrm{u}_k^n - r\mathrm{u}_k^n - r(\mathrm{u_x})_k^n(\Delta x)\\
&- r(\mathrm{u_{xx}})_k^n\frac{(\Delta x)^2}{2} - r(\mathrm{u_{xxx}})_k^n\frac{(\Delta x)^3}{6} + \mathcal{O}\left((\Delta x)^4\right) + 2r\mathrm{u}_k^n - r\mathrm{u}_k^n\\
&+ r(\mathrm{u_x})_k^n(\Delta x) - r(\mathrm{u_{xx}})_k^n\frac{(\Delta x)^2}{2} + r(\mathrm{u_{xxx}})_k^n\frac{(\Delta x)^3}{6} + \mathcal{O}\left((\Delta x)^4\right).
\end{aligned}$$

Hence we have:

$$\Delta t\,\tau_k^n = (u_t - \beta u_{xx})_k^n\Delta t + \mathcal{O}\left((\Delta t)^2\right) + \mathcal{O}\left(\Delta t\,(\Delta x)^2\right).$$

since, $u_t - \beta u_{xx} = 0$, then we have:

$$\tau_k^n = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)^2. \tag{14}$$

Finally, taking the supremum and the expectation operator, one gets,

$$\|\tau_k\|^2 = \mathbb{E}\left[\left(\sup_k|\tau_k^n|\right)^2\right] \to 0, \qquad \Delta x, \Delta t \to 0.$$

Hence, the RFDS (2) is mean square consistent. □

2.2 | Mean square stability of the random finite difference scheme

Definition 5. A random difference scheme $\mathrm{L_k^n u_k^n} = \mathrm{G_k^n}$ ($\mathrm{L_k^n}$ is the discretization operator) that approximates a random PDE $\mathrm{Lv} = \mathrm{G}$ (L denotes the differential operator) is said to be mean square stable, if there exist some positive constants ϵ, δ, non-negative constants η, ξ and u^0 is initial data such that:

$$\mathbb{E}\left[\sup_k\left|u_k^{n+1}\right|^2\right] \le \eta e^{\xi t}\mathbb{E}\left[\sup_k\left|u_k^0\right|^2\right] \tag{15}$$

for $t = (n+1)\Delta t$, $0 < \Delta x \le \epsilon$, $0 < \Delta t \le \delta$.

Theorem 2. Under the condition

$$\Delta t \le \frac{(\Delta x)^2}{2\beta_1}, \qquad 0 < \beta(\omega) \le \beta_1, \quad \omega \in \Omega, \tag{16}$$

the random finite difference scheme (8) for the random partial differential equation (1)–(3) is mean square stable.

Proof. From RFDS (8) we have

$$\begin{aligned}
\mathbb{E}\left[\sup_k\left|u_k^{n+1}\right|^2\right] &= \mathbb{E}\left[\sup_k\left|(1-2r)u_k^n + ru_{k+1}^n + ru_{k-1}^n\right|^2\right]\\
&\le \mathbb{E}\left[\left[(1-2r)^2 + r^2 + r^2 + 2r\,|1-2r| + 2r\,|1-2r| + 2r^2\right]\sup_k|u_k^n|^2\right].
\end{aligned}$$

If $0 < r \le \frac{1}{2}$ then we have $|1-2r| = 1-2r$. Hence,

$$\mathbb{E}\left[\sup_k\left|u_k^{n+1}\right|^2\right] \le \mathbb{E}\left[\sup_k|u_k^n|^2\right] \le \cdots \le \mathbb{E}\left[\sup_k\left|u_k^0\right|^2\right].$$

Therefore, the RFDS (8) is mean square stable with $\eta = 1$, $\xi = 0$ and under the condition (16).

□

Remark 1. The hypothesis of boundedness assumed in (16) for the random variable β allows us to generalize the corresponding results that are well-known in the deterministic scenario. This provides a consistent connection between the deterministic and the random setting. It must be pointed out that this hypothesis is not new at all since this kind of condition has been imposed by other authors in dealing with the study of both random ordinary and partial differential equations to extend important deterministic results to the random scenario[28,29,30].

3 | A NUMERICAL EXAMPLE

In this section we illustrate our previous theoretical results with an example. With this aim, we need to choose the distribution of the random variable β. As β must be positive we have chosen for β a beta distribution with parameters 1 and 3, i.e., $\beta \sim \text{Be}(1,3)$, whose density is given by $f_\beta(\beta) = 3(1-\beta)^2, 0 < \beta < 1$. Observe that the mean of this random variable is $1/4$ and its variance is $3/80 \approx 0.0375$. Thus, according to (16), $\beta_1 = 1$. For the initial condition, we take

$$u(x,0) = \sin(\pi x). \tag{17}$$

With this condition is easy to check that the analytical solution of problem (1)–(3) is given by

$$u(x,t) = \mathrm{e}^{\pi^2 t\beta} \sin(\pi x). \tag{18}$$

Solving a random problem implies not only to obtain the solution but also its statistical moments, as the mean and variance. In this case, we can compute the analytical mean, that is given by

$$\mathbb{E}[u(x,t)] = \frac{3\,\mathrm{e}^{-\pi^2 t}\left(\mathrm{e}^{\pi^2 t}\left(\pi^4 t^2 - 2\pi^2 t + 2\right) - 2\right)\sin(\pi x)}{\pi^6 t^3}. \tag{19}$$

This allows us to compare the proposed numerical solution with the analytical solution.

We have solved the numerical scheme (8)–(11) for several meshes. In particular, we present the results for three different meshes at the level time $t_N = 1/2$. For the first mesh, we fix $\Delta x = \frac{1}{32} \approx 0.0625$. In order to guarantee the stability, Δt is chosen so that it satisfies condition (16). As

$$0.00042 \approx \frac{1}{2400} = \Delta t \le \frac{(\Delta x)^2}{2\beta_1} = \frac{1}{2048} \approx 0.00049,$$

if we take $\Delta t = \frac{1}{2400}$ stability is guaranteed. This corresponds with $M = 32$ spatial intervals and $N = 1200$ levels of time. To check the order of the method, the successive meshes are constructed multiplying by two the spatial increment, Δx, and by four the temporal increment, Δt. This corresponds to $M = 16$ and $M = 8$, respectively. In all cases stability is reached.

In Fig. 1, we have plotted the expectation of the analytical solution in solid lines and the expectation of the numerical solution in points for the different meshes corresponding to $M = 32$, $M = 16$, $M = 8$. In this figure we can observe that results are very satisfactory, being a little worst at the bottom figure. In order to appreciate better the difference between the different meshes, we have calculated the error corresponding to the difference in absolute value between the analytical and numerical solutions. These errors are drawn in Fig. 2.

In Table1 we show the meshes we have used for computations and the error for the mean defined as the maximum of the difference in absolute value between the analytical mean and the numerical one.

From the data displayed in Table 1, we can observe that the numerical method we have developed is of order one in time and order two in space, i.e., (14) fulfils.

A similar study have been performed for the variance. The analytical variance to (18) is given by

$$\mathbb{V}\text{ar}[u(x,t)] = \sigma^2[u(x,t)] = \frac{3\,\mathrm{e}^{-2\pi^2 t}\left(\mathrm{e}^{2\pi^2 t}\left(2\pi^4 t^2 - 2\pi^2 t + 1\right) - 1\right)\sin^2(\pi x)}{4\pi^6 t^3}. \tag{20}$$

In Fig. 3, we have plotted the variance of the analytical solution in solid lines and the variance of the numerical solution for the different meshes in points. Again we can observe that results are very satisfactory, being a little worst at the center points of the bottom figure. Similarly to the expectation, we have calculated the error corresponding to the difference in absolute value of the variance between the analytical and numerical solutions. These errors are drawn in Fig. 4.

In Table 2 we show the meshes we have used for computations and the error defined as the difference in absolute value between the variance analytical solution and the numerical one.

As in the case of the expectation, from the data displayed in Table 2 we can observe that the numerical method we have developed is of order one in time and order two in space. Numerical results agree with theoretical findings about the order of the method given by equation (14).

If we solve problem (1)–(3) with the initial condition (17) and $\beta \sim \text{Be}(1,3)$ by Monte Carlo method, we obtain the results displayed in Table 3. Comparing Table 1, Table 2 and Table 3 we can observe that the method we have developed improves the obtained via Monte Carlo method. If a dishonest method[31] is applied (substituting the random variable β by its mean) an error for the mean of 0.12 is obtained.

Now, we complete the previous numerical analysis highlighting the key role that the random variable β plays in problem (1)–(3). To show this important fact, now we assume that $\beta \sim \text{Be}(2,6)$, whose density is given by $f_\beta(\beta) = 42(1-x)^5, 0 < \beta < 1$, so that random variable β has the same mean as in the previous numerical experiment, i.e., $1/4$, but a slightly smaller variance $1/48 \approx 0.0208333$. In spite of this small change, we can see its the remarkable effect on the mean (Fig. 5) which is reduced, and particularly on the variance (Fig. 6) of the solution stochastic process, that is reduced approximately by half. In Table 4 and Table 5, we show the absolute errors for the mean and the variance, respectively, while the values of these errors via Monte Carlo simulations are reported in Table 6.

4 | CONCLUSIONS

In this paper we have studied a randomized heat equation with deterministic initial and zero-boundary conditions. A random difference scheme has been constructed and consistency and sufficient conditions to guarantee stability has been established in the mean square sense. Theoretical findings have been tested with a test numerical example. Results have been compared successfully with an analytical solution. Also, the proposed method has been compared with other random numerical methods obtaining satisfactory results.

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CONFLICT OF INTEREST STATEMENT

The authors declare that there is no conflict of interest regarding the publication of this article. The authors express their deepest thanks and respect to the editor and reviewers for their valuable comments.

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TABLE 1 Size of the meshes and numerical scheme absolute error for the mean at $t = 1/2$ applied to the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \text{Be}(1, 3)$.

M	N	Δx	Δt	error for the mean
32	1200	$1/32$	$1/2400$	0.000080
16	300	$1/16$	$1/600$	0.00032
8	75	$1/8$	$1/150$	0.0013

TABLE 2 Size of the meshes and numerical scheme absolute error for the variance at $t = 1/2$ for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(1, 3)$.

M	N	Δx	Δt	error for the variance
32	1200	$1/32$	$1/2400$	0.000029
16	300	$1/16$	$1/600$	0.00012
8	75	$1/8$	$1/150$	0.00047

TABLE 3 Absolute errors for Monte Carlo method for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(1, 3)$.

simulations	error for the mean	error for the variance
1000	0.017	0.0022
10000	0.0054	0.000056
100000	0.00045	0.000316
1000000	0.00019	0.000051

TABLE 4 Size of the meshes and numerical scheme absolute error for the mean at $t = 1/2$ for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(2, 6)$.

M	N	Δx	Δt	error for the mean
32	1200	$1/32$	$1/2400$	0.000093
16	300	$1/16$	$1/600$	0.00037
8	75	$1/8$	$1/150$	0.0015

TABLE 5 Size of the meshes and numerical scheme absolute error for the variance at $t = 1/2$ for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(2, 6)$.

M	N	Δx	Δt	error for the variance
32	1200	$1/32$	$1/2400$	0.000026
16	300	$1/16$	$1/600$	0.00010
8	75	$1/8$	$1/150$	0.00042

TABLE 6 Absolute errors for Monte Carlo method applied to the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(2, 6)$.

simulations	error for the mean	error for the variance
1000	0.0061	0.0035
10000	0.00032	0.00054
100000	0.00021	0.00074
1000000	0.00013	0.00017

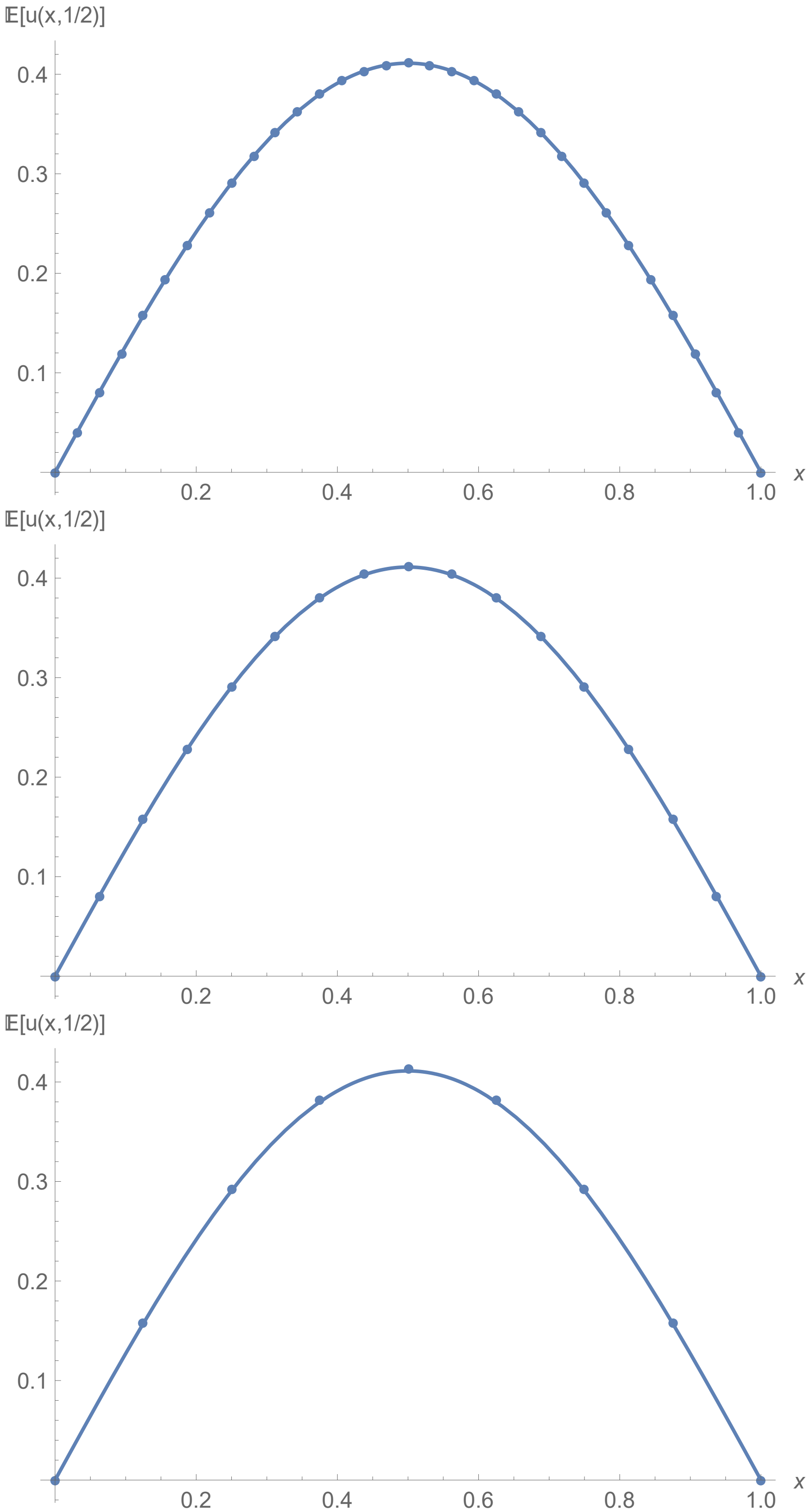

FIGURE 1 Expectation of analytical solution (solid lines) and expectation of numerical solutions (blue points) for different meshes at $t = 1/2$. Top: $M = 32$; Middle: $M = 16$; Bottom: $M = 8$, for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(1, 3)$.

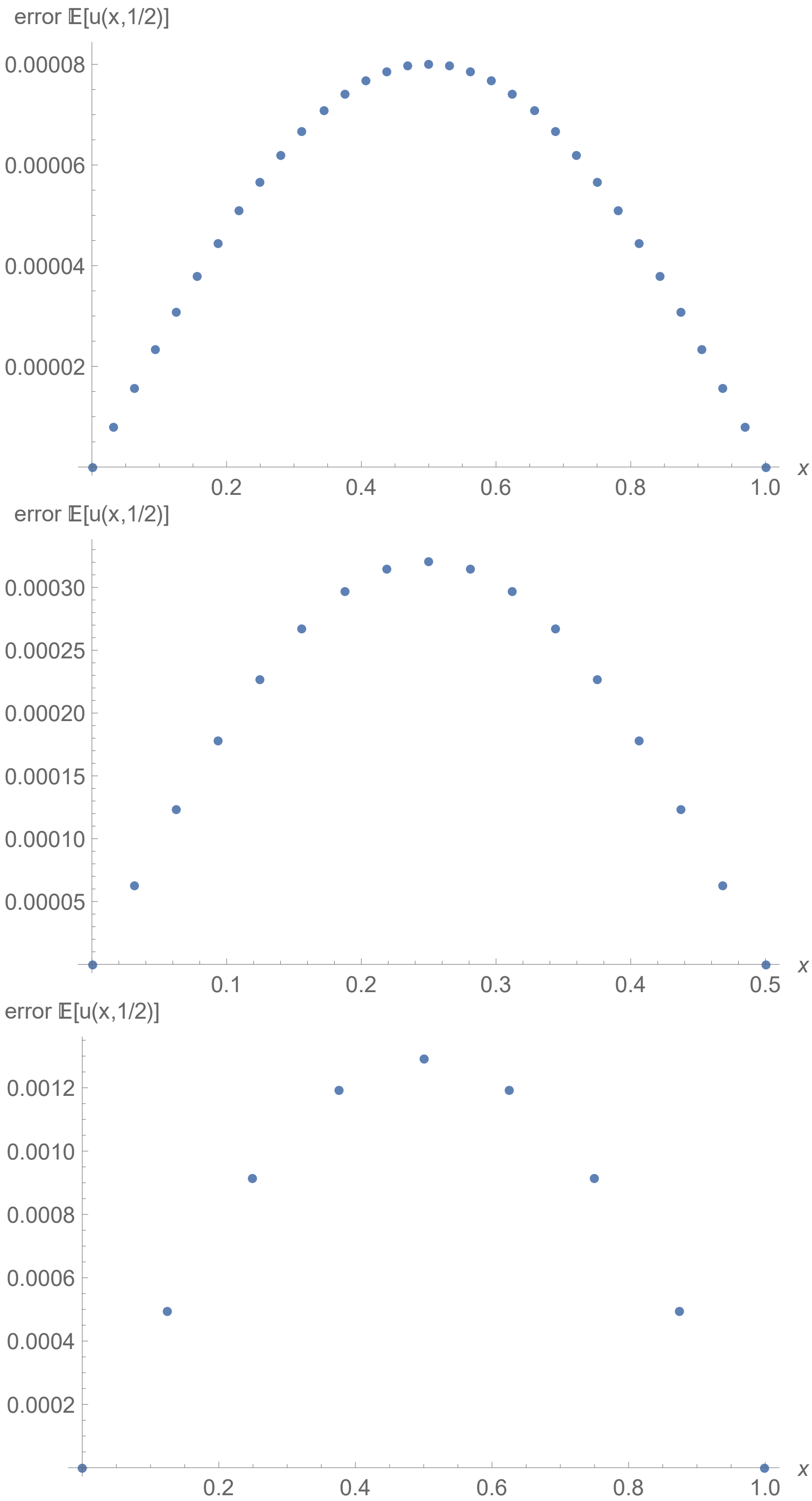

FIGURE 2 Absolute error between expectation of analytical solution and expectation of numerical solutions for different meshes at $t = 1/2$. Top: $M = 32$; Middle: $M = 16$; Bottom: $M = 8$, for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(1, 3)$.

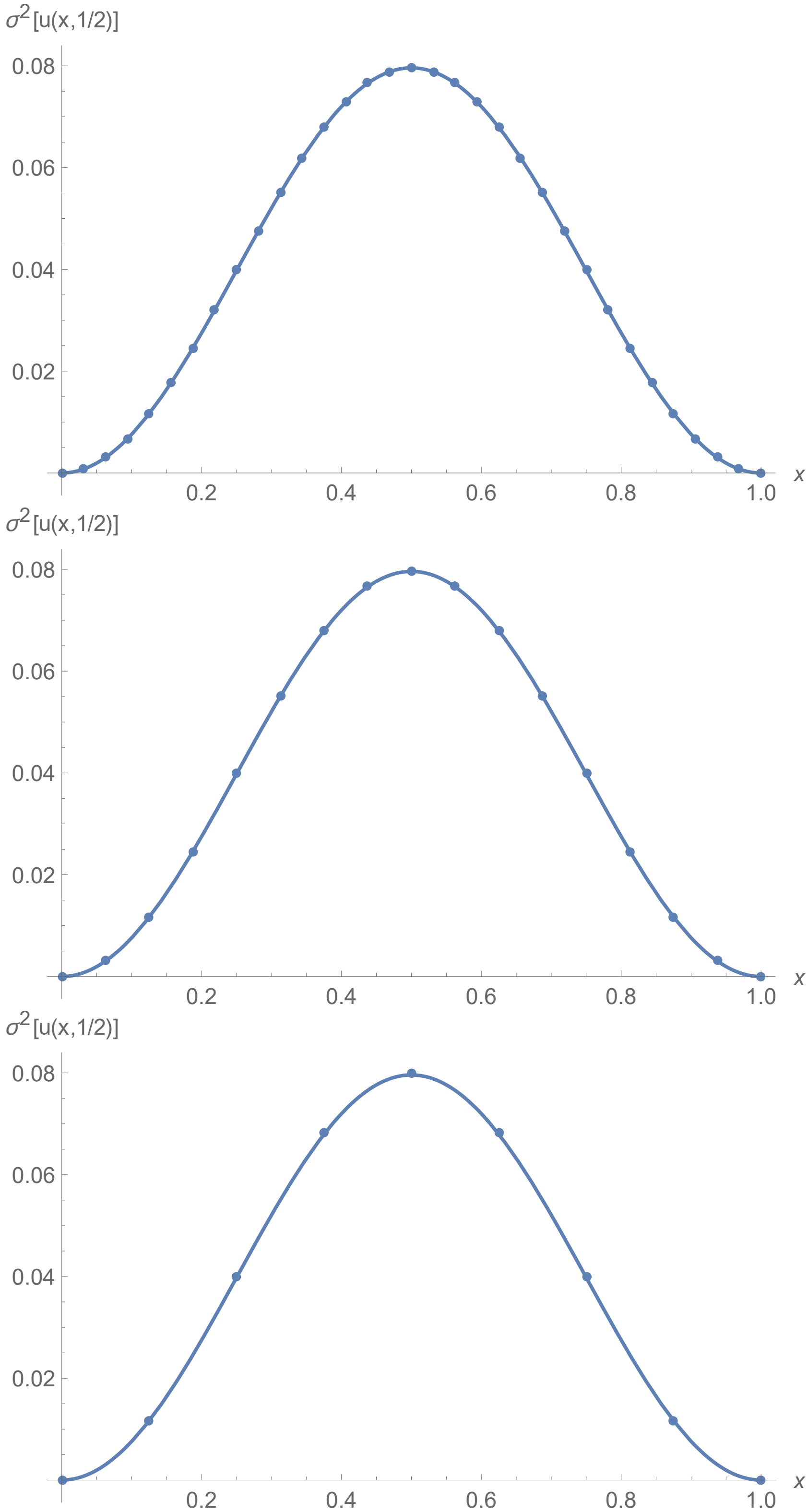

FIGURE 3 Variance of analytical solution (solid lines) and variance of numerical solutions (blue points) for different meshes at $t = 1/2$. Top: $M = 32$; Middle: $M = 16$; Bottom: $M = 8$, for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(1,3)$.

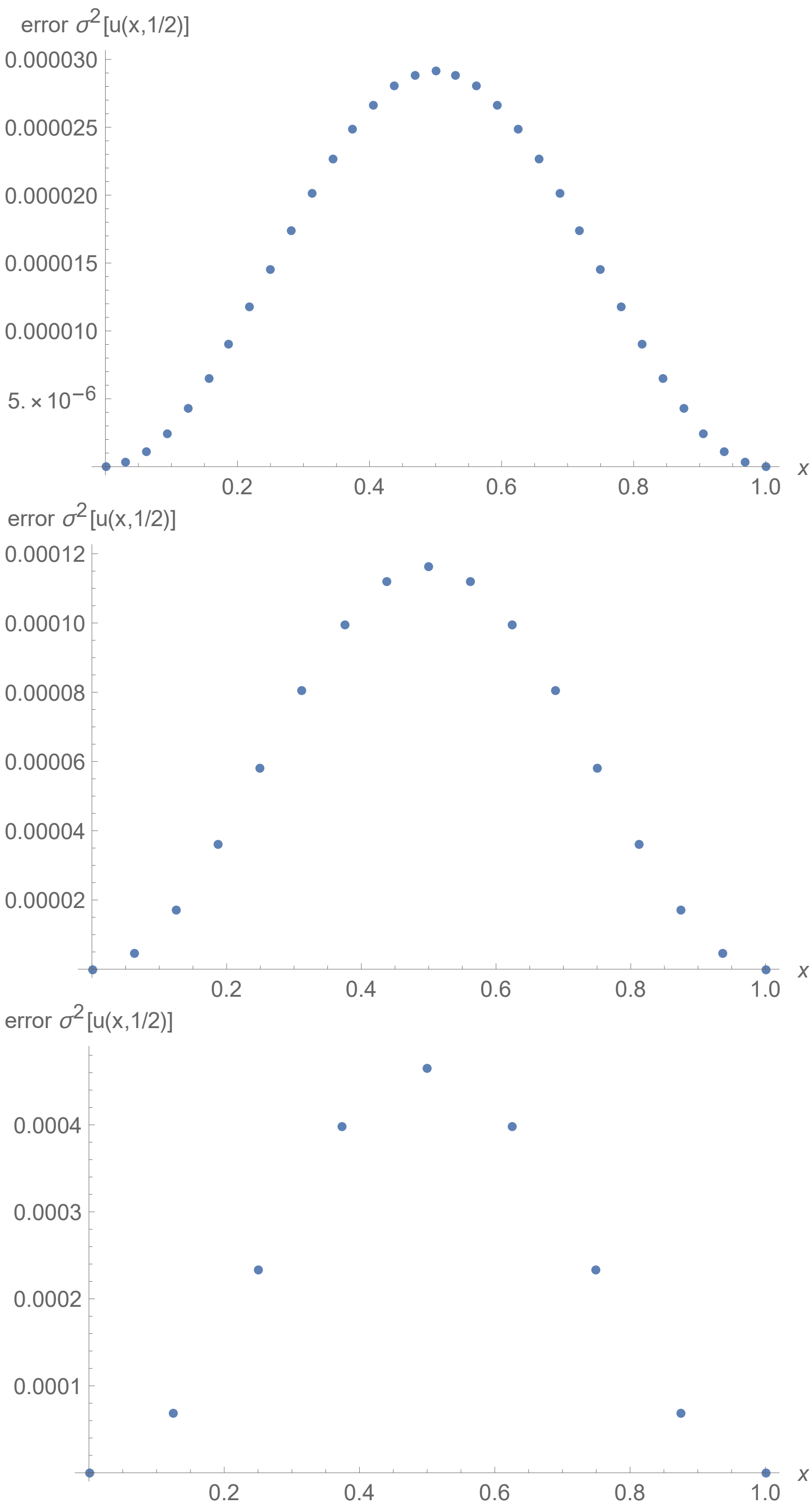

FIGURE 4 Absolute error between variance of analytical solution and variance of numerical solutions for different meshes at $t = 1/2$. Top: $M = 32$; Middle: $M = 16$; Bottom: $M = 8$, for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(1, 3)$.

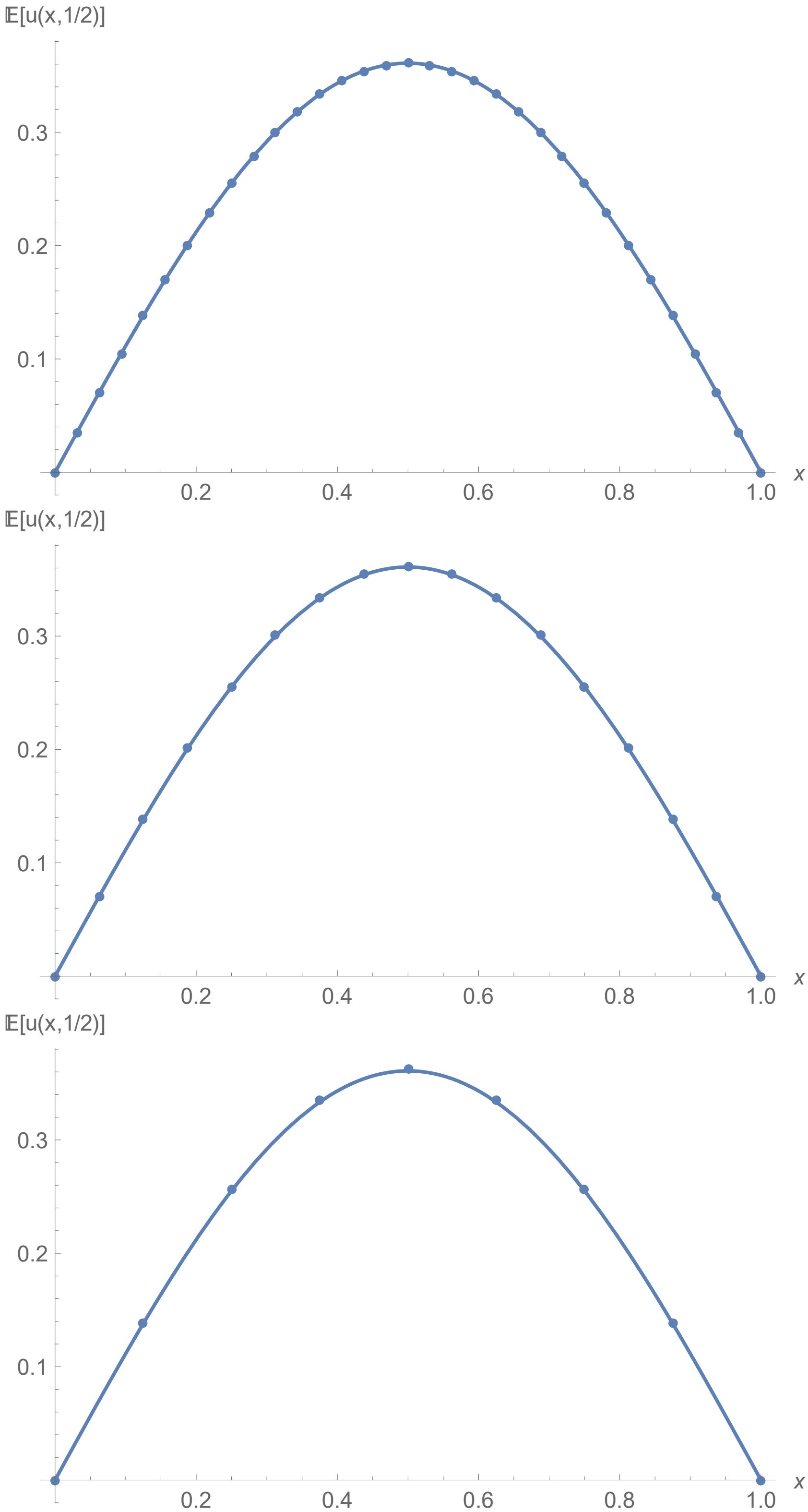

FIGURE 5 Expectation of analytical solution (solid lines) and expectation of numerical solutions (blue points) for different meshes at $t = 1/2$. Top: $M = 32$; Middle: $M = 16$; Bottom: $M = 8$, for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(2, 6)$.

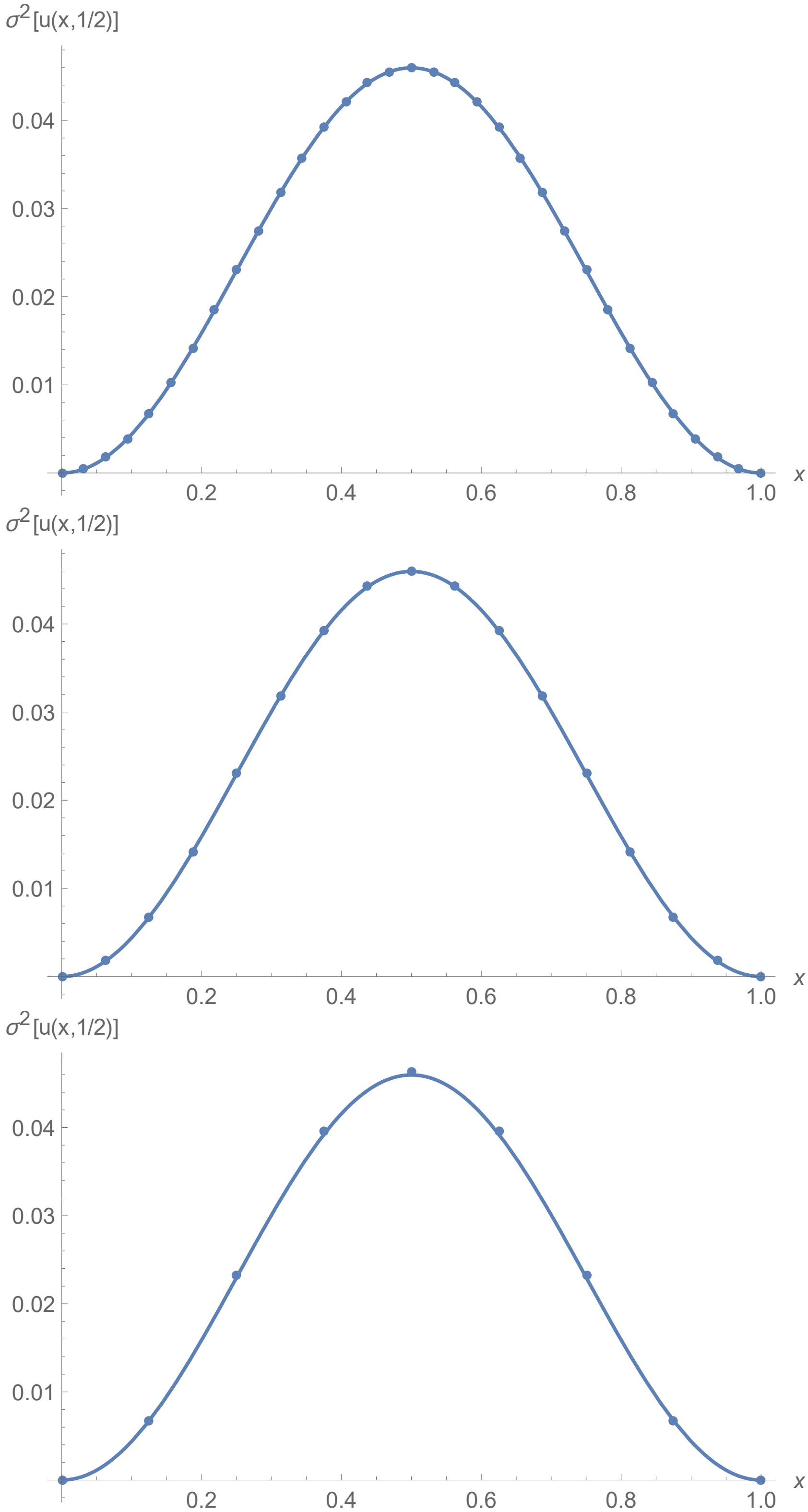

FIGURE 6 Variance of analytical solution (solid lines) and variance of numerical solutions (blue points) for different meshes at $t = 1/2$. Top: $M = 32$; Middle: $M = 16$; Bottom: $M = 8$, for the IBVP (1)–(3) with initial condition (17) and being $\beta \sim \mathrm{Be}(2, 6)$.