Noname manuscript No.
(will be inserted by the editor)

The Railway Line Frequency and Size Setting Problem

Alicia De-Los-Santos · Gilbert Laporte ·
Juan A. Mesa · Federico Perea

Received: date / Accepted: date

Abstract The problem studied in this paper takes as input data a set of lines forming a railway network, and an origin-destination (OD) matrix. The OD pairs may use either the railway network or an alternative transportation mode. The objective is to determine the frequency/headway of each line as well as its number of carriages, so that the net profit of the railway network is maximized. We propose a mixed integer non-linear programming program for this problem. Because of the computational intractability of this model, we develop four algorithms: a mixed integer linear programming (MIP) model, a MIP-based iterative algorithm, a shortest-path based algorithm, and a local search. These four algorithms are tested and compared over a set of randomly generated instances. An application over a case study shows that only the local search heuristic is capable of dealing with large instances.

Keywords Railway line planning · Mathematical programming · Heuristics

1 Introduction

The sequential railway planning process is based on the knowledge of travel patterns, and mainly consists of four stages: network design, line planning,

A. De-Los-Santos
Departamento de Estadística, Econometría, Investigación Operativa, Organización de Empresas y Economía Aplicada, Universidad de Córdoba, Spain, E-mail: aliciasantos@uco.es

J.A. Mesa
Departamento de Matemática Aplicada II, Universidad de Sevilla, Spain, E-mail: jmesa@us.es

Gilbert Laporte
CIRRELT and Canada Research Chair in Distribution Management, HEC Montréal, Canada, E-mail: gilbert.laporte@cirrelt.ca

Federico Perea (Corresponding Author)
Instituto Tecnológico de Informática, Universitat Politècnica de València, Spain, E-mail: perea@eio.upv.es

timetabling, and vehicle and crew scheduling [8]. Network design consists of choosing, possibly from an underlying network, the stations and the lines connecting them. Line planning aims at selecting, for each line, the two terminal stations, the itinerary, the size of the trains, and the frequency of services [12]. Rapid transit systems have characteristics belonging to both railway networks and public transit ([2], [4]). In public transit, and particularly in rapid transit, the first step of the line planning process is to determine the locations of the lines, which implies the construction of the infrastructure (stations and tracks). The frequencies of trains are chosen in the second step. These two problems are often solved jointly but, since the demand is time-dependent and elastic, the second problem also needs to be solved for the different periods of the day, days of the week, seasons, or whenever the demand changes. Because of these demand changes, it is sometimes necessary to modify the frequency and the composition of the trains. In this paper we will assume service regularity, which is an important quality characteristic of transit systems [9]. Frequency and fleet size settings are two intertwined problems, and their joint resolution seems reasonable. [1] discusses how making headways flexible and adapting capacity of the lines makes railway networks more attractive.

The objective functions in line planning can be grouped into two classes: customer oriented and cost oriented [13]. The objectives of the first class are based on the number of direct travelers, the traveling and riding time, and the number of transfers. Cost oriented objectives include fixed and variable costs. However, for operating companies costs are not the only factor to consider since minimum requirements about mobility are imposed by governments. Moreover, in recent years we have witnessed an increasing concern for sustainability in transport planning. Thus, the relationship between cost and revenues becomes relevant in decisions regarding transportation projects. One way of evaluating this relationship is the net profit, defined as the difference between the revenue and the total cost. Moreover, since revenue depends on ridership, maximizing profit usually contributes to increasing the ridership, which is one of the most common efficiency indicators in transportation networks [14]. The demand is given by an origin-destination (OD) matrix, and we also assume that a competing mode is available. This is one of the features that distinguishes our work from most of the line planning papers. Indeed, we do not assume that the demand is captive, but that it depends on the utility of the railway network relative to that of the competing modes. In other words, ridership depends on the level of quality of the service offered. This competition between transportation modes has previously been addressed. For example, [6] design infrastructure railway networks competing with other transportation modes, and [10] locate new stations on a railway network, also competing with a road mode.

The revenue depends on the number of trips captured by the railway systems, which is a function of the frequencies of the lines through the utility function of the railway and its comparison with that of the alternative mode. The cost depends both on the frequency of the lines and on the composition of the trains. In general, the higher the frequency the lower the number of carriages of the trains, but this relationship between both variables is not linear. Therefore, in order to evaluate the profit we have to consider both the frequencies and the number of carriages of the trains as variables. This is one of the reasons why it is interesting to jointly optimize the frequency and size of trains.

In this paper we consider that a set of lines is given. The objective is to assign train frequencies as well as the number of carriages of each line, so that the net profit of the operator is maximized. In the net profit we consider a number of terms, such as ticket fares, locomotive and carriage acquisition costs, crew costs, etc. The set of feasible frequencies/headways is given. We also consider that the number of carriages that a train can have is unlimited, as often railway lines are planned to cover all the demand forecasted in a usual day or period of time, even at peak times. This way, no congestion is possible. This problem is called the uncapacitated frequency and size setting problem (UFSP). We here assume that all passengers of the same OD pair follow the same path. Assuming passengers may take other routes to reach their destination is an even more complex problem, recently treated in [11].

The remaining of this paper is structured as follows. Section 2 formally describes our line frequency and size setting problem. Section 3 models the problem as a mixed integer non-linear program (MINLP), which is intractable even for small instances. In Section 4, four approaches to more efficiently solve this problem are proposed: a unique MIP model, a sequence of smaller MIP models, a heuristic algorithm based on shortest paths, and a local search heuristic (HLSA). All of them are tested and compared in Section 5, including a case study. This is followed by conclusions in Section 6.

2 Description of the problem

In this section, we present the input data needed to define the uncapacitated frequency and size setting problem.

- We are given a set of connected lines $\mathcal{L} = \{\ell_1, \ldots, \ell_{|\mathcal{L}|}\}$ in the railway network RN. Let $N = \{v_1, \ldots, v_n\}$ be the set of nodes corresponding to the stations that constitute the lines in $\mathcal{L}$. For the sake of readability, node v_i will be denoted by its subscript i whenever this creates no confusion.
- Let $A \subset \{(i,j) : i,j \in N, i \neq j\}$ be a (directed) arc set, and let $E = \{\{i,j\} : (i,j) \in A$, or $(j,i) \in A\}$ be the set of (undirected) edges defined from A.
- From these sets, we describe a RN as triplet $(N, E, \mathcal{L})$. In this paper we assume that (N, E) is a connected graph.
- Each line $\ell \in \mathcal{L}$ can be represented by a set of arcs $\{(i_1, i_2), (i_2, i_3), \ldots, (i_{n_\ell - 1}, i_{n_\ell})\}$, where i_1, i_{n_ℓ} are the terminal stations of the line, and $\{i_1, i_2, i_3, ..., i_{n_\ell}\}$ and $\{i_{n_\ell}, i_{n_\ell - 1}, \ldots, i_1\}$ define the two directed maximal paths of this line, n_ℓ being the number of stations of line ℓ.
- Let d_{ij} be the length of arc $(i,j) \in A$. The parameter d_{ij} can also represent the time needed to traverse arc (i,j), transforming distances in times by

means of the parameter λ_ℓ, which represents the average distance traveled by a train of line ℓ in a hour (commercial speed of the trains of this line). We assume the same value of λ_ℓ for all trains of line ℓ. We consider a parameter ν_ℓ representing the cycle time of line ℓ, measured as the time needed for a train of line ℓ to go from the initial station to the final station and returning to the initial station. Thus, $\nu_\ell = 2 \cdot L_\ell / \lambda_\ell$, where L_ℓ is the length of line ℓ.
- Let $\Delta_i^{\ell\ell'}$ be the average time spent by a traveler transferring at station i from line ℓ to line ℓ'.
- Let $W = \{w^1, \dots, w^{|W|}\} \subseteq N \times N$ be the set of ordered OD pairs, $w = (w_s, w_t)$. For each OD pair $w \in W$, g_w is the expected number of passengers per hour for an average day and u_w^{ALT} is the travel time associated to w using the alternative transportation mode, respectively.
- The ticket fare plus the passenger subsidy (the price that the government pays to the operator company for each trip) constitute the revenue per trip, and is denoted by η. The total number of hours that a train is operating per year is denoted by ρ. The analysis will be done assuming a time horizon of $\hat{\rho}$ years.
- The cost of operating one locomotive is c_{loc}, and the cost of operating one carriage is c_{carr}, both per unit of length. The crew cost per train and year is also given, and denoted by c_{crew}.
- The purchase cost of one locomotive is I_{loc}, and that of one carriage is I_{carr}. We consider a minimum number $y^{\min}$ of carriages for each train.
- The capacity of a carriage is given by parameter Θ, measured in number of passengers.
- A finite set of possible headways $\mathcal{H} \subset \mathcal{Z}^+$ for lines of the RN is given, measured in minutes.
- We assume that all costs refer to the same planning period and are discounted to the beginning of the time horizon.

3 A mixed integer non-linear programming model

In this section we model the problem as a mixed integer non-linear program (MINLP). The following decision variables are needed:

- $x_\ell \in \mathcal{H}$ is an integer variable representing the headway of line ℓ (time between two successive services, expressed in minutes). Here we note the correspondence between headway and frequency (number of services per unit of time). For this reason, along the paper we use both headway and frequency, depending on which is the most convenient.
- $y_\ell \geq y^{\min}$ and integer is the number of carriages used by each train of line ℓ.
- $u_w^{RN} > 0$ represents the travel time of pair w using the RN.
- $p_w^{RN} \in [0, 1]$ is the proportion of passengers of OD pair w using the RN, which depends on the travel time using the RN (variable u_w^{RN}) and on the travel time using the alternative mode (parameter u_w^{ALT}).

- $f_{ij}^{w\ell} = 1$ if the OD pair w traverses arc $(i,j) \in A$ using line ℓ, 0 otherwise. Note that these variables are set to zero whenever $(i,j) \notin \ell$, to reduce the size of the problem.
- $t_k^{w\ell\ell'} = 1$ if OD pair w transfers at station k from line ℓ to line ℓ', 0 otherwise. Note that these variables are set to zero whenever k does not belong to the two lines, or when k is the origin or destination of pair w, in order to reduce the size of the problem.
- $B_\ell \geq 0$ and integer is the required fleet of line ℓ, measured in number of trains.

Some of these variables can be explicitly described:

$$p_w^{RN} = \frac{1}{1 + e^{(\alpha - \beta(u_w^{ALT} - u_w^{RN}))}},\ w \in W, \tag{1}$$

$$u_w^{RN} = \sum_{\ell \in \mathcal{L}} \sum_{j:\{w_s,j\} \in \ell} \frac{x_\ell f_{w_s j}^{w\ell}}{2} + \sum_{\ell \in \mathcal{L}} \sum_{\{i,j\} \in \ell} f_{ij}^{w\ell} d_{ij} (60/\lambda_\ell)$$
$$+ \sum_{\ell \in \mathcal{L}} \sum_{\ell':\ell' \neq \ell} \sum_{k \in \ell \cap \ell'} t_k^{w\ell\ell'} (\frac{x_{\ell'}}{2} + \Delta_k^{\ell\ell'}),\ w = (w_s, w_t) \in W, \tag{2}$$

$$B_\ell = \lceil 120 \cdot L_\ell / x_\ell \lambda_\ell \rceil,\ \ell \in \mathcal{L}. \tag{3}$$

Equation (1) represents the modal split, which uses the travel time described in (2). The modal split is described through a logit function, with α, β two positive parameters that need to be calibrated. This function assumes that if the travel time using the RN is much higher than the travel time using the alternative mode, then the proportion of users who choose the RN is close to zero. On the other hand, this proportion is close to one if the travel time using the RN is much lower than the travel time using the alternative mode. This is depicted in Figure 1. Equation (2) defines the travel time of OD pair w using the RN, which is the sum of the average waiting time at the origin station, the in-vehicle time, and the average transfer time between lines. Note that since we assume that passengers arrive at stations following uniform distributions, the average waiting time at the origin station is half of the time between the services (the headway). Equation (3) defines the required fleet as a function of the headway, for each line, where $\lceil \cdot \rceil$ is the ceiling function.

The objective is the maximization of the net profit z_{NET}, defined as

$$\begin{aligned} \text{Maximize} \Big[& \rho\hat{\rho}\eta \sum_{w \in W} g_w p_w^{RN} \\ & - \rho\hat{\rho} \sum_{\ell \in \mathcal{L}} \lambda_\ell B_\ell (c_{loc} + y_\ell \cdot c_{carr}) \\ & - \sum_{\ell \in \mathcal{L}} B_\ell (I_{loc} + I_{carr} \cdot y_\ell) \\ & - \hat{\rho}\, c_{crew} \sum_{\ell \in \mathcal{L}} B_\ell \Big]. \end{aligned} \tag{4}$$

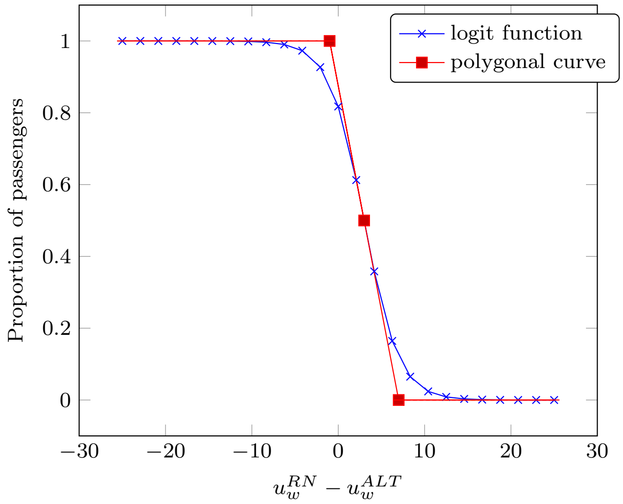

Fig. 1 Representation of the logit function and a polygonal curve to approximate it by a piecewise linear function, for $u_w^{ALT} = 3$, $\alpha = 0$ and $\beta = 0.5$. On the horizontal axis, the difference between the travel time using the RN and the travel time using the alternative mode. On the vertical axis, the proportion of users taking the RN.

The first term in (4) is the revenue z_{REV}, which depends on the number of passengers using the RN, times the number of years in the planning horizon, the hours operating per year, and the benefit of the operator (ticket fare plus subsidy) in the RN. The second term computes the rolling stock cost: the cost of operating the trains, which depends on the number of trains and on the number of carriages. The last two terms are the fleet acquisition cost and the crew operating cost, respectively.

The constraints of the problem are

$$t_k^{w\ell\ell'} \geq \sum_{j:(k,j)\in\ell} f_{kj}^{w\ell} + \sum_{i:(i,k)\in\ell'} f_{ik}^{w\ell'} - 1,\ w \in W, \ell \neq \ell' \in \mathcal{L}, k \in \ell \cap \ell',\ k \neq w_s, w_t, \tag{5}$$

$$\sum_{\ell\in\mathcal{L}} \sum_{i:(i,k)\in\ell} f_{ik}^{w\ell} - \sum_{\ell\in\mathcal{L}} \sum_{j:(k,j)\in\ell} f_{kj}^{w\ell} = \begin{cases} 0,\ k \in N \setminus \{w_s, w_t\}, \\ -1,\ k = w_s, \\ +1,\ k = w_t, \end{cases} \tag{6}$$

$$x_\ell \sum_{w\in W} g_w p_w^{RN} f_{ij}^{w\ell} \leq 60 \cdot \Theta \cdot y_\ell,\ \ \ell \in \mathcal{L}, \{i,j\} \in E. \tag{7}$$

Constraints (5) ensure that if an OD pair w enters station $k \in N$ using a line, and exits from k using another line, then a transfer takes place (variable $t_k^{w\ell\ell'} = 1$). Constraints (6) are the flow conservation constraints to find a path for every OD pair. Constraints (7) impose an upper bound on the maximum number of passengers that each line can carry per hour, which depends on the number of carriages and on the headway of this line.

The maximization of (4), subject to constraints (1)–(3) and (5)–(7), is an MINLP. Unfortunately, this model is intractable even in small instances. The non-linearities of this model will be specified, as well as some ways to avoid them, in Section 4, in order to have a more efficient model.

4 Algorithms

In this section we introduce several approaches to more efficiently solve our problem. The first one linearizes the MINLP program of Section 3. Because the number of new variables and constraints needed for such linearization is enormous, an iterative process based on solving a sequence of smaller linear models is proposed. Two other algorithms that do not rely on mathematical programming will be proposed: one based on shortest paths, and a local search heuristic.

4.1 MIP model

The MINLP introduced in Section 3 contains several non-linearities. In the following, we describe these non-linearities as well as the way in which they are linearized, in order to propose a unique MIP model to solve our problem.

1. The proportion of passengers using the RN defined by constraints (1), uses the non-linear logit function. We avoid this non-linearity by approximating the logit function by a piecewise linear function made up of three pieces. Let z be the variable u_w^{RN} representing the travel time in the RN and let $F(z) = 1/(1+\exp(\alpha - \beta(u_w^{ALT} - z))$ be the corresponding logit function for z. The piecewise linear function is defined as

$$\mathcal{P}(z) := \begin{cases} 1, & z < u_w^{ALT} - 2/\beta, \\ -\beta/4z + (2+\beta u_w^{ALT})/4, & z \in [u_w^{ALT} - 2/\beta, u_w^{ALT} + 2/\beta], \\ 0, & z \geq u_w^{ALT} + 2/\beta. \end{cases} \quad (8)$$

   Figure 1 depicts this approximation. The reader may note that increasing the number of pieces improves the approximation, but this would make the model more difficult to solve. We chose three intervals since preliminary tests showed a good trade-off between the efficiency of the model and the quality of the solution returned.
2. In constraints (7), the binary variable $f_{ij}^{w\ell}$ multiplies the continuous variable p_w^{RN}. This product can easily be linearized by defining a new set of positive variables $q_{ij}^{w\ell}$ to represent it, and by adding the following set of constraints:

$$q_{ij}^{w\ell} \leq f_{ij}^{w\ell}, \ell \in \mathcal{L}, (i,j) \in \ell, w \in W, \quad (9)$$

$$p_w^{RN} - (1 - f_{ij}^{w\ell}) \leq q_{ij}^{w\ell},\ \ell \in \mathcal{L}, (i,j) \in \ell, w \in W, \quad (10)$$

$$q_{ij}^{w\ell} \leq p_w^{RN},\ \ell \in \mathcal{L}, (i,j) \in \ell, w \in W. \quad (11)$$

Note that constraints (9) and (11) ensure that if $f_{ij}^{w\ell} = 0$ or $p_w^{RN} = 0$, then $q_{ij}^{w\ell}$ is also zero. From (10), if $f_{ij}^{w\ell} = 1$ and $p_w^{RN} = 1$, then $q_{ij}^{w\ell} = 1$.
So, the capacity constraint is redefined as:

$$\sum_{w \in W} g_w x_\ell q_{ij}^{w\ell} \leq 60 \cdot \Theta \cdot y_\ell.$$

We now express the headway $x_\ell \in \mathbb{Z}^+$ as a convex combination of binary variables x_ℓ^k, which take value one if the headway of line ℓ is the k-th in the set of feasible headways $\mathcal{H}$, denoted by H^k:

$$x_\ell = \sum_{k=1}^{|\mathcal{H}|} H^k \cdot x_\ell^k,\ \ell \in \mathcal{L}, \tag{12}$$

$$\sum_{k=1}^{|\mathcal{H}|} x_\ell^k = 1,\ \ell \in \mathcal{L}. \tag{13}$$

Thus, we define new variables $r_{ij}^{w\ell k}$ representing the products $q_{ij}^{w\ell} x_\ell^k$, which are linearized as follows:

$$r_{ij}^{w\ell k} \leq q_{ij}^{w\ell}, \ell \in \mathcal{L},\ (i,j) \in \ell, w \in W, k \in \{1, \ldots, |\mathcal{H}|\}, \tag{14}$$

$$r_{ij}^{w\ell k} \leq x_\ell^k, \ell \in \mathcal{L},\ (i,j) \in \ell, w \in W, k \in \{1, \ldots, |\mathcal{H}|\}, \tag{15}$$

$$q_{ij}^{w\ell} - (1 - x_\ell^k) \leq r_{ij}^{w\ell k},\ \ell \in \mathcal{L}, (i,j) \in \ell, w \in W, k \in \{1, \ldots, |\mathcal{H}|\}. \tag{16}$$

Therefore, the capacity constraints (7) are linearly expressed as:

$$\sum_{w \in W} g_w \sum_{k=1}^{|\mathcal{H}|} r_{ij}^{w\ell k} \cdot H^k \leq 60 \cdot \Theta \cdot y_\ell,\ \ell \in \mathcal{L}, (i,j) \in \ell.$$

3. In the definition of the travel time (2), we need to linearize the terms $x_\ell \cdot f_{w_s j}^{w\ell}$ and $t_k^{w\ell\ell'} \cdot x_{\ell'}$. To this end, we include two new integer variables $z_{w_s j}^{w\ell}$ and $\bar{z}_k^{w\ell\ell'}$ representing both products, respectively. Thus, we also need to add the following new constraints:

$$z_{w_s j}^{w\ell} \leq \mathcal{H}^{\max} f_{w_s j}^{w\ell},\ \ell \in \mathcal{L}, w = (w_s, w_t) \in W, (w_s, j) \in \ell, \tag{17}$$

$$z_{w_s j}^{w\ell} \leq x_\ell,\ \ell \in \mathcal{L}, w = (w_s, w_t) \in W, (w_s, j) \in \ell, \tag{18}$$

$$x_\ell - \mathcal{H}^{\max}(1 - f_{w_s j}^{w\ell}) \leq z_{w_s j}^{w\ell},\ \ell \in \mathcal{L}, w = (w_s, w_t) \in W, (w_s, j) \in \ell, \tag{19}$$

$$\bar{z}_k^{w\ell\ell'} \leq \mathcal{H}^{\max} \cdot t_k^{w\ell\ell'},\ \ell, \ell' \in \mathcal{L}, w \in W, k \in N, \tag{20}$$

$$\bar{z}_k^{w\ell\ell'} \leq x_\ell,\ \ell, \ell' \in \mathcal{L}, w \in W, k \in N, \tag{21}$$

$$x_{\ell'} - \mathcal{H}^{\max}(1 - t_k^{w\ell\ell'}) \leq \bar{z}_k^{w\ell\ell'},\ \ell, \ell' \in \mathcal{L}, w \in W, k \in N, \tag{22}$$

where $\mathcal{H}^{\max} = \max_{k=1,\ldots,|\mathcal{H}|} H^k$.

4. In the constraint (3) that defines the required fleet, since B_ℓ is an integer variable, after some algebra we linearize the ceiling function as:

$$120 \cdot L_\ell/\lambda_\ell \leq x_\ell \cdot B_\ell \leq 120 \cdot L_\ell/\lambda_\ell + x_\ell.$$

Since $x_\ell = \sum_{k=1}^{|\mathcal{H}|} x_\ell^k \cdot H^k$, we can describe $x_\ell \cdot B_\ell = \sum_{k=1}^{|\mathcal{H}|} x_\ell^k B_\ell \cdot H^k$. Each term of this sum can be linearized by means of integer variables $\sigma_\ell^k = x_\ell^k B_\ell$:

$$\sigma_\ell^k \leq B_\ell,\ \ell \in \mathcal{L}, k \in \{1, \ldots, |\mathcal{H}|\}, \tag{23}$$

$$\sigma_\ell^k \leq \mathcal{H}^{\max} x_\ell^k,\ \ell \in \mathcal{L}, k \in \{1, \ldots, |\mathcal{H}|\}, \tag{24}$$

$$B_\ell - \mathcal{H}^{\max}(1 - x_\ell^k) \leq \sigma_\ell^k,\ \ell \in \mathcal{L}, k \in \{1, \ldots, |\mathcal{H}|\}. \tag{25}$$

5. In the objective function (4) the product of two integer variables, $B_\ell \cdot y_\ell$, appears twice. To solve these non-linearities we define new binary variables y_ℓ^k, which take value one if the number of carriages of line ℓ is equal to γ, for $\gamma = \{y^{\min}, \ldots, y^{\max}\}$, $y^{\max}$ being the maximum number of carriages. These variables help describe the number of carriages for each line $y_\ell \in \mathbb{Z}^+$ as a convex combination of binary variables:

$$y_\ell = \sum_{\gamma=y^{\min}}^{y^{\max}} y_\ell^\gamma \cdot \gamma,\ \ell \in \mathcal{L}, \tag{26}$$

$$\sum_{\gamma=1}^{y^{\max}} y_\ell^\gamma = 1,\ \ell \in \mathcal{L}. \tag{27}$$

Thus, we include a new variable $\bar{\sigma}_\ell^\gamma$ representing the product $B_\ell \cdot y_\ell^\gamma$ which is linearized as follows:

$$\bar{\sigma}_\ell^\gamma \leq B_\ell,\ \ell \in \mathcal{L}, \gamma \in \{y^{\min}, \ldots, y^{\max}\}, \tag{28}$$

$$\bar{\sigma}_\ell^\gamma \leq y^{\max} x_\ell^\gamma,\ \ell \in \mathcal{L}, \gamma \in \{y^{\min}, \ldots, y^{\max}\}, \tag{29}$$

$$B_\ell - y^{\max}(1 - y_\ell^\gamma) \leq \bar{\sigma}_\ell^\gamma,\ \ell \in \mathcal{L}, \gamma \in \{y^{\min}, \ldots, y^{\max}\}. \tag{30}$$

The result is a unique MIP that solves our problem. As we will see in the computational experiments section, this model is only able to handle small-size instances. We now propose three more algorithms that aim at more efficiently solve our problem.

4.2 Mixed integer programming-based algorithm

Due to the large number of variables needed to linearize the MINLP model as a unique MIP, we now propose another algorithm in which the headway variables are fixed as parameters. This way we directly avoid several of the non-linearities presented in Section 4.1, a new drawback being that we now have to solve a sequence of MIP models, as many as the possible number headway combinations.

The reader may note that if x_ℓ is a parameter instead of a variable, and we add the linearizations described in constraints (8) to (11) to the MINLP defined in Section 3, the corresponding model $MIP(x_1, ..., x_{|\mathcal{L}|})$, is an integer linear programming model.

The algorithm presented in this section solves $MIP(x_1, ..., x_{|\mathcal{L}|})$, for all feasible combinations of headways $(x_1, ..., x_{|\mathcal{L}|}) \in \mathcal{H}^{|\mathcal{L}|}$, keeping as a final output the best solution found. This solution procedure is described in Algorithm 1.

Data: Input data for the UFSP problem
for *each combination of headways* $(x_1, ..., x_{|\mathcal{L}|})$ **do**
  solve $MIP(x_1, ..., x_{|\mathcal{L}|})$;
end
Result: $\arg \max_{(x_1,...,x_{|\mathcal{L}|})} MIP(x_1, ..., x_{|\mathcal{L}|})$.

Algorithm 1: Pseudo-code for the integer linear programming-based algorithm for the UFSP.

This algorithm is significantly more efficient than the MINLP model presented in Section 3. Nevertheless, it has proven incapable of dealing with medium-large instances, as will be shown in Section 5.

4.3 A shortest path-based algorithm

In this section, we present a heuristic that solves UFSP taking into account the passenger point of view, since all OD pairs are routed via their shortest paths, as opposed to sections 4.1 and 4.2, which route passengers via the more profitable paths for the operator. The idea is to iteratively check all possible combinations of headways as in Section 4.2, and once the headways are fixed, to assign demand in the RN taking into account the shortest path associated with each OD pair. The reader may note that the shortest paths depend on the frequencies. We then compute the number of passengers traveling on each line and on each arc. For each line, the arc with the largest number of passengers defines the minimum capacity that such line should have. Once the minimum required number of carriages for each line has been calculated, we can easily compute the profit of the RN. Algorithm 2 provides the corresponding pseudocode.

The reader may note that this algorithm does not always yield the same solution as the previously proposed approaches. This is because we now consider that all OD pairs will use their shortest path, which is not necessarily the case of a solution maximizing the profit. Consider the following example:

Data: Input data for the UFSP problem
for *each possible combination of headways* **do**
  Compute the shortest path for each OD pair and the number of passengers traveling on each line and arc, by using the logit function and the traveling time by the alternative mode;
  for *each line ℓ* **do**
    Find the arc e_ℓ of ℓ with maximum load;
    Find the minimum number of carriages needed to transport all passengers traversing e_ℓ;
  end
  Compute the profit z_{NET} and keep this solution.
end
Return the solution with the maximum net profit;
Result: The combination of headways and capacities that yield the maximum profit.

Algorithm 2: Pseudo-code for the shortest path-based algorithm.

two nodes, and only one OD-pair between these two nodes, which has three passengers. The ticket fare is 2/3 monetary units, and the cost of moving one carriage is one monetary unit. Each carriage can transport up to two passengers. Assume that all other parameters are non-relevant. The shortest path heuristic would propose as a solution to transport the three passengers, using two carriages, whereas the MIP models would propose to transport two passengers in one carriage, and the third one leaves the RN system to use another transport mode (transporting him/her would give you a revenue of 2/3, but moving the extra carriage you need costs one m.u., so not worth it). These two solutions are different.

4.4 A local search heuristic

The shortest path-based algorithm presented in Section 4.3 fails to solve large-size instances, due to the exponential number of potential frequency combinations to be checked. For this reason, in this section we present another heuristic based on the local search heuristic (HLSA) introduced by [5]. Note that this algorithm does not check all possible headway combinations, and consists of four different phases:

1. Compute the optimal configuration of frequencies assuming all lines have the same frequency.
2. Explore the neighborhood of the solution obtained in the first phase. We consider that two frequency configurations are neighbors if all lines but one have the same frequency, and the difference in the frequencies of the different line is as small as possible. We find the best neighbor, and keep this solution.
3. Perform another local search around the current solution. First, increase the frequency of one line to the next feasible frequency (operation denoted as *mov+*), as long as the solution improves. Then go back to the original solution and start decreasing the frequency of the line to the next feasible

frequency (operation denoted as $mov-$), as long as the solution improves. This is repeated for all for all lines without taking into account the improvement obtained at the previous lines but starting at the output of Step 2, and the best solution among those obtained moving the frequency of each line is kept.
4. Apply a steepest ascent algorithm. First, apply $mov+$ and $mov-$ to the first line. Keep the best solution found and, for this solution, apply $mov+$ and $mov-$ to the second line. Repeat this operation for all lines. The best solution is stored.

Note the difference between phases 3 and 4. Whereas in phase 3 we change the frequencies of the lines independently, and the solution in phase 2 and the solution in phase 3 will only differ in the frequency of one line, after phase 4 the new solution may differ in more than one frequency with respect to the solution after phase 3. The reader should also note that once the frequencies (or equivalently headways) of all lines are known, so are the travel times for OD pairs, and therefore so is the expected proportion of passengers using the RN. Hence, the capacities of each line are easily computed as the minimum number of carriages needed to transport all passengers. A pseudo-code of this algorithm is presented in Algorithm 3.

The following example illustrates this algorithm. Consider two lines ℓ_1, ℓ_2 and four possible frequencies for each line, namely, $\{1, 2, 3, 4\}$. We now describe the four phases of the local search heuristic applied to this example:

1. Compute the net profit of the RN assuming the following frequencies for the two lines: $[1, 1], [2, 2], [3, 3], [4, 4]$. Assume that the best profit is given by configuration $[3, 3]$, that is, both lines have a frequency of three services per hour.
2. In this phase, the neighbors of $[3, 3]$ are $\{[4, 3], [2, 3], [3, 4], [3, 2]\}$. Assume that the configuration $[3, 4]$ yields the highest profit, so we keep this solution, and use it in the next phase.
3. From solution $[3, 4]$, we increase the frequency of the first line (note that $[4, 4]$ does not have to be analyzed again, since it was computed in phase 1). We then analyze $[1, 4]$ (since frequency 5 is not feasible), then $[2, 4]$, and so on, until we stop improving the profit. If at the first iteration of $mov+$ ($[1, 4]$), a better profit is not obtained, we decrease the frequency of the first line in the same way as $mov+$. We do the same for the second line ℓ_2 and find [3,1] as the best solution obtained with movements on this line. Then we choose between [1,4] and [3,1] the frequency combination that has the best profit. Assume this is [3,1]. Then, the best solution in this phase is $[3, 1]$.
4. We now apply $mov+$ or $mov-$ to line ℓ_1 in solution $[3, 1]$ as in phase 3, iteratively while the profit improves. Assume that the best solution found is $[4, 1]$. We now apply $mov+$ or $mov-$ to the second line ℓ_2 in solution $[4, 1]$ while the profit improves. The solution obtained after these steps is the final solution of the algorithm, which could be, for example, $[4, 2]$.

Data: Input data for the UFSP problem
Phase 1: **for** *all possible frequencies* φ **do**
  Compute the value of the solution given by the combination of frequencies $(\varphi, ..., \varphi) \in \mathbb{R}^{|\mathcal{L}|}$;
end
Let $(\varphi^*, ..., \varphi^*) \in \mathbb{R}^{|\mathcal{L}|}$ be the solution with the maximum profit;
Phase 2: **for** $\ell = 1$ *to* $|\mathcal{L}|$ **do**
  Compute the value of the solution in which the frequency of line ℓ is $mov+(\varphi^*)$, and $mov-(\varphi^*)$;
end
Keep the best solution, denoted by $\varphi^* = (\varphi_1^*, ..., \varphi_{|\mathcal{L}|}^*)$;
Phase 3: **for** ℓ=*1 to* $|\mathcal{L}|$ **do**
  Apply $mov+$ to φ_ℓ^*, and compute the value of this solution;
  Keep doing this until the solution stops improving;
  Apply $mov-$ to φ_ℓ^*, and compute the value of this solution;
  Keep doing this until the solution stops improving;
end
Keep the best solution, denoted by $\varphi^* = (\varphi_1^*, ..., \varphi_{|\mathcal{L}|}^*)$;
Phase 4: **for** $\ell = 1$ *to* $|\mathcal{L}|$ **do**
  Modify φ^* by applying $mov+$ to φ_ℓ^* until the solution stops improving;
  Modify φ^* by applying $mov-$ to φ_ℓ^* until the solution stops improving;
  Update φ^* to the best solution found;
end
Result: Solution: frequencies given by $\varphi^* = (\varphi_1^*, ..., \varphi_{|\mathcal{L}|}^*)$ and minimum capacities to transport all passengers attracted, given the line frequencies φ^*.

Algorithm 3: Pseudo-code for the local search heuristic.

5 Computational experiments

We first report a computational experiment conducted over five different topologies, called 6×2, 7×3, 8×3, 15×5, and 20×6, where $n \times m$ stands for a network with n nodes and m lines. These topologies are described in Figures 2 to 6.

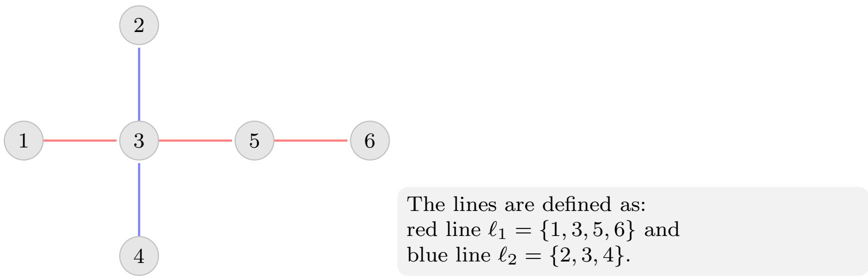

Fig. 2 Representation of a 6×2 configuration.

Ten instances were randomly generated for each of the four configurations, yielding 40 instances. The number of passengers of each OD pair w was

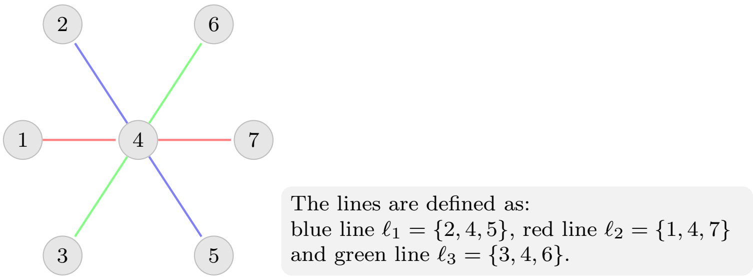

Fig. 3 Representation of a 7×3 configuration.

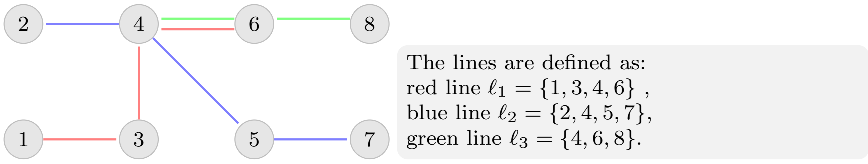

Fig. 4 Representation of a 8×3 configuration.

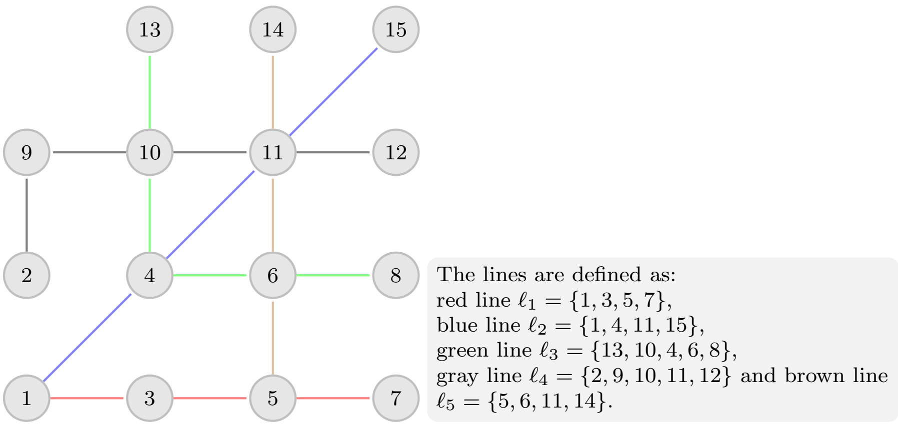

Fig. 5 Representation of a 15×5 configuration.

obtained following a discrete uniform distribution $\mathrm{U}(5a, 15a)$, where a is an integer between 140 and 300, different for each instance. To define each arc length, the coordinates of each station were set randomly by means of another uniform distribution defined over previously fixed cells, so both coordinates were uniformly chosen in $\mathrm{U}(a, b) \times \mathrm{U}(c, d)$, where a, b, c, d are the corners of the corresponding cell. Therefore, the arc lengths differ from one instance to another, since the station locations are also different. The travel times u_w^{ALT} using the alternative mode were obtained by means of the Euclidean distance on the plane and a speed of 20 km/h, whereas the travel times in the RN were obtained according to the Euclidean distances on the graph, assuming a speed of 30 km/h, plus the waiting times at the origin stations, plus the transfer times if any. Costs (both purchase and operations) are based on the specific

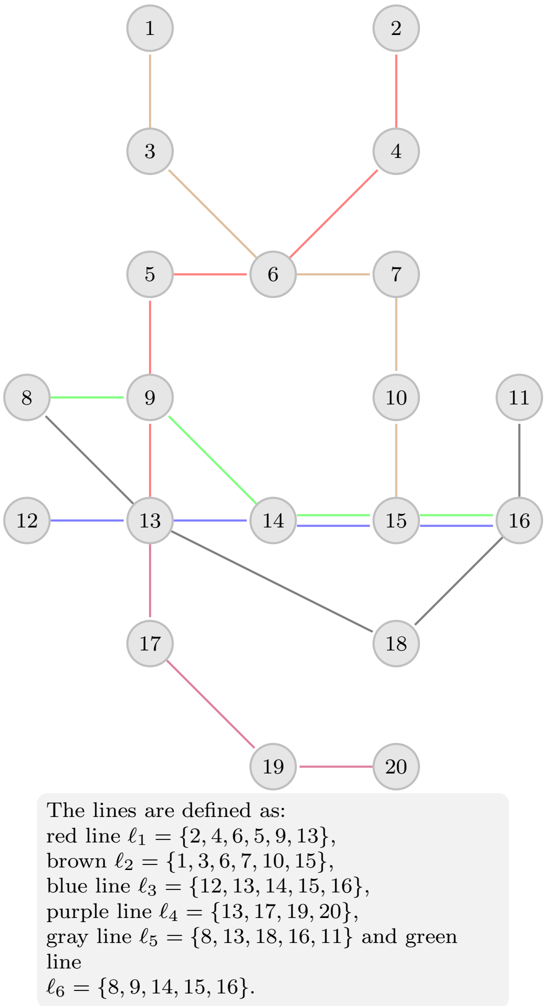

Fig. 6 Representation of a 20×6 configuration.

train model Civia. as in [3]. The parameters of the logit function were set to $\alpha = -0.3$ and $\beta = 1$. (In [7], $\alpha = 0.3$ and $\beta = 1$). Four possible headways were considered for our experiments, namely: $\{5, 10, 15, 20\}$. Table 1 summarizes the values given to the other input parameters. Complete data can be obtained from the authors upon request.

All computations for Algorithm 1 were performed in GAMS/CPLEX, on a computer with 8 GB of RAM memory and 2.8 GHz CPU and four cores.

The complete results of the other four configurations are given in Table 2. Column "Instance" refers to the instance tested, indicating the configuration $n \times m$ as well as the replicate, which is the number after the hyphen. Afterwards, we show the CPU time, in seconds, of the MIP-model, in column Time.

Parameters		
Name	**Description**	**Value**
$\hat{\rho}$	years to recover the purchase	20
ρ	number of operative hours per year	6935
c_{loc}	costs for operating one locomotive per kilometer [€/km]	34
c_{carr}	operating cost of a carriage per kilometer [€/km]	2
c_{crew}	per crew and year for each train [€/ year]	$75 \cdot 10^3$
I_{loc}	purchase cost of one locomotive in €	$2.5 \cdot 10^6$
I_{carr}	purchase cost of one carriage in €	$0.9 \cdot 10^6$
Θ	capacity of each carriage (number of passengers)	$2 \cdot 10^2$
x_ℓ	possible values	{5,10,15,20}
η	Ticket fare + subsidy	3.5

Table 1 Input parameters for the computational experiments.

For the iterated MIP-algorithm, we show the CPU time in column "Time". For the Shortest-Paths algorithm we show the percentage deviation with respect to the best solution found (in all cases this is achieved by the MIP model) and the CPU time (in columns "Gap" and "Time"), and the same information is shown for the HLSA algorithm. We note that the gap is computed as:

$$gap = 100\frac{Value_{MIP} - Value_{alg}}{Value_{MIP}},$$

where $Value_{MIP}$ is the optimal value found by the MIP model, and $Value_{alg}$ is the value of the solution returned by the corresponding algorithm.

These results are summarized by configuration in Table 3. In this table we also show the number of possible headway combinations in column "Number", as an indication of how difficult it would be for the MIP-iterative and shortest path algorithm to check all such possible headway combinations. Interesting aspects to note are the following:

- Although both algorithms yield the same solutions, solving the MIP model is much more efficient than applying the MIP iterative algorithm. As a matter of fact, the MIP iterative approach fails to solve the instances corresponding to the largest configuration within the given time limit of one hour. On average, only 45 combinations out of the 1024 potential ones were solved within this time limit. On the other hand, the MIP model solves these instances to optimality in 1587.08 seconds on average.
- The algorithms that do not rely on mathematical programming, namely shortest path algorithm and HLSA, obtain solutions much more efficiently than the other two. The gaps of these two algorithms are quite small. For the smallest sets of instances, they both yield the optimal solutions. Only in the 8×3 (HLSA) and 20×6 (HLSA and shortest path algorithm) configurations they present strictly positive gaps. Even in such cases, these gaps are quite small, on average 0.89 % and 1.36% for the shortest path algorithm and for the HLSA, respectively. Out of these two algorithm, the HLSA is by far the most efficient.

Another interesting aspect of the MIP-based algorithm is how close are the optimal travel times of the OD pairs using the railway network, with respect to

Table 2 Complete computational results over random instances.

	MIP-Model	MIP-Iterative	Shortest-Paths		HLSA	
Instance	Time	Time	Gap	Time	Gap	Time
6 × 2-01	0.27	2.35	0.00	0.04	0.00	0.03
6 × 2-02	0.27	1.94	0.00	0.02	0.00	0.01
6 × 2-03	0.25	1.96	0.00	0.01	0.00	0.01
6 × 2-04	0.24	1.88	0.00	0.02	0.00	0.01
6 × 2-05	0.26	1.88	0.00	0.01	0.00	0.01
6 × 2-06	0.28	1.85	0.00	0.05	0.00	0.01
6 × 2-07	0.26	2.65	0.00	0.01	0.00	0.01
6 × 2-08	0.27	2.27	0.00	0.01	0.00	0.01
6 × 2-09	0.33	1.93	0.00	0.01	0.00	0.01
6 × 2-10	0.29	2.34	0.00	0.01	0.00	0.00
7 × 3-01	0.40	11.92	0.00	0.90	0.00	0.09
7 × 3-02	0.38	10.33	0.00	0.22	0.00	0.03
7 × 3-03	0.38	10.36	0.00	0.23	0.00	0.07
7 × 3-04	0.38	10.53	0.00	0.88	0.00	0.03
7 × 3-05	0.42	10.59	0.00	0.24	0.00	0.03
7 × 3-06	0.45	10.45	0.00	0.20	0.00	0.03
7 × 3-07	0.37	10.41	0.00	0.87	0.00	0.03
7 × 3-08	0.39	11.60	0.00	0.19	0.00	0.03
7 × 3-09	0.45	10.52	0.00	0.22	0.00	0.03
7 × 3-10	0.42	11.60	0.00	0.19	0.00	0.03
8 × 3-01	0.72	18.88	0.00	0.47	4.52	0.06
8 × 3-02	0.69	17.41	0.00	0.11	1.87	0.02
8 × 3-03	0.82	18.09	0.00	0.12	6.71	0.03
8 × 3-04	0.95	18.44	0.00	0.11	3.11	0.02
8 × 3-05	0.73	18.10	0.00	0.11	0.00	0.02
8 × 3-06	0.74	17.67	0.00	0.11	0.00	0.02
8 × 3-07	0.76	17.93	0.00	0.11	0.00	0.02
8 × 3-08	0.77	17.52	0.00	0.11	7.10	0.02
8 × 3-09	0.69	17.91	0.00	0.11	0.00	0.02
8 × 3-10	0.85	17.71	0.00	0.11	0.00	0.02
15 × 5-01	62.62	3026.82	0.00	25.52	0.00	0.55
15 × 5-02	56.83	2831.12	0.00	25.43	0.00	0.50
15 × 5-03	78.29	2778.12	0.00	25.49	0.00	0.50
15 × 5-04	45.66	2759.19	0.00	25.47	0.00	0.50
15 × 5-05	20.81	2741.67	0.00	25.85	0.00	0.51
15 × 5-06	80.22	2808.74	0.00	25.13	0.00	0.50
15 × 5-07	44.54	2781.74	0.00	25.77	0.00	0.51
15 × 5-08	85.42	2882.96	0.00	25.45	0.00	0.51
15 × 5-09	65.95	2861.29	0.00	25.50	0.00	0.50
15 × 5-10	163.10	2895.27	0.00	25.33	0.00	0.49
20 × 6-01	1514.95	3600.00	8.27	335.89	8.27	2.04
20 × 6-02	1600.01	3600.00	4.81	337.57	4.81	2.00
20 × 6-03	2219.47	3600.00	3.21	333.59	3.21	1.97
20 × 6-04	1322.53	3600.00	5.27	332.66	5.27	1.97
20 × 6-05	1072.00	3600.00	1.53	335.41	1.53	2.05
20 × 6-06	1569.84	3600.00	8.50	331.73	8.50	1.96
20 × 6-07	1468.50	3600.00	3.03	337.19	3.03	2.00
20 × 6-08	1739.32	3600.00	4.49	335.06	4.49	1.98
20 × 6-09	1362.42	3600.00	5.61	338.98	5.61	2.02
20 × 6-10	2001.76	3600.00	0.00	335.51	0.00	1.94

Table 3 Average results by configuration, and global averages.

		MIP-Model	MIP-Iterative	Shortest-Paths		HLSA	
Config	Number	Time	Time	Gap	Time	Gap	Time
6×2	16	0.27	2.11	0.00	0.02	0.00	0.01
7×3	64	0.40	10.83	0.00	0.42	0.00	0.04
8×3	64	0.77	17.97	0.00	0.15	2.33	0.03
15×5	1024	70.34	2836.69	0.00	25.49	0.00	0.50
20×6	4096	61587.08	–	4.47	335.36	4.47	1.99
Total Average	–	331.77	–	0.89	72.29	1.36	0.51

the closest change point in the approximation of the logit function defined by (8). This somehow measures the robustness of the MIP models with respect to the linearization chosen for the logit function. Note that this distance is defined as the minimum percentage change in the travel times, so that the new travel time is in another sector of the piecewise linear approximation defined in (8). We note that significant changes in the travel times would be needed in order to jump to the next (or previous) sector defined in (8), which on average would be an 85 % change in the travel times.

5.1 A case study

We now show our insights into a case study over the suburban Madrid commuter network, depicted in Figure 7. The input data for this case-study are based on real data: the OD demand is provided by the operator and the rest of parameters are based on the specific train model Civia, used for regional railways in Madrid. In our study, we have considered an average OD matrix over all time intervals. This network has 87 stations (nodes), 90 links (edges), forming 12 lines. As for the potential headways, we assumed two different cases: like in the random experiments the potential frequencies are the set $\{5,10,15,20\}$, yielding a total of $4^{12} = 16,777,216$ possible headway combinations and we also increased this set assuming eight possible headways are possible, namely $\{3,4,5,6,10,12,15,20\}$, yielding a total of $8^{12} \sim 6.8 \cdot 10^{11}$ possible headway combinations.

Since the MIP problems to be solved are far too large, neither the MIP-iterative algorithm, nor the MIP model, were able to return a feasible solution. We actually tried solving the case study with these two solver-dependent algorithms, but after one hour we had to stop the computation because the models had not been built. Therefore, we only show the performance of the path-based algorithm and the local-search heuristic. We first remark that the shortest path-based algorithm was not able to check all possible headway combinations within three days of computational time because of their huge number. Table 4 reports the number of headway combinations checked, and the average time to obtain the solution of each iteration, when running the shortest-path algorithm for three days.

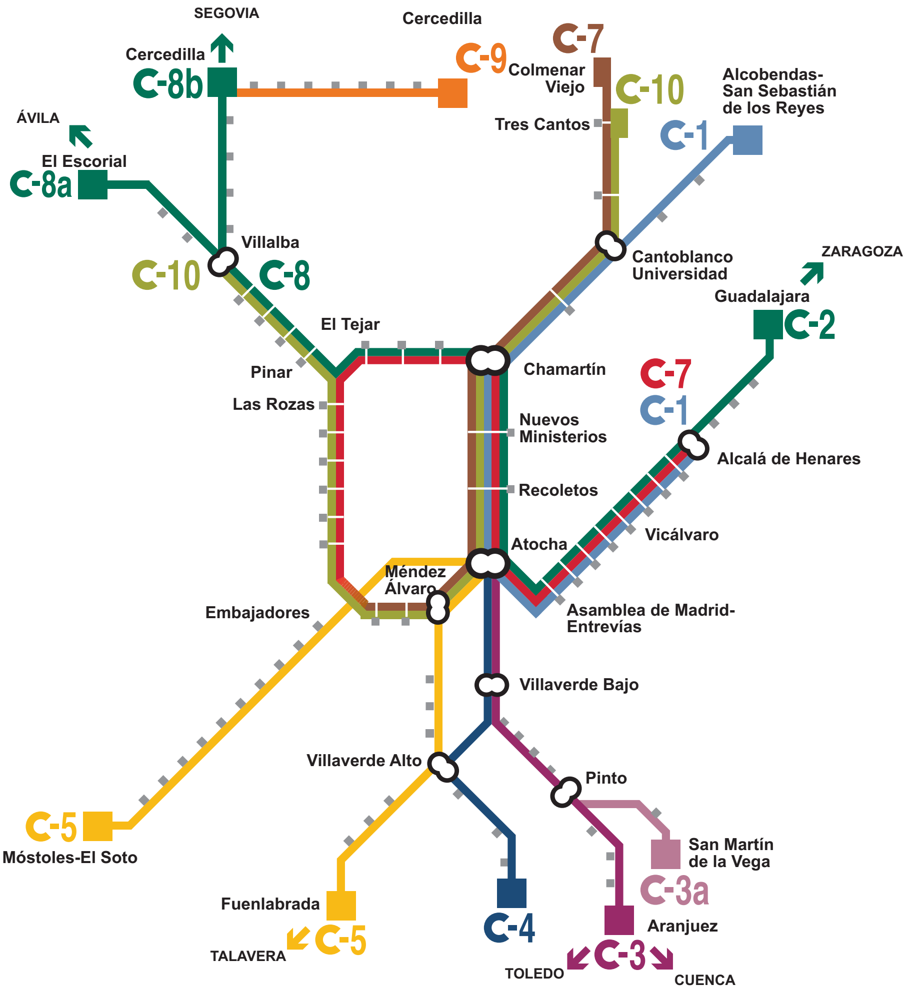

Fig. 7 The Madrid commuter system.

Table 4 Results of the shortest path-based algorithm after three days.

	Number of iterations	Average time (seconds)
Four headways	19781	13.10
Eight headways	19725	13.14

In contrast, the local search heuristic was able to find a solution within a relatively short amount of time. We now elaborate on the solutions obtained by the local search heuristic. The frequencies and number of vehicles for each of the 12 lines, for the two instances solved, are as follows:

- Four possible frequencies: frequencies: [12, 4, 4, 4, 12, 12, 4, 4, 4, 4, 3, 3], number of carriages: [10,4,9,1,6,8,9,5,8,2,1,1].
- Eight possible frequencies: frequencies: [12, 4, 4, 6, 15, 10, 4, 4, 4, 3, 4, 3]; number of carriages: [10,4,9,1,6,8,9,5,8,2,1,1]

Table 5 Results of the HLSA algorithm over the case study.

	Profit	Trips	Seconds
Four frequencies	23824255498	183223.27	1168.91
Eight frequencies	24148849723	182506.06	2238.63

Table 5 shows the profit, the ridership (measured as the number of expected trips attracted by the railway network), and the CPU time of the solutions to the two problems proposed for this case study. We first note that the computational time to solve the 8-frequency case is roughly twice the computational time needed to solve the 4-frequency case. Second, these CPU times are under one hour, which is highly satisfactory, in view of the fact that the other three algorithms failed to find a solution within three days of CPU time. Third, we note that the second solution has a higher profit (as expected, since the set of potential frequencies of the second instance contains the set of potential frequencies of the first one), but a lower number of attracted trips. A characteristic of our problem is that it may be beneficial to attract fewer passengers, since the extra cost needed to transport them may exceed the revenue generated by the ridership.

6 Conclusions

We have introduced the railway line frequency and size setting problem, in which the frequency of the lines and their number of cars are simultaneously determined, with the objective of maximizing the net profit of the network. We have assumed competition between modes, and therefore the network has to provide shorter trip durations in order to attract ridership. This problem was first modeled as an MINLP. Because of the intractability of this model, an MIP model that linearizes the MINLP originally proposed is introduced. In an attempt to reduce the size of the MIP model, an algorithm based on solving a sequence of smaller MIP models is also presented. Besides, a shortest path-based algorithm and a local search heuristic were proposed. Experimental results on small random instances have shown that the the local search heuristic shows a good tradeoff between efficiency and solution quality. We have also solved an instance derived from a case study based on the Madrid commuter network. The local search heuristic was able to find solutions within less than one hour, whereas the other three algorithms proposed in this paper could not find a solution within 72 hours.

Acknowledgements This research was partly funded by the Canadian Natural Sciences and Engineering Research Council under grant 2015-06189, by the *Ministerio de Economía y Competitividad (Spain)/FEDER* under projects MTM2012-37048, MTM2015-67706-P and DPI2012-36243-C02-01, and by *Junta de Andalucía (Spain)/FEDER* under excellence project P10-FQM-5849. Part of this research was done while Federico Perea was enjoying a research visit to CIRRELT, funded by the Universitat Politècnica de València, under program PAID-00-15. This support is gratefully acknowledged. Thanks are due to the referees for their valuable comments.

References

1. Albrecht, T.: Automated timetable design for demand-oriented service on suburban railways. Public Transport **1**(1), 5–20 (2009)
2. Caprara, A., Kroon, L., Monaci, M., Peeters, M., Toth, P.: Passenger Railway optimization. (C. Barnhart and G. Laporte, eds.), Handbooks in Operations Research and Management Science 14, Transportation, chapter 3, pp. 129–187. North-Holland, Amsterdam (2007)
3. De-Los-Santos, A., Laporte, G., Mesa, J., Perea, F.: Simultaneous frequency and capacity setting in uncapacitated metro lines in presence of a competing mode. Transportation Research Procedia **3**, 289–298 (2014)
4. Desaulniers, G., Hickman, M. (C. Barnhart and G. Laporte, eds.), Handbook in Operations Research and Management Science 14, Transportation, chapter 2, Public Transport, pp. 69–127. North-Holland, Amsterdam (2007)
5. Gallo, M., Montella, B., D'Acierno, L.: The transit network design problem with elastic demand and internalisation of external costs: An application to rail frequency optimisation. Transportation Research Part C: Emerging Technologies **19**(6), 1276–1305 (2011)
6. Laporte, G., Marín, A., Mesa, J.A., Perea, F.: Designing robust rapid transit networks with alternative routes. Journal of Advanced Transportation **45**(1), 54–65 (2011)
7. Marín, A., García-Ródenas, R.: Location of infrastructure in urban railway networks. Computers & Operations Research **36**(5), 1461–1477 (2009)
8. Michaelis, M., Schöbel, A.: Integrating line planning, timetable, and vehicle scheduling: a customer oriented heuristic. Public Transport **1**(3), 211–232 (2009)
9. van Oort, N., van Nes, R.: Regularity analysis for optimizing urban transit network design. Public Transport **1**(2), 155–168 (2009)
10. Perea, F., Mesa, J.A., Laporte, G.: Adding a new station and a road link to a road-rail network in the presence of modal competition. Transportation Research Part B: Methodological **68**, 1–16 (2014)
11. Schmidt, M., Schöbel, A.: The complexity of integrating passenger routing decisions in public transportation models. Networks **65**(3), 228–243 (2015)
12. Schmidt, M.E.: Integrating Routing Decisions in Public Transportation Problems. Springer-Verlag, New York (2014)
13. Schöbel, A.: Line planning in public transportation. OR Spectrum **34**, 491–510 (2012)
14. Vuchic, V.R.: Urban Transit: Operations, Planning, and Economics. Wiley, Hoboken, New Jersey (2005)