The Windy General Routing Polyhedron: A global view of many known Arc Routing Polyhedra

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Abstract

The Windy Postman Problem consists of finding a minimum cost traversal of all the edges of an undirected graph with two costs associated with each edge, representing the costs of traversing it in each direction. In this paper we deal with the Windy General Routing Problem (WGRP), in which only a subset of edges must be traversed and a subset of vertices must be visited. This is also an NP-hard problem that generalizes many important Arc Routing Problems (ARP's) and has some interesting real-life applications. Here we study the description of the WGRP polyhedron, for which some general properties and some large families of facet-inducing inequalities are presented. Moreover, since the WGRP contains many well-known routing problems as special cases, this paper also provides a global view of their associated polyhedra. Finally, for the first time, some polyhedral results for several ARP's defined on mixed graphs formulated using two variables per edge are presented.

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1 Introduction

Arc Routing Problems (ARP's) have their origin in the celebrated Königsberg Bridge Problem solved by Euler. They basically consist of finding a traversal on a graph satisfying some conditions related to the links of the graph. ARP's have been studied in depth during the last 40 years due to the large number of real situations that can be modelled in this way: collection or delivery of goods, mail distribution, network maintenance (electrical lines or gas mains inspection), snow removal, garbage collection, etc. The book edited by Dror [2000] summarizes the state of the art and real-life applications of the ARPs. Other interesting papers are those by Assad & Golden [1995] and Eiselt, Gendreau & Laporte [1995a, 1995b].

The first ARP presented in the literature is the well-known Chinese Postman Problem (CPP, Guan 1962). It consists of finding a shortest tour (closed walk) traversing all the *links* of a given graph G. Since then, this problem has been generalized in several aspects. Orloff (1974) proposed the Rural Postman Problem (RPP), in which not all the links of the graph have to be traversed by the tour but only those in a given subset of 'required' links. Orloff also proposed

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the General Routing Problem (GRP), in which the graph has a given set of 'required' links to be traversed and a given set of 'required' vertices to be visited. The GRP can also be considered as a generalization of the Graphical Traveling Salesman Problem (GTSP), studied by Cornuèjols et al. (1985), Fleischmann (1985) and Naddef & Rinaldi (1991). The GTSP consists of finding a shortest tour visiting all the vertices of a given graph G at least once. All these problems were defined on *undirected* graphs, where all the links are *edges* that can be traversed in both directions with the same cost. These four problems are shown in figure 1 (box labeled 'undirected'), where an arc between two problems means that the first one is generalized by the second one.

Alternatively, the CPP can be defined on a *directed* graph, where all the links are *arcs* that must be traversed in a given direction. This problem is known as the Directed Chinese Postman Problem (DCPP, Edmonds & Johnson 1973). Obviously the RPP and the GRP can also be defined on a *directed* graph to obtain their 'directed' versions, the DRPP (Christofides et al. 1986, Savall 1990 and Gun 1993) and the DGRP. The GTSP on a directed graph is known as the Graphical Asymmetric TSP (GATSP, Chopra & Rinaldi 1996). These four problems are also represented in figure 1 (box labeled 'directed').

A graph G is called a *mixed* graph if it has edges and arcs simultaneously. The CPP on a mixed graph, the Mixed Chinese Postman Problem (MCPP, Edmonds & Johnson 1973, Christofides et al. 1984, Grötschel & Win 1992, Nobert & Picard 1996) is then a generalization of both the CPP and the DCPP. Note that any routing problem defined on a mixed graph generalizes the corresponding problems defined on undirected and directed graphs. This is the case for the Mixed RPP (MRPP, Romero 1997 and Corberán, Romero & Sanchis 1997) and for the Mixed GRP (MGRP, Corberán, Mejía & Sanchis 2005). Note that since the GTSP is a pure Node Routing Problem, the GTSP defined on a mixed graph is equivalent to the GATSP. These four 'mixed' problems are also shown in figure 1 (box labeled 'mixed'). An arc between two 'boxes' means that each problem in the first box is generalized by the corresponding problem in the second box.

Finally, we have the Windy Postman Problem (WPP), proposed by Minieka (1979) and also studied by Guan (1984), Win (1987) and Grötschel & Win (1988, 1992). This problem consists of finding a minimum cost traversal of all the edges of an undirected graph in which the cost of traversing an edge (i, j) in a given direction, c_{ij}, can be different to the cost of traversing it in the opposite direction, c_{ji}. Obviously, the WPP generalizes the undirected CPP (when $c_{ij} = c_{ji}$), but also the DCPP (each arc (i, j) with cost c can be modelled as an edge with costs $c_{ij} = c$ and $c_{ji} = \infty$) and therefore the MCPP. As before, windy routing problems are also represented in figure 1 (box labeled 'windy'). As for the mixed case, the windy version of the GTSP is equivalent to the GATSP.

In this paper we deal with the Windy General Routing Problem (WGRP). This is the most general routing problem among those represented in figure 1 and contains all the other problems as special cases. Consequently, most of the theoretical results obtained for the WGRP and the algorithms designed for its approximate or exact resolution can be applied to many other ARP's.

Except for the CPP and the DCPP, which can be solved in polynomial time, all the other problems in figure 1 are NP-hard problems and therefore the WGRP is also NP-hard. The most successful approach to optimally solving these difficult routing problems is the Polyhedral Combinatorics. It is based on the description of the polyhedron defined by the convex hull of all the solutions associated with a formulation of the problem. This approach has recently been used in Corberán, Plana and Sanchis (2005b) to optimally solve WGRP instances of large size and its success is mostly due to the partial description of the WGRP polyhedron we present here. This description is the subject of this paper.

On the other hand, all the formulations proposed in the literature for ARP's defined on

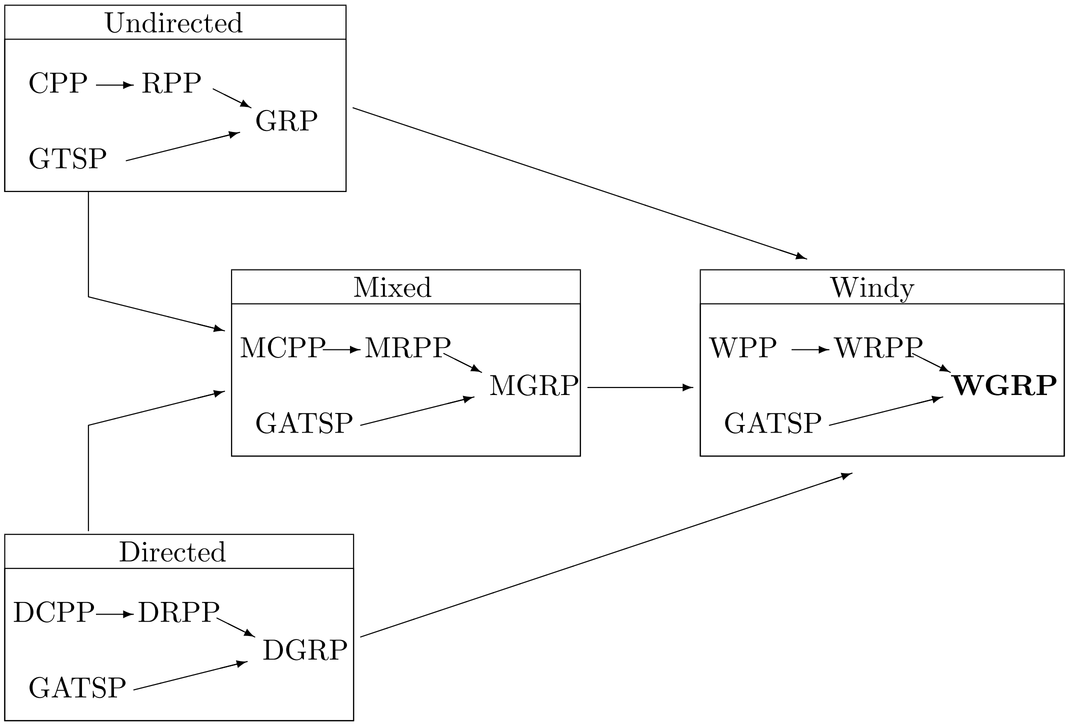


Figure 1: Relationship among routing problems

undirected or directed graphs use only one variable per link. This variable represents, in the directed case, the number of times its associated arc is traversed (in the fixed direction) while, in the undirected case, it represents only the number of times the edge is traversed in any of the two possible directions. All windy problems are formulated using two variables associated with each edge, representing the number of times the edge is traversed in each direction. However, ARP's defined on mixed graphs can be formulated using one variable per edge (Nobert & Picard 1996 and Corberán et al. 2003 and 2005) or two (Christofides et al. 1984 and Ralphs 1993). And although the polyhedra associated with ARP's defined on mixed graphs and formulated using only one variable per edge have already been studied (Corberán et al. 2003 and 2005), this is not the case when these problems are formulated using two variables per edge. Then, as a by-product of the polyhedral study of the WGRP presented here, we also obtain some results on the polyhedra associated with ARP's defined on mixed graphs whose formulation uses two variables per edge.

Although the facet inducing inequalities presented in this paper are associated with a formulation with 2 variables per edge, they can be easily transformed into facets of the polyhedra of the ARP's shown in figure 1. Hence, a final contribution of this work is that it also gives a global view of the polyhedra associated with many well-known ARP's.

More specifically, the WGRP is formally defined and some notation is presented in section 2. Section 3 introduces the WGRP polyhedron and presents some results characterizing its facet defining inequalities. In section 4 several families of valid inequalities and their facet defining property are described. Section 5 discusses the application of these results to mixed ARP's formulated with 2 variables per edge. Finally, section 6 presents the conclusions.

2 Problem definition and notation

The Windy General Routing Problem can be defined as follows. Let $G = (V, E)$ be an undirected and connected graph with two non-negative costs, c_{ij} and c_{ji}, associated with each edge $(i, j) \in E$, corresponding to the costs of traversing it from i to j and from j to i, respectively. Given a subset $E_R \subseteq E$ of 'required' edges and a subset $V_R \subseteq V$ of 'required' vertices, the problem is to find a minimum cost tour (closed walk) on G traversing each required edge and visiting each required vertex at least once.

Note that the vertices incident with required edges can also be considered as required vertices and then included in V_R. Without loss of generality, we will assume that the original graph has been transformed to satisfy $V = V_R$. This is not a serious restriction as it is easy to transform WGRP instances which do not satisfy the assumption into equivalent instances which do. Hence, we will assume, in what follows, that we are working with a connected graph $G = (V, E) := (V_R, E_R \cup E_{NR})$.

If we remove all the non required edges from graph $G = (V, E)$, the resulting graph $G_R = (V, E_R)$ is, in general, not connected. Let p be the number of connected components of graph G_R and let $V_1, V_2, \ldots, V_p$ denote the vertex sets of these connected components, which will be called *R-sets*. We will call the subgraphs $G(V_i)$ of G *R-connected components*. Note that each isolated required vertex itself defines an R-set.

If S_1 and S_2 are two vertex sets, (S_1, S_2) represents the set of edges with an endpoint in S_1 and another endpoint in S_2, while $(S_1, S_2)_R$ and $(S_1, S_2)_{NR}$ represent, respectively, the sets of required and non-required edges in (S_1, S_2). Given a vertex subset S, $\delta(S)$ denotes the cutset $(S, V \setminus S)$ i.e., the set of edges with an endpoint in S and another endpoint not in S, while $E(S)$ denotes the set of edges with both endpoints in S. A vertex is called R-even (R-odd) if it is incident with an even (odd) number of required edges and a subset of vertices $S \subseteq V$ is R-even (R-odd) if it contains an even (odd) number of R-odd vertices.

A tour for the WGRP is a closed walk on the edges of G traversing each required edge and visiting each required vertex at least once. Each tour is represented by an incidence vector $x \in \mathbb{R}^{2|E|}$ with two variables x_{ij}, x_{ji} associated with each edge $e = (i, j) \in E$ representing the number of times e is traversed in each direction. Given an edge set $F \subseteq E$, $x(F)$ denotes the sum of the variables in x corresponding to the edges in F, i.e. $x(F) := \sum_{(i,j)\in F}(x_{ij} + x_{ji})$. Moreover, a tour for the WGRP can be represented by a directed graph (V, A), where A contains as many copies of arc (i, j) as the number of times the original edge (i, j) has been traversed from i to j. In a similar way to that given in Benavent et al. (2005) for the WRPP, the WGRP can be formulated as:

$$\text{Minimize} \quad \sum_{(i,j)\in E} (c_{ij}x_{ij} + c_{ji}x_{ji})$$

$$\text{s.t.:} \quad x_{ij} + x_{ji} \geq 1, \quad \forall (i, j) \in E_R \tag{1}$$

$$\sum_{(i,j)\in\delta(i)} (x_{ij} - x_{ji}) = 0, \quad \forall i \in V \tag{2}$$

$$\sum_{i\in S,\ j\notin S} x_{ij} \geq 1, \quad \forall S = \bigcup_{k\in Q} V_k, \quad Q \subset \{1, \ldots, p\} \tag{3}$$

$$x_{ij}, x_{ji} \geq 0, \quad \forall (i, j) \in E \tag{4}$$

$$x_{ij}, x_{ji} \ \text{integer}, \quad \forall (i, j) \in E \tag{5}$$

Traversing inequalities (1) oblige the tour x to traverse each required edge at least once (in any direction). *Symmetry* equations (2) force the tour x to depart from each vertex as many times as it arrives at it. Given any edge cutset of G, $(S, V \setminus S)$, by combining some of the equations (2),

$$\sum_{i \in S,\ j \notin S} x_{ij} \;=\; \sum_{i \in S,\ j \notin S} x_{ji}$$

is obtained. *Connectivity* inequalities (3) prevent the formation of subtours. Since inequalities (1) guarantee that x traverses each edge cutset containing required edges, inequalities (3) are only needed for edge cutsets joining different R-sets. Finally, (4) and (5) assure that all the variables are non-negative and integer.

3 The WGRP polyhedron

Let WGRP(G) denote the convex hull of all the tours for the WGRP on graph G, i.e., all the integer vectors $x \in \mathbb{R}^{2|E|}$ satisfying (1) to (4). WGRP(G) is an unbounded polyhedron which has the following properties:

Theorem 1 *If G is a connected graph, the dimension of WGRP(G) is* $2|E| - |V| + 1$.

Proof: The matrix defined by the coefficients of equations (2) has rank $|V| - 1$, therefore it follows that $dim(WGRP(G)) \leq 2|E| - (|V| - 1)$. On the other hand, it is not difficult to see that $aff(WGRP(G)) = \{x \in \mathbb{R}^{2|E|} : \ x \text{ satisfies } (2)\}$. Therefore, $dim(WGRP(G)) \geq 2|E| - |V| + 1$ and the result follows. ♦

Note that WGRP(G) is not of full dimension and different inequalities can induce the same facet of it. Such inequalities are called equivalent. In this section we show that inequalities (4), (1) and (3) from the formulation of the problem are facet-inducing for WGRP(G).

Theorem 2 *If edge $e = (i, j) \in E$ is not a bridge of G, then inequalities $x_{ij} \geq 0$ and $x_{ji} \geq 0$ are facet-inducing for WGRP(G). If in addition $e = (i, j) \in E_R$, then inequality $x_{ij} + x_{ji} \geq 1$ is facet-inducing for WGRP(G).*

Proof: We will only study the case when $e \in E_R$. If e is not a bridge of G, graph $G \setminus \{e\}$ is connected and there are $K-1$ affinely independent tours for the WGRP on $G \setminus \{e\}$, $x^1, x^2, \ldots, x^{K-1}$, where $K = \dim(\text{WGRP}(G))$. All these tours are not tours for the WGRP on G because they do not traverse the required edge e. Let C^1 be a cycle on G traversing e once from j to i. Then, $x^1 + C^1, x^2 + C^1, \ldots, x^{K-1} + C^1$, $x^1 + 2C^1$ are K affinely independent tours for the WGRP on G satisfying $x_{ij} = 0$. Hence, inequality $x_{ij} \geq 0$ is facet-inducing for WGRP(G).

Consider now the inequality $x_{ij} + x_{ji} \geq 1$ and let C^2 be a cycle on G traversing e once from i to j. Then, $x^1 + C^1, x^2 + C^1, \ldots, x^{K-1} + C^1, x^1 + C^2$ are K affinely independent tours for the WGRP on G satisfying $x_{ij} + x_{ji} = 1$. ♦

Before discussing the facet-defining property of connectivity and other families of inequalities, we will present some properties that are common to all of them in what follows.

3.1 Weak Configuration inequalities

The polyhedra based on standard formulations using only one variable per edge corresponding to the routing problems in figure 1 satisfy the following property. All the facet-inducing inequalities (except the trivial ones) are *configuration inequalities* (Naddef & Rinaldi, 1991). An inequality $F(x) \geq b_0$ is a configuration inequality if there is a partition $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r\}$ of V such that the subgraphs $G(\mathcal{B}_k)$ are connected, the variables associated with links in the sets $E(\mathcal{B}_k)$ have coefficient zero in the inequality and the variables associated with links from a given set $\mathcal{B}_p$ to another given set $\mathcal{B}_q$ have the same coefficient in the inequality.

It is easy to see that WGRP(G) does not satisfy the above property. Connectivity inequalities (3), for example, are configuration inequalities with $\mathcal{B} = \{S, V \setminus S\}$, while traversing inequalities (1), and others that will be presented later, are not configuration inequalities. In what follows we will show that all the facet-inducing inequalities for WGRP(G), except the trivial and the traversing ones, share some of the properties described above for the configuration inequalities. This will allow us to define a new wider class, the *weak configuration inequalities.*

Lemma 1 *Let $F(x) \geq b_0$ be a facet-inducing inequality for WGRP(G) different from the trivial and the traversing ones.*
(a) For each variable x_{ij} there is a tour x satisfying $F(x) = b_0$ and $x_{ij} \geq 1$.
(b) For each required edge $e = (i, j) \in E_R$ there is a tour x satisfying $F(x) = b_0$ and $x_{ij} + x_{ji} \geq 2$.

Proof: If (a) is not satisfied, every tour x such that $F(x) = b_0$ satisfies $x_{ij} = 0$. Then,

$$\{x \ \text{ WGRP-tour on G}: \ F(x) = b_0\} \subseteq \{x \ \text{ WGRP-tour on G}: \ x_{ij} = 0\}$$

and, hence, either $F(x) \geq b_0$ is equivalent to a trivial inequality or it is not facet-inducing. In a similar way, if (b) is not satisfied, every tour x such that $F(x) = b_0$ satisfies $x_{ij} + x_{ji} = 1$ and either $F(x) \geq b_0$ is equivalent to a traversing inequality or it is not facet-inducing. ♦

Part (a) in lemma 1 implies that for each variable x_{ij} associated with a *non-required* edge (i, j), there is a tour x such that $F(x) = b_0$ and uses x_{ij} *an extra time.* This means that if a copy of the arc (i, j) is removed from the directed graph associated with x and is replaced by any other path from i to j, another WGRP tour is obtained. Part (a) also implies that for each variable x_{ij} associated with a *required* edge (i, j), there is a tour x such that $F(x) = b_0$ and uses x_{ij} at least once, although not necessarily *an extra time.* Finally, part (b) implies that at least for one variable of each pair x_{ij}, x_{ji} associated with a *required* edge (i, j), there is a tour x such that $F(x) = b_0$ and uses it *an extra time.*

An inequality $F(x) \geq b_0$ will be called a *weak configuration inequality* if there is a partition $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r\}$ of V such that the subgraphs $G(\mathcal{B}_k)$ are connected, the variables associated with edges in the sets $E(\mathcal{B}_k)$ have coefficient zero in the inequality and the variables associated with *non-required* edges from a given set $\mathcal{B}_p$ to another given set $\mathcal{B}_q$ have the same coefficient in the inequality. Note that if, in addition, the variables associated with *required* edges from $\mathcal{B}_p$ to $\mathcal{B}_q$ also have the same coefficient in the inequality, we have a (strong) configuration inequality.

Theorem 3 *All the facet-inducing inequalities for the MGRP(G), except those equivalent to the trivial or the traversing ones, are weak configuration inequalities.*

Proof: Let $F(x) := \sum_{(i,j) \in E}(a_{ij}x_{ij} + a_{ji}x_{ji}) \geq b_0$ be such an inequality and let $E^0 = \{e = (i, j) \in E : \ a_{ij} = a_{ji} = 0\}$. Let S be the node set of a connected component of $G^0 = (V, E^0)$.

Assume there is an edge $e = (i, j) \in E(S)$ with $a_{ij} \neq 0$ (necessarily $a_{ij} > 0$). Let x^* be a tour for the WGRP on G such that $F(x^*) = b_0$ and $x^*_{ij} \geq 1$ (it exists from lemma 1). Let us replace in x^* arc (i, j) by a path from i to j in $E^0(S)$. If x^* uses x_{ij} an extra time, we obtain a tour

x^{**} for the WGRP on G satisfying $F(x^{**}) < b_0$, which is a contradiction. From lemma 1, this is true for all the variables associated with non-required edges and for one variable of each pair associated with a required edge. Then, if x^* does not use x_{ij} an extra time, we add to x^* the arcs in a cycle formed by the arc from j to i (whose associated variable x_{ji}, which is used an extra time, has coefficient zero) and a path in E^0 from i to j, obtaining a tour for the WGRP on G violating the inequality. Therefore, if $\mathcal{B}=\{\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_r\}$ are the vertex sets of the connected components of G^0, the induced graphs $G(\mathcal{B}_k)$ are connected and $a_{ij}=a_{ji}=0$, $\forall e=(i,j)\in E(\mathcal{B}_k)$, $k = 1, 2, \cdots, r$.

Let us suppose now that there are two non-required edges $e=(i,j)$, $f=(u,v)$ with $i,u\in\mathcal{B}_p$ and $j,v\in\mathcal{B}_q$ such that $a_{ij} > a_{uv}$. Let x^* be a tour for the WGRP on G using x_{ij} an extra time and $F(x^*)=b_0$. Replace in x^* arc (i,j) by a path in $G(\mathcal{B}_p)$ from i to u, the edge f traversed from u to v and a path in $G(\mathcal{B}_q)$ from v to j. This way, a tour violating the inequality is obtained, which is a contradiction. ♦

Then, associated with each facet-inducing inequality for the WGRP(G) we have a *configuration graph*, $G_{\mathcal{C}} = (V^{\mathcal{C}}, E^{\mathcal{C}})$. This is the graph resulting from shrinking vertex sets $\mathcal{B}_k$, $k = 1, \dots, r$, into a single vertex each and, after that, shrinking each set of non-required parallel edges into one single non-required edge, but keeping all the required edges. If two costs equal to the coefficients of the corresponding variables in the inequality are associated with each edge in $G_{\mathcal{C}}$, this graph keeps all the information about the inequality, since b_0 is the length of the shortest tour for the WGRP on graph $G_{\mathcal{C}}$. Hence, each facet-inducing inequality for the WGRP(G) can be represented by means of its corresponding configuration graph $G_{\mathcal{C}}$. The following theorem implies that in order to show that a given inequality is facet-inducing for WGRP(G) it will suffice to prove it for the polyhedron associated with the configuration graph, which is simpler in general.

Theorem 4 *If a weak configuration inequality is facet-inducing for WGRP($G_{\mathcal{C}}$), it is also facet-inducing for WGRP(G).*

Proof: It suffices to prove the result for a graph G obtained from $G_{\mathcal{C}}$ by replacing vertex $\mathcal{B}_1$ (for example) by a connected graph $G(\mathcal{B}_1) = (V^{\mathcal{B}_1}, E^{\mathcal{B}_1})$. For simplicity, $F(x) \geq b_0$ will denote both the inequality on G and the inequality on $G_{\mathcal{C}}$. Note that given a tour x for the WGRP on G, by shrinking $G(\mathcal{B}_1)$ into a single vertex we obtain a tour x^* for the WGRP on $G_{\mathcal{C}}$ satisfying $F(x) = F(x^*) \geq b_0$.

Let $K = \dim(\text{WGRP}(G_{\mathcal{C}}))$ and let us suppose that $b_0 \neq 0$. Then, there are K linearly independent tours for the WGRP on $G_{\mathcal{C}}$ satisfying $F(x)=b_0$, say $x^1, x^2, \dots, x^K$. After replacing $\mathcal{B}_1$ by the graph $G(\mathcal{B}_1)$, some non-required edges may appear between the vertices of $G(\mathcal{B}_1)$ and nodes B_i, $i \neq 1$. Let us call $G'_{\mathcal{C}}$ the graph obtained from $G_{\mathcal{C}}$ by adding Q non-required edges $e'_1, e'_2, \dots, e'_Q$, where each e'_i is parallel to a non-required edge (not necessarily distinct) e_j of $G_{\mathcal{C}}$.

For each e'_i, $i = 1, 2, \dots, Q$, we will construct two tours in $G'_{\mathcal{C}}$. Let $e_j = (\mathcal{B}_1, \mathcal{B}_q)$ be the original edge of $G_{\mathcal{C}}$ to which e'_i is parallel. From lemma 1, there is a tour in $G_{\mathcal{C}}$ satisfying $F(x)=b_0$ and that traverses e_j from $\mathcal{B}_1$ to $\mathcal{B}_q$, and another one that traverses it in the opposite direction. By replacing the edge e_j by the edge e'_i in these two tours, we obtain two new tours on $G'_{\mathcal{C}}$ satisfying $F(x)=b_0$. Then we have $K+2Q$ linearly independent tours on $G'_{\mathcal{C}}$ satisfying $F(x) = b_0$. These tours can be completed with edges of $G(\mathcal{B}_1)$ to obtain $K+2Q$ tours on G, $x^1, x^2, \dots, x^{K+2Q}$, also satisfying $F(x) = b_0$ because the variables associated with the edges of $G(\mathcal{B}_1)$ do not appear in the inequality.

Consider now the WGRP defined on graph $G(\mathcal{B}_1)$. Let $M = \dim(\text{WGRP}(G(\mathcal{B}_1)))$ and let $y^1, y^2, \dots, y^M$ be M linearly independent tours for the WGRP on $G(\mathcal{B}_1)$. Then, x^1+y^1, x^1+

$y^2, \ldots, x^1+y^M$ are also tours for the WGRP on G satisfying $F(x) = b_0$. Expressing the $K+2Q+M$ tours as rows of a matrix and subtracting the first row x^1 from the last M rows, we obtain a full rank matrix. Then, since $K = 2|E^{\mathcal{C}}| - |V^{\mathcal{C}}| + 1$ and $M = 2|E^{\mathcal{B}_1}| - |V^{\mathcal{B}_1}| + 1$, we have $K+2Q+M = 2(|E^{\mathcal{C}}| + |E^{\mathcal{B}_1}| + Q) - (|V^{\mathcal{C}}| - 1 + |V^{\mathcal{B}_1}|) + 1 =$dim(WGRP($G$)) linearly independent tours satisfying $F(x) = b_0$. ♦

As mentioned before, strong and weak configuration inequalities only differ in the fact that, in the first one, 'parallel' variables (variables x_{ij}, x_{km} associated with edges in $(\mathcal{B}_p, \mathcal{B}_q)$, with $i, k \in \mathcal{B}_p$ and $j, m \in \mathcal{B}_q$) have equal coefficients, while the last one can have 'parallel' variables with different coefficients. By arguing as in the proof of theorem 3 above it can be shown that if for each variable x_{ij} there is a tour satisfying $F(x) = b_0$ and using x_{ij} an extra time, then $F(x) \geq b_0$ is a strong configuration inequality, with all the variables in $(\mathcal{B}_p, \mathcal{B}_q)$ having a coefficient equal to a_{pq} or a_{qp}. A variable with a different coefficient in a weak configuration inequality is then a variable x_{ij} associated with a required edge $(i, j) \in (\mathcal{B}_p, \mathcal{B}_q)$, such that there is no tour satisfying $F(x) = b_0$ which uses x_{ij} an extra time. It can be seen that, in this case, the coefficients of variables x_{ij} and x_{ji} are, respectively, $a_{pq}+k$ and a_{qp}, for some k satisfying $0 < k \leq a_{pq} + a_{qp}$.

Therefore, if a given facet-inducing inequality $F(x) \geq b_0$ for WGRP(G) is not a strong configuration inequality, then there is at least one variable x_{ij} (associated with a required edge) such that every tour x satisfying $F(x) = b_0$ also satisfies either $x_{ij} = 0$ or $x_{ij} + x_{ji} = 1$. In other words, if a given facet $\mathcal{F}$ of WGRP(G) is not induced by a strong configuration inequality, then every tour for the WGRP on G which lies in $\mathcal{F}$ also lies either in a trivial facet or in a traversing facet of WGRP(G). Hence, we think that there must be relatively few such facets in WGRP(G). In fact, the only case of this type that we have found is a subset of the class of Zigzag inequalities. All the other inequalities presented in the following sections are strong configuration inequalities.

3.2 Connectivity inequalities

To finish this section, we will study the conditions under which connectivity inequalities (3) are facet-inducing for WGRP(G). Furthermore, the proof of theorem 5 shows the usefulness of the configuration issue and illustrates the use of theorem 4 above.

Theorem 5 *Connectivity inequalities (3) are facet-inducing for WGRP(G) if subgraphs $G(S)$ and $G(V \setminus S)$ are connected.*

Proof: The configuration graph $G_{\mathcal{C}}$ associated with connectivity inequalities has only two nodes, say 1 and 2, corresponding to S and to $V \setminus S$, respectively, joined by a single non-required edge. Therefore, dim(WGRP($G_{\mathcal{C}}$))= 1 and, since the tour $x_{12} = x_{21} = 1$ satisfies $\sum_{i \in S,\ j \notin S} x_{ij} = x_{12} = 1$, the inequality is facet-inducing for WGRP($G_{\mathcal{C}}$). By applying theorem 4, connectivity inequalities (3) are facet-inducing for WGRP(G). ♦

4 Other Facets of the WGRP polyhedron

In the presence of the integrality conditions (5), equations (2) and inequalities (4), (1) and (3) in the formulation are enough to describe all the tours for the WGRP on G. Nevertheless, this is not the case when the integrality conditions are relaxed. In this section we study some other families of inequalities which are facet-inducing for WGRP(G). In particular, we obtain valid inequalities for the WGRP from known valid inequalities for the undirected GRP.

Let $G=(V,E)$ be an undirected graph and let GRP(G) be the polyhedron associated with the formulation of the undirected GRP using one variable y_e per edge $e=(i,j)\in E$, representing the number of times the edge is traversed in any direction. Consider now the same graph $G=(V,E)$ but with two non-negative costs associated with each edge and let WGRP(G) be the polyhedron studied in this paper. In Benavent et al. (2003) it is shown that if $\sum_{e\in E}\beta_e y_e \geq b_0$ is a valid inequality for GRP(G), the inequality $\sum_{e=(i,j)\in E}\beta_e(x_{ij}+x_{ji})\geq b_0$ is valid for WGRP(G). Therefore, the R-odd cut, K-C, Path-Bridge and Honeycomb inequalities presented in what follows are valid inequalities for the WGRP(G).

4.1 R-odd cut inequalities

An edge cutset $\delta(S)$ is called R-odd if it contains an odd number of required edges. Given that any tour must cross any given edge cutset an even number of times, the following *R-odd cut inequalities*

$$x(\delta(S)) \geq |\delta_R(S)|+1, \quad \forall S\subset V \quad \text{such that } \delta(S) \text{ is } R\text{-odd} \tag{6}$$

are valid for WGRP(G). It can be proved (Plana, 2005) that they are facet-inducing for WGRP(G) if subgraphs $G(S)$ and $G(V\setminus S)$ are connected.

4.2 K-C inequalities

K-C inequalities were introduced in Corberán & Sanchis (1994) for the undirected RPP and in Corberán, Romero & Sanchis (2003) they were generalized to the Mixed GRP in two families, the K-C and the K-C$_{02}$ inequalities. In this section we will study their generalization to the WGRP.

Consider a partition of V into $K+1$ subsets $\{M_0,M_1,M_2,\ldots,M_{K-1},M_K\}$, $K\geq 3$, such that each R-set V_i, is contained by one of the node sets $M_0\cup M_K, M_1, M_2,\ldots,M_{K-1}$, the induced subgraphs $G(M_i)$, $i=0,1,2,\ldots,K$ are connected and (M_0,M_K) contains a positive and even number of required edges. Furthermore, we will assume that sets (M_i,M_{i+1}) are non-empty, $i=0,1,\ldots,K-1$. Such a partition defines the configuration graph shown in figure 2a, where the costs of its edges are $c(M_0,M_K)=c(M_K,M_0)=K-2$ and $c(M_i,M_j)=|i-j|$ for all i,j such that $\{i,j\}\neq\{0,K\}$. These costs, shown in figure 2a, correspond to the coefficients of the variables in the inequality. Each number represents the coefficient of the variable associated with the traversal of the edge from the nearest node to the farthest one. The *K-C inequality* is then

$$(K-2)\,x\Big(M_0,M_K\Big)+\sum_{\substack{0\leq i<j\leq K\\ \{i,j\}\neq\{0,K\}}}|i-j|\,x\Big(M_i,M_j\Big)\geq 2(K-1)+(K-2)|(M_0,M_K)_R| \tag{7}$$

Since K-C inequalities are a particular case of Path-Bridge inequalities, which will be presented later, the proof that K-C inequalities are facet-inducing is omitted here.

4.3 K-C$_{02}$ inequalities

With the same underlying configuration graph, K-C$_{02}$ inequalities differ from the standard ones in that integer K can now take value 2 and in that the coefficients of the variables and the RHS change as follows:

- $c(M_0,M_K)=c(M_K,M_0)=K-1$,

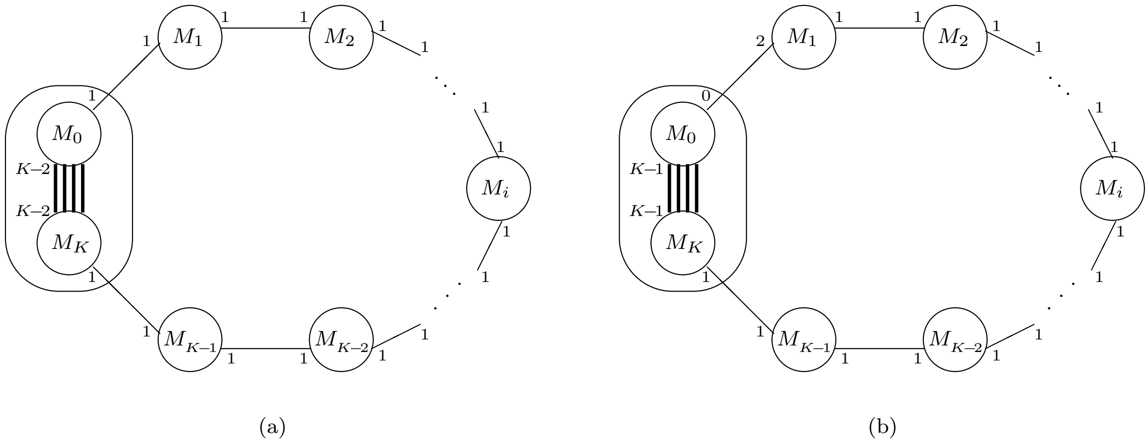


Figure 2: K-C and K-C_{02} configurations.

- $c(M_0, M_1) = 0$, $c(M_1, M_0) = 2$
- $c(M_i, M_{i+1}) = c(M_{i+1}, M_i) = 1, \qquad i = 1, 2, \ldots, K-1$
- otherwise, $c(M_i, M_j)$ is the shortest path cost from M_i to M_j in $G_{\mathcal{C}}$, i.e.
  - $c(M_0, M_j) = j - 1, \qquad 1 \le j \le K-1$
  - $c(M_j, M_0) = j + 1, \qquad 1 \le j \le K-1$
  - $c(M_i, M_j) = |i - j|, \qquad 1 \le i, j \le K$

The corresponding *K-C_{02} inequality* is then

$$\sum_{(i,j)\in E} (a_{ij}x_{ij} + a_{ji}x_{ji}) \ge 2(K-1) + (K-1)|(M_0, M_K)_R| \tag{8}$$

where $a_{ij} = c(M_p, M_q)$ if $i \in M_p$ and $j \in M_q$. These inequalities are also a special case of the Path-Bridge$_{02}$ inequalities presented next.

4.4 Path-Bridge inequalities

Path-Bridge inequalities were introduced by Letchford (1997) for the undirected GRP and are based on the Path inequalities for the GTSP proposed by Cornuèjols et al. (1985). In this section we study their extension to the WGRP. Let $P \ge 1$ and $B \ge 0$ be integers such that $P + B \ge 3$ is an odd number and let $n_i \ge 2$, $i = 1, \ldots, p$ be integer numbers. Let us define a partition $\{A, Z, \{V_j^i\}_{j=1,\ldots,n_i}^{i=1,\ldots,P}\}$ of V satisfying that each R-set is contained either in $A \cup Z$ or in a set V_j^i, the induced subgraphs $G(V_j^i)$ are connected, $i = 1, 2, \ldots, P$, $j = 0, 1, 2, \ldots, n_i+1$ (where for the sake of simplicity we identify A with V_0^i and Z with $V_{n_i+1}^i$ for all i) and (A, Z) contains a number B of required edges. Furthermore, we will suppose that sets (V_j^i, V_{j+1}^i) are non-empty. Then, the configuration graph has P paths from A to Z, each of them with $n_i + 2$ vertices and $n_i + 1$ edges. The edges joining the sets A and Z form the *bridge*. This configuration graph is shown in figure 3, where only one number near an edge means that the coefficients of its two corresponding variables are equal. These coefficients are:

- $c(A, Z) = c(Z, A) = 1$
- $c(V_j^i, V_k^i) = \frac{|j-k|}{n_i - 1} \qquad \forall j, k \in \{0, 1, \ldots, n_{i+1}\}, \quad i \in \{1, \ldots, P\}$

- $c(V_j^i, V_k^r) = \frac{1}{n_i - 1} + \frac{1}{n_r - 1} + \left| \frac{j-1}{n_i-1} - \frac{k-1}{n_r-1} \right|, \quad \forall\, i \neq r \in \{1, \dots, P\}, \quad j \in \{1, \dots, n_i\},$ $k \in \{1, \dots, n_r\}$

The Path-Bridge inequality, PB, is then

$$\sum_{(i,j)\in E} (a_{ij}x_{ij} + a_{ji}x_{ji}) \geq 1 + \sum_{i=1}^{P} \frac{n_i + 1}{n_i - 1} + |(A, Z)_R| \tag{9}$$

where, again, $a_{ij} = c(V_p^k, V_q^l)$ if $i \in V_p^k$ and $j \in V_q^l$.

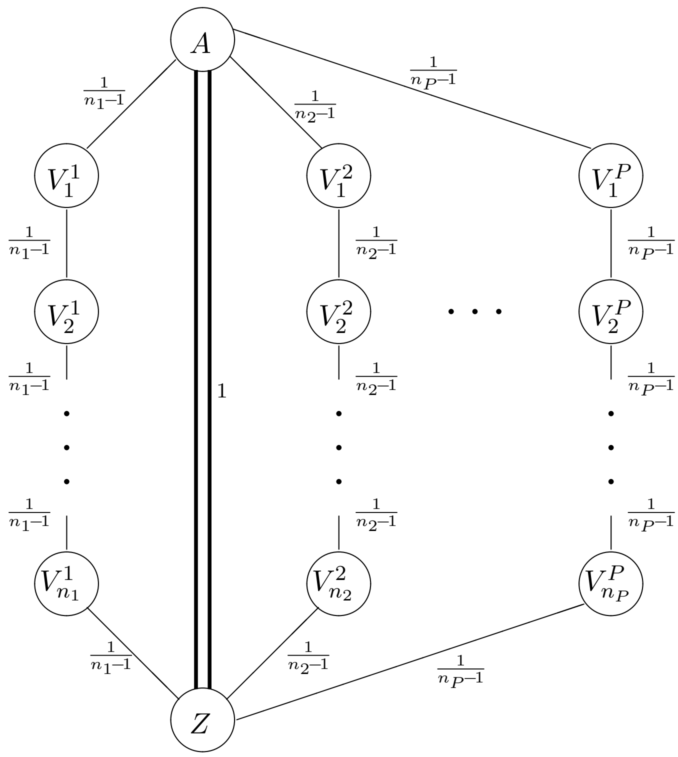


Figure 3: Path-Bridge configuration graph.

A Path-Bridge inequality is called *n-regular* if all the paths have the same length n. In this case, by multiplying it by $n-1$, the inequality (9) can be rewritten in an easier way. Note that a K-C inequality is a particular case of a Path-Bridge inequality with $P = 1$ and that Path-Bridge inequalities reduce to the *s-Path inequalities* for the GATSP, introduced by Chopra & Rinaldi (1996), when there are no required links crossing from A to Z.

Theorem 6 *Path-Bridge inequalities (9) are facet-inducing for WGRP(G).*

Proof: Let us first suppose that the number B of required edges joining A and Z is even. Then, the number P of paths from A to Z is odd. If we consider in $G_{\mathcal{C}}$ all the required edges as non-required and we replace each edge by a pair of opposite arcs, we obtain an instance $G'_{\mathcal{C}}$ of the GATSP. By subtracting $|(A, Z)_R|$ from the RHS of the PB inequality, an *s-path* inequality is obtained, which is facet-inducing for GATSP$(G'_{\mathcal{C}})$ (see Chopra & Rinaldi, 1996). The dimension of this polyhedron is $2|E_{\mathcal{C}}| - |V_{\mathcal{C}}| + 1$ and there is this number of affinely independent tours for the GATSP on $G'_{\mathcal{C}}$ satisfying the inequality as an equality. Each one of these tours can be easily transformed into tours for the WGRP on $G_{\mathcal{C}}$ (by adding to them one arc per each required edge in (A,Z), half of them directed from A to Z and the other half from Z to A) satisfying the PB inequality (9) as an equality. Hence, PB inequalities are facet-inducing for WGRP($G_{\mathcal{C}}$) and, from theorem 4, for WGRP(G).

Let us now suppose that B is odd and P is even. Let $\bar{e}$ be a given required edge in (A,Z). We construct the tour x^1 as the one traversing edge $\bar{e}$ once in each direction, half of the edges in $(A,Z)_R \backslash \{\bar{e}\}$ from A to Z and the other half from Z to A. Tour x^1 traverses half of the paths from A to Z and the other half from Z to A. From x^1 we construct the following tours:

(a) For each edge $e \in (A,Z) \setminus \{\bar{e}\}$ we define a tour traversing e from A to Z once more than tour x^1 does and traversing $\bar{e}$ only from Z to A. All the other edges are traversed as in x^1. Another similar tour traversing e from Z to A and $\bar{e}$ from A to Z is considered. Then we have $2|(A,Z)|-2$ tours.

(b) For each edge $e = (V_i^k, V_{i+1}^k)$, we construct a tour x that traverses all the edges in the path k, except e, in both directions. Edge $\bar{e}$ is traversed by x in the same direction as path k is traversed by x^1. All the other edges are traversed by x as they are traversed by x^1. Then we have $\sum_{i=1}^{P}(n_i+1)$ tours.

(c) For each path k, it is possible to build n_i+1 different tours using the opposite orientations to those used in (b) for the other $P-1$ paths. From among them we select, for each path k, only the tour that does not traverse the edge (V_0^k, V_1^k) and we then have P tours.

(d) For each of the remaining edges (if any) there are always two tours for the WGRP in $G_{\mathcal{C}}$ satisfying (9) as an equality, traversing this edge exactly once in one direction each, edges in (A,Z) and edges in the P paths. Then, if E' is the set of such remaining edges, we have obtained $2|E'|$ tours.

Including x^1 we have $1+2|(A,Z)|-2+\sum_{i=1}^{P}(n_i+1)+P+2|E'|$ tours, all of them satisfying the PB inequality as an equality. Hence we have $2|E_{\mathcal{C}}| - \sum_{i=1}^{P} n_i - 1 =$dim(WGRP($G_{\mathcal{C}}$)) tours. If we subtract x^1 from all the other tours and express them as the rows of a matrix (sorted as (a), (b)+(c) and (d) above) and the variables associated with the edges as columns (sorted as $\bar{e}$, $(A,Z) \setminus \{\bar{e}\}$, (V_i^k, V_{i+1}^k) for all i, k and E'), we obtain the matrix shown in figure 4. Submatrix B

	$\bar{e}$	$(A,Z) \setminus \{\bar{e}\}$	Paths	E'
(a)	*	I	0	0
(b)+(c)	*	0	B	0
(d)	*	*	*	I

Figure 4: Matrix appearing in the proof of theorem 6

is a full rank matrix (see Plana 2005 for details) and, hence, also the complete matrix in figure 4. Therefore, the PB inequality is facet-inducing for WGRP($G_{\mathcal{C}}$) and then it is also facet-inducing for WGRP(G). ♦

4.5 Path-Bridge$_{02}$ inequalities

Path-Bridge$_{02}$ inequalities were presented in Corberán, Mejía and Sanchis (2005) for the MGRP. The configuration graph corresponding to a Path-Bridge$_{02}$ inequality is similar to the previous one except that integers n_i can now take value 1 and the paths are classified into two types: AZ paths and ZA paths. The associated configuration graph is shown in figure 5, where the two coefficients associated with an edge are represented only if they are different.

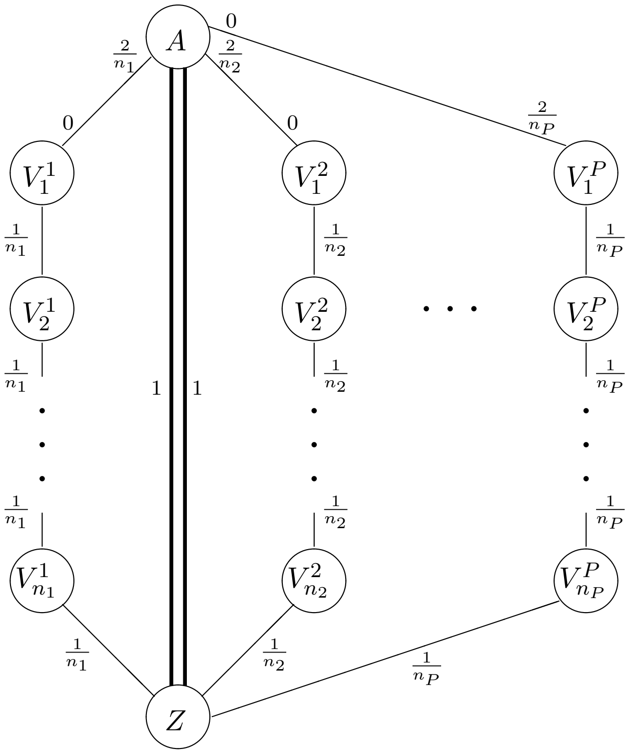


Figure 5: PB$_{02}$ configuration graph.

The coefficients are:

- $c(A, Z) = c(Z, A) = 1$
- $c(V_0^i, V_1^i) = 0$ and $c(V_1^i, V_0^i) = \frac{2}{n_i}$ for each AZ path i
- $c(V_0^i, V_1^i) = \frac{2}{n_i}$ and $c(V_1^i, V_0^i) = 0$ for each ZA path i
- $c(V_j^i, V_{j+1}^i) = c(V_{j+1}^i, V_j^i) = \frac{1}{n_i}$.
- $c(V_j^i, V_k^i)$, with $|j-k|>1$, is the shortest path cost from V_j^i to V_k^i using edges of the path i, i.e.,
  - $c(V_j^i, V_k^i) = \frac{|j-k|}{n_i} \quad \forall\, j \neq k \in \{1, 2, \ldots, n_i+1\}$
  - $c(V_0^i, V_k^i) = \frac{k-1}{n_i}$ and $c(V_k^i, V_0^i) = \frac{k+1}{n_i}$ if path i is of type AZ, $1 \leq k \leq n_i$.
  - $c(V_0^i, V_k^i) = \frac{k+1}{n_i}$ and $c(V_k^i, V_0^i) = \frac{k-1}{n_i}$ if path i is of type ZA, $1 \leq k \leq n_i$.
- Order the remaining edges (edges in (V_j^i, V_l^k), $i \neq k$, if any) in an arbitrary way $e_1, e_2, \ldots, e_h$. For $i=1$ to h, if $e_i=(u, v)$, we assign to c_{uv} the maximum value such that there is a WGRP

tour on $G_{\mathcal{C}}$ with cost $P+1+|(A,Z)_R|$ traversing e_i just from u to v but not traversing the edges $\{e_{i+1},\ldots,e_h\}$. Similarly, we assign to c_{vu} the maximum value such that there is a tour of c-length $P+1+|(A,Z)_R|$ traversing e_i from v to u (perhaps also from u to v) but not traversing the edges $\{e_{i+1},\ldots,e_h\}$ (sequential lifting).

The Path-Bridge$_{02}$ inequality, PB$_{02}$, is then

$$\sum_{(i,j)\in E}(a_{ij}x_{ij}+a_{ji}x_{ji})\geq P+1+|(A,Z)_R| \qquad (10)$$

Note that the main difference between PB$_{02}$ and PB inequalities is that variables in sets (A,V_1^i) have different coefficients for its two associated variables. One of these coefficients is $\frac{2}{n_i}$, twice the cost of the other edges in path i, while the other coefficient is 0 (see figure 5). The direction represented by the variable with coefficient 0 in each path determines whether the path is of type AZ or ZA. In a Path-Bridge$_{02}$ configuration graph, the pair of asymmetric coefficients $(\frac{2}{n_i},0)$ for each path i can be associated with any edge (V_j^i,V_{j+1}^i) in the path, thus obtaining different but equivalent inequalities (given the same ordering in the sequential lifting process). Finally, when $P=1$ the Path-Bridge$_{02}$ inequality (10) reduces to a K-C$_{02}$ inequality.

Theorem 7 *Path-Bridge$_{02}$ inequalities (10) are valid for WGRP(G)*

Proof: The proof is similar to that in Corberán et al. (2005) for the MGRP and is omitted here for the sake of brevity. ♦

Theorem 8 *Let $F(x)\geq b_0$ be a Path-Bridge$_{02}$ inequality (10) and let P_{AZ} and P_{ZA} be the number of paths of types AZ and ZA, respectively. Then, $F(x)\geq b_0$ is facet-inducing of WGRP(G) if $|(A,Z)_R|\geq|P_{AZ}-P_{ZA}|+1$.*

Proof: The proof is similar to that of theorem 6 and is omitted here. Note only that condition $|(A,Z)_R|\geq|P_{AZ}-P_{ZA}|+1$ is needed in order to construct WGRP tours traversing the paths of type AZ (type ZA) from A to Z (from Z to A) and satisfying $F(x)=b_0$. ♦

As for the (standard) PB inequalities, when all the paths are of the same length n, we obtain the n-regular PB$_{02}$ inequalities, which can be written in an easier way.

4.6 Honeycomb inequalities

Honeycomb inequalities were first proposed by Corberán & Sanchis (1998) for the undirected GRP and by Corberán et al. (2005) for the MGRP. These inequalities also generalize the K-C inequalities. Consider a partition $\{A_1,A_2,\ldots,A_L,\ A_{L+1},\ldots,A_K\}$ of V, $3\leq K\leq p$, $1\leq L\leq K$, such that each R-set is contained by one A_i and the induced subgraphs $G(A_i)$ are connected. Each set A_i, $i=1,2,\ldots,L$, is divided into $\gamma_i\geq 2$ subsets, $A_i=B_i^1\cup B_i^2\cup\ldots\cup B_i^{\gamma_i}$, satisfying that each B_i^j contains an even number of R-odd nodes and the induced subgraphs $G(B_i^j)$ are connected, $j=1,2,\ldots,\gamma_i$, and the graph defined by nodes $B_i^1,B_i^2,\ldots,B_i^{\gamma_i}$ and the required edges (B_i^j,B_i^k) is connected (and even).

Note that when A_i consists of several R-sets, the last condition implies that the partition of A_i into the B_i^j is made by cutting the R-sets. For notational convenience, we denote $B_i^0=A_i$, $i=L+1,\ldots,K$. Therefore we have the following partition of V:

$$\mathcal{B}=\{B_1^1,B_1^2,\ldots,B_1^{\gamma_1},B_2^1,B_2^2,\ldots,B_2^{\gamma_2},\ \ldots\ ,B_L^1,B_L^2,\ldots,B_L^{\gamma_L},\ B_{L+1}^0,\ldots,B_K^0\}$$

This partition $\mathcal{B}$ defines the configuration graph $G_{\mathcal{C}} = (\mathcal{B}, \mathcal{E})$. Let T be a set of non-required edges in $G_{\mathcal{C}}$ joining nodes corresponding to different A_j such that $(\mathcal{B}, T)$ is a spanning tree (see figure 6, where the arcs in T are represented by thin lines and the required links by bold lines). Then, for each pair of nodes B_i^j, B_p^q in $\mathcal{B}$, let $d(B_i^j, B_p^q)$ denote the number of edges in the unique path in $(\mathcal{B}, T)$ joining B_i^j to B_p^q. We will also assume that $d(B_i^j, B_i^q) \geq 3 \quad \forall\ i = 1, \ldots, L$ and $\forall\ j \neq q$.

We divide the set $\mathcal{E}$ into 3 subsets: the set $\mathcal{C}$ formed by the edges joining nodes B_i^p, B_i^q with $p, q \neq 0$ (the nodes obtained by 'cutting' the sets A_i, $i = 1, \ldots, L$), the set of edges in T and the set formed by the remaining edges, which will be called $\mathcal{I}n$. The costs of the edges of graph

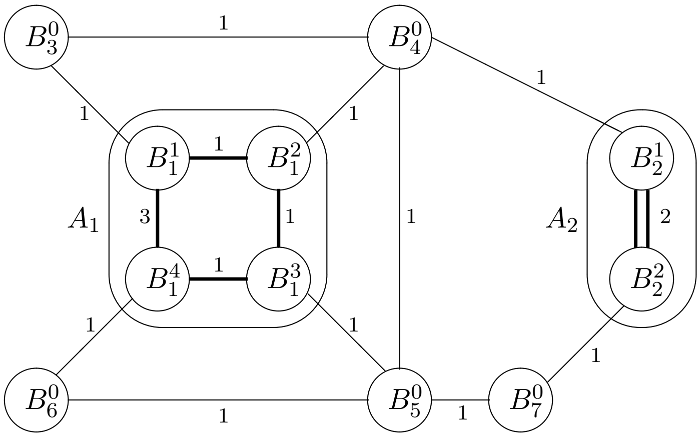


Figure 6: Honeycomb Configuration

$G_{\mathcal{C}} = (\mathcal{B}, \mathcal{E})$ are (see figure 6):

I) For the edges $(B_q^i, B_q^j) \in \mathcal{C}$, such that the path in $(\mathcal{B}, T)$ joining B_q^i and B_q^j does not contain more than one node related to the same A_s (except the nodes B_q^i and B_q^j), $c(B_q^i, B_q^j) = c(B_q^j, B_q^i) = d(B_q^i, B_q^j) - 2$.

II) For the edges $(B_r^i, B_q^j) \in T \cup \mathcal{I}n$, $r \neq q$, such that the path in $(\mathcal{B}, T)$ joining B_r^i and B_q^j does not contain more than one node related to the same A_s, $c(B_r^i, B_q^j) = c(B_q^j, B_r^i) = d(B_r^i, B_q^j)$.

III) For the remaining edges (if any), we compute their two costs with a process of sequential lifting like the one described for PB_{02} inequalities.

The corresponding *Honeycomb inequality* is then

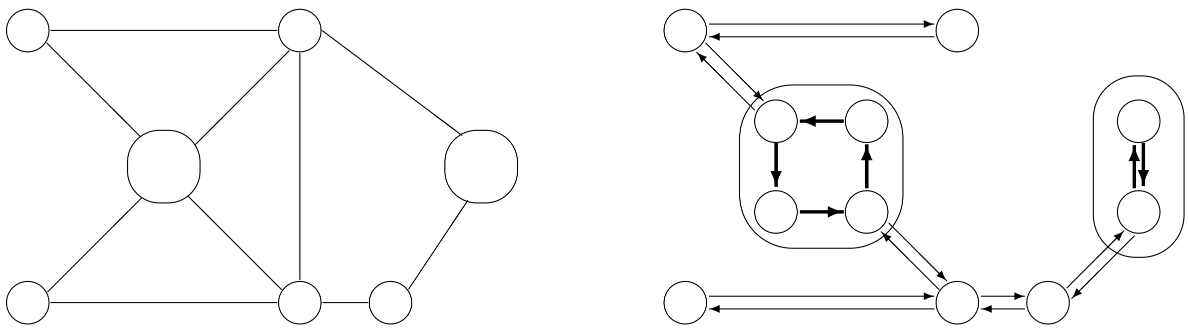
$$\sum_{(i,j) \in E} (a_{ij} x_{ij} + a_{ji} x_{ji}) \geq 2(K-1) + \sum_{(B_q^i, B_q^j) \in \mathcal{E}_R} c(B_q^i, B_q^j) \tag{11}$$

where $a_{ij} = c(B_r^i, B_q^j)$ if $i \in B_r^i$ and $j \in B_q^j$.

Note that when the configuration graph has no edges of type III, all the coefficients in the Honeycomb inequality can be computed in terms of the shortest distances in the graph $(\mathcal{B}, T)$. This occurs, for example, when every node B_q^i, $i \neq 0$, has degree 1 in $(\mathcal{B}, T)$. Given that the sequential lifting process for a set of edges guarantees the validity of an inequality if it is valid without this set of edges, and that this is also true for the facet inducing property, in what follows we will assume that the Honeycomb configuration has no edges of type III.

Before proving that the Honeycomb inequalities are facet-inducing for WGRP(G), let us show how to build tours for the WGRP on $G_{\mathcal{C}} = (\mathcal{B}, \mathcal{E})$ satisfying the Honeycomb inequality,

$F(x) \geq b_0$, as an equality. Let $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ denote the graph with node set $\bar{\mathcal{A}} = \{A_1, A_2, \ldots, A_L, A_{L+1}, \ldots, A_K\}$ and having an edge (A_i, A_j) for each edge (B_i^p, B_j^q) in T (figure 7a). Then, by traversing all the $K-1$ edges of any spanning tree of $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ once in each direction, and each required edge in $G_{\mathcal{C}}$ once, we obtain a tour of cost b_0 (figure 7b).



(a) (b)

Figure 7: Graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ and a WGRP tour on $G_{\mathcal{C}}$ satisfying (11) as an equality.

Theorem 9 *Honeycomb inequalities (11) are facet-inducing of WGRP(G) if the shrunk graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ is 2-connected.*

Proof: The dimension of WGRP($G_{\mathcal{C}}$) is $2|\mathcal{E}| - |\mathcal{B}| + 1 = 2|\mathcal{E}| - |T| = 2|\mathcal{C}| + 2|\mathcal{I}n| + |T|$ and this is the number of linear independent WGRP tours x satisfying the inequality as an equality that we build in what follows. Let x^d be the incidence vector of a set of L cycles traversing each required edge exactly once (such cycles exist since the required edges in $G_{\mathcal{C}}$ form even graphs).

For each variable x_{uv} associated with an edge in $\mathcal{C} \cup \mathcal{I}n$, consider the tour that uses x_{uv} exactly once, the path in T from v to u and the required edges as in x^d. The nodes still not visited are connected with edges in T used once in each direction. Then we have $2|\mathcal{C}|+2|\mathcal{I}n|$ tours.

Since the graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ is 2-connected, it can be proved that the only equation satisfied by the incidence vectors of all its spanning trees is $\sum_{e \in T_{\bar{\mathcal{A}}}} x_e = |\bar{\mathcal{A}}| - 1$. Hence, $|T_{\bar{\mathcal{A}}}| = |T|$ different spanning trees can be selected in such a way that their incidence vectors are linearly independent. For each spanning tree of graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ we can define a WGRP tour, such as the one shown in figure 7b, which uses each edge in the tree once in each direction and the required edges as in x^d.

If we subtract x^d from the $2|\mathcal{C}| + 2|\mathcal{I}n| + |T|$ tours above and we express them as rows of a matrix, then we obtain a full rank matrix. Hence, the Honeycomb inequality is facet-inducing for WGRP($G_{\mathcal{C}}$) and, by applying theorem 4, they are facet-inducing for WGRP(G). ♦

4.7 Honeycomb$_{02}$ inequalities

As for K-C and Path-Bridge inequalities, there is another version of the Honeycomb inequalities in which some edges have coefficients 0 and 2. Honeycomb$_{02}$ inequalities were presented in Corberán, Mejía & Sanchis (2005) for the MGRP. As in that paper, we only consider here the case $L = 1$, i.e. when only one R-set (or cluster of R-sets) is divided into γ_1 parts. We have not studied the general case because it is quite a bit more complicated. The reason is that after replacing some pairs of coefficients $(1, 1)$ in a Honeycomb configuration with $L > 1$ by coefficients $(0, 2)$, a non-valid inequality can be obtained.

With a similar configuration graph $G_{\mathcal{C}} = (\mathcal{B}, E^{\mathcal{C}})$ and tree T (see figure 8), the nodes in set A_1 are classified into two types, nodes of type $\mathcal{O}$ (those that will be incident with edges in T with coefficients 0 and 2) and nodes of type $\mathcal{I}$, in such a way that there is at least one node of each type. We will assume that $d(B_1^i, B_1^j) \geq 3$ if B_1^i, B_1^j are nodes of the same type, while $d(B_1^i, B_1^j) \geq 2$ if B_1^i, B_1^j are of different types. The coefficients are defined as:

- $c(B_1^i, B_1^j) = d(B_1^i, B_1^j) - 2$ if B_1^i, B_1^j are nodes of the same type.
- $c(B_1^i, B_1^j) = d(B_1^i, B_1^j) - 1$ if B_1^i, B_1^j are nodes of different types.
- $c(B_1^i, B_q^0) = 0$ and $c(B_q^0, B_1^i) = 2$ if B_1^i is of type $\mathcal{O}$ and $(B_1^i, B_q^0) \in T$
- for the remaining edges in T, $c(B_q^i, B_r^j) = c(B_r^j, B_q^i) = 1$
- otherwise, $c(B_q^i, B_r^j)$ is the shortest path cost in T from B_q^i to B_r^j

With these coefficients, the Honeycomb$_{02}$ inequality is expressed as in (11).

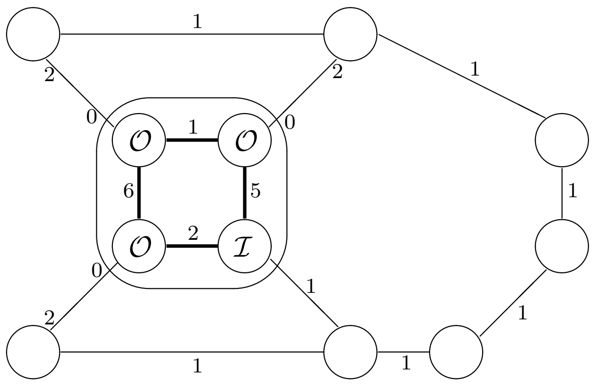


Figure 8: Honeycomb$_{02}$ Configuration graph with $L=1$

Theorem 10 *Honeycomb$_{02}$ inequalities are valid for WGRP(G)*

Proof: The proof is similar to that in Corberán Mejía & Sanchis (2005) for the MGRP and is omitted here for the sake of brevity. ♦

Theorem 11 *Honeycomb$_{02}$ inequalities are facet-inducing for WGRP(G) if the shrunk graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ is 2-connected.*

Proof: The proof is similar to that of theorem 9 and the details can be found in Plana (2005). The main difference is the way the tours associated with variables that correspond to traversing an edge $e = (u, v)$ in $\mathcal{C}$ from a node u of type $\mathcal{O}$ to a node v of type $\mathcal{I}$ are defined. If (u, v) is required, we construct a tour traversing each edge in the path in T from u to v twice and some edges in T once in each direction to connect those nodes B_i^0 that have still not been visited. In this case not all the required edges can be traversed as in x^d and some of them are redirected to obtain a symmetric graph. If e is not required, the tour is constructed in a similar way. ♦

4.8 Zigzag inequalities

Zigzag inequalities have been fully described in Corberán, Plana & Sanchis (2005). Here we briefly present the two families of Zigzag inequalities: the Even and Odd Zigzag inequalities.

Consider a partition of set V into 4 subsets, M^1, M^2, M^3 and M^4. Let α_{ij} denote the number of required edges in (M^i, M^j) and suppose one of the following conditions is satisfied:

Even case: M^1, M^2, M^3 and M^4 are R-even and $\alpha_{12} = \alpha_{34} = \alpha_{14} = \alpha_{23} = 0$

Odd (simple) case: M^1, M^2, M^3 and M^4 are R-odd and $\alpha_{12} + \alpha_{34} = \alpha_{14} + \alpha_{23}$

The configuration graphs $G_{\mathcal{C}}$ associated with Even and Odd (simple) Zigzag inequalities are shown in figures 9a and 9b (where the required edges are represented in bold lines) and are defined by the partition of V above and by the following coefficients:

$$c(M^1, M^2) = c(M^3, M^4) = 0, \quad c(M^2, M^1) = c(M^4, M^3) = 2, \quad c(M^i, M^j) = 1 \text{ otherwise.}$$

Then, the Even and Odd (simple) Zigzag inequalities can be expressed as:

$$x(\delta(M^1 \cup M^2)) + 2x(M^2 \!:\! M^1) + 2x(M^4 \!:\! M^3) \geq \alpha_{13} + \alpha_{24} + \alpha_{14} + \alpha_{23} + 2 \qquad (12)$$

where, given any set of edges (S_1, S_2), $x(S_1 \!:\! S_2) = \sum_{(i,j):\ i \in S_1, j \in S_2} x_{ij}$.

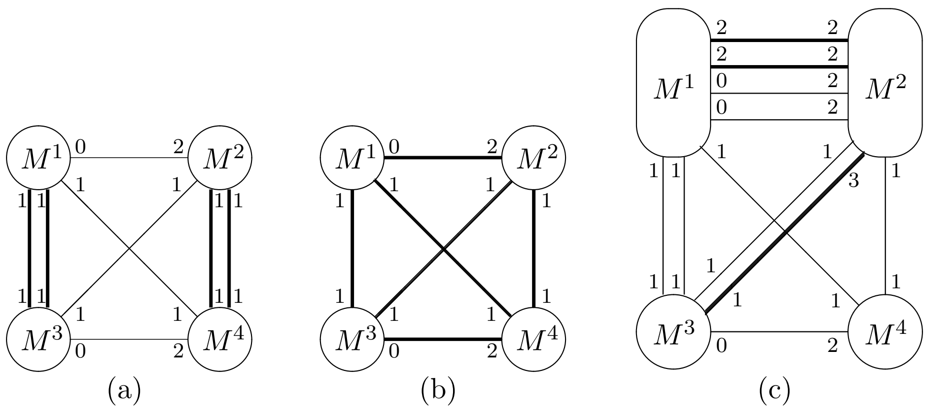


Figure 9: Zigzag configuration graphs.

In the above mentioned paper it is shown that Even and Odd (simple) Zigzag inequalities (12) are valid and facet-inducing for WGRP(G) if the following conditions hold: If $G_{\mathcal{C}}$ is a complete graph and $\alpha_{13}, \alpha_{24} \geq 2$ (even case) or $\alpha_{13} \geq |\alpha_{12} - \alpha_{14}| + 1$ and $\alpha_{24} \geq |\alpha_{12} - \alpha_{23}| + 1$ (odd case). All these inequalities are strong configuration inequalities. Furthermore, it is also proved that Odd (simple) Zigzag inequalities are equivalent to *3-wheel* inequalities for the WPP (Win, 1987).

Condition $\alpha_{12} + \alpha_{34} = \alpha_{14} + \alpha_{23}$ for the Odd (simple) case above is very strong and can be relaxed as follows to obtain a more general version of Odd Zigzag inequalities. Let M^1, M^2, M^3 and M^4 be a partition of V into R-odd sets. Let us represent the horizontal edges by $\mathcal{H} = (M^1, M^2) \cup (M^3, M^4)$ and the diagonal ones by $\mathcal{D} = (M^2, M^3) \cup (M^1, M^4)$ and note that $\mathcal{H} \cup \mathcal{D} = \delta(M^1 \cup M^3)$. Let $\mathcal{F} \subset (\mathcal{H} \cup \mathcal{D})_R$ be a subset of required edges (shown as bold lines in figure 9c, where all the edges are required) satisfying

$$|\mathcal{H}_R \backslash \mathcal{F}| + |\mathcal{D}_R \cap \mathcal{F}| = |\mathcal{D}_R \backslash \mathcal{F}| + |\mathcal{H}_R \cap \mathcal{F}| \qquad (13)$$

The more general Odd Zigzag inequality is

$$x(\delta(M^1 \cup M^2)) + 2x(M^2 \!:\! M^1) + 2x(M^4 \!:\! M^3) + 2x(F_{zz}) \geq \alpha_{13} + \alpha_{24} + \alpha_{14} + \alpha_{23} + 2|\mathcal{H} \cap \mathcal{F}| + 2 \quad (14)$$

where $x(F_{zz})$ denotes the variables associated with the edges in $\mathcal{F}$ in the direction given by the zigzag, i.e. $M^1 - M^2 - M^3 - M^4 - M^1$. The pairs of coefficients associated with each edge are shown in figure 9c. Odd Zigzag inequalities (14) are valid and facet-inducing for WGRP(G) if $G_{\mathcal{C}} \backslash \mathcal{F}$ is a complete graph, there are two required edges $e_{13} \in (M^1, M^3)$ and $e_{24} \in (M^2, M^4)$ and the remaining required edges in $G_{\mathcal{C}}$ can be oriented to induce a symmetric graph satisfying that all the edges in $(\mathcal{H} \cup \mathcal{D})_R \backslash \mathcal{F}$ are oriented in the direction of the zigzag and all the edges in $\mathcal{F}$ are oriented in the opposite direction.

Odd Zigzag inequalities (14) are weak configuration inequalities when $\mathcal{F} \neq \emptyset$. If $\mathcal{F} = \emptyset$ they reduce to inequalities (12) and condition (13) becomes $\alpha_{12} + \alpha_{34} = \alpha_{14} + \alpha_{23}$.

5 The Mixed GRP Polyhedron

As mentioned in the Introduction, the MGRP is the General Routing Problem defined on a mixed graph $G = (V, E, A)$, where V is the set of vertices, E is the set of edges and A is the set of arcs, and the MCPP is a special case of it.

Nobert & Picard (1996) proposed a formulation for the MCPP using only one variable per edge expressing the number of times a given edge is traversed in any direction. Such an approach has been also used for the MGRP in Corberán et al. (2003, 2005), where a wide polyhedral study is carried out. We call this kind of formulation F1. On the other hand, different formulations using two variables per edge were proposed for the MCPP by Christofides et al. (1984) and Ralphs (1993). They will be called F2 formulations. A theoretical and computational comparison of both formulations for ARP's on mixed graphs can be found in Corberán, Mota & Sanchis (2006). Although formulation F2 is perhaps a more intuitive approach to a problem defined on a mixed graph, polyhedral investigations on it have not been presented up to now. That is the subject of this section, where we will take advantage of the previous WGRP study.

As for the WGRP, we will assume again that the original graph has been transformed to satisfy $V = V_R$. In addition, it can also be assumed that $E \backslash E_R = \emptyset$, because each non-required edge can be replaced by a pair of opposite non-required arcs (Corberán Romero & Sanchis 2003). Although this last transformation is not as important for formulation F2 as it is for F1, in what follows we suppose we are working on a strongly connected graph $G = (V, E, A) := (V_R, E_R, A_R \cup A_{NR})$.

An MGRP instance can be transformed into a WGRP instance in which the cost of traversing the edges in the forbidden directions are set to infinity. Then, the variables associated with the forbidden directions can be removed from the WGRP formulation, thus obtaining an MGRP formulation with just one variable associated with each arc and two variables associated with each edge. Hence, formulation F2 for the MGRP is exactly the one presented for the WGRP in section 2 except that some variables do not exist (all the variables x_{ji} associated with arcs (i, j)). Therefore, constraints (1) and (4) are replaced by

$$
\begin{aligned}
x_{ij} + x_{ji} &\geq 1 \qquad \forall (i, j) \in E (= E_R) && (15)\\
x_{ij} &\geq 1 \qquad \forall (i, j) \in A_R && (16)\\
x_{ij} &\geq 0 \qquad \text{otherwise} && (17)
\end{aligned}
$$

Let MGRP(G) represent the convex hull of all the feasible solutions of the F2 formulation for the MGRP on $G = (V, E, A)$ (MGRP tours). From the results obtained for the WGRP polyhedron, it can be shown that, if G is a strongly connected graph, the dimension of MGRP(G) is $2|E| + |A| - |V| + 1$. Furthermore, *traversing* and *trivial* inequalities are facet-inducing for MGRP(G) if the corresponding link (edge or arc) $e = (i, j)$ satisfies that $G \backslash \{e\}$ is a strongly connected graph. All other facet-inducing inequalities for the MGRP(G) are *weak configuration inequalities* and their associated configuration graphs can have 'parallel' variables with different coefficients associated with their edges.

Obviously any valid inequality for the WGRP is also a valid inequality for the MGRP. Furthermore, the facet-inducing inequalities for WGRP(G) described in this paper are also facet-inducing for MGRP(G) if the arcs in G have the 'appropriate direction', i.e. if the removed variables do not avoid the existence of all the tours needed to define a facet.

Before presenting the following inequalities, we need some more notation. If $G = (V_R, E_R, A_R \cup A_{NR})$ is a mixed graph and S_1, S_2 are two vertex sets, we denote

$$E(S_1, S_2) = \{e = (i,j) \in E : \ i \in S_1, \ j \in S_2 \quad \text{or} \quad j \in S_1, \ i \in S_2\}$$

$$A(S_1, S_2) = \{a = (i,j) \in A : \ i \in S_1, \ j \in S_2\}$$

$$A_R(S_1, S_2) = \{a = (i,j) \in A_R : \ i \in S_1, \ j \in S_2\}$$

A mixed graph is called *balanced* if, for every $S \subset V$, the difference between the number of arcs leaving S and the number of arcs entering S is less than or equal to the number of edges in $E(S, V \setminus S)$.

It can be shown that the following inequalities are facet-inducing for MGRP(G):

- Connectivity inequalities (3), if graphs $G(S)$ and $G(V \setminus S)$ are strongly connected.
- R-odd cut inequalities (6), if $G(S)$ and $G(V \setminus S)$ are strongly connected and $\epsilon > |\alpha - \beta|$, where $\epsilon = |E(S, V \setminus S)|$, $\alpha = |A_R(S, V \setminus S)|$ and $\beta = |A_R(V \setminus S, S)|$.
- K-C inequalities (7) if $\epsilon \geq |\alpha - \beta|$ and K-C$_{02}$ inequalities (8) if $\epsilon \geq |\alpha + 1 - \beta| + 1$, where $\epsilon = |E(M_0, M_K)|$, $\alpha = |A_R(M_0, M_K)|$ and $\beta = |A_R(M_K, M_0)|$. Here we assume that all the sets $A(M_i, M_{i+1})$, $A(M_{i+1}, M_i)$ are non-empty.
- Path-Bridge inequalities (9), if $\epsilon \geq |\alpha - \beta|$ for P odd and $\epsilon \geq |\alpha - \beta| + 1$ for P even, and Path-Bridge$_{02}$ inequalities (10), if $\epsilon \geq |\alpha + P_{AZ} - \beta - P_{ZA}| + 1$, where P_{AZ} and P_{ZA} are the number of paths of types AZ and ZA, respectively, $\epsilon = |E(A, Z)|$, $\alpha = |A_R(A, Z)|$ and $\beta = |A_R(Z, A)|$. The existence of all the pairs of opposite (non-required) arcs forming the P paths is assumed here.
- Honeycomb inequalities (11), if the subgraph of $G_{\mathcal{C}}$ induced by the required links $E \cup A_R$ is balanced and the graph $(\bar{A}, T_{\bar{A}})$ is 2-connected. Honeycomb$_{02}$ inequalities if, in addition, for each edge or arc (u, v), with node u of type $\mathcal{O}$ and node v of type $\mathcal{I}$, the subgraph of $G_{\mathcal{C}}$ induced by the required links plus two extra arcs from u to v is a balanced graph. We assume the existence of all the pairs of opposite (non-required) arcs forming the tree T.
- Zigzag inequalities (12) and (14), if there are 4 links defining a zigzag M^1-M^2-M^3-M^4-M^1 and,
  (a) for the even case: the subgraph of $G_{\mathcal{C}}$ induced by the required links is balanced.
  (b) for the odd case: there are two required edges $e_{13} \in (M^1, M^3)$ and $e_{24} \in (M^2, M^4)$, and the remaining required edges in $G_{\mathcal{C}}$ can be oriented in such a way that, together with the required arcs in $G_{\mathcal{C}}$, they induce a symmetric graph in which all the links in $(\mathcal{H} \cup \mathcal{D})_R \setminus \mathcal{F}$ are oriented in the direction of the zigzag and all the links in $\mathcal{F}$ are oriented in the opposite direction.

Given that the MCPP and MRPP are special cases of the MGRP, the above inequalities are also facet-inducing for the polyhedra associated with these problems when they are formulated with two variables.

6 Conclusions

Given a 'windy' graph (an undirected graph with two costs associated with each edge, representing the costs of traversing it in each direction) and given a subset of 'required' edges and a subset of 'required' vertices, the Windy General Routing Problem consists of finding a minimum cost tour traversing each required edge and visiting each required vertex at least once. Since undirected, directed and mixed graphs can be modelled with a windy graph, and the GRP generalizes the CPP, RPP and GTSP, all the routing problems represented in figure 1 are special cases of the WGRP. Therefore, most of the theoretical and practical results obtained for the WGRP can be applied to these important problems.

In this paper we have presented a polyhedral study of the WGRP. In addition to describing some basic properties of its associated polyhedron, WGRP(G), we have shown that all its facet-inducing inequalities, except those equivalent to the trivial or the traversing ones, are *weak* configuration inequalities. This is a generalization of the (strong) configuration property related to the polyhedra associated with the problems in figure 1. Then, each facet-inducing inequality can be represented by means of a configuration graph and we have proved that, if a given inequality is facet-inducing for the polyhedron defined on this simple configuration graph, then it is also facet-inducing for the whole polyhedron WGRP(G).

We have also described wide families of facet-inducing inequalities. This partial description of WGRP(G) has been recently used in Corberán, Plana and Sanchis (2005b) to develop a branch & cut algorithm capable of solving WGRP instances of large size and other difficult instances of some of the problems shown in figure 1. Although slightly different versions of most of these inequalities were already known about for other routing problems polyhedra, it is in this paper that they have been generalized for the WGRP and proved to be facet-inducing of WGRP(G).

On the other hand, a partial description of the polyhedra associated with ARP's defined on mixed graphs whose formulation uses two variables per edge has also been obtained. This polyhedral study had not been done before.

A final contribution of this paper is that it also provides a global view of most of the known facet-inducing inequalities for the polyhedra associated with the problems in figure 1, since versions of facet-inducing inequalities for other problems with formulations using only one variable per link are very similar.

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References

[1] A.A. Assad & B.L. Golden, Arc Routing Methods and Applications, in: M.O. Ball, T.L. Magnanti, C.L. Monma & G.L. Nemhauser (Eds.), Network Routing. Handbooks of Operations Research and Management Science 8, North Holland, Amsterdam, 1995, pp. 375-483.

[2] E. Benavent, A. Carrotta, A. Corberán, J.M. Sanchis & D. Vigo (2003): "Lower Bounds and Heuristics for the Windy Rural Postman Problem". Technical Report TR03-2003. Department of Statistics and OR, University of Valencia (Spain). Submitted to *EJOR*.

[3] S. Chopra & G. Rinaldi (1996): "The Graphical Asymmetric Traveling Salesman Polyhedron: Symmetric Inequalities". *SIAM J. Discrete Math.* 9, 602-624.

[4] N. Christofides, E. Benavent, V. Campos, A. Corberán & E. Mota (1984): "An optimal method for the mixed postman problem". In P. Thoft-Christensen (Ed.) *System Modelling and Optimization*, Lecture Notes in Control and Inf. Sciences, 59. Berlin: Springer.

[5] N. Christofides, V. Campos, A. Corberán & E. Mota (1986): "An algorithm for the Rural Postman Problem on a directed graph". *Mathematical Programming Study* 26, 155-166.

[6] A. Corberán, G. Mejía & J.M. Sanchis (2005): "New Results on the Mixed General Routing Problem". *Operations Research* 53, 363-376.

[7] A. Corberán, E. Mota & J.M. Sanchis (2006): "A Comparison of Two Different Formulations for Arc Routing Problems on Mixed Graphs". To appear in *Computers & Operations Research.*

[8] A. Corberán, I. Plana & J.M. Sanchis (2005a): "Zigzag inequalities: A new class of facet-inducing inequalities for Arc Routing Problems". To appear in *Mathematical Programming.*

[9] A. Corberán, I. Plana & J.M. Sanchis (2005b): "A Branch & Cut Algorithm for the Windy General Routing Problem". Technical Report TR04-2005. Department of Statistics and OR, University of Valencia (Spain). Submitted to *Networks.*

[10] A. Corberán, A. Romero & J.M. Sanchis (2003): "The Mixed General Routing Problem Polyhedron". *Mathematical Programming* 96, 103-137.

[11] A. Corberán & J.M. Sanchis (1994): "A polyhedral approach to the Rural Postman Problem". *European Journal of Operational Research* 79, 95-114.

[12] A. Corberán & J.M. Sanchis (1998):"The General Routing Problem: facets from the RPP and GTSP polyhedra". *European Journal of Operational Research* 108, 538-550.

[13] G. Cornuèjols, J. Fonlupt & D. Naddef (1985): "The traveling salesman problem on a graph and some related integer polyhedra". *Mathematical Programming* 33, 1-27.

[14] M. Dror (Ed.), ARC Routing: Theory, Solutions and Applications, Kluwer Academic Publishers, Massachusetts, 2000.

[15] J. Edmonds & E. Johnson (1973):"Matching, Euler Tours and the Chinese Postman Problem". *Mathematical Programming* 5, 88-124.

[16] H.A. Eiselt, M. Gendreau & G. Laporte (1995a): "Arc-Routing Problems, Part 1: the Chinese Postman Problem". *Operations Research* 43, 231-242 .

[17] H.A. Eiselt, M. Gendreau & G. Laporte (1995b): "Arc-Routing Problems, Part 2: the Rural Postman Problem". *Operations Research* 43, 399-414.

[18] B. Fleischmann (1985): "A cutting-plane procedure for the traveling salesman problem on a road network". *European Journal of Operational Research* 21, 307-317.

[19] M. Grötschel & Z. Win (1988): "On the windy postman polyhedron". *Report No. 75*, Schwerpunkt-progam der Deutschen Forschungsgeneinschaft, Universität Augsburg, Germany.

[20] M. Grötschel & Z. Win (1992): "A Cutting Plane Algorithm for the Windy Postman Problem". *Mathematical Programming* 55, 339-358.

[21] M. Guan (1962): "Graphic Programming using odd and even points". *Chinese Mathematics* 1, 273-277.

[22] M. Guan (1984): "On the Windy Postman Problem". *Discrete Applied Mathematics*, 9, 41-46.

[23] H. Gun (1993): "Polyhedral structure and efficient algorithms for certain classes of directed rural postman problem". *PhD Dissertation*, University of Maryland at College Park, USA.

[24] A. Letchford (1997): "New inequalities for the General Routing Problem". *European Journal of Operational Research* 96, 317-322.

[25] E. Minieka (1979): "The Chinese Postman Problem for Mixed Networks". *Management Science* 25, 643-648.

[26] D. Naddef & G. Rinaldi (1991): "The Symmetric Traveling Salesman Polytope and its Graphical Relaxation: Composition of Valid Inequalities". *Mathematical Programming* 51, 359-400.

[27] Y. Nobert & J.C. Picard (1996): "An optimal algorithm for the mixed chinese postman problem". *Networks* 27, 95-108.

[28] C. Orloff (1974):"A fundamental problem in vehicle routing". *Networks* 27, 95-108.

[29] I. Plana (2005): "The Windy General Routing Problem". *PhD Dissertation*, University of Valencia, Spain.

[30] T. Ralphs (1993): "On the Mixed Chinese Postman Problem". *Operations Research Letters* 14, 123–127.

[31] A. Romero (1997): "On the Mixed Rural Postman Problem". *PhD Dissertation*, University of Valencia, Spain.

[32] J.V. Savall (1990): "Polyhedral results and approximate algorithms for the Directed Rural Postman Problem". *PhD Dissertation*, University of Valencia, Spain.

[33] Z. Win (1987): "Contributions to Routing Problems". *PhD Dissertation*, University of Augsburg, Germany.