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Additional Information

Robust Compensation of Delay and Diffusive Actuator Dynamics Without Distributed Feedback

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Abstract—This paper deals with robust observer-based output-feedback stabilization of systems whose actuator dynamics can be described in terms of partial differential equations (PDEs). More specifically, delay dynamics (first-order hyperbolic PDE) and diffusive dynamics (parabolic PDE) are considered. The proposed controllers have a PDE observer-based structure. The main novelty is that stabilization for an arbitrarily large delay or diffusion domain length is achieved, while distributed integral terms in the control law are avoided. The exponential stability of the closed-loop in both cases is proved using Lyapunov functionals, even in the presence of small uncertainties in the time delay or the diffusion coefficient. The feasibility of this approach is illustrated in simulations using a second-order plant with an exponentially unstable mode.

I. INTRODUCTION

TIME-delay systems have received growing attention from researchers over the past years as they are ubiquitous in engineering applications such as rolling mills, chemical reactors, oil or gas factories and networked control systems, among others [1]. Large delays often lead to closed-loop instability if they are not taken into account, and limit the achievable performance of conventional controllers [2]. More recently, the ability to manipulate flow properties has also become a question of major technological importance, in which convection (hyperbolic PDE dynamics) and/or diffusion (parabolic PDE dynamics) occur [3]. Topics on compensating infinite-dimensional actuator dynamics are introduced in [4].

Traditional predictor-based controllers for time-delay systems, as developed in [5], [6], [7], use control laws in the form of integral equations, whose discretization may cause problems in their practical implementation [8], [9]. The application of new backstepping techniques developed for first-order hyperbolic PDEs has also led to equivalent results, when applied to time-delay systems [10]. Modeling the delay phenomenon as a transport PDE has been shown to provide a solid framework with ample tools for analysis and design [11]. In this context, input-delay systems are just a particular case of a broader class of systems with infinite-dimensional actuator dynamics, which have attracted attention recently, and whose stabilizing controllers also involve distributed (sometimes double) integrals of the actuator state [12], [13].

Stabilization of input-delayed systems without distributed terms has been pursued in different directions. A successful

approach consists of ignoring the distributed terms in the traditional predictor leading to a static feedback control law [14], [15]. Another approach is based on designing observers to estimate the predicted state, rather than explicitly computing it. The problem is then translated to that of state observation via a delayed output measurement. The latter has been recurrently approached in the literature without the need for integral terms [16], [17], [18]. However, it has been only recently that this fact has been used to deal with input-delayed systems. This idea was first devised in [19] and further extended with an LMI-based design methodology in [20]. In the past few years, this technique has been extended to systems with input/output delays and time-varying delays [21], [22]. In these works, a chain of sequential observers is used, in which each of the components estimates a prediction of the state over an interval, whose length equals a fraction of the delay, achieving asymptotic stability for arbitrarily large delays as the number of sequential predictors goes to infinity.

The present work extends the ideas introduced recently in [23], where the predicted state (for the delay case) or the "anti-diffused" state (for the diffusive case) are estimated using suitable observers. Instead of using an infinite chain of sequential observers, the infinite dimensionality in our approach stems from the fact that the observer is given as a PDE. The backstepping observer design techniques developed in [10], [12] are exploited. Furthermore, uncertainties in the delay or the diffusion coefficient are considered, which is a departure from [23] and makes the analysis substantially more complicated. Robustness to delay uncertainties in the PDE framework has been previously investigated in [24], [25], [26]. Upper bounds on the uncertainties that guarantee exponential stability of the closed-loop system are derived. In the nominal case, stabilization is achieved for any arbitrarily large delay or diffusion domain, even for unstable systems. Moreover, the controller design as simple as that of a conventional observer-based state feedback. The proposed methodology is illustrated using a second-order system with an exponentially unstable mode.

Notation: The state of a PDE is represented by a function $u(x, t)$, where t is time and x is referred to as the spatial variable. The 2-norm of a finite-dimensional vector $X(t)$ is denoted by $|X(t)|$. The spaces $L_2([a, b])$ and $H_1([a, b])$ are used, defined, respectively, as the space of square-integrable functions and the space of functions whose derivative is square-integrable, in the interval $[a, b]$. If $u \in L_2$, the corresponding norm is simply denoted and computed by $\|u\|^2 = \int_a^b |u(x, t)|^2 \, \mathrm{d}x$, whereas if $u \in H_1$, then the Sobolev norm is defined by $\|u\|_{H_1}^2 = \|u\|^2 + \|u_x\|^2$. The minimum and maximum eigenvalues of a symmetric positive definite matrix,

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$Q > 0$, are denoted by $\underline{\lambda}(Q)$ and $\overline{\lambda}(Q)$, respectively.

II. PRELIMINARIES

This paper deals with a class of systems whose actuator dynamics can be described in terms of PDEs. The first type of systems considered in this work are those described by

$$\dot{X}(t) = AX(t) + Bu(\Delta D, t), \qquad (1)$$
$$u_t(x, t) = u_x(x, t), \qquad (2)$$
$$u(D_0, t) = U(t), \qquad (3)$$
$$Y(t) = CX(t), \qquad (4)$$

where $X \in \mathbb{R}^n$ is the ODE state, A, B and C are matrices with appropriate dimensions and $u \in \mathcal{C}^1$ is the PDE state, whose spatial domain is given by

$$x \in [\underline{x}, D_0], \qquad \underline{x} = \min\{0, \Delta D\}.$$

The system (1)-(3) is equivalent to an LTI system with an input delay of $D = D_0 - \Delta D$ units of time, where $D_0 \geq 0$ is the assumed plant delay and ΔD is a bounded delay mismatch. To see this, note that the solution of (2)-(3) is given by

$$u(x, t) = U(t - (D_0 - x)), \qquad (5)$$

and thus $u(\Delta D, t) = U(t - D)$. Clearly, it is assumed that $\Delta D \leq D_0$ so that the total delay $D \geq 0$ remains positive. If the whole state is available and ΔD is known, the global asymptotic stabilization to zero of (1)-(3) can be achieved by the predictive feedback control law $U(t) = KP(t)$, where

$$P(t) = e^{AD}X(t) + \int_{\Delta D}^{D_0} e^{A(D_0 - y)}Bu(y, t)\, dy, \qquad (6)$$

and the vector K is such that $A + BK$ is Hurwitz. This follows from the fact that the right-hand side of (6) equals $X(t + D)$, which can be seen using (5) and applying a change of variables.

The second type of systems treated here are those described by

$$\dot{X}(t) = AX(t) + Bu(0, t), \qquad (7)$$
$$u_t(x, t) = \epsilon u_{xx}(x, t), \qquad (8)$$
$$u_x(0, t) = 0, \qquad (9)$$
$$u(D, t) = U(t), \qquad (10)$$
$$Y(t) = CX(t). \qquad (11)$$

where $D \geq 0$ is the spatial domain length, $\epsilon = \epsilon_0 + \Delta\epsilon$ is the diffusion coefficient, in which $\epsilon_0 \neq 0$ is known and $\Delta\epsilon > -\epsilon_0$ is a small additive uncertainty. In this case, the actuator dynamics (8) is governed by a parabolic PDE, the so-called heat equation. Therefore, the control action undergoes a diffusive process before reaching the ODE.

The similarities between (1)-(3) and (7)-(10) go beyond the obvious ones. It turns out that, if the whole state is available and ϵ is known, a stabilizing control law for (7)-(10) is given by $U(t) = K\Pi(t)$, where

$$\Pi(t) = M(D)X(t) + \int_0^D m(D - y)Bu(y, t)\, dy, \qquad (12)$$

the vector K is again to be chosen such that $A + BK$ is Hurwitz, and

$$m(s) = \frac{1}{\epsilon} \int_0^s M(\xi)\, d\xi, \qquad (13)$$

$$M(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & \frac{A}{\epsilon} \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad (14)$$

being $I \in \mathbb{R}^{n \times n}$ the identity matrix. This result is a slightly modified version of Theorem 1 in [12].

Remark 1. *While $P(t)$ in (6) is the "predicted" state D units of time ahead, i.e., $P(t) = X(t + D)$, we shall refer to $\Pi(t)$ in (12) as the "anti-diffused" state.*

A handicap of the control laws (6) and (12) lies in the fact that they are actually integral equations, since the control action appears explicitly on the left-hand side and under an integral sign on the right-hand side. Therefore, the discretization of the integral term for its implementation can lead to instability [9]. Furthermore, the whole state needs to be accessible, which is often not the case in practice. In what follows, an output-based control strategy is introduced, by means of which exponential stabilization is achieved. The key idea behind the proposed control laws is to design observers to estimate the predicted state $P(t)$ for a system with delay actuator dynamics, or the "anti-diffused" state $\Pi(t)$, for a system with diffusive actuator dynamics. To this end, it is assumed that the pair (A, B) is controllable and the pair (A, C) is observable.

III. DELAY ACTUATOR DYNAMICS

Theorem 1 below introduces an observer-based controller for (1)-(4) and guarantees the closed-loop exponential stability. First, the closed-loop equations (31)-(36) are obtained, which are composed of two systems (the state and the observer error, as it usual in observer-based controllers). Because of the uncertainty, these are coupled to a third one, given by (37)-(39). A suitable (invertible) backstepping transformation is proposed to map these systems into target systems whose stability is proved via Lyapunov analysis. Exponential stability of the target system is established by (55). After that, the stability of the original systems is proved using the inverse transformations.

Theorem 1. *Consider the closed-loop system composed of (1)-(4) and the observer-based controller*

$$\dot{\hat{P}}(t) = A\hat{P}(t) + BU(t) + e^{AD_0}L(Y(t) - \hat{v}(0, t)), \qquad (15)$$
$$\hat{v}_t(x, t) = \hat{v}_x(x, t) + Ce^{Ax}L(Y(t) - \hat{v}(0, t)), \qquad (16)$$
$$\hat{v}(D_0, t) = C\hat{P}(t), \qquad (17)$$
$$U(t) = K\hat{P}(t), \qquad (18)$$

where K and L are such that $A + BK$ and $A - LC$ are Hurwitz. Then, there exists a $\delta > 0$ such that for all $|\Delta D| \leq \delta$, i.e., for all $D \in [D_0 - \delta, D_0 + \delta]$, the zero solution of the $(X, u, \hat{P}, \hat{v})$-system is exponentially stable, that is, there exist positive constants R and ρ such that for all initial conditions

$(X_0, u_0, \hat{P}_0, \hat{v}_0) \in \mathbb{R}^n \times L_2(\underline{x}, D_0) \times \mathbb{R}^n \times H_1(\underline{x}, D_0),$ *the following holds:*

$$\Upsilon(t) \leq R\Upsilon(0)e^{-\rho t},$$

where

$$\Upsilon(t) = |X(t)|^2 + \|u(t)\|^2_{L_2[\underline{x}, D_0]} + |\hat{P}(t)|^2 + \|\hat{v}(t)\|^2_{H_1[\underline{x}, D_0]}.$$

Proof. Let us define a distributed output prediction by

$$v(x,t) = Ce^{A(x-\Delta D)}X(t) + C \int_{\Delta D}^{x} e^{A(x-y)} Bu(y,t)\, \mathrm{d}y. \tag{19}$$

Computing the time derivative of (6) and the spatial and temporal derivatives of (19), and using (1)-(4), one arrives at the following ODE-PDE cascade system

$$\dot{P}(t) = AP(t) + BU(t), \tag{20}$$
$$v_t(x,t) = v_x(x,t), \tag{21}$$
$$v(D_0, t) = CP(t), \tag{22}$$
$$Y(t) = v(\Delta D, t), \tag{23}$$

where an integration by parts in the variable y and the fact that A and e^{Ax} commute for all x was used in (20)-(21); and (22)-(23) follow simply by evaluating (19) at $x = D_0$ and $x = \Delta D$, respectively. The original input-delay system (1)-(4) has been then mapped into the virtual system (20)-(23), in which the delay is affecting the output. Let us introduce the error variables

$$\tilde{P}(t) \triangleq P(t) - \hat{P}(t), \tag{24}$$
$$\tilde{v}(x,t) \triangleq v(x,t) - \hat{v}(x,t). \tag{25}$$

Differentiating (24)-(25), using (15)-(17) and (20)-(23), and adding and subtracting $v(0,t)$, the observer error system can be written as

$$\dot{\tilde{P}}(t) = A\tilde{P}(t) - e^{AD_0}L\tilde{v}(0,t) - e^{AD_0}L\mathcal{I}(t), \tag{26}$$
$$\tilde{v}_t(x,t) = \tilde{v}_x(x,t) - Ce^{Ax}L\tilde{v}(0,t) - Ce^{Ax}L\mathcal{I}(t), \tag{27}$$
$$\tilde{v}(D_0, t) = C\tilde{P}(t). \tag{28}$$

where $\mathcal{I}(t) = v(\Delta D, t) - v(0,t) = \int_0^{\Delta D} v_x(x,t)\, \mathrm{d}x$, which follows from the Newton-Leibniz formula. Now, let us introduce the mappings $(X, u) \mapsto (X, w)$ and $(\tilde{P}, \tilde{v}) \mapsto (\tilde{P}, \tilde{w})$, defined by the backstepping transformations

$$w(x,t) = u(x,t) - Ke^{A(x-\Delta D)}X(t)$$
$$\quad - \int_{\Delta D}^{x} Ke^{A(x-y)} Bu(y,t)\, \mathrm{d}y, \tag{29}$$
$$\tilde{w}(x,t) = \tilde{v}(x,t) - Ce^{A(x-D_0)}\tilde{P}(t). \tag{30}$$

Using (18) and the transformations (29)-(30), the systems (1)-(3), (26)-(28) are mapped into

$$\dot{X}(t) = (A + BK)X(t) + Bw(\Delta D, t), \tag{31}$$
$$w_t(x,t) = w_x(x,t), \tag{32}$$
$$w(D_0, t) = -K\tilde{P}(t), \tag{33}$$

$$\dot{\tilde{P}}(t) = (A - e^{AD_0}LCe^{-AD_0})\tilde{P}(t) - e^{AD_0}L\tilde{w}(0,t)$$
$$\quad - e^{AD_0}L\mathcal{I}(t), \tag{34}$$
$$\tilde{w}_t(x,t) = \tilde{w}_x(x,t), \tag{35}$$
$$\tilde{w}(D_0, t) = 0, \tag{36}$$

respectively, where (32) followed from an integration by parts, (34) used (30) with $x = 0$, and (35) used the fact that A and e^{Ax} commute for all x. Also, using (18) and (24), the system (20)-(22) can be written as

$$\dot{P}(t) = (A + BK)P(t) - BK\tilde{P}(t), \tag{37}$$
$$v_t(x,t) = v_x(x,t), \tag{38}$$
$$v(D_0, t) = CP(t). \tag{39}$$

Gathering previous expressions and after some straightforward manipulations, the overall transformation $(X, u, \hat{P}, \hat{v}) \mapsto (X, w, \tilde{P}, \tilde{w}, P, v)$ can be written as

$$w(x,t) = u(x,t) - Ke^{A(x-\Delta D)}X(t)$$
$$\quad - \int_{\Delta D}^{x} Ke^{A(x-y)} Bu(y,t)\, \mathrm{d}y, \tag{40}$$

$$\tilde{P}(t) = e^{AD}X(t) + \int_{\Delta D}^{D_0} e^{A(D_0-y)} Bu(y,t)\, \mathrm{d}y$$
$$\quad - \hat{P}(t), \tag{41}$$

$$\tilde{w}(x,t) = Ce^{A(x-D_0)}\hat{P}(t) - \hat{v}(x,t)$$
$$\quad - C\int_{x}^{D_0} e^{A(x-y)} Bu(y,t)\, \mathrm{d}y, \tag{42}$$

$$P(t) = e^{AD}X(t) + \int_{\Delta D}^{D_0} e^{A(D_0-y)} Bu(y,t)\, \mathrm{d}y, \tag{43}$$

$$v(x,t) = Ce^{A(x-\Delta D)}X(t) + C\int_{\Delta D}^{x} e^{A(x-y)} Bu(y,t)\, \mathrm{d}y. \tag{44}$$

The inverse transformation is given by

$$u(x,t) = w(x,t) + Ke^{(A+BK)(x-\Delta D)}X(t)$$
$$\quad + \int_{\Delta D}^{x} Ke^{(A+BK)(x-y)} Bw(y,t)\, \mathrm{d}y, \tag{45}$$
$$\hat{P}(t) = P(t) - \tilde{P}(t), \tag{46}$$
$$\hat{v}(x,t) = v(x,t) - \tilde{w}(x,t) - Ce^{A(x-D_0)}\tilde{P}(t). \tag{47}$$

where the fact that (45) is the inverse of (40) is proved in Appendix A.

In order to assess stability, let us choose the Lyapunov functional

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{48}$$

where

$$V_1(t) = X(t)^T S_1 X(t) + \frac{a_1}{2} \int_{\Delta D}^{D_0} e^x w(x,t)^2\, \mathrm{d}x$$
$$\quad + \frac{a_1}{4} \int_{\underline{x}}^{\Delta D} e^x w(x,t)^2\, \mathrm{d}x,$$

$$V_2(t) = b_0 P(t)^T S_1 P(t) + \frac{b_1}{2} \int_{\underline{x}}^{D_0} e^x v(x,t)^2\, \mathrm{d}x$$
$$\quad + \frac{b_2}{2} \int_{\underline{x}}^{D_0} e^x v_x^2(x,t)\, \mathrm{d}x,$$

$$V_3(t) = c_0 \tilde{P}(t)^T T^T S_2 T \tilde{P}(t) + \frac{c_1}{2} \int_0^{D_0} e^x \tilde{w}(x,t)^2 \, \mathrm{d}x$$
$$+ \frac{c_1}{4} \int_{\underline{x}}^0 e^x \tilde{w}^2(x,t) \, \mathrm{d}x + \int_{\underline{x}}^{D_0} e^x \tilde{w}_x^2(x,t) \, \mathrm{d}x,$$

the constants $a_i, b_i, c_i > 0$ are specified in the subsequent analysis, $T = e^{-AD_0}$ is defined for the sake of brevity, and $S_1 = S_1^T > 0$, $S_2 = S_2^T > 0$ are the solutions to the Lyapunov equations

$$S_1(A+BK) + (A+BK)^T S_1 = -Q_1, \tag{49}$$
$$S_2(A-LC) + (A-LC)^T S_2 = -Q_2, \tag{50}$$

for some symmetric positive definite matrices Q_1 and Q_2, respectively. Using integration by parts[1], the time derivative of $V_1(t)$ along the trajectories of (31)-(36) is given by

$$\dot{V}_1(t) = -X^T Q_1 X + 2X^T S_1 B w(\Delta D, t)$$
$$+ \frac{a_1}{2} e^{D_0} w(D_0, t)^2 - \frac{a_1}{2} e^{\Delta D} w(\Delta D, t)^2$$
$$- \frac{a_1}{2} \int_{\Delta D}^{D_0} e^x w(x,t)^2 \, \mathrm{d}x + \frac{a_1}{4} e^{\Delta D} w(\Delta D, t)^2$$
$$- \frac{a_1}{4} e^{\underline{x}} w(\underline{x}, t)^2 - \frac{a_1}{4} \int_{\underline{x}}^{\Delta D} e^x w^2(x,t) \, \mathrm{d}x$$
$$\leq -\frac{\underline{\lambda}(Q_1)}{2} |X|^2 + \left(\frac{2|S_1 B|^2}{\underline{\lambda}(Q_1)} - \frac{a_1}{4} e^{\Delta D} \right) w(\Delta D, t)^2$$
$$+ \frac{a_1}{2} e^{D_0} |K|^2 |\tilde{P}|^2 - \frac{a_1}{2} \int_{\Delta D}^{D_0} e^x w(x,t)^2 \, \mathrm{d}x$$
$$- \frac{a_1}{4} \int_{\underline{x}}^{\Delta D} e^x w(x,t)^2 \, \mathrm{d}x \tag{51}$$

where (49) was used and Young's inequality was employed to upper bound the second term. Proceeding in a very similar fashion, the derivative of $V_2(t)$ along the trajectories of (37)-(39) is obtained as

$$\dot{V}_2 = -b_0 P^T Q_1 P - 2b_0 P^T S_1 BK \tilde{P}$$
$$+ \frac{b_1}{2} e^{D_0} v(D_0, t)^2 - \frac{b_1}{2} e^{\underline{x}} v^2(\underline{x}, t)$$
$$- \frac{b_1}{2} \int_{\underline{x}}^{D_0} e^x v(x,t)^2 \mathrm{d}x + \frac{b_2}{2} e^{D_0} v_t(D_0, t)^2$$
$$- \frac{b_2}{2} e^{\underline{x}} v_t(\underline{x}, t)^2 - \frac{b_2}{2} \int_{\underline{x}}^{D_0} e^x v_x(x,t)^2 \mathrm{d}x$$
$$\leq \left(-\frac{b_0 \underline{\lambda}(Q_1)}{2} + \frac{b_1}{2} e^{D_0} |C|^2 + \frac{b_2}{2} e^{D_0} \kappa_1 \right) |P|^2$$
$$+ \left(\frac{2b_0 |S_1 BK|^2}{\underline{\lambda}(Q_1)} + \frac{b_2}{2} e^{D_0} \kappa_2 \right) |\tilde{P}|^2$$
$$- \frac{b_1}{2} \int_{\underline{x}}^{D_0} e^x v(x,t)^2 \mathrm{d}x - \frac{b_2}{2} \int_{\underline{x}}^{D_0} e^x v_x(x,t)^2 \mathrm{d}x, \tag{52}$$

where the bound

$$v_t(D_0, t)^2 \leq \kappa_1 |P|^2 + \kappa_2 |\tilde{P}|^2,$$

[1]By the differentiation under the integral sign rule one has that $\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b e^x w(x,t)^2 \, \mathrm{d}x = \int_a^b 2e^x w(x,t) w_x(x,t) \, \mathrm{d}x$, where (35) was used. Then, applying integration by parts leads to $\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b e^x w(x,t)^2 \, \mathrm{d}x = e^b w(b,t)^2 - e^a w(a,t)^2 - \int_a^b e^x w(x,t)^2 \, \mathrm{d}x$.

with $\kappa_1 = 2|C(A+BK)|^2$ and $\kappa_2 = 2|CBK|^2$ was employed, which follows by differentiating (23), plugging (20), squaring both sides and then using Young's inequality. Similarly, the time derivative of $V_3(t)$ along the trajectories of (31)-(36) can be written as

$$\dot{V}_3 = -c_0 \tilde{P}^T T^T Q_2 T \tilde{P} - 2c_0 \tilde{P}^T T^T S_2 L \tilde{w}(0,t)$$
$$- 2c_0 \tilde{P}^T T^T S_2 L \mathcal{I}(t) - \frac{c_1}{2} \tilde{w}(0,t)^2$$
$$- \frac{c_1}{2} \int_0^{D_0} e^x \tilde{w}(x,t)^2 \, \mathrm{d}x + \frac{c_1}{4} \tilde{w}(0,t)^2$$
$$- \frac{c_1}{4} e^{\underline{x}} \tilde{w}(\underline{x}, t)^2 - \frac{c_1}{4} \int_{\underline{x}}^0 e^x \tilde{w}(x,t)^2 \, \mathrm{d}x$$
$$- e^{\underline{x}} \tilde{w}_x(\underline{x}, t)^2 - \int_{\underline{x}}^{D_0} e^x \tilde{w}_x^2(x,t) \, \mathrm{d}x$$
$$\leq -\frac{c_0 \underline{\lambda}(T^T Q_2 T)}{4} |\tilde{P}|^2 + \frac{4c_0 |T^T S_2 L|^2}{\underline{\lambda}(T^T Q_2 T)} \mathcal{I}(t)^2$$
$$+ \left(\frac{2c_0 |T^T S_2 L|^2}{\underline{\lambda}(T^T Q_2 T)} - \frac{c_1}{4} \right) \tilde{w}(0,t)^2$$
$$- \frac{c_1}{2} \int_0^{D_0} e^x \tilde{w}(x,t)^2 \, \mathrm{d}x - \frac{c_1}{4} \int_{\underline{x}}^0 e^x \tilde{w}(x,t)^2 \, \mathrm{d}x$$
$$- \int_{\underline{x}}^{D_0} e^x \tilde{w}_x^2(x,t) \, \mathrm{d}x \tag{53}$$

where the fact that T and A commute was employed; (50) and $\tilde{w}_x(D_0, t) = 0$ were used, where the latter follows by differentiating (36) in time and using the resulting expression into (35) evaluated at $x = D_0$; and Young's inequality was employed to upper bound the second term $(2a^T b \leq |a|^2/2 + 2|b|^2)$ and the third one $(2a^T b \leq |a|^2/4 + 4|b|^2)$. Gathering (51), (52) and (53), and choosing

$$b_0 = \frac{c_0 \underline{\lambda}(T^T Q_2 T) \underline{\lambda}(Q_1)}{16 |S_1 BK|^2}, \quad c_0 = \frac{8a_1 e^{D_0} |K|^2}{\underline{\lambda}(T^T Q_2 T)},$$
$$a_1 = \frac{8|S_1 B|^2}{\underline{\lambda}(Q_1) e^{\Delta D}}, \quad b_1 = \frac{b_0 \underline{\lambda}(Q_1)}{2e^{D_0} |C|^2}, \quad c_1 = \frac{8c_0 |T^T S_2 L|^2}{\underline{\lambda}(T^T Q_2 T)},$$

the derivative of (48) is given by

$$\dot{V} \leq -\frac{\underline{\lambda}(Q_1)}{2} |X|^2 - \frac{a_1}{2} \int_{\Delta D}^{D_0} e^x w(x,t)^2 \, \mathrm{d}x$$
$$- \frac{a_1}{4} \int_{\underline{x}}^{\Delta D} e^x w(x,t)^2 \, \mathrm{d}x$$
$$- \left(\frac{b_0 \underline{\lambda}(Q_1)}{4} - \frac{b_2}{2} e^{D_0} \kappa_1 \right) |P|^2$$
$$- \frac{b_1}{2} \int_{\underline{x}}^{D_0} e^x v(x,t)^2 \, \mathrm{d}x$$
$$- \left(\frac{c_0 \underline{\lambda}(T^T Q_2 T)}{16} - \frac{b_2}{2} e^{D_0} \kappa_2 \right) |\tilde{P}|^2$$
$$- \frac{c_1}{2} \int_0^{D_0} e^x \tilde{w}(x,t)^2 \, \mathrm{d}x - \frac{c_1}{4} \int_{\underline{x}}^0 e^x \tilde{w}(x,t)^2 \, \mathrm{d}x$$
$$+ \left(\frac{4c_0 |T^T S_2 L|^2}{\underline{\lambda}(T^T Q_2 T)} \delta e^\delta - \frac{b_2}{2} \right) \int_{\underline{x}}^\delta e^x v_x^2(x,t) \, \mathrm{d}x$$
$$- \frac{b_2}{2} \int_\delta^{D_0} e^x v_x^2(x,t) \, \mathrm{d}x - \int_{\underline{x}}^{D_0} e^x \tilde{w}_x^2(x,t) \, \mathrm{d}x \tag{54}$$

in which the following bound was used

$$\mathcal{I}(t)^2 \leq |\Delta D| \int_{\min\{0,\Delta D\}}^{\max\{0,\Delta D\}} v_x^2(x,t)\,\mathrm{d}x$$

$$\leq \delta \int_{\underline{x}}^{\delta} v_x^2(x,t)\,\mathrm{d}x \leq \delta e^{\delta} \int_{\underline{x}}^{\delta} e^x v_x^2(x,t)\,\mathrm{d}x,$$

where the first inequality follows from Jensen's, the second holds because the integral of a positive function is an increasing function of its upper limit and it assumed that $|\Delta D| \leq \delta$, and the third one follows from the fact that $e^{\delta} e^x \geq 1, \forall x \in [\underline{x}, \delta]$. Next, choosing

$$b_2 < \frac{1}{4e^{D_0}} \min\left\{ \frac{b_0 \underline{\lambda}(Q_1)}{\kappa_1}, \frac{c_0 \underline{\lambda}(T^T Q_2 T)}{4\kappa_2} \right\},$$

and selecting δ such that

$$\delta e^{\delta} < \frac{b_2 \underline{\lambda}(T^T Q_2 T)}{8 c_0 |T^T S_2 L|^2},$$

it follows from (54) and (48) that

$$\dot{V}(t) \leq -\mu V(t), \tag{55}$$

where

$$\mu = \min\left\{ \frac{\underline{\lambda}(Q_1)}{8\overline{\lambda}(S_1)}, \frac{\underline{\lambda}(T^T Q_2 T)}{32\overline{\lambda}(S_2)}, \left(\frac{8 c_0 |T^T S_2 L|^2}{b_2 \underline{\lambda}(T^T Q_2 T)} \delta e^{\delta} - 1 \right) \right\}.$$

From (48), one can find that

$$\psi_1 \Xi(t) \leq V(t) \leq \psi_2 \Xi(t), \tag{56}$$

where

$$\Xi(t) = |X|^2 + \|w\|^2 + |P|^2 + \|v\|_{H_1}^2 + |\tilde{P}|^2 + \|\tilde{w}\|_{H_1}^2,$$

and

$$\psi_1 = \max\left\{ \underline{\lambda}(S_1), b_0 \underline{\lambda}(S_1), c_0 \underline{\lambda}(T^T S_2 T), \frac{a_1 e^x}{4}, \right.$$
$$\left. \frac{b_1 e^x}{2}, \frac{b_2 e^x}{2}, \frac{c_1 e^x}{4}, e^x \right\},$$

$$\psi_2 = \max\left\{ \overline{\lambda}(S_1), b_0 \overline{\lambda}(S_1), c_0 \overline{\lambda}(T^T S_2 T), \frac{a_1 e^{D_0}}{2}, \right.$$
$$\left. \frac{b_1 e^{D_0}}{2}, \frac{b_2 e^{D_0}}{2}, \frac{c_1 e^{D_0}}{2}, e^{D_0} \right\}.$$

Integrating (55) and then using (56), the following exponential stability estimate is obtained for the transformed system

$$\Xi(t) \leq \frac{\psi_2}{\psi_1} \Xi(0) e^{-\mu t}, \forall t \geq 0. \tag{57}$$

Now, it is necessary to show the exponential stability of the original system, that is, in the sense of the norm

$$\Upsilon(t) = |X|^2 + \|u\|^2 + |\hat{P}|^2 + \|\hat{v}\|_{H_1}^2.$$

Using (40)-(42) and (45)-(47), one can show there exist constants α_i and β_i in $[1, \infty)$ such that

$$\Xi(t) \leq \alpha_1 |X|^2 + \alpha_2 \|u\|^2 + \alpha_3 |\hat{P}|^2 + \alpha_4 \|\hat{v}\|_{H_1}^2,$$
$$\Upsilon(t) \leq \beta_1 |X|^2 + \beta_2 \|w\|^2 + \beta_3 |P|^2 + \beta_4 \|v\|_{H_1}^2$$
$$+ \beta_5 |\tilde{P}|^2 + \beta_6 \|\tilde{w}\|_{H_1}^2,$$

for all $t \geq 0$, from which it follows that

$$\phi_1 \Upsilon(t) \leq \Xi(t) \leq \phi_2 \Upsilon(t), \tag{58}$$

being $\phi_1 = 1/\max \beta_i$ and $\phi_2 = \max \alpha_i$. Therefore, one gets the exponential stability estimate

$$\Upsilon(t) \leq \frac{\psi_2}{\psi_1} \frac{\phi_2}{\phi_1} \Upsilon(0) e^{-\mu t}, \forall t \geq 0,$$

thus completing the proof. $\square$

Remark 2. *Some similarities between the observer* (15)-(17) *and the sequential observers from [16] can be observed, although the exact relation is not clear. The continuous spatial variable x in this formulation seems to play the role of the discrete index j, used therein to denote each of the observer components.*

IV. DIFFUSIVE ACTUATOR DYNAMICS

Theorem 2 below introduces the proposed observer-based controller for (7)-(11) and guarantees the closed-loop exponential stability. The proof is very similar to that of Theorem 1. In this case, the two systems that compose the closed-loop are given by (78)-(85), which are also coupled to a third one because of the uncertainty, given by (86)-(89). Again, a suitable (invertible) backstepping transformation is proposed to map these systems into target systems whose stability is proved via Lyapunov analysis. Then, the stability of the original systems is proved using the inverse transformations.

Theorem 2. *Consider the closed-loop system composed of* (7)-(11) *and the observer-based controller*

$$\dot{\hat{\Pi}}(t) = A\hat{\Pi}(t) + BU(t) + M_0(D)L\big(Y(t) - \hat{\nu}(0,t)\big), \tag{59}$$
$$\hat{\nu}_t(x,t) = \epsilon_0 \hat{\nu}_{xx}(x,t) + CM_0(x)L\big(Y(t) - \hat{\nu}(0,t)\big), \tag{60}$$
$$\hat{\nu}_x(0,t) = 0, \tag{61}$$
$$\hat{\nu}(D,t) = C\hat{\Pi}(t), \tag{62}$$
$$U(t) = K\hat{\Pi}(t), \tag{63}$$

where

$$M_0(x) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & \frac{A}{\epsilon_0} \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

the matrices K and L are such that $A + BK$ and $A - LC$ are Hurwitz and $\epsilon_0 > 0$. Then, there exists a $\delta > 0$ such that for all $|\Delta\epsilon| \leq \delta$, i.e., for all $\epsilon \in [\epsilon_0 - \delta, \epsilon_0 + \delta]$, the zero solution of the $(X, u, \hat{\Pi}, \hat{\nu})$-system is exponentially stable, that is, there exist positive constants R and ρ such that for all initial conditions $(X_0, u_0, \hat{\Pi}_0, \hat{\nu}_0) \in \mathbb{R}^n \times H_1(0, D) \times \mathbb{R}^n \times H_1(0, D)$, the following holds:

$$\Upsilon(t) \leq R\Upsilon(0)e^{-\rho t},$$

where

$$\Upsilon(t) = |X(t)|^2 + \|u(t)\|_{H_1[0,D]}^2 + |\hat{P}(t)|^2 + \|\hat{v}(t)\|_{H_1[0,D]}^2.$$

Proof. Let us define

$$\nu(x,t) = CM(x)X(t) + C\int_0^x m(x - y)Bu(y,t)\,\mathrm{d}y. \tag{64}$$

Computing the time derivative of (12) and the spatial and temporal derivatives of (64), one arrives at the following ODE-PDE cascade system

$$\dot{\Pi}(t) = A\Pi(t) + BU(t), \qquad (65)$$
$$\nu_t(x,t) = \epsilon\nu_{xx}(x,t), \qquad (66)$$
$$\nu(D,t) = C\Pi(t), \qquad (67)$$
$$\nu_x(0,t) = 0, \qquad (68)$$
$$Y(t) = \nu(0,t), \qquad (69)$$

See Appendix B for details. Analogously to the input-delay case, the original system with diffusive actuator dynamics (7)-(11) has been mapped into the virtual system (65)-(69), in which the diffusive dynamics is affecting the output. Let us define the error variables as

$$\tilde{\Pi}(t) \triangleq \Pi(t) - \hat{\Pi}(t), \qquad (70)$$
$$\tilde{\nu}(x,t) \triangleq \nu(x,t) - \hat{\nu}(x,t). \qquad (71)$$

Differentiating (70)-(71) and using (59)-(62), the observer error system is obtained as

$$\dot{\tilde{\Pi}}(t) = A\tilde{\Pi}(t) - M_0(D)L\tilde{\nu}(0,t), \qquad (72)$$
$$\tilde{\nu}_t(x,t) = \epsilon_0\tilde{\nu}_{xx}(x,t) - CM_0(x)L\tilde{\nu}(0,t) + \Delta\epsilon\nu_{xx}(x,t), \quad (73)$$
$$\tilde{\nu}_x(0,t) = 0, \qquad (74)$$
$$\tilde{\nu}(D,t) = C\tilde{\Pi}(t). \qquad (75)$$

Now, let us introduce the backstepping transformations

$$w(x,t) = u(x,t) - KM(x)X(t) \\ - K\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y, \qquad (76)$$
$$\tilde{w}(x,t) = \tilde{\nu}(x,t) - CM_0(x)M_0(D)^{-1}\tilde{\Pi}(t), \qquad (77)$$

Using (63) and the transformations (76)-(77) the systems (7)-(10) and (72)-(75) are mapped into

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \qquad (78)$$
$$w_t(x,t) = \epsilon w_{xx}(x,t), \qquad (79)$$
$$w_x(0,t) = 0, \qquad (80)$$
$$w(D,t) = -K\tilde{\Pi}(t), \qquad (81)$$
$$\dot{\tilde{\Pi}}(t) = (A - M_0(D)LCM_0(D)^{-1})\tilde{\Pi}(t) \\ - M_0(D)L\tilde{w}(0,t), \qquad (82)$$
$$\tilde{w}_t(x,t) = \epsilon_0\tilde{w}_{xx}(x,t) + \Delta\epsilon\nu_{xx}, \qquad (83)$$
$$\tilde{w}_x(0,t) = 0, \qquad (84)$$
$$\tilde{w}(D,t) = 0 \qquad (85)$$

Most of the calculations involved in the transformation above are the same as those carried out in Appendix B. Some hints follow: (78) employed (76) evaluated at $x=0$ and (139); (79) followed after subtracting the first-in-time and second-in-space derivatives of (76), applying integration by parts twice, and using (133)-(136) and (138)-(139); (140) was used to obtain (80) while $M_0'(0)=0$ was used to obtain (84); and finally, (83) used that $M_0''(x) = \epsilon_0^{-1}AM_0(x)$ and the fact that A and

$M_0(x)$ commute for all x. On the other hand, using (63) and (70), the system (65)-(68) can be written as

$$\dot{\Pi}(t) = (A+BK)\Pi(t) - BK\tilde{\Pi}(t), \qquad (86)$$
$$\nu_t(x,t) = \epsilon\nu_{xx}(x,t), \qquad (87)$$
$$\nu_x(0,t) = 0, \qquad (88)$$
$$\nu(D,t) = C\Pi(t). \qquad (89)$$

Gathering previous equations, the overall transformation $(X,u,\hat{\Pi},\hat{\nu}) \mapsto (X,w,\tilde{\Pi},\tilde{w},\Pi,\nu)$ can be written as

$$w(x,t) = u(x,t) - KM(x)X(t) \\ - K\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y, \qquad (90)$$
$$\tilde{\Pi}(t) = M(D)X(t) + \int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y \\ - \hat{\Pi}(t), \qquad (91)$$
$$\tilde{w}(x,t) = CM_0(x)M_0(D)^{-1}\hat{\Pi}(t) - \hat{\nu}(x,t) \\ + CM(x)X(t) + C\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y \\ - CM_0(x)M_0(D)^{-1}\Big(M(D)X(t) \\ + \int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y\Big), \qquad (92)$$
$$\Pi(t) = M(D)X(t) + \int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y, \qquad (93)$$
$$\nu(x,t) = CM(x)X(t) + C\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y, \qquad (94)$$

while the inverse transformation is given by

$$u(x,t) = w(x,t) + KN(x)X(t) \\ + K\int_0^x n(x-y)Bw(y,t)\,\mathrm{d}y, \qquad (95)$$
$$\hat{\Pi}(t) = \Pi(t) - \tilde{\Pi}(t), \qquad (96)$$
$$\hat{\nu}(x,t) = \nu(x,t) - CM_0(x)M(D)^{-1}\tilde{\Pi}(t) - \tilde{w}(x,t). \qquad (97)$$

where

$$n(s) = \frac{1}{\epsilon}\int_0^s N(\xi)\,\mathrm{d}\xi,$$
$$N(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & \frac{A+BK}{\epsilon} \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

The fact that (95) is the inverse of (90) is proved in Appendix C. In order to assess stability, let us choose the Lyapunov functional

$$V(t) = V_1(t) + V_2(t) + V_3(t) \qquad (98)$$

where

$$V_1(t) = a_0 X^T S_1 X + \frac{a_1}{2\epsilon}\|w\|^2 + \frac{a_2}{2\epsilon}\|w_x\|^2$$
$$V_2(t) = b_0 \Pi^T S_1 \Pi + \frac{b_1}{2\epsilon}\|\nu\|^2 + \frac{b_2}{2\epsilon}\|\nu_x\|^2$$
$$V_3(t) = \tilde{\Pi}^T M_0^{-T} S_2 M_0^{-1}\tilde{\Pi} + \frac{c_1}{2\epsilon_0}\left(\|\tilde{w}\|^2 + \|\tilde{w}_x\|^2\right),$$

where $M_0 = M_0(D)$ for the sake of brevity and $S_1 = S_1^T > 0$, $S_2 = S_2^T > 0$ are the solutions to the Lyapunov equations

$$S_1(A + BK) + (A + BK)^T S_1 = -Q_1, \tag{99}$$
$$S_2(A - LC) + (A - LC)^T S_2 = -Q_2. \tag{100}$$

for some symmetric positive definite matrices Q_1 and Q_2. Using integration by parts[2], the time derivative of $V_1(t)$ along the trajectories of (78)-(81) can be written as

$$\begin{aligned}
\dot{V}_1(t) &= -a_0 X^T Q_1 X + 2a_0 X^T S_1 B w(0,t) \\
&\quad + a_1 w(D,t) w_x(D,t) - a_1 \|w_x\|^2 \\
&\quad + a_2 w_x(D,t) w_{xx}(D,t) - a_2 \|w_{xx}\|^2 \\
&\leq -\frac{a_0 \underline{\lambda}(Q_1)}{2}|X|^2 + \frac{2a_0 |S_1 B|^2}{\underline{\lambda}(Q_1)} w(0,t)^2 \\
&\quad + \frac{Da_1^2}{a_2} w(D,t)^2 + \frac{a_2}{4D} w_x(D,t)^2 \\
&\quad + \frac{a_2}{4D} w_x(D,t)^2 + Da_2 w_{xx}(D,t)^2 \\
&\quad - a_1 \|w_x\|^2 - a_2 \|w_{xx}\|^2
\end{aligned} \tag{101}$$

where Young's inequality was used conveniently used multiple times. To proceed, some inequalities are derived next. By the fundamental theorem of calculus and Jensen's inequality, $\left(w(D,t) - w(0,t)\right)^2 = \left(\int_0^D w_x \, dx\right)^2 \leq D\|w_x\|^2$, and then expanding the squared difference and employing Young's inequality to upper bound the cross term leads to

$$w(0,t)^2 \leq 2w(D,t)^2 + 2D\|w_x\|^2, \tag{102}$$

Proceeding in a similar way with w_x and $\tilde{w}$, and using (80) and (85), respectively, yields

$$w_x(D,t)^2 \leq D\|w_{xx}\|^2, \tag{103}$$
$$\tilde{w}(0,t)^2 \leq D\|\tilde{w}_x\|^2. \tag{104}$$

Integrating $\|w\|^2$ by parts and using Young's inequality conveniently, leads to $\|w\|^2 \leq 2Dw(D,t)^2 + 4D^2\|w_x\|^2$, from which we get

$$-\|w_x\|^2 \leq \frac{1}{2D}w(D,t)^2 - \frac{1}{4D^2}\|w\|^2, \tag{105}$$

follows. Using the same procedure with $\|w_x\|^2$ and taking (80) into account yields

$$-\|w_{xx}\|^2 \leq -\frac{1}{4D^2}\|w_x\|^2. \tag{106}$$

Using (102)-(103) into (101) and selecting

$$a_1 = \frac{8Da_0 |S_1 B|^2}{\underline{\lambda}(Q_1)},$$

yields

$$\begin{aligned}
\dot{V}_1(t) &\leq -\frac{a_0 \underline{\lambda}(Q_1)}{2}|X|^2 + Da_2 w_{xx}(D,t)^2 \\
&\quad + \left(\frac{4a_0 |S_1 B|^2}{\underline{\lambda}(Q_1)} + \frac{Da_1^2}{a_2}\right) w(D,t)^2
\end{aligned}$$

[2]By the differentiation under the integral sign rule one has that $\frac{d}{dt}\frac{1}{2\epsilon}\|w(t)\|^2 = \int_0^D w(x,t) w_{xx}(x,t)\,dx$, where (79) was used. Then, applying integration by parts leads to $\frac{d}{dt}\frac{1}{2\epsilon}\|w(t)\|^2 = w(D,t)w_x(D,t) - w(0,t)w_x(0,t) - \|w_x(t)\|^2$.

$$-\frac{a_1}{2}\|w_x\|^2 - \frac{a_2}{2}\|w_{xx}\|^2 \tag{107}$$

Now, using (105)-(106) into (107), one can write

$$\begin{aligned}
\dot{V}_1(t) &\leq -\frac{a_0 \underline{\lambda}(Q_1)}{2}|X|^2 + Da_2 w_{xx}(D,t)^2 \\
&\quad + \left(\frac{4a_0 |S_1 B|^2}{\underline{\lambda}(Q_1)} + \frac{Da_1^2}{a_2} + \frac{a_1}{4D}\right) w(D,t)^2 \\
&\quad - \frac{a_1}{8D^2}\|w\|^2 - \frac{a_2}{8D^2}\|w_x\|^2
\end{aligned} \tag{108}$$

Furthermore, using (79), (81) and (82),

$$w_{xx}(D,t)^2 \leq \kappa_1 |\tilde{\Pi}|^2 + \kappa_2 \tilde{w}(0,t)^2 \tag{109}$$

where $\kappa_1 = 2\epsilon^{-2}|K(A - M_0 L C M_0^{-1})|^2$ and $\kappa_2 = 2\epsilon^{-2}|KM_0 L|^2$. Using (81), (104) and (109) into (108) yields

$$\begin{aligned}
\dot{V}_1(t) &\leq -\frac{a_0 \underline{\lambda}(Q_1)}{2}|X|^2 + D^2 a_2 \kappa_2 \|\tilde{w}_x\|^2 \\
&\quad + \left(\left(\frac{4a_0 |S_1 B|^2}{\underline{\lambda}(Q_1)} + \frac{Da_1^2}{a_2} + \frac{a_1}{4D}\right)|K|^2 \right. \\
&\quad \left. + Da_2 \kappa_1\right)|\tilde{\Pi}|^2 - \frac{a_1}{8D^2}\|w\|^2 - \frac{a_2}{8D^2}\|w_x\|^2
\end{aligned} \tag{110}$$

Similarly as before, using integration by parts and Young's inequality, the time derivative of $V_2(t)$ along the trajectories of (86)-(88) can be bounded by

$$\begin{aligned}
\dot{V}_2(t) &= -b_0 \Pi^T Q_1 \Pi - 2b_0 \Pi^T S_1 BK \tilde{\Pi} \\
&\quad + b_1 \nu(D,t)\nu_x(D,t) - b_1 \|\nu_x\|^2 \\
&\quad + b_2 \nu_x(D,t)\nu_{xx}(D,t) - b_2 \|\nu_{xx}\|^2 \\
&\leq -\frac{b_0 \underline{\lambda}(Q_1)}{2}|\Pi|^2 + \frac{2b_0 |S_1 BK|^2}{\underline{\lambda}(Q_1)}|\tilde{\Pi}|^2 \\
&\quad + \frac{Db_1^2}{b_2}\nu(D,t)^2 + \frac{b_2}{4D}\nu_x(D,t)^2 \\
&\quad + \frac{b_2}{4D}\nu_x(D,t)^2 + Db_2 \nu_{xx}(D,t)^2 \\
&\quad - b_1 \|\nu_x\|^2 + b_2 \|\nu_{xx}\|^2 \\
&\leq -\left(\frac{b_0 \underline{\lambda}(Q_1)}{2} + \frac{Db_1^2 |C|^2}{b_2} + \frac{b_1 |C|^2}{2D} + Db_2 \kappa_3\right)|\Pi|^2 \\
&\quad + \left(\frac{2b_0 |S_1 BK|^2}{\underline{\lambda}(Q_1)} + Db_2 \kappa_4\right)|\tilde{\Pi}|^2 \\
&\quad - \frac{b_1}{4D^2}\|\nu\|^2 - \frac{b_2}{2}\|\nu_{xx}\|^2
\end{aligned} \tag{111}$$

where the inequalities

$$\nu_x(D,t)^2 \leq D\|\nu_{xx}\|^2 \tag{112}$$
$$-\|\nu_x\|^2 \leq \frac{1}{2D}\nu(D,t)^2 - \frac{1}{4D^2}\|\nu\|^2, \tag{113}$$
$$\nu_{xx}(D,t)^2 \leq \kappa_3 |\Pi|^2 + \kappa_4 |\tilde{\Pi}|^2, \tag{114}$$

with $\kappa_3 = 2\epsilon^{-2}|C(A + BK)|^2$ and $\kappa_4 = 2\epsilon^{-2}|CBK|^2$ were used. Note that (112) and (113) follow by the same procedures used to derive (103) and (105), respectively, whereas (114) follows from (86)-(89). Choosing

$$b_1 = \min\left\{\sqrt{\frac{b_0 \underline{\lambda}(Q_1)b_2}{8D|C|^2}}, \frac{b_0 \underline{\lambda}(Q_1)D}{4|C|^2}\right\}, \tag{115}$$

in (111) yields

$$\dot{V}_2(t) \leq -\left(\frac{b_0\underline{\lambda}(Q_1)}{4} + Db_2\kappa_3\right)|\Pi|^2$$
$$+ \left(\frac{2b_0|S_1BK|^2}{\underline{\lambda}(Q_1)} + Db_2\kappa_4\right)|\tilde{\Pi}|^2$$
$$- \frac{b_1}{4D^2}\|\nu\|^2 - \frac{b_2}{2}\|\nu_{xx}\|^2. \tag{116}$$

Again, integrating by parts, using Young's inequality and (104), the derivative of $V_3(t)$ along the trajectories of (82)-(85) can be bounded by

$$\dot{V}_3(t) = -\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})|\tilde{\Pi}|^2 - 2M_0^{-T}S_2L\tilde{w}(0,t)$$
$$- c_1\|\tilde{w}_x\|^2 + c_1\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}(x,t)\nu_{xx}(x,t)\,\mathrm{d}x$$
$$- c_1\|\tilde{w}_{xx}\|^2 + c_1\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}_x(x,t)\nu_{xx}(x,t)\,\mathrm{d}x$$
$$\leq -\frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{2}|\tilde{\Pi}|^2$$
$$+ \left(\frac{8D|M_0^{-T}S_2L|^2}{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})} - c_1\right)\|\tilde{w}_x\|^2$$
$$+ c_1\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}(x,t)\nu_{xx}(x,t)\,\mathrm{d}x$$
$$- c_1\|\tilde{w}_{xx}\|^2 + c_1\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}_x(x,t)\nu_{xx}(x,t)\,\mathrm{d}x \tag{117}$$

Integrating $\|\tilde{w}\|^2$ and $\|\tilde{w}_x\|^2$ by parts, using Young's inequality and taking (84)-(85) into account, one can show that

$$\|\tilde{w}\|^2 \leq 4D^2\|\tilde{w}_x\|^2, \qquad \|\tilde{w}_x\|^2 \leq 4D^2\|\tilde{w}_{xx}\|^2. \tag{118}$$

Using Cauchy-Schwartz, Young and (118), the following bounds are derived

$$\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}\nu_{xx}\,\mathrm{d}x \leq \frac{1}{4}\|\tilde{w}_x\|^2 + 4D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2\|\nu_{xx}\|^2$$
$$\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}_x\nu_{xx}\,\mathrm{d}x \leq \frac{1}{2}\|\tilde{w}_{xx}\|^2 + 2D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2\|\nu_{xx}\|^2,$$

which plugged into (117) and after choosing

$$c_1 = \frac{16D|M_0^{-T}S_2L|^2}{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})},$$

yield

$$\dot{V}_3(t) \leq -\frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{2}|\tilde{\Pi}|^2$$
$$- \frac{c_1}{4}\|\tilde{w}_x\|^2 - \frac{c_1}{2}\|\tilde{w}_{xx}\|^2$$
$$+ 6D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2 c_1\|\nu_{xx}\|^2 \tag{119}$$

Gathering (110), (116) and (119), and selecting

$$a_1 = \min\left\{\sqrt{\frac{a_2\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{24D|K|^2}}, \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})D}{6|K|^2}\right\}$$
$$a_0 = \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})\underline{\lambda}(Q_1)}{72|S_1B|^2|K|^2},$$

$$b_0 = \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})\underline{\lambda}(Q_1)}{24|S_1BK|^2},$$

leads to

$$\dot{V}(t) = -\frac{a_0\underline{\lambda}(Q_1)}{2}|X|^2 - \left(\frac{b_0\underline{\lambda}(Q_1)}{4} - Db_2\kappa_3\right)|\Pi|^2$$
$$- \left(\frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{4} - Db_2\kappa_4 - Da_2\kappa_1\right)|\tilde{\Pi}|^2,$$
$$- \frac{a_1}{8D^2}\|w\|^2 - \frac{a_2}{8D^2}\|w_x\|^2$$
$$- \frac{b_1}{4D^2}\|\nu\|^2 - \left(\frac{b_2}{2} - 6D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2 c_1\right)\|\nu_{xx}\|^2$$
$$- \left(\frac{c_1}{4} - D^2a_2\kappa_2\right)\|\tilde{w}_x\|^2 - \frac{c_1}{2}\|\tilde{w}_{xx}\|^2 \tag{120}$$

Now, choosing

$$a_2 = \frac{1}{8D}\min\left\{\frac{c_1}{D\kappa_2}, \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{2\kappa_1}\right\},$$
$$b_2 = \frac{1}{8D}\min\left\{\frac{b_0\underline{\lambda}(Q_1)}{\kappa_3}, \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{2\kappa_4}\right\},$$

into (120) yields

$$\dot{V}(t) = -\frac{a_0\underline{\lambda}(Q_1)}{2}|X|^2 - \frac{b_0\underline{\lambda}(Q_1)}{8}|\Pi|^2$$
$$- \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{8}|\tilde{\Pi}|^2$$
$$- \frac{a_1}{8D^2}\|w\|^2 - \frac{a_2}{8D^2}\|w_x\|^2$$
$$- \frac{b_1}{4D^2}\|\nu\|^2 - \left(\frac{b_2}{2} - 6D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2 c_1\right)\|\nu_{xx}\|^2$$
$$- \frac{c_1}{8}\|\tilde{w}_x\|^2 - \frac{c_1}{2}\|\tilde{w}_{xx}\|^2. \tag{121}$$

Integrating $\|\nu_x\|^2$ by parts, using Young's inequality and taking (84) into account, one gets $\|\nu_x\|^2 \leq 4D^2\|\nu_{xx}\|^2$, which can be used, along with (118), to further bound (121) as

$$\dot{V}(t) = -\frac{a_0\underline{\lambda}(Q_1)}{2}|X|^2 - \frac{b_0\underline{\lambda}(Q_1)}{8}|\Pi|^2$$
$$- \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{8}|\tilde{\Pi}|^2$$
$$- \frac{a_1}{8D^2}\|w\|^2 - \frac{a_2}{8D^2}\|w_x\|^2$$
$$- \frac{b_1}{4D^2}\|\nu\|^2 - \left(\frac{b_2}{2} - 6D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2 c_1\right)\frac{1}{4D^2}\|\nu_x\|^2$$
$$- \frac{c_1}{32D^2}\|\tilde{w}\|^2 - \frac{c_1}{8D^2}\|\tilde{w}_x\|^2. \tag{122}$$

Assuming $|\Delta\epsilon| \leq \delta$ and selecting

$$\delta < \frac{\epsilon_0}{2D}\sqrt{\frac{b_2}{3c_1}},$$

it follows from (98) and (IV) that

$$\dot{V}(t) \leq \mu V(t)$$

where

$$\mu = \min\left\{\frac{\underline{\lambda}(Q_1)}{8\overline{\lambda}(S_1)}, \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{8\overline{\lambda}(M_0^{-T}S_2M_0^{-1})},\right.$$

$$\left(\frac{1}{2} - \frac{6D^2(\Delta\epsilon/\epsilon_0)^2 c_1}{b_2}\right)\frac{\epsilon}{2D^2}, \frac{\epsilon_0}{16D^2}\right\}.$$

Now, from (98), one can find that

$$\psi_1 \Xi(t) \le V(t) \le \psi_2 \Xi(t), \qquad (123)$$

where

$$\Xi(t) = |X|^2 + \|w\|_{H_1}^2 + |\tilde{\Pi}|^2 + \|\tilde{w}\|_{H_1}^2 + |\Pi|^2 + \|\nu\|_{H_1}^2,$$

and

$$\psi_1 = \min\left\{a_0\underline{\lambda}(S_1), b_0\underline{\lambda}(S_1), \underline{\lambda}(M_0^{-T}S_2 M_0^{-1}),\right.$$
$$\left.\frac{1}{2\epsilon}\min\{a_1,a_2,b_1,b_2\}, \frac{c_1}{2\epsilon_0}\right\},$$
$$\psi_2 = \max\left\{a_0\overline{\lambda}(S_1), b_0\overline{\lambda}(S_1), \overline{\lambda}(M_0^{-T}S_2 M_0^{-1}),\right.$$
$$\left.\frac{1}{2\epsilon}\max\{a_1,a_2,b_1,b_2\}, \frac{c_1}{2\epsilon_0}\right\}.$$

Hence, the following exponential stability estimate is obtained for the transformed system

$$\Xi(t) \le \frac{\psi_2}{\psi_1}\Xi(0)e^{-\mu t}, \forall t \ge 0. \qquad (124)$$

Now, an estimate is derived in terms of

$$\Upsilon(t) = |X|^2 + \|u\|_{H_1}^2 + |\hat{\Pi}|^2 + \|\hat{\nu}\|_{H_1}^2.$$

Using (90)-(94) and (95)-(97), one can show there exist constants α_i, β_i such that

$$\Xi(t) \le \alpha_1|X|^2 + \alpha_2\|u\|_{H_1}^2 + \alpha_3|\hat{\Pi}|^2 + \alpha_4\|\hat{\nu}\|_{H_1}^2,$$
$$\Upsilon(t) \le \beta_1|X|^2 + \beta_2\|w\|_{H_1}^2 + \beta_3|\tilde{\Pi}|^2 + \beta_4\|\tilde{w}\|_{H_1}^2$$
$$+ \beta_5|\Pi|^2 + \beta_6\|\nu\|_{H_1}^2,$$

for all $t \ge 0$, from which it follows that

$$\phi_1\Upsilon(t) \le \Xi(t) \le \phi_2\Upsilon(t), \qquad (125)$$

being $\phi_1 = 1/\max\beta_i$, $\phi_2 = \max\alpha_i$. Therefore, from (124)-(125), one gets the exponential stability estimate

$$\Upsilon(t) \le \frac{\psi_2}{\psi_1}\frac{\phi_2}{\phi_1}\Upsilon(0)e^{-\mu t}, \forall t \ge 0,$$

completing the proof. $\qquad\square$

V. SIMULATIONS

The proposed control strategies are illustrated in this section using a second-order system defined by

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

which has an exponentially unstable mode, provided that the poles of the system are located at $s = \pm 1$. The feedback gain matrices are chosen as $K = L^T = [-2, -2]$, which guarantee $A + BK$ and $A - LC$ Hurwitz, being all their poles located at $s = -1$.
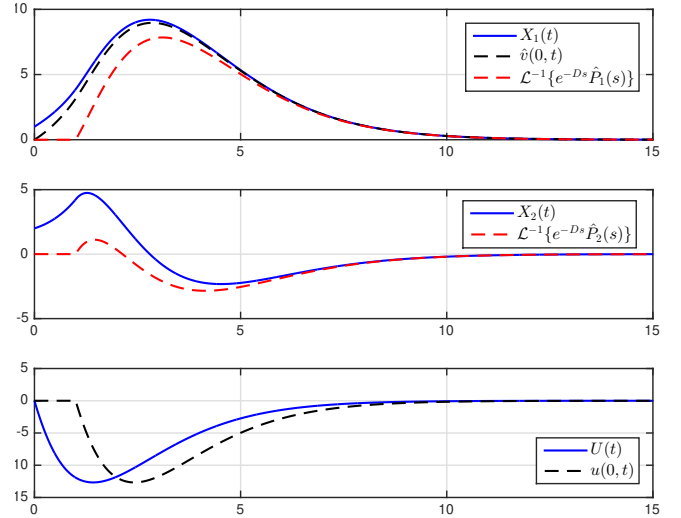

Fig. 1. Nominal simulation of a plant with delay actuator dynamics, where $X(0) = [1, 2]^T$, $\hat{P}(0) = [0, 0]^T$ and $\hat{v}(x, 0) = 0$, $\forall x \in [0, 1)$

A. Delay case

First, we consider the case modeled by (1)-(4), in which the input is affected by a delay $D = 1$. The control law (15)-(18) is implemented using an upwind scheme (first order accurate both in time and space) for the PDE discretization. Simulation results are shown in Fig. 1 for the nominal case, that is, with $D_0 = 1$. Note that the system (solid blue) runs in an open-loop fashion until the control action reaches the system at $t = D$. The observer estimates $\hat{P}_1$ and $\hat{P}_2$, which are actually D units of time ahead predictions, are shown delayed by D units of time (dashed red), to match the actual state (blue). One can also see that the value of $\hat{v}$ at the spatial location $x = 0$ contains an actual estimation of the output (dashed black), as expected. The bottom plot shows the control law (blue) and the actual signal that reaches the ODE (black), which is simply delayed by D units of time.

Robustness is also illustrated in Fig. 2, where a $+5\%$ additive disturbance in the time delay is considered, that is, $D = 1.05$. One can see that the asymptotic stability is preserved in spite of the uncertainty.

B. Diffusion case

Now, we consider the case modeled by (7)-(11), in which the input undergoes a diffusive process through a domain of length $D = 1$ with a diffusive coefficient $\epsilon = 1$. The control law (59)-(63) is implemented using a first-order-in-time and second-order-in-space discretization for the PDE. Simulation results are shown in Fig. 3 for the nominal case, that is $\epsilon_0 = \epsilon = 1$. The system states are depicted at the top and central plots (blue). Recall that $\hat{\Pi}_1$ and $\hat{\Pi}_2$, are actually the "anti-diffused" state estimates, as discussed in Remark 1. Then, we plot the observer estimates after undergoing a diffusion process through a domain of length $D = 1$ and with $\epsilon = 1$ (dashed red), to see that they match the actual state (blue). One can also see that the value of $\hat{v}$ at the spatial location $x = 0$ contains an actual estimation of the output (dashed black), as expected.
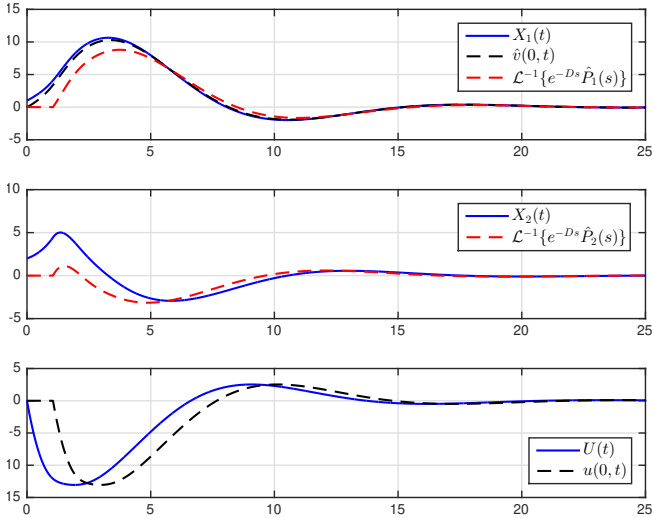
Fig. 2. Robust simulation of a plant with delay actuator dynamics, where $X(0) = [1, 2]^T$, $\hat{P}(0) = [0, 0]^T$ and $\hat{v}(x, 0) = 0$, $\forall x \in [0, 1)$
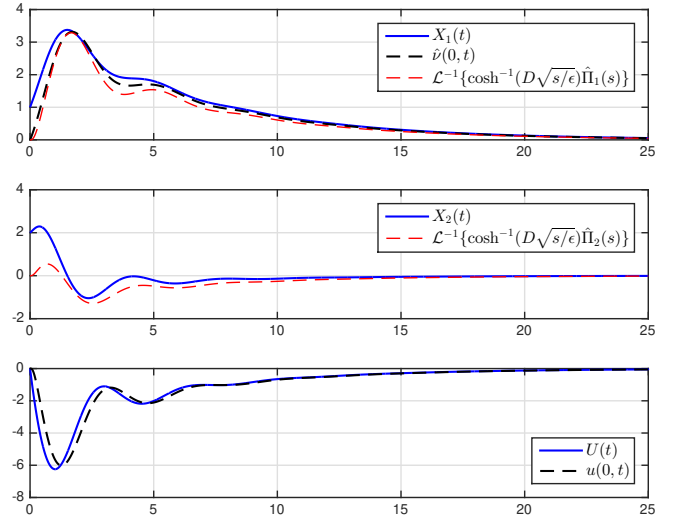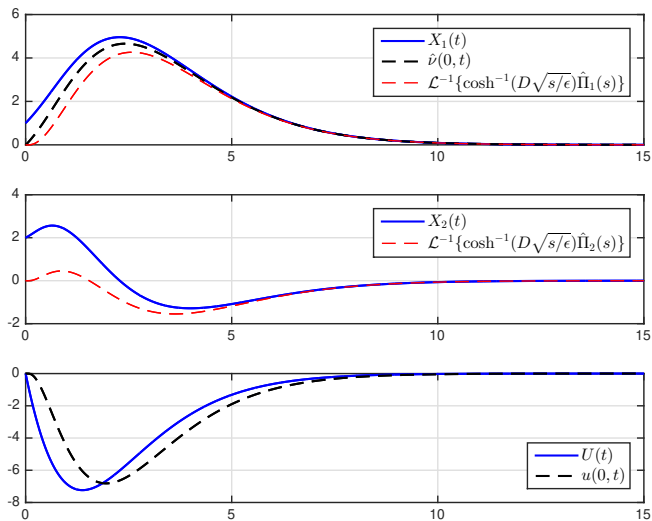

Fig. 3. Nominal simulation of a plant with diffusive actuator dynamics, where $X(0) = [1, 2]^T$, $\hat{\Pi}(0) = [0, 0]^T$ and $\hat{\nu}(x, 0) = 0$, $\forall x \in [0, 1)$

The bottom plot shows the control law (blue) and the actual signal that reaches the ODE (black).

Robustness is also illustrated for this case, performing one more simulation in which $\epsilon = 2$ while we keep $\epsilon_0 = 1$. The results are shown in Fig. 4, where it can be seen that small oscillations appear but stability is preserved.

VI. Conclusions and future work

The problem of compensating delay or diffusive actuator dynamics via output measurement has been addressed in this work. Furthermore, the compensation is achieved for any arbitrarily large delay or diffusion domain length, respectively, while avoiding integral terms. Robustness under small variations in the delay size or the diffusion coefficient has been also proved. The proposed control laws may be of interest in practice as one only needs to take care of the PDE discretization. Although it may not be an straightforward task,

Fig. 4. Robust simulation of a plant with diffusive actuator dynamics, where $X(0) = [1, 2]^T$, $\hat{\Pi}(0) = [0, 0]^T$ and $\hat{\nu}(x, 0) = 0$, $\forall x \in [0, 1)$

ample tools are available for that purpose. Future work may include extending the same procedures to actuators governed by wave dynamics or more general types of PDEs.

Acknowledgment

Appendix A

The transformation (40) can be compactly written as

$$w(x,t) = u(x,t) - f(x - \Delta D)X(t) - (g \star u)(x,t) \quad (126)$$

where $f(x) = Ke^{Ax}$, $g(x) = Ke^{Ax}B$ and $\star$ denotes the convolution operator in the x variable, i.e., $(g \star u)(x,t) = \int_{-\infty}^{\infty} g(x - y)u(y,t)\,\mathrm{d}y$. Note that the limits of the integral can be truncated assuming that $g : [0, \infty)$ and provided that $u : [\Delta D, D_0] \times [0, \infty)$. Taking the Laplace transform of (126) yields

$$w(\sigma, t) = \Gamma u(\sigma, t) - K(\sigma I - A)^{-1}e^{-\Delta D\sigma}X(t) \quad (127)$$

where σ is the Laplace argument and $\Gamma = I - K(\sigma I - A)^{-1}B$. Solving (127) for $u(\sigma, t)$ yields

$$u(\sigma, t) = \Gamma^{-1}w(\sigma, t) + \Gamma^{-1}K(\sigma I - A)^1 e^{-\Delta D\sigma}X(t) \quad (128)$$

where $\Gamma^{-1} = I + K(\sigma I - A - BK)^{-1}B$, which follows by the Woodbury identity. Adding and subtracting $K(\sigma I - A - BK)^{-1}$ to $\Gamma^{-1}K(\sigma I - A)^{-1}$ and using the identity $(\sigma I - A - BK)^{-1}(I - BK(\sigma I - A)^{-1}) = (\sigma I - A)^{-1}$ leads to

$$\Gamma^{-1}K(\sigma I - A)^1 = K(\sigma I - A - BK)^{-1}. \quad (129)$$

Finally, plugging (129) into (128) and taking the inverse Laplace transform yields (45), which completes the proof.

APPENDIX B

The proof of this transformation requires several intermediate results that are derived first. Let us start by defining

$$q(x,y) = \frac{1}{\epsilon} \int_0^{x-y} M(\xi) B \, d\xi, \qquad (130)$$

to rewrite (12) and (64) in a more compact notation

$$\Pi(t) = M(D)X(t) + \int_0^D q(D,y)u(y,t)\,dy, \qquad (131)$$

$$\nu(x,t) = CM(x)X(t) + C\int_0^x q(x,y)u(y,t)\,dy. \qquad (132)$$

Note that (130) satisfies the following relations

$$q_{xx}(x,y) = q_{yy}(x,y), \qquad (133)$$
$$q(x,x) = 0, \qquad (134)$$
$$\epsilon q_x(x,y) = M(x-y)B, \qquad (135)$$
$$\epsilon q_y(x,y) = -M(x-y)B. \qquad (136)$$

Let us define $R = \begin{bmatrix} 0 & A/\epsilon \\ I & 0 \end{bmatrix}$ and $\Phi^T = \begin{bmatrix} I & 0 \end{bmatrix}$ so that $M(\xi)$ in (14) can be expressed as $M(\xi) = \Phi^T e^{R\xi} \Phi$. Direct computations then show that

$$M'(\xi) = \Phi^T R e^{R\xi} \Phi, \qquad (137)$$
$$M''(\xi) = \Phi^T R^2 e^{R\xi} \Phi = \epsilon^{-1} A M(\xi), \qquad (138)$$

where the last equality follows from the fact that $\Phi^T R^2 = \epsilon^{-1} A \Phi^T$, which is readily verified. Evaluating (14) and its derivative (137) at $\xi = 0$, the following holds

$$M(0) = I, \qquad (139)$$
$$M'(0) = 0. \qquad (140)$$

Now, integrating (138) from 0 to $x-y$ on both sides, post-multiplying by B and using (140) yields

$$M'(x-y)B = \epsilon^{-1} A \int_0^{x-y} M(\xi)B\,d\xi = Aq(x,y), \quad (141)$$

where the last equality follows from (130). Differentiating (136) and using (141) leads to

$$\epsilon q_{yy}(x,y) = Aq(x,y). \qquad (142)$$

The equations derived so far in this appendix are instrumental, which are next used to show how to obtain the transformed system (65)-(69). Differentiating (131), using (7)-(8) and integrating twice by parts, yields

$$\dot{\Pi}(t) = M(D)[AX(t) + Bu(0,t)] + \Big(q_y(D,0)u(0,t)$$
$$- q_y(D,D)u(D,t) - q(D,0)u_x(0,t)$$
$$+ \int_0^D q_{yy}(D,y)u(y,t)\,dy \Big)\epsilon. \qquad (143)$$

Using (9), (136) and (139), one can simplify (143) to

$$\dot{\Pi}(t) = M(D)AX(t) + Bu(D,t)$$
$$+ \int_0^D \epsilon q_{yy}(D,y)u(y,t)\,dy. \qquad (144)$$

Plugging (142) evaluated at $x = D$ into (144), using (131) and the fact that $M(D)$ and A commute, yields (65). On the other hand, (66) can be obtained by computing the first-in-time and second-in-space derivatives of (132), integrating twice by parts in the former, subtracting the resulting expressions and using (133)-(136), (138)-(139). Finally, (67) follows simply by evaluating (64) at $x = D$, while (68) follows by evaluating the spatial derivative of (64) at $x = 0$ and using (140).

APPENDIX C

First, it is shown that $M(x)$ can be alternatively rewritten in terms of a hyperbolic function as

$$M(x) = \cosh(x\sqrt{A/\epsilon}). \qquad (145)$$

Let us recall the matrices R and Φ, defined in Appendix B. Because of their structure, it is verified (by direct computations) that $\Phi^T R^{2j+1} \Phi = 0$ and $\Phi^T R^{2j} \Phi = (A/\epsilon)^j$, for all $j \in \{0,1,\ldots,\infty\}$. Therefore, using the Taylor expansion of the matrix exponential, one has that $M(x) = \Phi^T e^{Rx} \Phi = \sum_{n=0}^\infty (\Phi^T R^n \Phi)x^n/n! = \sum_{n=0}^\infty (A/\epsilon)^n x^{2n}/(2n)! = \sum_{n=0}^\infty (x\sqrt{A/\epsilon})^{2n}/(2n)! = \cosh(x\sqrt{A/\epsilon})$, which proves (145). Using (145), the backstepping transformation (90) can be compactly rewritten as

$$w(x,t) = u(x,t) - f(x)X(t) - (g \star u)(x,t), \qquad (146)$$

where $f(x) = K\cosh(x\sqrt{A/\epsilon})$, $g(x) = \epsilon^{-1}K\int_0^x f(\xi)\,d\xi B$ and $\star$ denotes the convolution operator in the x variable. Taking the Laplace transform of (146) yields

$$w(\sigma,t) = \Gamma u(\sigma,t) - K\sigma(\sigma^2 I - A/\epsilon)^{-1}X(t), \qquad (147)$$

where σ is the Laplace argument and $\Gamma = I - \epsilon^{-1}K(\sigma^2 I - A/\epsilon)^{-1}B$. Solving (147) for $u(\sigma,t)$ yields

$$u(\sigma,t) = \Gamma^{-1}w(\sigma,t) + \Gamma^{-1}K\sigma(\sigma^2 I - A/\epsilon)^{-1}X(t), \quad (148)$$

where $\Gamma^{-1} = I + \epsilon^{-1}K(\sigma^2 I - (A + BK)/\epsilon)^{-1}B$, which follows by the Woodbury identity. Now, adding and subtracting $K\sigma(\sigma^2 I - (A+BK)/\epsilon)^{-1}$ to $\Gamma^{-1}K\sigma(\sigma^2 I - A/\epsilon)^{-1}$ and using the identity $(\sigma^2 I - (A+BK)/\epsilon)^{-1}\big[I - \epsilon^{-1}BK(\sigma^2 I - A\epsilon)^{-1}\big] = (\sigma^2 I - A\epsilon)^{-1}$ leads to

$$\Gamma^{-1}K\sigma(\sigma^2 I - A/\epsilon)^{-1} = K\sigma(\sigma^2 I - (A+BK)/\epsilon)^{-1}. \quad (149)$$

Finally, plugging (149) into (148) and taking the inverse Laplace transform of the resulting expression yields (95), which completes the proof.

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