Power bounded composition operators on spaces of meromorphic functions [☆]

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We study composition operators with holomorphic symbols defined on spaces of meromorphic functions, when endowed with their natural locally convex topology. First, we show that such operators are well-defined, continuous and never compact. Then, we study the dynamics and prove that a composition operator is power bounded or mean ergodic if and only if the symbol is a nilpotent element in the group of automorphisms.

1. Introduction and notation

Let U be a open connected subset (=*domain*) of $\mathbb{C}$ and $\varphi : U \to U$ a holomorphic map of U into itself. The purpose of this brief note is to study the behavior of the orbits of composition operators $C_\varphi(f) := f \circ \varphi$, on the space $M(U)$ of meromorphic functions defined on U. We are interested in the case when the orbits of all the elements under C_φ are bounded. If this happens, the operator C_φ is called *power bounded*. We prove that, in this case, it is equivalent to C_φ be *mean ergodic*.

Given a subset $D \subset U$ we say that it is *discrete in* U whenever its accumulation set is contained in $\mathbb{C} \setminus U$ (i.e. it is discrete and closed in U). A *meromorphic function* f in U is a complex valued function f such

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that there exists a subset $D \subset U$ discrete in U with $f \in H(U \setminus D)$ and such that for each $u \in D$ there is $k \in \mathbb{N}$ such that $(z-u)^k f$ admits a holomorphic extension in u. The minimum value of k satisfying this condition is called *the order of f at u* and is denoted by $o_u(f)$. Let $P_f = D$ denote the set of *poles* of the meromorphic function f.

One natural way of endowing the space $M(U)$ of meromorphic functions with a topology is to consider it as a subspace of $C(U, \widehat{\mathbb{C}})$, $\widehat{\mathbb{C}}$ being the Alexandroff compactification of $\mathbb{C}$. In $\widehat{\mathbb{C}}$ it is considered the chordal metric and in $C(U, \widehat{\mathbb{C}})$ it is considered the topology τ_{chor} of locally uniform convergence. This is a metrizable non-locally convex topology, and moreover, a result of Cima and Schober [7] asserts that no comparable topology with τ_{chor} in $M(U)$ is complete. In 1995, Grosse-Erdmann studied deeply in [8] the locally convex topology introduced by Holdgrün in [9], giving a complete description of the seminorms, the properties of the topology, and showing that this topology, namely τ_{hol}, solved in the affirmative a conjecture of Tietz [15]. We describe briefly this topology.

A *positive divisor* δ on U is a map $\delta : U \to \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $P_\delta = \{z \in U : \ \delta(u) \neq 0\}$ is discrete in U. As a consequence of the Laurent integral formula, the space

$$M(U,\delta) = \{f \in M(U) : \ P_f \subset P_\delta \text{ and } o_z(f) \leq \delta(z) \text{ for all } z \in P_\delta\}$$

is a closed subspace of $H(U \setminus P_\delta)$ endowed with the compact open topology. In fact, it is shown in [11] that $M(U, \delta)$ is isomorphic to $H(U)$. Hence it is Fréchet Montel (see Ref. [4]). We denote by $PD(U)$ the set of positive divisors on U, endowed with the natural order $\delta_1 \leq \delta_2$ when $\delta_1(u) \leq \delta_2(u)$ for all $u \in U$. The Holdgrün topology τ_{hol} is defined as the inductive limit

$$\operatorname*{ind}_{\delta \in PD(U)} M(U,\delta),$$

with respect to the inclusions $M(U, \delta_1) \hookrightarrow M(U, \delta_2)$ whenever $\delta_1 \leq \delta_2$. Grosse-Erdmann showed in [8] that endowed with this topology $M(U)$ is an ultrabornological, Montel and complete Hausdorff locally convex space. He also proved that each $M(U, \delta)$ is a topological subspace of $M(U)$, and hence, in particular, $H(U)$ endowed with the compact open topology (that we will denote by τ_c throughout this paper) is a closed topological subspace of $M(U)$, and that $M(U)$ is not separable. The product in $M(U)$ is separately but not jointly continuous, hence it is not an algebra. The projections over the terms in the Laurent development are continuous. Altogether these facts permit us to assert that Holdgrün's topology is the natural locally convex topology in $M(U)$. A fundamental system of seminorms was also given. However, we do not need to write them explicitly. In the following it will be important to note that a linear operator $T : M(U) \to M(U)$ is continuous if and only if the restriction of T to each step $M(U, \delta)$ is continuous. Moreover, each bounded set $B \subset M(U)$ is contained (and then bounded) in some step $M(U, \delta)$. The vector valued analogues of this topology have been studied in [6,10,11].

Let X be a locally convex Hausdorff space and $T : X \to X$ a continuous and linear operator from X to X. The iterates of T are denoted by $T^n := T \circ \cdots \circ T$, $n \in \mathbb{N}$. For $x \in X$ we write $Orb(T, x) := \{T^n x, n \in \mathbb{N}_0\}$ as the *orbit* of x by T. If the sequence $(T^n)_{n\in\mathbb{N}}$ is equicontinuous in the space $L(X)$ of all continuous and linear operators from X to X, T is called *power bounded*. In case $X = M(U)$, it is a Montel space, and hence barrelled. Consequently, an application of the uniform boundedness principle can be applied to conclude that T is power bounded if and only if the orbit $\{T^n(x) : \ n \in \mathbb{N}\}$ is bounded for every $x \in X$.

Given any $T \in L(X)$, we introduce the notation

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N},$$

for the Cesàro means of T. The operator T is *mean ergodic* precisely when $\{T_{[n]}\}_{n=1}^{\infty}$ converges pointwise, i.e., for all $x \in X$, the limit $\lim_{n\to\infty} T_{[n]}(x)$ exists. If $\{T_{[n]}\}_{n=1}^{\infty}$ converges uniformly on the bounded sets of X then, T will be called *uniformly mean ergodic*.

In 2011 Bonet and Domański characterized those composition operators C_φ defined on $H(U)$ which are power bounded and proved that this condition is equivalent to the composition operator being mean ergodic or uniformly mean ergodic, see [5, Proposition 1].

We characterize those composition operators C_φ, $C_\varphi(f) := f \circ \varphi$, on the space of meromorphic functions $M(U)$ on a domain U in $\mathbb{C}$, $\varphi : U \to U$ holomorphic, such that C_φ is power bounded. We consider in $M(U)$ the locally convex topology introduced by Holdgrün in [9] and deeply studied by Grosse-Erdmann in [8]. We show that such a composition operator is well defined, continuous but it cannot be compact. We show that power boundedness is equivalent to uniform mean ergodicity. Moreover, we describe precisely the form of φ such that C_φ is power bounded. More precisely, we prove that only for symbols $\varphi \in H(U)$ for which there exists $n \in \mathbb{N}$ such that $\varphi^n = id_U$ the composition operator is power bounded or mean ergodic. Here id_U stands for the identity in U.

Our notation for topology and functional analysis is standard, see for example [14]. In what follows, given $x \in \mathbb{C}$ and $r > 0$ we denote by $B(x, r)$ the open ball centered at x with radius r.

2. Composition operators on spaces of meromorphic functions

Proposition 2.1. *Let U_1 and U_2 be two domains in $\mathbb{C}$ and let $\varphi : U_2 \to U_1$ be a non-constant holomorphic function. Then $C_\varphi : M(U_1) \to M(U_2)$, $f \mapsto f \circ \varphi$ is a well-defined and continuous linear operator.*

Proof. A function f is meromorphic in U_1 if and only if there are $f_1, f_2 \in H(U_1)$ such that $f = f_1/f_2$. Hence the operator is well defined. Since $M(U_1)$ is an inductive limit of the Fréchet spaces $M(U_1, \delta)$ for $\delta \in PD(U_1)$, to prove that C_φ is continuous, it is enough to show that for every positive divisor $\delta \in PD(U_1)$ in U_1, there exists $\tilde{\delta} \in PD(U_2)$ such that $C_\varphi(M(U_1, \delta)) \subseteq M(U_2, \tilde{\delta})$. Once this is established, since $M(U_2, \delta)$ is a topological subspace of $M(U_2)$, the continuity follows from the closed graph theorem, because in this situation the composition operator is continuous if we endow $M(U_2, \tilde{\delta})$ with the topology of pointwise convergence in $U_2 \setminus P_{\tilde{\delta}}$. Now fix $\delta \in PD(U_1)$. By Weierstrass's Theorem [13, Theorem 8.3.2], we can choose g such that its set of zeros $Z_g = P_\delta$ and with order of each zero α equal to $\delta(\alpha)$. This implies that every $f \in M(U_1, \delta)$ can be written as $f = f_1/g$, with $f_1 \in H(U_1)$. Since $g \circ \varphi$ is holomorphic, if we define $P_{\tilde{\delta}} := Z_{g\circ\varphi}$ then $P_{\tilde{\delta}}$ is discrete in U_2. For $a \in P_{\tilde{\delta}}$, let $k(a)$ be the order as a zero at a of the holomorphic function $g \circ \varphi$. We can define a positive divisor $\tilde{\delta} : U_2 \to \mathbb{N}_0$ in the following way:

$$\tilde{\delta}(z) = \begin{cases} 0 & \text{if } z \notin P_{\tilde{\delta}}, \\ k(a) & \text{if } z \in P_{\tilde{\delta}}. \end{cases}$$

Now it is clear that $C_\varphi(M(U_1, \delta)) \subset M(U_2, \tilde{\delta})$. $\square$

To state the following result, we recall that, if X and Y are locally convex spaces, a linear operator $T : X \to Y$ is said to be *compact* whenever there exists a 0-neighborhood V in X such that $T(V)$ is relatively compact in Y.

Proposition 2.2. *Let $\varphi : U_2 \to U_1$ be a non-constant holomorphic function. Then the composition operator $C_\varphi : M(U_1) \to M(U_2)$ is not compact.*

Proof. If C_φ is compact, there exists a 0-neighborhood V such that $C_\varphi(V)$ is a relatively compact subset of $M(U_2)$. In particular, $C_\varphi(V)$ is bounded in $M(U_2)$. Then there exists $\delta \in PD(U_2)$ such that $C_\varphi(V) \subseteq M(U_2, \delta)$. Hence, we obtain

$$C_\varphi M(U_1) \subseteq M(U_2, \delta). \tag{2.1}$$

For every $z_0 \in U_1$, we can consider $\omega_0 = \varphi(z_0)$ and $f(z) = \frac{1}{z-\omega_0} \in M(U_1)$. Then $C_\varphi(f)(z) = \frac{1}{\varphi(z)-\omega_0}$ has a pole in z_0, which is a contradiction with (2.1). $\square$

3. Dynamics of composition operators in $M(U)$

We begin the last section by observing the following fact. If X is a Montel space, then it is barrelled and reflexive. If $T : X \to X$ is a power bounded operator, by [3, Proposition 3.3] (see also [1,2]), T is mean ergodic, which implies that the sequence $(1/n \sum_{m=1}^{n} T^m)_{n\in\mathbb{N}}$ is equicontinuous, since X is barrelled. Hence the sequence $(1/n \sum_{m=1}^{n} T^m)_{n\in\mathbb{N}}$ converges uniformly on the compact subsets of X and, then uniformly on the bounded subsets of X, since X is Montel. Therefore, T is uniformly mean ergodic.

Let $\varphi : U \to U$ be a continuous self-map on a topological space U. We say that φ has *stable orbits* on U if for every compact subset K of U there is a compact subset L of U such that $\varphi^n(K) \subset L$ for all $n \in \mathbb{N}$. Bonet and Domański [5] showed that C_φ is power bounded on $H(U)$ if and only if it is mean ergodic and if and only if the map φ has stable orbits on U. In our framework, the condition of stability of the orbits of φ is only a necessary condition (see after the next result).

Theorem 3.1. *Let $U \subset \mathbb{C}$ be a domain and let $\varphi : U \to U$ be a non-constant holomorphic function. The following assertions are equivalent for the composition operator $C_\varphi : M(U) \to M(U)$:*

(a) *C_φ is power bounded.*
(b) *C_φ is (uniformly) mean ergodic.*
(c) *There exists $n \in \mathbb{N}$ such that $\varphi^n = id_U$.*

Proof. Since $M(U)$ is a Montel space, by the observation at the beginning of this section, (a) implies (b). Condition (c) implies that the orbits of C_φ are finite, and then (a) is fulfilled.

If we assume (b), then an easy calculation shows that the sequence $((1/n)C_{\varphi^n})_n$ is pointwise convergent to 0 (see, for example, the beginning of Section 2 in [2]). Since $H(U)$ is a closed topological subspace of $M(U)$ and φ is holomorphic, the operator C_φ is also mean ergodic in $H(U)$. Then, an application of [5, Proposition 1] implies that φ has stable orbits on U. Now, we apply a result due to Abate (see [12, Theorem 5.5.4]) to conclude that we have two possibilities:

(1) There is a fixed point $z_0 \in U$ of φ such that $(\varphi^n)_n$ converges to the constant function $\alpha(z) := z_0$ in $(H(U), \tau_c)$, or
(2) There exists a subsequence $(\varphi^{n_k})_k$ which converges to id_U in $(H(U), \tau_c)$.

Suppose that (1) happens, and let $0 < r < 1$ such that $B(z_0, r) \subset U$. Let $f(z) := 1/(z - z_0)$. We consider the set of poles of the orbit $P(Orb(C_\varphi, f)) := \cup_n P_{f\circ\varphi^n}$, i.e., the union of the poles of all the functions $f \circ \varphi^n$, $n \in \mathbb{N}_0$. This set is discrete and closed in U since the boundedness of $((1/n)C_{\varphi^n}(f))_n$ implies that there exists a positive divisor δ of U such that $\varphi^n(f) \subset M(U, \delta)$ for each $n \in \mathbb{N}$. The compact open convergence of $(\varphi^n)_n$ to the constant function α permits to get n_0 big enough such that $\varphi^{n_0}(B(z_0, r)) \subset B(z_0, r/2)$. Let $g := \varphi^{n_0}|_{B(z_0,r)}$, which satisfies $g^n(B(z_0, r)) \subset B(z_0, r/2)$ for each $n \in \mathbb{N}$. Applying the Maximum Modulus Principle to the function $(g(z) - z_0)/(z - z_0)$ (taking its holomorphic extension in z_0) we get $|g(z) - z_0| \le (1/2)|z - z_0|$ for any $z \in B(z_0, r)$. An iteration of this inequality gives

$$|g^n(z) - z_0| \le \frac{r}{2^n} \le \frac{1}{2^n},$$

for each $n \in \mathbb{N}$ and $z \in B(z_0, r)$. Let $n_k := n_0 k$. For $z_1 \in B(z_0, r) \setminus P(Orb(C_\varphi, f))$ and $k \geq 1$, we have

$$\left|\frac{1}{n_k} f \circ \varphi^{n_k}(z_1)\right| = \left|\frac{1}{n_0 k(g^k(z_1) - z_0)}\right| \geq \frac{2^k}{n_0 k},$$

and this yields that $(|\frac{1}{n_k} f \circ \varphi^{n_k}(z_1)|)_k$ tends to infinity as k goes to infinity, a contradiction with the boundedness of $((1/n)C_{\varphi^n}(f))_n$ in $(H(U \setminus P(Orb(C_\varphi, f)), \tau_c)$.

Thus only (2) is possible. We see that this implies that φ is an automorphism and $\varphi^{-n} := (\varphi^n)^{-1}$ belongs to the closure of $\{\varphi^n : n \in \mathbb{N}\}$ in the compact open topology for each $n \in \mathbb{N}$. To see this, let $(\varphi^{n_k})_k$ be a convergent subsequence of $(\varphi^n)_n$ to id_U. For a fixed $n \in \mathbb{N}$, let $t_k = n_k - n$, which is a natural number for k big enough. The stability of the orbits permits to select a subsequence $(t_{k_j})_j$ such that $(\varphi^{t_{k_j}})_j$ is convergent in $(H(U), \tau_c)$ to some function g, and hence $\varphi^n g = g\varphi^n = id_U$. Therefore, it is easy to see that φ is an automorphism of U such that φ^{-1} has stable orbits. Suppose now that there exists some $u \in U$ with infinite orbit $Orb(\varphi^{-1}, u) = \{\varphi^{-n}(u) : n \in \mathbb{N}\}$. This set is relatively compact because φ^{-1} has stable orbits. Let $f(z) := 1/(z-u)$. The sequence $((1/n)C_{\varphi^n}(f))_n = ((1/n)f \circ \varphi^n)_n$ has poles in $Orb(\varphi^{-1}, u)$, which is infinite and relatively compact. Hence $((1/n)C_{\varphi^n}(f))_n$ is not bounded in $M(U)$, a contradiction. Thus we have that for each $z \in U$ the orbit of φ^{-1} is finite, and consequently also the orbit of φ is finite for every $z \in U$. Since φ is an automorphism, this yields that each $z \in U$ is a periodic point for the orbit of φ, and hence

$$U = \bigcup_n (\varphi^n - id_U)^{-1}(0).$$

The discreteness of the zeros of non-constant holomorphic functions and an application of Baire category theorem give (c), which concludes the proof. □

Example 3.2. If $U = \mathbb{D}$, the unit ball in $\mathbb{C}$, Bonet and Domański [5] gave a precise description of the maps φ such that $C_\varphi : H(U) \to H(U)$ is power bounded. In particular, when φ is a rotation in the unit ball defined as $\varphi(z) = e^{i\theta\pi} z$ for θ an irrational number, then $C_\varphi : H(U) \to H(U)$ is power bounded (and uniformly mean ergodic) but $C_\varphi : M(U) \to M(U)$ is not power bounded nor mean ergodic.

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