New Results on the Mixed General Routing Problem

A. Corberán[1]*, G. Mejía[2] and J.M. Sanchis[2]

[1] Dept. d'Estadística i Investigació Operativa, Universitat de València, Spain
[2] Dept. de Matemática Aplicada, Universidad Politécnica de Valencia, Spain

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Abstract

In this paper we deal with the polyhedral description and the resolution of the Mixed General Routing Problem. This problem, in which the service activity occurs both at some of the nodes and at some of the arcs and edges of a mixed graph, contains a large number of important arc and node routing problems as special cases. Here, a large family of facet defining inequalities, the Honeycomb inequalities, is described. Furthermore, a cutting plane procedure for this problem that incorporates new separation procedures for the K-C and Honeycomb inequalities is presented. Extensive computational experiments over different sets of instances are included.

Key Words: Polyhedral Combinatorics, Rural Postman Problem, General Routing Problem, Mixed Rural Postman Problem.

1 Introduction

Routing Problems arise in several areas of distribution management and logistics and their practical significance is widely known. While in Node Routing Problems the service activity occurs at all (or at some subset of) the nodes, Arc Routing Problems are routing problems where a single vehicle or a fleet of vehicles must service all (or some subset of) the arcs (and/or edges) of a transportation network. Although Arc Routing Problems have historically received less attention than Node Routing Problems, they have been extensively studied in recent years. The papers by Assad & Golden (1995), Eiselt, Gendreau & Laporte (1995a,b) and the recent book edited by Dror (2000a) summarize the state of the art and practical applications of these problems.

In this paper we deal with a more general routing problem in which the service activity occurs both at some of the nodes and at some of the arcs and edges of a mixed graph. Given a strongly connected mixed graph $G = (V, E, A)$ with vertex set V, edge set E, arc set A, a cost c_e for each *link* $e \in E \cup A$, a subset $E_R \subseteq E$ of *required edges*, a subset $A_R \subseteq A$ of *required arcs* and a subset $V_R \subseteq V$ of *required vertices*, the *Mixed General Routing Problem* (MGRP) is the problem of finding a minimum cost tour passing through each $e \in E_R \cup A_R$ and through each $i \in V_R$ at least once.

*corresponding author: angel.corberan@uv.es

The MGRP contains most of the best known routing problems with a single vehicle as special cases:

- When $A = \emptyset$, the General Routing Problem (GRP) is obtained (Orloff (1974), Letchford (1997,1999), Corberán & Sanchis (1998), Corberán, Letchford & Sanchis (2001)).
  - If, in addition, $V_R = \emptyset$, the Rural Postman Problem (RPP) is obtained (Orloff (1974), Christofides et al. (1981), Corberán & Sanchis (1994), Ghiani & Laporte (2000), Fernández et al. (2001)). Similarly, when $E_R = E$, the RPP reduces to the well known Chinese Postman Problem or CPP (Guan (1962), Edmonds (1963), Edmonds & Johnson (1973)).
  - On the other hand, if $E_R = \emptyset$ the GRP reduces to the Steiner Graphical Traveling Salesman Problem (Cornuèjols, Fonlupt & Naddef, 1985), also called the Road Traveling Salesman Problem (Fleischman, 1985); if, in addition, $V_R = V$, the Graphical Traveling Salesman Problem is obtained (Cornuèjols, Fonlupt & Naddef, 1985).
- When $E = \emptyset$ and $V_R = \emptyset$, the MGRP reduces to the Directed Rural Postman Problem (Christofides et al., 1986). If, in addition, $A_R = A$, the Directed Chinese Postman Problem (DCPP) is obtained (Edmonds & Johnson, 1973). Also, as in the undirected case, the Graphical Asymmetric Traveling Salesman Problem (GATSP) is obtained when $E = \emptyset$, $A_R = \emptyset$ and $V_R = V$ (Chopra & Rinaldi, 1996).
- Finally, when $A \neq \emptyset$, $E \neq \emptyset$ and $V_R = \emptyset$, the Mixed Rural Postman Problem (MRPP) is obtained (Romero (1997), Corberán, Romero & Sanchis (2002)). If, in addition, $E_R = E$ and $A_R = A$, the MRPP reduces to the Mixed Chinese Postman Problem (Edmonds & Johnson (1973), Christofides et al. (1984), Grötschel & Win (1992), Nobert & Picard (1996)).

While the CPP and the DCPP can be solved in polynomial time, all the other problems are NP-hard. These results can be found in some of the previous references and in the papers by Papadimitriou (1976), Lenstra & Rinnooy-Kan (1976) and Dror (2000b). In what refers to the polyhedral descriptions and solution methods for these problems, the reader is referred to the above mentioned papers as well as to the chapters by Eglese & Letchford, Hertz & Mittaz and Benavent, Corberán & Sanchis in the book edited by Dror (2000a).

Very recently, the Mixed General Routing Problem has been studied in Corberán, Romero & Sanchis (2002). In this paper, the authors present a formulation of the problem and a partial description of its associated polyhedron. Some basic facet defining inequalities are introduced (trivial, connectivity, balanced-set and R-odd cut constraints), as well as a large family of inequalities, the Path Bridge (PB) inequalities. Furthermore, some computational results are presented with a preliminary cutting-plane algorithm. This procedure, which includes separation algorithms for the connectivity, balanced-set and R-odd cut constraints, has produced very good lower bounds over a set of 100 randomly generated instances of the MRPP.

In this paper, we present a new family of facet defining inequalities, the Honeycomb inequalities, first introduced by Corberán and Sanchis (1998) for the undirected GRP. New separation procedures for these inequalities and for an important subset of the PB constraints, the K-C inequalities, have allowed us to improve the previous cutting-plane algorithm considerably. More precisely, in the next section we define the problem and we present the notation to be used. Section 3 briefly summarizes the known results on the MGRP polyhedron, while sections 4 and 5 describe a new family of facet-inducing inequalities, the Honeycomb inequalities. Finally, section 6 describes the implemented cutting-plane algorithm and the computational results obtained.

2 Problem definition and notation

In the MGRP, as in the undirected case, it is helpful to assume, without loss of generality, that the vertices incident to any required link (arc or edge) are also required. Moreover, we will also assume that the original graph has been transformed to satisfy that $V = V_R$ and $E \setminus E_R = \emptyset$. This is not a serious restriction as there is a simple way to transform MGRP instances which do not satisfy the assumption into instances which do (see, for instance, Christofides et al. (1986) or Eiselt, Gendreau & Laporte (1995b)). Hence, we will assume, in what follows, that we are working with a (simplified) strongly connected graph $G = (V, E, A) := (V_R, E_R, A_R \cup A_{NR})$, in which all the vertices are either required or incident to a required link and $E \setminus E_R = \emptyset$.

The first part of the polyhedral study of the MGRP is given in Corberán, Romero & Sanchis (2002). In this paper we use the same notation. Let $G^R = (V, E, A_R)$ be the graph obtained by deleting in G all the non-required arcs. This graph is, in general, non-connected. Let us denote by p the number of its connected components and by $V_1, V_2, \ldots, V_p$, with $V_1 \cup \ldots \cup V_p = V$, their corresponding vertex sets (R-sets). The subgraphs of G induced by the R-sets will be represented by $C_i = G(V_i)$, $i = 1, \ldots, p$, and they will be referred to as R-connected components. Given $S, T \subset V$, $(S : T)$ will denote the set of links with an endpoint in S and the other in T, $E(S : T)$ will be the set of edges between S and T, while $A(S : T)$ will represent the set of arcs from S to T. Hence, $(S : T) = E(S : T) \cup A(S : T) \cup A(T : S)$. Furthermore, $\delta(S) = (S : V \setminus S)$, $A^+(S) = A(S : V \setminus S)$, $A^-(S) = A(V \setminus S : S)$, $A(S) = A^+(S) \cup A^-(S)$, $E(S) = E(S : V \setminus S)$ and $\gamma(S)$ will denote the set of links with both endpoints in S. These sets are defined in a similar way with respect only to the required links or to the non-required links: $(S : T)_R$, $A_{NR}(S : T)$, $A^+_R(S)$, $\delta_R(S)$, etc.

Applying the necessary and sufficient conditions for a mixed graph to be Eulerian (Ford and Fulkerson, 1962), a family $\mathcal{F}$ of links of G will be a tour for the MGRP if $\mathcal{F}$ contains all the required links, and graph $(V, E^{\mathcal{F}} \cup A^{\mathcal{F}})$ is even, connected and balanced. The balanced condition means that for every $S \subset V$, the difference between the number of arcs in $A^+(S)$ (leaving S) and the number of arcs in $A^-(S)$ (entering S) is less than or equal to the number of edges in $E(S)$.

In Corberán, Romero & Sanchis (2002), the MGRP is formulated with respect to semitours, i.e., the family of links obtained from any tour for the MGRP by deleting one copy of every required link. An incidence vector $x = (x_e : \; e \in E \cup A) \in \mathbb{R}^{E \cup A}$ is associated with each tour (semitour), where x_e denotes the number of times a link $e \in E \cup A$ appears in the tour (semitour). If x^R denotes the incidence vector of the required links, x is a semitour for the MGRP on G if, and only if, $x + x^R$ is a tour. A vertex $v \in V$ is *R-odd* if it is incident to an odd number of required links, otherwise it is *R-even*. Note that every isolated required vertex is R-even. Given $S \subset V$, let $u_S = |A^+_R(S)| - |A^-_R(S)| - |E(S)|$. The set of semitours for the MGRP is the set of vectors $x \in \mathbb{R}^{E \cup A}$ satisfying:

$$x_e \geq 0 \;\; and \;\; integer, \qquad \forall e \in E \cup A \tag{1}$$

$$x(\delta(\{i\})) \equiv 0 \bmod 2, \qquad \forall i \in V \; : \; v \; is \; R-even \tag{2}$$

$$x(\delta(\{i\})) \equiv 1 \bmod 2, \qquad \forall i \in V \; : \; v \; is \; R-odd \tag{3}$$

$$x(A^+(S)) \geq 1, \qquad \forall S = \cup_{k \in Q} V_k, \;\; Q \subset \{1, \ldots, p\} \tag{4}$$

$$-x(A^+(S)) + x(A^-(S)) + x(E(S)) \geq u_S, \qquad \forall S \subset V \tag{5}$$

where conditions (2) and (3), (4) and (5) force the graph represented by the tour to be, respectively, even, connected and balanced.

The above system (1) to (5) includes an equation associated with each set $S \subset V$ with $E(S) = \emptyset$. Let us represent by $K_1, K_2, \ldots, K_q$ the sets of vertices of the connected components of the graph (V, E). Some sets K_i could consist of a single vertex only. We will call the q subgraphs of G induced by sets K_i *edge-connected components* of G. The q equations

$$x(A^+(K_i)) + u_{K_i} = x(A^-(K_i)), \quad i = 1, 2, \dots, q \tag{6}$$

will be referred to as *the system equations* and any $q-1$ of them are linearly independent.

3 MGRP Polyhedron

Let MGRP(G) be the convex hull of all the semitours $x \in \mathbb{R}^{E \cup A}$ satisfying (1) to (5). Corberán, Romero and Sanchis (2002) show that MGRP(G) is an unbounded polyhedron with dim(MGRP$(G)) = |E \cup A| - q + 1$, where q is the number of the connected components of the graph (V, E), and that the following inequalities are, under mild conditions, facet-inducing for MGRP(G):

- Trivial inequalities
- Connectivity inequalities (4)
- Balanced-set inequalities (5)
- The R-odd cut inequalities

$$x(\delta(S)) \geq 1, \ \forall \delta(S) \quad R\text{-odd cutset of } G \tag{7}$$

- The (standard) Path-Bridge (PB) and the Path-Bridge$_{02}$ (PB$_{02}$) inequalities.

Moreover, it is also shown that all facet-inducing inequalities for the MGRP(G), except those equivalent to the trivial ones, are *configuration inequalities*, i.e., each inequality is defined by a partition of V, $\mathcal{B} = \{B_1, B_2, \dots, B_r\}$, and by some costs associated with the links among these node sets B_i. The variables associated with the links in $\gamma(B_i)$ have coefficient zero in the inequality, and the variables associated with the links in $(B_i : B_j)$ have coefficient equal to the cost defined between B_i and B_j. Then, associated with an inequality, we have a *configuration graph*, $G_{\mathcal{C}}$, having node set $\mathcal{B}$, a required edge (B_i, B_j) for each required edge (u, v) of G with $u \in B_i$, $v \in B_j$, a required arc (B_i, B_j) for each required arc (u, v) of G with $u \in B_i$, $v \in B_j$ and a non-required arc (B_i, B_j) for each pair B_i, B_j such that $A_{NR}(B_i : B_j) \neq \emptyset$. In other words, $G_{\mathcal{C}}$ is the graph resulting after shrinking node sets B_i, $i = 1, \dots, r$, into a single vertex each, and shrinking each set of non-required parallel arcs into one single arc, but keeping all the required links.

To illustrate the above definitions, we describe here the K-C inequalities and their configuration graph. For these inequalities, which are a particular case of PB inequalities, separation algorithms are presented in section 6.1.

A *K-C configuration* (see figure 1a) is defined by an integer $K \geq 3$, a partition of V into $K + 1$ subsets $\{M_0, M_1, M_2, \dots, M_{K-1}, M_K\}$ such that each R-set V_i, $1 \leq i \leq p$, is contained in exactly one of the node sets $M_0 \cup M_K, M_1, M_2, \dots, M_{K-1}$, the induced subgraphs $G(M_i)$, $i = 0, 1, 2, \dots, K$, are strongly connected and $(M_0 : M_K)$ contains a positive and even number of required links, and by the cost functions defined as $c(M_0, M_K) = K - 2$ and $c(M_i, M_j) = |i - j|$, $\forall i, j \ : \ \{i, j\} \neq \{0, K\}$.

The partition $\mathcal{B} = \{M_0, M_1, M_2, \dots, M_{K-1}, M_K\}$ and the above costs define the configuration graph $G_{\mathcal{C}}$ whose skeleton is shown in figure 1a. Internal arcs (M_i, M_j) not represented in

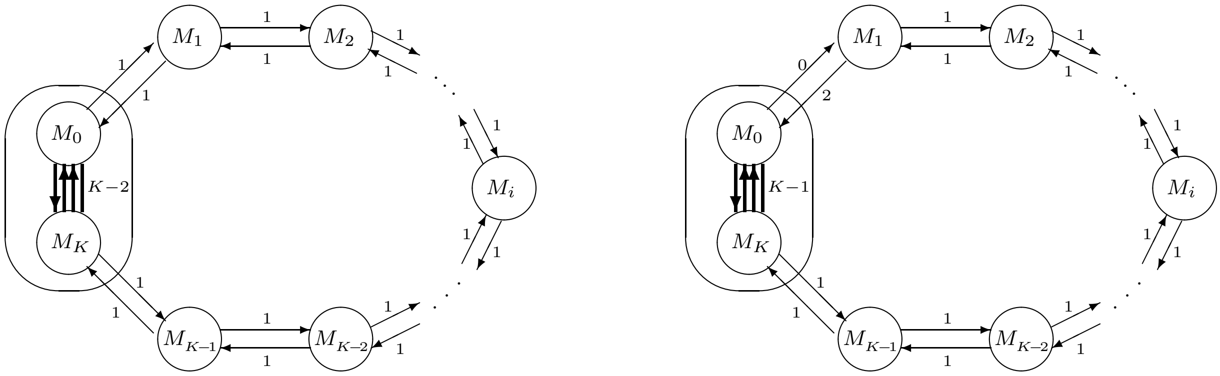

Figure 1: K-C and K-C_{02} configurations.

figure 1 have a cost equal to the length of the shortest path from M_i to M_j using external arcs. The associated *K-C inequality* is:

$$(K-2)\ x\Big((M_0 : M_K)\Big) + \sum_{\substack{0 \le i < j \le K \\ (i,j) \ne (0,K)}} |i-j|\ x\Big((M_i : M_j)\Big) \ge 2(K-1) \tag{8}$$

Other facet-inducing inequalities closely related to the above (standard) K-C inequalities are the K-C_{02} inequalities, whose skeleton is shown in figure 1b. Note that the opposite external arcs in $(M_0 : M_1)$ now have coefficients 0 and 2 (these coefficients could be associated with the opposite arcs in any set $(M_i : M_{i+1})$, for $i = 1, 2, \ldots, K-1$, but the corresponding inequalities would be equivalent), and that links in (M_0, M_K) now have coefficient $K-1$. Again, internal arcs (M_i, M_j) have a cost equal to the length of the shortest path from M_i to M_j using external arcs.

All the above described inequalities, except the trivial ones, have been proved to be facet-inducing for MGRP(G) in Corberán, Romero and Sanchis (2002) by using two 'lifting' theorems. These theorems state conditions for a given configuration inequality which is facet-inducing for MGRP$(G_{\mathcal{C}})$ to be also facet-inducing for MGRP(G). Below, a third lifting theorem that will be used in the next sections is presented.

Theorem 1 *Let G be a mixed graph and let $\mathcal{C}$ be a configuration on G. The associated configuration inequality is facet-inducing for MGRP(G) if the configuration inequality $F(x) \ge c_0$ associated with $\mathcal{C}$ on graph $G_{\mathcal{C}}$ is facet-inducing for MGRP($G_{\mathcal{C}}$) and the following conditions are satisfied:*

(a) *Given any proper subset of edges in $G_{\mathcal{C}}$, it is possible to orient each edge in such a way that inequality $F(x) \ge c_0$ is also facet-inducing for the MGRP polyhedron associated with the resulting graph $G'_{\mathcal{C}}$.*

(b) *Given any edge in $G_{\mathcal{C}}$ that has been oriented in (a) to obtain graph $G'_{\mathcal{C}}$, there exists at least one semitour x^* for the MGRP traversing edge e in the opposite direction to that given in $G'_{\mathcal{C}}$ satisfying $F(x^*) = c_0$.*

Proof: The proof is similar to that of the second lifting theorem in Corberán, Romero and Sanchis (2002) and is omitted here for the sake of brevity. ♦

4 Honeycomb inequalities

The *Honeycomb* inequalities were introduced by Corberán & Sanchis (1998) for the undirected GRP. Like PB inequalities, they are also a generalization of K-C inequalities. However, the generalization is in a different direction and neither class contains the other.

In a K-C configuration (see figure 1), a R-connected component (or a cluster of R-connected components) is divided into two parts (M_0 and M_K). In this section we generalize this configuration simultaneously both in the number of parts a R-connected component is divided into and in the number of R-connected components we divide.

As an illustration, consider the MGRP instances in figure 2, where the required links and the required vertices are represented in bold lines. It can be seen that the vector x such that $x_a = 0.5$ for each $a \in A_{NR}$ and $x_e = 0$ for each $e \in E_R \cup A_R$ is an extreme point of the polyhedron defined by all the inequalities mentioned before, but x is not a semitour. An inequality that is not satisfied by x is $\sum_{e \in E \cup A} x_e \geq 6$, which could be considered as a more general K-C inequality dividing a R-connected component into 3 parts (this would be the case of figure 2a) or dividing two R-connected components simultaneously (figure 2b). We show below that these (honeycomb) inequalities are facet-inducing of MGRP(G).

Figure 2: Two MGRP instances

Consider a partition of the set of vertices V into K vertex sets $\{A_1, A_2, \ldots, A_L, A_{L+1}, \ldots, A_K\}$, $3 \leq K \leq p$, $1 \leq L \leq K$, in such a way that each V_j, $1 \leq j \leq p$, is contained in exactly one A_i and the induced subgraphs $G(A_i)$, $i = 1, 2, \ldots, K$, are strongly connected.

Suppose we can now partition each set A_i, $i = 1, 2, \ldots, L$, into $\gamma_i \geq 2$ subsets, $A_i = B_i^1 \cup B_i^2 \cup \ldots \cup B_i^{\gamma_i}$, satisfying the following conditions:

H1) Each B_i^j contains an even number of R-odd nodes, $j = 1, 2, \ldots, \gamma_i$.

H2) The induced subgraphs $G(B_i^j), j = 1, 2, \ldots, \gamma_i$, are strongly connected.

H3) The graph with node set $B_i^1, B_i^2, \ldots, B_i^{\gamma_i}$ and having an edge (B_i^j, B_i^k) for every pair of nodes $B_i^j \neq B_i^k$ such that $(B_i^j : B_i^k)_R \neq \emptyset$, is connected.

Condition H3 is obviously satisfied when A_i is a single R-set. When A_i consists of several R-sets, condition H3 implies that the partition of A_i into the B_i^j is made by cutting the R-connected components.

For notational convenience, we denote $B_i^0 = A_i$, $i = L+1, \ldots, K$. We have therefore the following partition of V:

$$\mathcal{B} = \{B_1^1, B_1^2, \ldots, B_1^{\gamma_1}, B_2^1, B_2^2, \ldots, B_2^{\gamma_2}, \ \ldots \ , B_L^1, B_L^2, \ldots, B_L^{\gamma_L}, \ B_{L+1}^0, \ldots, B_K^0\}$$

This partition $\mathcal{B}$ defines a configuration graph $G_{\mathcal{C}} = (\mathcal{B}, \mathcal{E} \cup \mathcal{A})$ with a set of nodes $\mathcal{B}$ and a set of links $\mathcal{E} \cup \mathcal{A}$ formed by a required link (edge or arc) (B_r^i, B_q^j) for each required link $e \in (B_r^i : B_q^j)_R$ and a non-required arc (B_r^i, B_q^j) between each couple of nodes B_r^i, B_q^j such that $A_{NR}(B_r^i : B_q^j) \neq \emptyset$.

Let us suppose that there is a set T of pairs of opposite non-required arcs in $G_{\mathcal{C}}$ joining nodes corresponding to different A_j, $j = 1, 2, \ldots K$, such that the undirected graph with node set $\mathcal{B}$ and having an edge (B_i^j, B_p^q) for each pair of opposite arcs (B_i^j, B_p^q), (B_p^q, B_i^j) in T, is a spanning tree. Then, for each pair of nodes B_i^j, B_p^q in $\mathcal{B}$, $d(B_i^j, B_p^q)$ will denote the number of arcs in the unique path in $(\mathcal{B}, T)$ joining B_i^j to B_p^q. We will assume that the following condition is also satisfied:

H4) $d(B_i^j, B_i^q) \geq 3 \quad \forall\, i = 1, \ldots, L$ and $\forall\, j \neq q$.

The graph $(\mathcal{B}, T)$ defines the skeleton of the configuration (see figure 3, where the arcs in T are represented in thin lines and the required links in bold lines) and looks like a honeycomb where each cell is a K-C configuration defined by a pair of nodes B_q^i, B_q^j (related to the same A_q) and by the unique path in T joining B_q^i and B_q^j ($K \geq 3$ since H4 is satisfied). We divide the set of links in the configuration graph $G_{\mathcal{C}} = (\mathcal{B}, \mathcal{E} \cup \mathcal{A})$ into 3 sets:

- The set $\mathcal{C}$ (from 'cut') formed by the links joining nodes B_i^p, B_i^q with $p, q \neq 0$ (the nodes obtained by 'cutting' the sets A_i, $i = 1, \ldots, L$).
- The set of arcs in T, which will be called external arcs.
- The set $\mathcal{I}n$ formed by the remaining arcs, which will be called internal arcs.

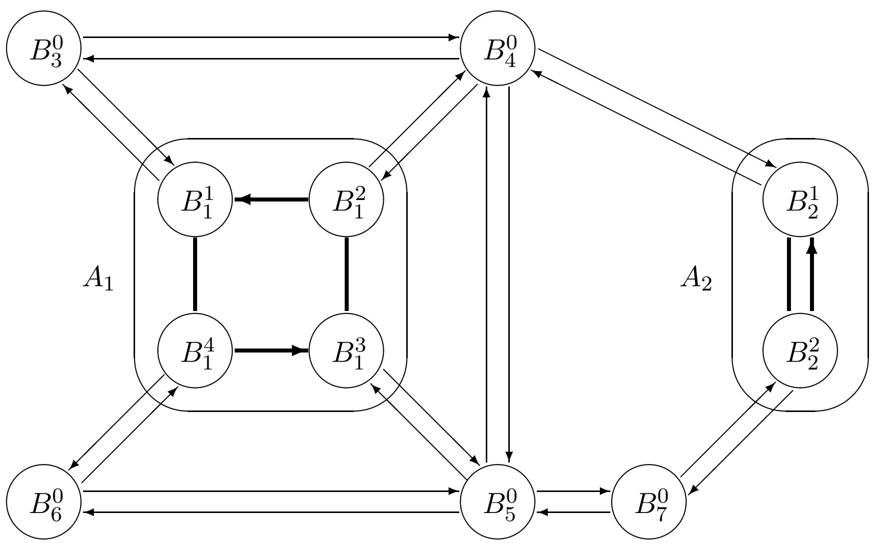

Figure 3: Honeycomb Configuration

We define the configuration costs on the links of $(\mathcal{B}, \mathcal{E} \cup \mathcal{A})$ as follows:

I) For the links $(B_q^i, B_q^j) \in \mathcal{C}$ such that the path in $(\mathcal{B}, T)$ joining B_q^i and B_q^j does not contain more than one node related to the same A_s (except the nodes B_q^i and B_q^j):

$$c(B_q^i, B_q^j) = d(B_q^i, B_q^j) - 2$$

(Note that, since H4 is satisfied, $c(B_q^i, B_q^j) \geq 1$).

II) For the arcs $(B_r^i, B_q^j) \in T \cup \mathcal{I}n$, $r \neq q$, such that the path in $(\mathcal{B}, T)$ joining B_r^i and B_q^j does not contain more than one node related to the same A_s:

$$c(B_q^i, B_q^j) = d(B_q^i, B_q^j)$$

III) The remaining links (if any), i.e. links (B_r^i, B_q^j), such that in the path in $(\mathcal{B}, T)$ joining B_r^i and B_q^j there is more than one node related to the same A_s (as distinct from pair B_r^i, B_q^j, when $r = q$), are ordered in an arbitrary way $a_1, a_2, \ldots, a_k$. For $h = 1, \ldots, k$, if $a = (B_r^i, B_q^j)$, let $c(B_r^i, B_q^j)$ be the maximum value such that a belongs to a semitour of c-cost $2(K-1)$ using only links from $(\mathcal{E} \cup \mathcal{A}) \setminus \{a_{h+1}, \ldots, a_k\}$ (sequential lifting).

A *honeycomb configuration* will be the pair formed by the partition $\mathcal{B}$ of V and the costs obtained from T. The following inequality will be called *honeycomb inequality*:

$$\sum_{a \in E \cup A} c_a x_a \geq 2(K-1), \quad \text{where} \quad c_a = c(B_r^i, B_q^j) \; \forall a \in (B_r^i : B_q^j), \quad c_a = 0 \; \forall a \in \gamma(B_r^i) \qquad (9)$$

It is interesting to note that, when the degree of every node B_q^i, $i \neq 0$, in $(\mathcal{B}, T)$ is equal to 1 (the nodes from parts of the R-connected components are all 'leaves' of the tree), the configuration graph $(\mathcal{B}, \mathcal{E} \cup \mathcal{A})$ *has no links of type III*. In this case, there is no need for the sequential lifting and all the coefficients in the honeycomb inequality can be computed in terms of the shortest distances in the graph $(\mathcal{B}, T)$.

Given that the sequential lifting process for a set of links guarantees the validity of an inequality if it is valid without this set of links, and that this is also true for a facet inducing inequality, in what follows we will assume that the honeycomb configuration has no links of type III. In other words, for each link (B_r^i, B_q^j), the path in $(\mathcal{B}, T)$ joining B_r^i and B_q^j does not contain more than one node related to the same A_s (except the nodes B_r^i and B_q^j, when $q = r$).

Before proving that the honeycomb inequalities are valid for MGRP(G) and that, under certain conditions, they are facet-inducing, let us show how to build semitours for the MGRP on the graph $(\mathcal{B}, \mathcal{E} \cup \mathcal{A})$. A family of links $S \subset \mathcal{E} \cup \mathcal{A}$ is a semitour for the MGRP on $G_{\mathcal{C}}$ if (1) graph $(\mathcal{B}, S)$ is even, (2) graph $(\mathcal{B}, S \cup \mathcal{E} \cup \mathcal{A}_R)$ is balanced and (3) the graph obtained by shrinking every node set $\{B_i^1, B_i^2, \ldots, B_i^{\gamma_i}\}$ in graph $(\mathcal{B}, S)$ into a single node A_i, $i = 1, 2, \ldots, L$, is (strongly) connected.

Condition (1) means that every node B_i^j should be incident with an even (or zero) number of links in S, and it is due to the fact that every node B_i^j contains an even number of R-odd nodes. With respect to condition (3), it is interesting to note that graph $(\mathcal{B}, S)$ may be connected or disconnected, since connectivity among different B_i^j, $j = 1, 2, \ldots, \gamma_i$, (within the same A_i) is assured by (H3) and within B_i^j by (H2). Nevertheless, S must connect at least one node B_i^j of each A_i, $i = 1, \ldots, K$, in such a way that after shrinking each node set $\{B_i^1, B_i^2, \ldots, B_i^{\gamma_i}\}$ into a single node A_i, we obtain a connected graph.

As an illustration, we show how to build semitours with c-cost $2(K-1)$. Let us denote by $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ the undirected graph with node set $\bar{\mathcal{A}} = \{A_1, A_2, \ldots, A_L, A_{L+1}, \ldots, A_K\}$ and having an edge (A_i, A_j) for each pair of opposite arcs (B_i^p, B_j^q), (B_j^q, B_i^p) in T (figure 4). Then, by taking a copy of each one of the $K-1$ pairs of opposite arcs corresponding to any spanning tree of $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$, we obtain a semitour of c-cost $2(K-1)$ (figure 4). These semitours play a crucial role in the proof that the honeycomb inequalities are facet-inducing of MGRP(G).

Given that Honeycomb inequalities are valid for the undirected GRP (Corberán and Sanchis,

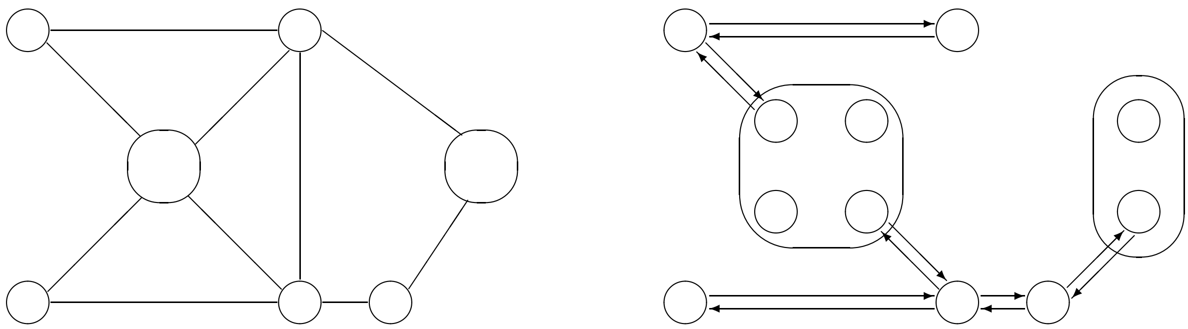

Figure 4: Graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ and a semitour with c-cost $2(K-1)$

1998), they are also valid for MGRP(G). To show that, under certain conditions, they are also facet-inducing, we will use the following well known result from Polyhedral Theory:

Lemma 1 *Let $P \subset \mathbb{R}^n$ be a polyhedron such that aff(P) $= \{x \in \mathbb{R}^n : \; Mx = b\}$, where M is a $q \times n$ matrix. Let $cx \leq \alpha$ be a valid inequality for P and let $F = P \cap \{x \in \mathbb{R}^n : \; cx = \alpha\}$. Then, F is a facet of P if, and only if, given another valid inequality $dx \leq \beta$, with $F \subseteq P \cap \{x \in \mathbb{R}^n : dx = \beta\}$, there exist $\mu \in \mathbb{R}^q$ and $\lambda \geq 0$ such that $d = \lambda c + \mu M$ and $\beta = \lambda\alpha + \mu b$.*

The following result will be also applied:

Lemma 2 *Let M be the matrix of the equation system associated with polyhedron MGRP(G). A given vector d' can be written as $d' = \mu M$ if:*

1. for every link e of G, joining two vertices belonging to the same edge-connected component, $d'(e) = 0$ is satisfied, and

2. for every subset W of arcs of G, joining vertices belonging to different edge-connected components and defining a cycle on the graph obtained when shrinking each edge-connected component of G into a single node, $d'(W) = 0$ is satisfied.

Proof: Let $K_1, K_2, \ldots, K_q$ be the vertex sets of the q edge-connected components of G. The equation system associated with the MGRP on G is $Mx = b$, where

i) M has q rows, one for each edge-connected component:

$$x(A^+(K_i)) - x(A^-(K_i)) = -u_{K_i}, \quad i = 1, 2, \ldots, q$$

ii) M has one column associated with each link of G:

- **a)** For each arc joining two different edge-connected components, say from K_i to K_j, its corresponding column has a 1 in row i and a -1 in row j.
- **b)** For each link joining two vertices in the same edge-connected component, its corresponding column has all its entries equal to zero.

Let $d' \in \mathbb{R}^{E \cup A}$ be a vector satisfying the two conditions described in the lemma. We will show that there exists $\mu \in \mathbb{R}^q$ such that $d' = \mu M$.

From (1) and (ii.b), both vectors d' and μM (for any μ) have their entries corresponding to the links in the edge-connected components equal to zero. Hence, it suffices to show the result for the arcs joining different edge-connected components.

Consider any arborescence rooted in K_1 (for example) in the directed graph obtained by considering the node sets K_i as single nodes (see figure 5).

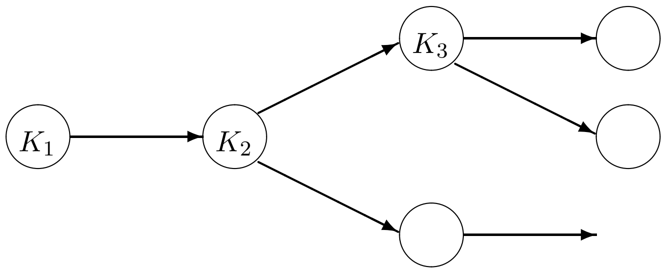

Figure 5: Arborescence rooted in K_1

Set $\mu_1 = 1$ and compute μ_j such that $\mu_1 - \mu_j = d'_{1j}$ for each K_j adjacent to K_1 in the arborescence. Repeat this process now with the nodes adjacent to K_j and so on. We have defined a vector $\mu \in \mathbb{R}^q$ satisfying $d'_{ij} = \mu_i - \mu_j$ for every arc (K_i, K_j) in the arborescence.

If we now consider the cycle $K_i \to K_j \to K_i$, condition (2) of the lemma implies that $d'_{ij} + d'_{ji} = 0$ and then $d'_{ji} = -d'_{ij} = \mu_j - \mu_i$ also for every arc (K_j, K_i) such that (K_i, K_j) is in the arborescence.

Finally, given any arc (K_r, K_p), let W be the cycle $K_p, K_{i_1}, K_{i_2}, \ldots, K_r, K_p$ that uses arcs in the arborescence or its opposite ones. Given that all these arcs satisfy $d'_{ij} = \mu_i - \mu_j$ and given that $d'(W) = 0$ from condition (2), we have

$$\begin{aligned} d'_{rp} &= -d'_{pi_1} - d'_{i_1 i_2} - \ldots - d'_{i_m r} = \\ &= -\mu_p + \mu_{i_1} - \mu_{i_1} + \mu_{i_2} - \mu_{i_2} + \ldots - \mu_{i_m} + \mu_r = \mu_r - \mu_p \end{aligned}$$

and, hence, $\mu M = d'$. ♦

Theorem 2 *Honeycomb inequalities (9) are facet-inducing of MGRP(G) if*

1. *the shrunk graph $(\bar{A}, T_{\bar{A}})$ is 2-connected, and*

2. *the subgraph of $G_{\mathcal{C}}$ induced by the required links is balanced.*

Proof: Let $cx \geq 2(K-1)$ be the Honeycomb inequality and let $dx \geq \beta$ be a valid inequality such that for every semitour x satisfying $cx = 2(K-1)$ would also satisfy $dx = \beta$. Given that the RHS is not zero, we can assume that $\beta = 2(K-1)$. Let us set $\lambda = 1$. As a consequence of the previous lemma, in order to show that $cx \geq 2(K-1)$ is facet-inducing of MGRP(G), it will suffice to show that the vector $d' = d - c$ satisfies conditions (1) and (2) in Lemma 2.

Given an arc $\overrightarrow{a}$, we will denote its opposite arc by $\overleftarrow{a}$. In what follows, the conditions satisfied by the components of vector d corresponding to the different links of $G_{\mathcal{C}}$ are presented.

a) <u>For the arcs in T</u>
Let $\overrightarrow{a} = (B^i_r, B^j_t) \in T$. Given that graph $(\bar{A}, T_{\bar{A}})$ is 2-connected, although node A_r is removed,

the graph remains connected, and we can find a spanning tree with $K-2$ edges. Let T' be the corresponding set of $K-2$ pairs of opposite arcs in $G_{\mathcal{C}}$. We define the semitour x^1 as:

$$x^1_{\overrightarrow{a}} = x^1_{\overleftarrow{a}} = 1, \quad x^1_e = 1 \ \ \forall e \in T', \quad x^1_e = 0 \text{ otherwise},$$

which satisfies $cx^1 = 2 + 2(K-2) = 2(K-1)$ and therefore $dx^1 = 2(K-1)$.

Let now $\overrightarrow{b} = (B^i_r, B^m_s) \in T$ be any other arc with $s \neq t$ (this arc exists because graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ is 2-connected). Given that the edges in $T_{\bar{\mathcal{A}}}$ corresponding to T' and to arc $\overrightarrow{b}$ also define a spanning tree, the semitour x^2 is defined in the following way:

$$x^2_{\overrightarrow{b}} = x^2_{\overleftarrow{b}} = 1, \quad x^2_e = 1 \ \ \forall e \in T', \quad x^2_e = 0 \text{ otherwise},$$

which also satisfies $cx^2 = 2 + 2(K-2) = 2(K-1)$ and therefore $dx^2 = 2(K-1)$.

We have $d(x^1 - x^2) = 0$ and then $d_{\overrightarrow{a}} + d_{\overleftarrow{a}} = d_{\overrightarrow{b}} + d_{\overleftarrow{b}}$. Hence, every pair of opposite arcs $\overrightarrow{a}, \overleftarrow{a}$ in T incident to node B^i_r have the same value for the sum $d_{\overrightarrow{a}} + d_{\overleftarrow{a}}$. By iterating this argument, we find that all the opposite pairs of arcs $\overrightarrow{a}, \overleftarrow{a}$ in T have the same value for the sum $d_{\overrightarrow{a}} + d_{\overleftarrow{a}}$. Furthermore, as $dx^1 = 2(K-1)$, we have

$$d_{\overrightarrow{a}} + d_{\overleftarrow{a}} = 2,$$

for every pair of opposite arcs $\overrightarrow{a}, \overleftarrow{a}$ in T.

b) <u>For the internal arcs</u>
Let now $\overrightarrow{a} = (B^i_r, B^j_q)$ be an internal arc with $r \neq q$ and where i, j can be zero. Let us call $(u, v) = (B^i_r, B^j_q)$. On graph $(\mathcal{B}, T)$ there exists a path joining u and v, say

$$u, \overrightarrow{a_1}, w_1, \overrightarrow{a_2}, \ldots, \overrightarrow{a_{H-1}}, w_{H-1}, \overrightarrow{a_H}, v,$$

where $H = c_{\overrightarrow{a}}$ and nodes $u, w_1, w_2, \ldots, w_{H-1}, w_H, v$ are associated with different sets A_i. The edges in graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ corresponding to arcs $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_{H-1}}, \overrightarrow{a_H}$ do not form a cycle, and we can add $K-1-H$ more edges in $T_{\bar{\mathcal{A}}}$ in order to obtain a spanning tree of $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$. Let T' be the corresponding set of $K-1-H$ pairs of opposite arcs in $G_{\mathcal{C}}$. We define the semitour x^1 as (see figure 6):

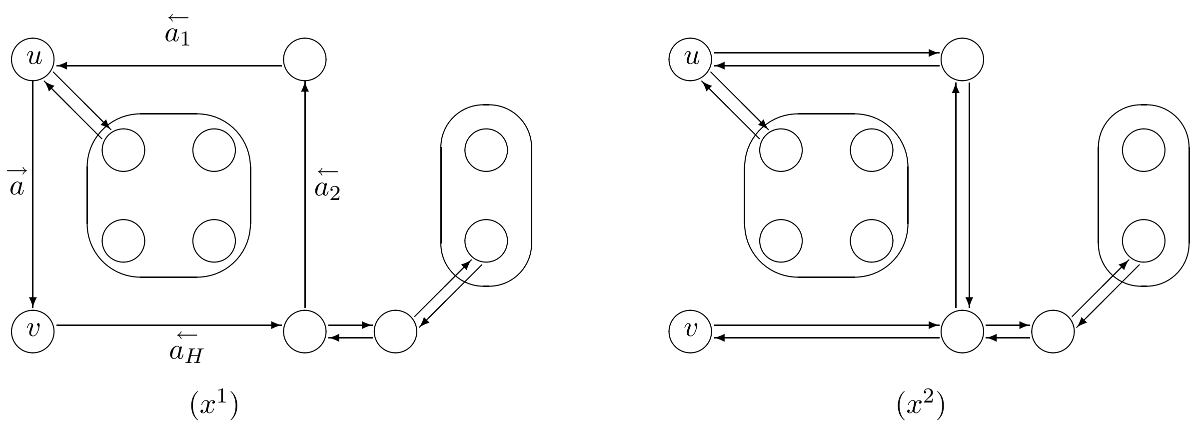

Figure 6: Semitours x^1 and x^2 associated with an internal arc $\overrightarrow{a}$.

$$x^1_{\overrightarrow{a}} = x^1_{\overleftarrow{a_1}} = x^1_{\overleftarrow{a_2}} = \ldots = x^1_{\overleftarrow{a_H}} = 1, \quad x^1_e = 1 \ \ \forall e \in T', \quad x^1_e = 0 \text{ otherwise},$$

which satisfies $cx^1 = H + H + 2(K-1-H) = 2(K-1)$. We define the semitour x^2 as:

$$x^2_{\overrightarrow{a}} = 0, \quad x^2_{\overrightarrow{a_1}} = x^2_{\overleftarrow{a_1}} = \ldots = x^2_{\overrightarrow{a_H}} = x^2_{\overleftarrow{a_H}} = 1, \quad x^2_e = 1 \ \ \forall e \in T', \quad x^2_e = 0 \text{ otherwise},$$

which also satisfies $cx^1 = 2H + 2(K-1-H) = 2(K-1)$. Then, $dx^1 = dx^2 = 2(K-1)$ and hence, $d(x^1 - x^2) = 0$ and we obtain

$$d_{\overrightarrow{a}} = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}}$$

c) <u>For the links in $\mathcal{C}$</u>
Let $e = (B^i_r, B^j_r) = (u, v)$ be a link in $\mathcal{C}$ (edge or arc) such that it can be traversed from u to v. On graph $(\mathcal{B}, T)$ there exists a path joining u and v, say

$$u, \overrightarrow{a_1}, w_1, \overrightarrow{a_2}, \ldots, \overrightarrow{a_{H-1}}, w_{H-1}, \overrightarrow{a_H}, v,$$

where $c_e = H-2$ and nodes $u, w_1, w_2, \ldots, w_{H-1}$ are associated with different sets A_i. The edges in graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ corresponding to the arcs $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_{H-1}}$ do not form a cycle, and we can add $K-H$ more edges in $T_{\bar{\mathcal{A}}}$ in order to obtain a spanning tree of $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$. Let T' be the corresponding set of $K-H$ pairs of opposite arcs in $G_{\mathcal{C}}$. We define the semitours x^1 and x^2 as (see figure 7):

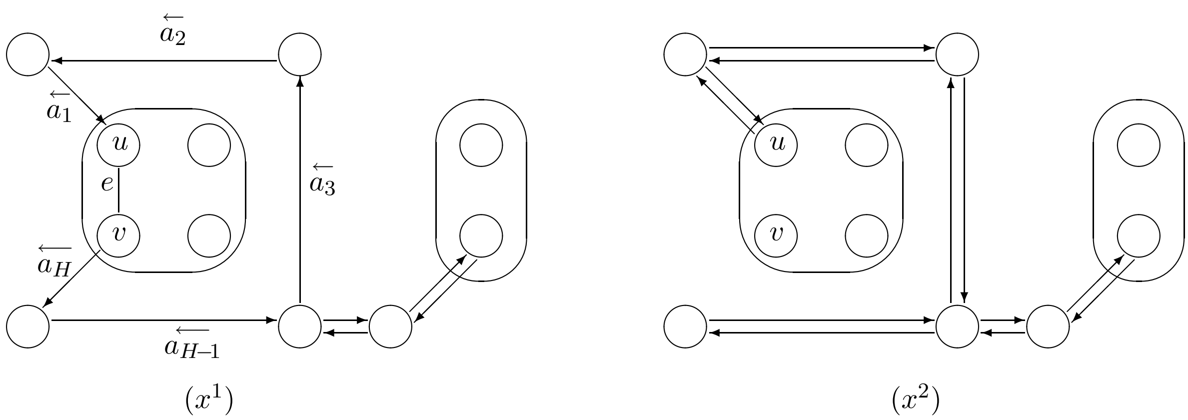

Figure 7: Semitours x^1 and x^2 associated with a link $e \in \mathcal{C}$.

$$x^1_e = x^1_{\overleftarrow{a_1}} = x^1_{\overleftarrow{a_2}} = \ldots = x^1_{\overleftarrow{a_H}} = 1, \quad x^1_a = 1 \ \ \forall a \in T', \quad x^1_a = 0 \text{ otherwise},$$

$$x^2_{\overrightarrow{a_1}} = x^2_{\overleftarrow{a_1}} = \ldots = x^2_{\overrightarrow{a_{H-1}}} = x^2_{\overleftarrow{a_{H-1}}} = 1, \quad x^2_a = 1 \ \ \forall a \in T', \quad x^2_a = 0 \text{ otherwise},$$

which satisfy $cx^1 = cx^2 = 2(K-1)$ and then $dx^1 = dx^2 = 2(K-1)$. From $d(x^1 - x^2) = 0$, we obtain

$$d_e + d_{\overleftarrow{a_H}} = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_{H-1}}} \quad \text{or} \quad d_e = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_{H-1}}} - d_{\overleftarrow{a_H}}$$

and

$$d_e = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}} - 2 \tag{10}$$

as $d_{\overrightarrow{a_H}} + d_{\overleftarrow{a_H}} = 2$.

Equation (10) is satisfied by every link in $\mathcal{C}$. Furthermore, for the edges and arcs with both endnodes in the same edge-connected component, we have:

(c.1) If $e = (u, v) \in \mathcal{C}$ is an edge, it can be traversed both from u to v and from v to u. Therefore,

$$d_e = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}} - 2 \qquad \text{and} \qquad d_e = d_{\overleftarrow{a_1}} + d_{\overleftarrow{a_2}} + \ldots + d_{\overleftarrow{a_H}} - 2$$

Hence, by adding both expressions and considering that, from (a), $d_{\overrightarrow{a_i}} + d_{\overleftarrow{a_i}} = 2$, we obtain

$$d_e = H - 2 = c_e$$

(c.2) If $\overrightarrow{a} = (B_r^i, B_r^j) \in \mathcal{C}$ is an arc parallel to an edge e in $\mathcal{C}$, given that edge e can be traversed in the opposite direction to that of $\overrightarrow{a}$, we obtain

$$d_{\overrightarrow{a}} = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}} - 2 \quad \text{and} \quad d_e = d_{\overleftarrow{a_1}} + d_{\overleftarrow{a_2}} + \ldots + d_{\overleftarrow{a_H}} - 2$$

By adding both expressions $d_e + d_{\overrightarrow{a}} = 2H - 4$ is obtained. Finally, given that, from (c.1), $d_e = H - 2$ we obtain

$$d_{\overrightarrow{a}} = H - 2 = c_{\overrightarrow{a}}$$

(c.3) Let now $\overrightarrow{a} = (B_r^i, B_r^j) = (u, v) \in \mathcal{C}$ be an arc within an edge-connected component. For the sake of simplicity, let us suppose that the two endnodes of $\overrightarrow{a}$ are connected by two edges $e_1 = (u, w)$ and $e_2 = (w, v)$ (see figure 8). On graph $(\mathcal{B}, T)$ there exists a path joining u and v, say

$$u, \overrightarrow{a_1}, u_1, \overrightarrow{a_2}, \ldots, u_{L-1}, \overrightarrow{a_L}, u_L, \overrightarrow{a_{L+1}}, u_{L+1}, \ldots, \overrightarrow{a_{H-1}}, u_{H-1}, \overrightarrow{a_H}, v,$$

where $c_{\overrightarrow{a}} = H - 2$, and a path joining w and a node in the previous path, say u_L:

$$w, \overrightarrow{b_1}, w_1, \overrightarrow{b_2}, w_2, \ldots, w_{m-1}, \overrightarrow{b_m}, w_m = u_L$$

where $c_{e_1} = L + m - 2$ and $c_{e_2} = H - L + m - 2$. Proceeding as before,

$$d_{\overrightarrow{a}} = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_L}} + d_{\overrightarrow{a_{L+1}}} + \ldots + d_{\overrightarrow{a_H}} - 2$$

$$d_{e_1} = d_{\overleftarrow{a_1}} + d_{\overleftarrow{a_2}} + \ldots + d_{\overleftarrow{a_L}} + d_{\overrightarrow{b_1}} + \ldots + d_{\overrightarrow{b_m}} - 2$$

$$d_{e_2} = d_{\overleftarrow{a_{L+1}}} + \ldots + d_{\overleftarrow{a_H}} + d_{\overleftarrow{b_1}} + \ldots + d_{\overleftarrow{b_m}} - 2$$

and by adding these 3 expressions and considering that, from (c.1), $d_{e_1} = c_{e_1}$ and $d_{e_2} = c_{e_2}$, we obtain

$$d_{\overrightarrow{a}} + (L + m - 2) + (H - L + m - 2) = 2m + 2H - 6$$

and

$$d_{\overrightarrow{a}} = H - 2 = c_{\overrightarrow{a}}$$

Hence, vector d satisfies:

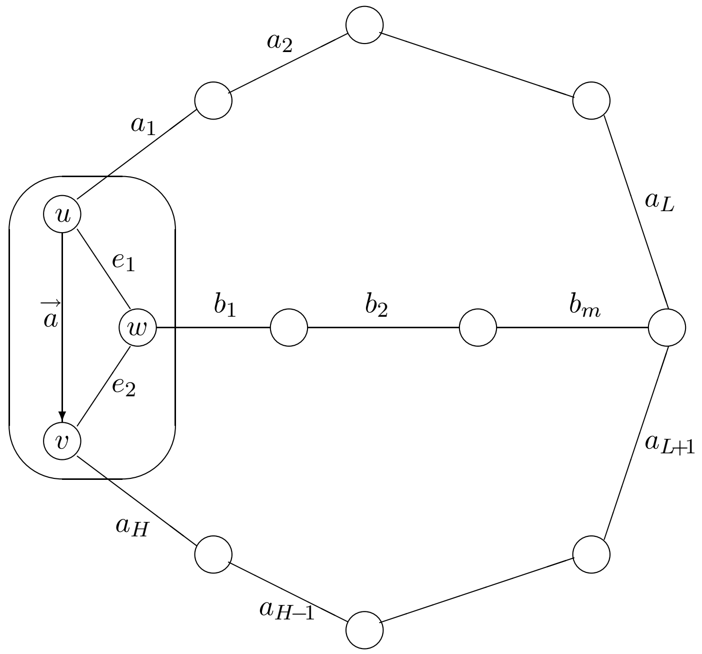

Figure 8: Arc $\overrightarrow{a} \in \mathcal{C}$ connected by two edges in the proof of theorem 2

1. $d_{\overrightarrow{a}} + d_{\overleftarrow{a}} = 2$ for every pair of opposite arcs $\overrightarrow{a}, \overleftarrow{a}$ in T,
2. $d_{\overrightarrow{a}} = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}}$ for every internal arc $\overrightarrow{a}$,
3. $d_e = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}} - 2$ for every link e in $\mathcal{C}$, and
4. $d_e = H - 2 = c_e$ for every link e in $\mathcal{C}$ with both endnodes in the same edge-connected component,

where, in each case, H is the number of arcs $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_H}$ of the unique path in T joining the two terminal nodes of the corresponding link.

In what follows we will show that vector $d' = d - c$ satisfies conditions (1) and (2) in Lemma 2:

(1) For every link e of G joining two vertices belonging to the same edge-connected component, from 4, $d_e = H - 2 = c_e$ and then $d'_e = 0$.

(2) Let W be a set of arcs in $G_{\mathcal{C}}$, joining vertices belonging to different edge-connected components and defining a cycle on the graph obtained when shrinking each edge-connected component into a single node. We have to prove that $d(W) = c(W)$ is satisfied.

If W contains an internal arc $\overrightarrow{a} = (u, v)$, we replace it by the arcs in the path in T joining u and v. Let us call this new cycle W'. Given that $d_{\overrightarrow{a}} = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}}$ and also $c_{\overrightarrow{a}} = c_{\overrightarrow{a_1}} + c_{\overrightarrow{a_2}} + \ldots + c_{\overrightarrow{a_H}}$, it follows that $d(W') = d(W)$ and $c(W') = c(W)$. In the same way, if W contains an arc $\overrightarrow{a} = (u, v)$ in $\mathcal{C}$ (necessarily between different edge-connected components), we replace it by the arcs in the path in T joining u and v. Let us call this new cycle W''. Again, as $d_{\overrightarrow{a}} = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}} - 2$ and $c_{\overrightarrow{a}} = c_{\overrightarrow{a_1}} + c_{\overrightarrow{a_2}} + \ldots + c_{\overrightarrow{a_H}} - 2$, it follows that

$d(W'') = d(W) + 2$ and $c(W'') = c(W) + 2$. Hence, we can assume that W is formed only by external arcs. We can also assume that W is a cycle of one of the following types:

- Type I: W is a cycle formed by a pair of opposite arcs $\overrightarrow{a}, \overleftarrow{a}$ in T. Then, $d(W) = d_{\overrightarrow{a}} + d_{\overleftarrow{a}} = 2$ and also $c(W) = c_{\overrightarrow{a}} + c_{\overleftarrow{a}} = 2$.

- Type II: W is a cycle starting on a node $u = B_i^r$ and ending on a node $v = B_i^s$ (related to the same R-set A_i) and without traversing any other node B_j^q, $q \neq 0$ (see figure 9). For

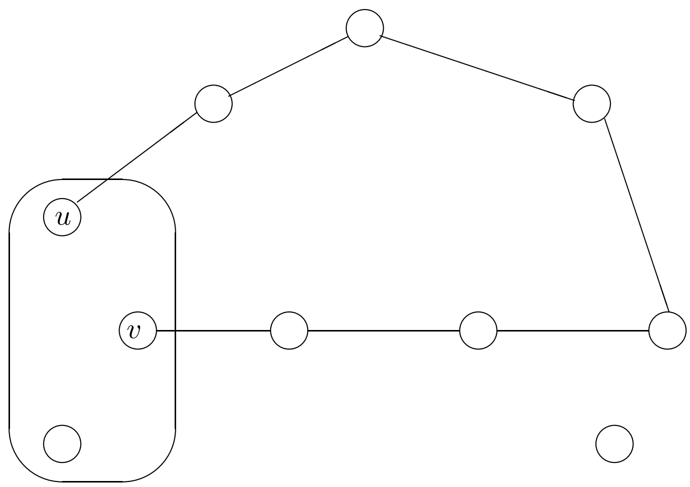

Figure 9: Cycle of type I in the proof of theorem 2

simplicity, let us suppose that there exists an edge $e = (u, v)$. Then, proceeding as in (c.1), $d_e = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}} - 2 = H - 2$ and then $d(W) = d_{\overrightarrow{a_1}} + d_{\overrightarrow{a_2}} + \ldots + d_{\overrightarrow{a_H}} = H$. Given that $c_e = c_{\overrightarrow{a_1}} + c_{\overrightarrow{a_2}} + \ldots + c_{\overrightarrow{a_H}} - 2 = H - 2$, we also obtain $c(W) = c_{\overrightarrow{a_1}} + c_{\overrightarrow{a_2}} + \ldots + c_{\overrightarrow{a_H}} = H$.

- Type III: W is a cycle joining nodes B_i^r with $r \neq 0$ corresponding to different R-sets A_i. For the sake of simplicity, we will assume that the number of such R-sets is 3, as in figure 10a. For the two nodes incident with arcs in W in each R-set, there exists a unique path in T joining them. These paths also define a cycle on the graph obtained when shrinking each edge-connected component into a single node. Let W_1, W_2 and W_3 be these cycles (see figure 10b). Each W_i is a cycle of type II above and then $c(W_i) = d(W_i)$ and $c(W_1) + c(W_2) + c(W_3) = d(W_1) + d(W_2) + d(W_3)$. The arcs in cycles W_1, W_2 and W_3 are the arcs in cycle W plus a set of arcs, say F. Then, $c(W) + c(F) = d(W) + d(F)$ and, given that $c(F) = d(F)$, as F is formed by cycles of type I, we obtain $c(W) = d(W)$.

The two conditions in lemma 2 are therefore satisfied and there exists a vector μ such that $\mu M = d' = d - c$ which implies $d = c + \mu M$ and the Honeycomb inequality is facet-inducing of MGRP($G_{\mathcal{C}}$). In order to show that it is also facet inducing of MGRP(G), we have to prove that the conditions (a) and (b) of theorem 1 are satisfied.

(a) Given a proper set F of E_R it is always possible, as in $G_{\mathcal{C}}$ the set of links $A_R \cup E_R$ induce an even and balanced subgraph, to orient its edges in such a way that the required links also induce an even and balanced subgraph in the resulting graph $G'_{\mathcal{C}}$. Then, the Honeycomb inequality is facet inducing of MGRP($G'_{\mathcal{C}}$).

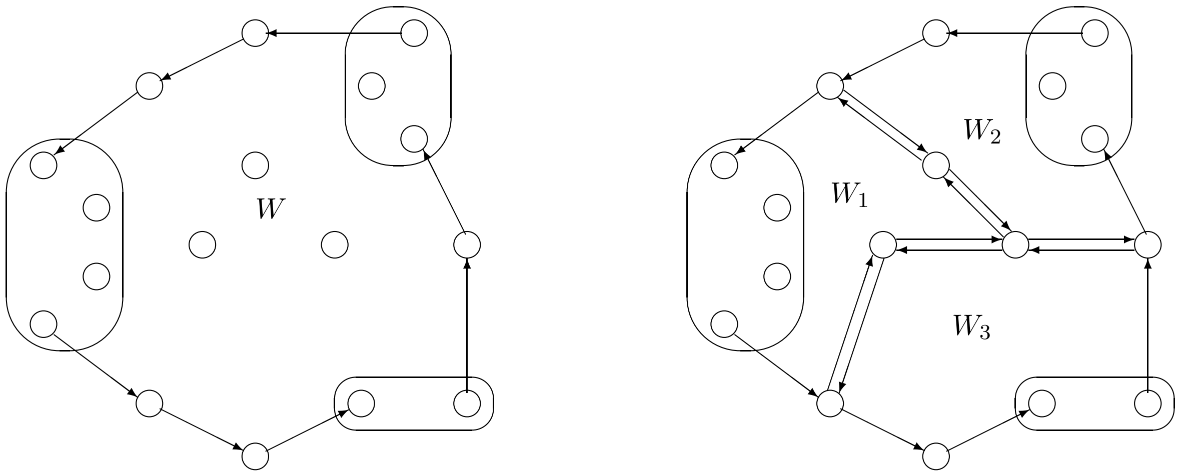

Figure 10: Cycle of type II in the proof of theorem 2

(b) Let $e = (u, v)$ be an edge in $G_{\mathcal{C}}$ that has been oriented from u to v in $G'_{\mathcal{C}}$. Associated with e there exists a path in T joining u to v, say

$$u, \vec{a_1}, w_1, \vec{a_2}, \ldots, \overrightarrow{a_{H-1}}, w_{H-1}, \vec{a_H}, v.$$

As usual, we can add a set T' of pairs of opposite arcs in T to the arcs in the path from u to v to obtain the vector x^*:

$$x^*_e = x^*_{\vec{a_1}} = x^*_{\vec{a_2}} = \ldots = x^*_{\vec{a_H}} = 1, \quad x^*_a = 1 \ \ \forall a \in T', \quad x^*_a = 0 \text{ otherwise.}$$

This vector is a semitour for the MGRP on $G_{\mathcal{C}}$ because the edge e (traversed from v to u) and the arcs $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_H}$ form a cycle, which satisfies $cx^* = 2(K-1)$ and the edge e is traversed, at least once, from v to u.

Hence, the Honeycomb inequality is facet-inducing of MGRP(G). ♦

5 Honeycomb$_{02}$ inequalities

In the previous section we have studied the Honeycomb inequalities. They have the same coefficients as the corresponding inequalities for the undirected GRP and will be referred to as 'standard' Honeycomb inequalities. In this section we will present a related set of inequalities, the Honeycomb$_{02}$ inequalities. Their study will not be as general as that done for the standard inequalities as we will consider only Honeycomb$_{02}$ inequalities with $L = 1$, i.e. when *only one* R-set is divided into γ_1 parts. Before their formal definition, let us first see several simple examples.

While a standard Honeycomb configuration can be seen as a combination of several standard K-C configurations, a Honeycomb$_{02}$ configuration can be seen as a combination of several standard K-C and K-C$_{02}$ configurations. Consider the structure of a Honeycomb$_{02}$ configuration with 6 nodes where a R-set has been divided into 3 parts (see figure 11a, where the number associated with each link represents the coefficient of the corresponding variable in the inequality). In the standard Honeycomb inequality all the variables associated with arcs in T have coefficient equal to 1. Assign now coefficients 0 and 2 to the pair of opposite arcs in T joining nodes 1 and 4 (but maintaining the other arcs in T with coefficient equal to 1). Notice that the

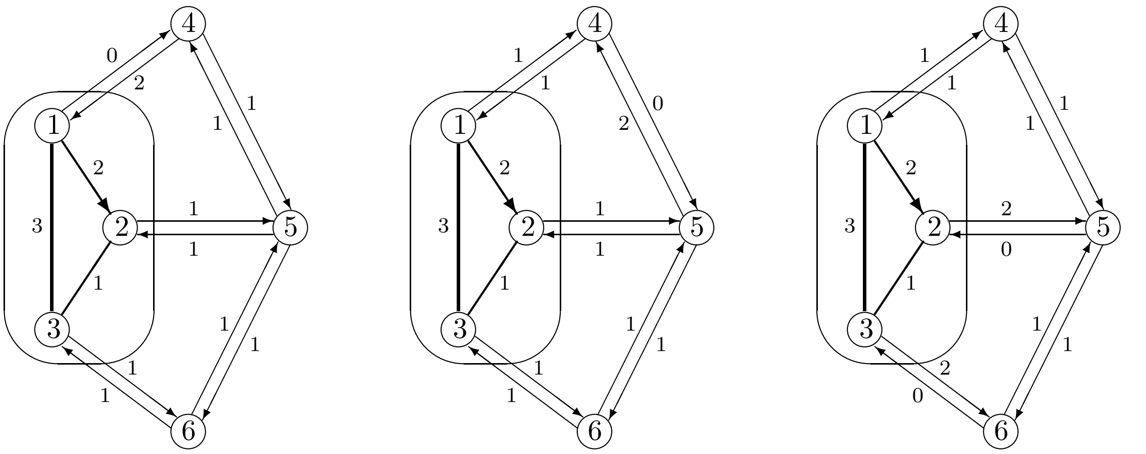

Figure 11: Three equivalent Honeycomb_{02} configurations.

coefficient of the arc joining nodes 1 and 2 is now 2 (in the standard Honeycomb inequality it would be 1) given that the path joining nodes 1 and 2 by arcs in T contains a pair of opposite arcs with coefficients 0 and 2, just like the external arcs of a K-C_{02}. Something similar occurs with the edge joining nodes 1 and 3, but not with that joining nodes 2 and 3, whose associated path would correspond to a standard K-C. We will show that, with these coefficients, inequality $F(x) \geq 6$ is valid and facet-inducing of MGRP(G).

Let us now consider node 4 in this configuration. This node has a system equation associated with it: $x(A^+(4)) = x(A^-(4))$. In the corresponding inequality $F(x) \geq 6$ we can replace the variables $x_{41}+x_{45}$ by the variables $x_{14}+x_{54}$ to obtain an equivalent inequality, whose coefficients are shown in figure 11b. If we proceed in a similar way with the equation associated with node 5 and then with that corresponding to node 6, we obtain another equivalent inequality, whose coefficients are shown in figure 11c. This last inequality has two arcs in the skeleton with coefficient 0, entering at nodes 2 and 3 respectively, whilst the initial inequality has only one such arc (leaving node 1). However, both inequalities are equivalent.

On the other hand, it is easy to see that if a Honeycomb_{02} configuration has, simultaneously, an arc with coefficient 0 leaving a node B_1^i and another arc (also with coefficient 0) entering at a node B_1^j, its corresponding inequality is not valid for MGRP(G).

Hence, given a Honeycomb_{02} configuration with $L = 1$, we will classify the γ_1 nodes B_1^i into two types: nodes of type $\mathcal{O}$ (from 'out') and nodes of type $\mathcal{I}$ (from 'in'). A coefficient 0 will be associated to each arc in T leaving from a node of type $\mathcal{O}$ (and coefficient 2 to its opposite arcs). There is no loss of generality in this step because the corresponding inequality is equivalent to that obtained by assigning a coefficient 0 to each arc in T entering at a node of type $\mathcal{I}$. The coefficient of a link $e \in \mathcal{C}$ is $H-2$ if e joins two nodes of the same type and is $H-1$ otherwise, where H represents the number of arcs in T in the unique path joining these nodes.

Finally, let us consider a Honeycomb_{02} configuration with all its nodes B_1^i of type $\mathcal{O}$. By using the equation corresponding to the set of nodes $B_1^1, B_1^2, \ldots, B_1^{\gamma_1}$, a standard Honeycomb inequality is obtained. Hence, in a Honeycomb_{02} configuration with $L = 1$, which we define next, there will be at least one node of type $\mathcal{O}$ and one node of type $\mathcal{I}$.

Let $G = (V, E, A)$ be a mixed graph. Consider now a standard Honeycomb configuration, as defined in Section 4, with $L=1$ (only the set A_1 is divided into γ subsets B_1^i). We have a partition of V, $\mathcal{B}=\{B_1^1, B_1^2, \ldots, B_1^{\gamma}, B_2^0, B_3^0, \ldots, B_K^0\}$, a set of arcs T forming the skeleton of the configuration and $d(B_i^p, B_j^q)$ will denote the number of arcs in the unique path joining B_i^p and B_j^q by arcs in T. We will suppose that in graph $(\mathcal{B}, T)$ the in-degree and the out-degree of each node B_1^j is equal to 1. These nodes will be labelled as nodes of type $\mathcal{O}$ and nodes of type $\mathcal{I}$.

As the number, K, of components in a K-C configuration must be greater than or equal to 3 in the standard case, while it is 2 for the K-C_{02} case, condition H4 of the standard Honeycomb configurations is replaced here by:

- $d(B_1^j, B_1^q) \geq 3$ if B_1^j, B_1^q are nodes of the same type, and
- $d(B_1^j, B_1^q) \geq 2$ if B_1^j, B_1^q are nodes of different types.

The configuration graph $G_{\mathcal{C}}=(\mathcal{B}, \mathcal{E} \cup \mathcal{A})$ is shown in figure 12. Again, we will suppose that this configuration graph has no links of type III and hence there is no need for the sequential lifting. We define the costs as follows:

- For the links in $\mathcal{C}$:
  - $c(B_1^i, B_1^j)=d(B_1^i, B_1^j)-2$, if B_1^i, B_1^j are nodes of the same type.
  - $c(B_1^i, B_1^j)=d(B_1^i, B_1^j)-1$, if B_1^i, B_1^j are nodes of different type.
- For the arcs in T (external arcs):
  - $c(B_1^i, B_q^0)=0, c(B_q^0, B_1^i)=2$, if B_1^i is a node of type $\mathcal{O}$.
  - $c(B_r^i, B_q^j)=1$ otherwise.
- For the arcs in $\mathcal{I}n$ (internal arcs):
  - $c(B_r^i, B_q^j)$ is the c-length of the shortest path from B_r^i to B_q^j using arcs in T.

A *Honeycomb*$_{02}$ *configuration* will be the pair $(\mathcal{B}, c)$ and its corresponding *Honeycomb*$_{02}$ *inequality* is:

$$\sum_{a \in E \cup A} c_a x_a \geq 2(K-1), \quad \text{where} \quad c_a=c(B_r^i, B_q^j) \ \forall a \in (B_r^i, B_q^j), \quad c_a=0 \ \forall a \in \gamma(B_r^i) \tag{11}$$

Theorem 3 *Honeycomb*$_{02}$ *inequalities (11) are valid for MGRP(G).*

Proof: We will consider, w.l.o.g., that $G_{\mathcal{C}}$ is a complete graph and that the set $\mathcal{C}$ contains only edges. Let S be any semitour for the MGRP on $G_{\mathcal{C}}$. We have to show that $c(S) \geq 2(K-1)$.

If S uses an internal arc, we can replace it by the path in $(\mathcal{B}, T)$ joining its terminal nodes and we obtain another semitour with the same c-cost. If S uses an edge (u, v) in $\mathcal{C}$ joining two nodes of different types, we can replace it by the path in $(\mathcal{B}, T)$ joining u and v and we obtain another semitour with the same c-cost. If S uses the edges in any cycle in $\mathcal{C}$, we can remove them to obtain a semitour with less c-cost. If S uses two adjacent edges in $\mathcal{C}$, say (u, v) and (v, w), replacing them by the edge (u, w) (also in $\mathcal{C}$), another semitour with less or equal c-cost is obtained.

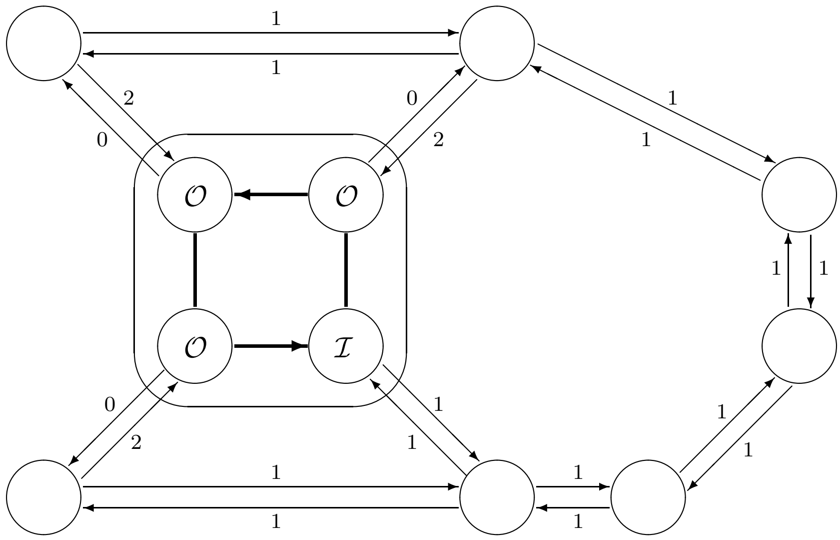

Figure 12: Honeycomb_{02} Configuration with $L = 1$

It can therefore be assumed that S uses only arcs in T and some isolated edges in $\mathcal{C}$ joining nodes of the same type. If S uses, for example, an edge $e = (B_1^i, B_1^j)$, it also uses external arcs joining the nodes B_1^i and B_1^j. Replacing the edge e by the path in T joining B_1^i and B_1^j, a semitour with c-cost equal to $c(S) + 2$ is obtained (an edge with c-cost $H - 2$ has been replaced by a path with c-cost H). Removing the two opposite arcs incident with B_1^i, a semitour with c-cost $c(S)$ is obtained.

Hence, we can finally consider that S uses only arcs in T. As every node has even degree with respect to the arcs in S, from the definition of T (if each pair of opposite arcs were replaced by an edge, T would be a tree), S consists of a set of pairs of opposite arcs in T and pairs of copies of arcs in T. On the other hand, given that S connects at least one node associated with each set $A_1, A_2, \ldots, A_K$, it contains at least $K - 1$ pairs of arcs in T. Each pair of opposite arcs in T has a c-cost of 2 (either $1 + 1$ or $0 + 2$). Each pair of copies of an arc in T has at least a c-cost of 2 (either $1 + 1$ or $2 + 2$) except the pairs of copies of an arc leaving a node of type $\mathcal{O}$, which has a c-cost of 0.

Given that the arcs in S plus the required links induce a balanced graph, if S uses a pair of copies of an arc from a node B_1^i (of type $\mathcal{O}$) to another node, say B_p^0, S also uses a pair of copies of an arc leaving B_p^0 and entering a node B_q^0, and so on. Notice that the arcs used by S have to be 'balanced' with the required links and that nodes B_p^0, are not incident to required links. Then, this 'double path' starting at B_1^i arrives at some node B_1^j. We can now remove from S the pair of copies of the arc leaving the node B_1^i to obtain a set of arcs that also connect at least one node associated with each set $A_1, A_2, \ldots, A_K$.

S then contains at least $K - 1$ pairs of arcs in T, each pair with a c-cost greater than or equal to 2. Hence, $c(S) \geq 2(K - 1)$ and the Honeycomb_{02} inequality is valid for MGRP(G). ♦

Some semitours having 'double paths', starting at a node B_1^i of type $\mathcal{O}$ and ending at a node B_1^j of type $\mathcal{I}$, similar to those described in the previous proof, are needed to show that the Honeycomb_{02} inequality is facet-inducing of MGRP(G). They are semitours for the MGRP if these 'double paths' together with the required links define a balanced graph. This is the purpose of condition (3) in the following theorem.

Theorem 4 *Honeycomb$_{02}$ inequalities (11) with $L=1$ are facet-inducing of MGRP(G) if*

1. *the shrunk graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ is 2-connected,*

2. *The subgraph of $G_{\mathcal{C}}$ induced by the required links is balanced and*

3. *For every edge or arc $(u,v) \in \mathcal{C}$, with node u of type $\mathcal{O}$ and node v of type $\mathcal{I}$, the subgraph of $G_{\mathcal{C}}$ induced by the required links plus two extra arcs from u to v, is a balanced graph.*

Proof: The proof is similar to that of theorem 2 and is omitted here for the sake of brevity. We would just like to briefly comment on the main difference between both proofs, which is related to condition (3) above. Let $e = (B_r^i, B_r^j) = (u,v) \in \mathcal{C}$ be a link (edge or arc) that can be traversed from u to v, where node u is of type $\mathcal{O}$ and node v is of type $\mathcal{I}$. Note that the semitour used in the proof of theorem 2 corresponding to link e above (see figure 7a) cannot be used here, as $cx = 2K$. A semitour satisfying $cx = 2(K-1)$, which now has to be associated to e, is that represented in figure 13a. It is a semitour for the MGRP on $G_{\mathcal{C}}$ because condition (3) is satisfied. Furthermore, semitours such as that represented in figure 13b are needed in the proof. They also satisfy $cx = 2(K-1)$ but, again, condition (3) has to be satisfied. ♦

Figure 13: Semitours used in the proof of theorem 4.

Condition (3) above implies the existence of (required) edges joining each pair of adjacent nodes of different type (see figure 14a). In fact, if all the required links are arcs, condition (3) cannot be satisfied, although it can be seen that, in this case, the Honeycomb$_{02}$ inequality is dominated by the standard Honeycomb inequality. Finally, condition (3) is satisfied if all the required links are edges, as the subgraph of $G_{\mathcal{C}}$ induced by the required links is even (see figure 14b).

6 Computational Experience

Corberán, Romero and Sanchis (2002) implemented a preliminary cutting-plane algorithm for the MGRP with separation procedures only for the connectivity, R-odd cut and balanced-set inequalities. Here, we improve this algorithm by adding to it separation procedures for the K-C and the Honeycomb inequalities.

The initial LP relaxation, LP$_0$, includes the system equations (6), one connectivity inequality (4) for each R-set, one balanced-set inequality (5) for each 'unbalanced' vertex, and one R-odd

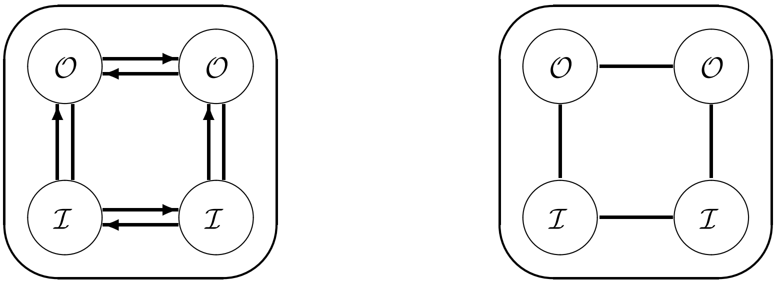

Figure 14: Required links satisfying condition (3) of theorem 4

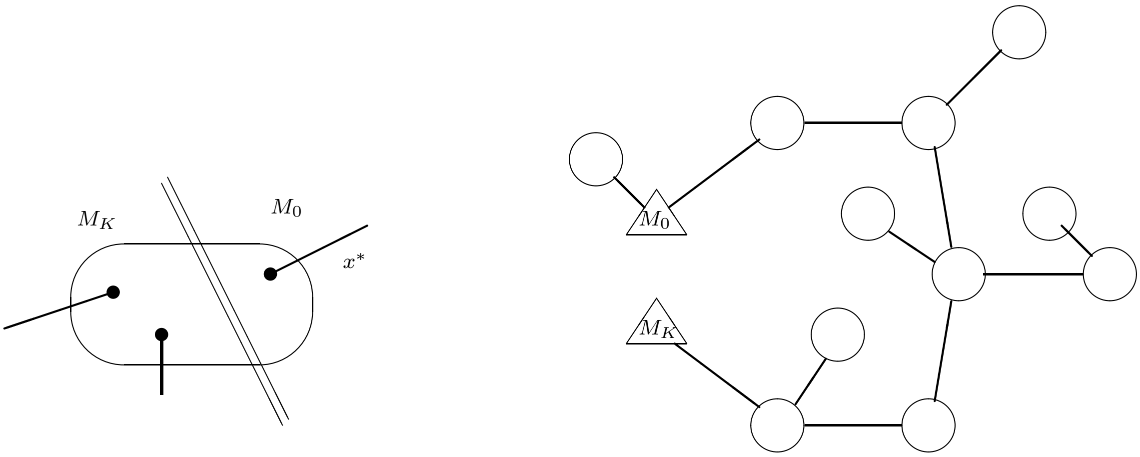

Figure 15: Phases I and II for K-C separation

cut inequality (7) for each R-odd 'balanced' vertex. Let $x^* \in \mathbb{R}^{|E \cup A|}$ be a given LP$_i$ solution. If x^* represents a semitour, it is an optimal solution for the MGRP. Otherwise, we consider the weighted graph $G^*(V, E, A, x^*)$ as the input to all of the separation procedures. The separation of connectivity, R-odd cut and balanced-set inequalities can be done in polynomial time. In the following subsections, we describe heuristic separation algorithms for the K-C and Honeycomb (with $L = 1$) inequalities described in the previous sections.

If the separation procedures find violated inequalities of the above mentioned classes, they are added to the current LP$_i$ relaxation to generate the LP$_{i+1}$ and so on. When the separation procedures do not find violated inequalities and the current LP solution x^* is not a semitour for the MGRP, we invoke branch and bound. If the resulting integer solution is a semitour for the MGRP, it is optimal. Otherwise, the procedure terminates with a tight lower bound, but no feasible MGRP solution.

6.1 K-C and K-C$_{02}$ separation

The separation procedures for K-C and K-C$_{02}$ inequalities are based on those developed for the undirected GRP in Corberán, Letchford & Sanchis (2001). For each R-set, the algorithm consists of three phases. At the two first phases, which are similar to those for the undirected GRP, we ignore the directions of the arcs.

In phase I (see figure 15a), we search to divide each R-set into two parts, say M_0 and M_K, both containing an even number of R-odd nodes and such that $x^*(\delta(M_0)) \cong 1$ but $x^*(M_0 :$

$M_K) \cong 0$.

Phase II (see figure 15b) consists of finding the (undirected) maximum x^*-weight tree spanning the nodes M_0, M_K (represented as triangles) and the $p-1$ remaining R-sets (represented as circles). Then, the tree is transformed into a path linking M_0 and M_K by iteratively shrinking each R-set with degree 1 on the tree into its unique adjacent node.

At this point, we have an initial K-C configuration with node sets $M_0, M_1, \ldots, M_K$. We consider the configuration mixed graph having node set $\{M_0, M_1, \ldots, M_K\}$ and having an arc (M_i, M_j) —an edge in the case (M_0, M_K)— with x^*-weight equal to the sum of the x^*-weights of all the arcs —links in the case (M_0, M_K)— (u, v) in G with $u \in M_i$ and $v \in M_j$. See figure 16a for an example with $K = 6$.

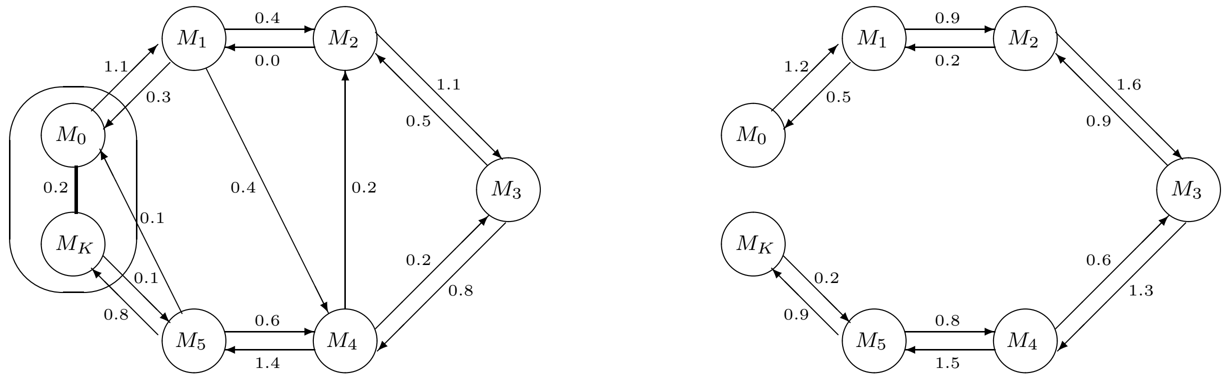

Figure 16: Skeleton graph for K-C separation

In phase III we check the corresponding K-C inequality for violation and, if not, we try to find a smaller violated K-C inequality. To do this, from the graph in figure 16a we build the skeleton graph in figure 16b. It consists of the opposite arcs (M_i, M_{i+1}), (M_{i+1}, M_i), for $i = 0, 1, \ldots, K-1$. Such a transformation is done in the following way. Each non-external arc (M_p, M_q) is removed and its x^*-weight is added to every arc in the unique path from M_p to M_q using external arcs. Given that the coefficient at the K-C inequality of each internal arc in $(M_p : M_q)$ is $|p-q|$, the value of its LHS, $F(x^*)$, remains unchanged. For example, we remove the arc (M_4, M_2) in graph 16a and 0.2 is added to the x^*-weight of arcs (M_4, M_3) and (M_3, M_2).

With respect to the links in $(M_0 : M_K)$, note that their orientations do not affect the computation of $F(x^*)$. Hence, they can all be considered as edges in graph 16a. Then, each link $e \in (M_0 : M_K)$ could be replaced either by the external arcs in the path from M_0 to M_K or by those in its opposite path. We add $\frac{x_e^*}{2}$ to each external arc in the skeleton (50% to each path). Given that the coefficient of link e is $K-2$ instead of K, this step increases the value of the LHS in $2x_e^*$ units.

Now, $F(x^*)$ can be easily computed as the sum of all the x^*-weights shown in figure 16b, minus $2x^*(M_0 : M_K)$. If $F(x^*) < 2(K-1)$, the K-C inequality is violated. In the example in figure 16, $F(x^*) = 10.6 - 0.4 = 10.2 \geq 10$ and, then, the K-C inequality is not violated and its associated slack is $+0.2$.

At this point, we check whether by shrinking some consecutive nodes M_i and M_{i+1} into a single node we obtain a smaller K-C configuration with an associated violated K-C inequality. For example, it is easy to see that by shrinking nodes M_2 and M_3 into a single node, we obtain a new K-C configuration (with $K = 5$) whose corresponding skeleton graph is exactly that obtained after shrinking nodes M_2 and M_3 in the skeleton graph of figure 16b. Notice that all the coefficients associated with the links in graph 16a, involved in the computation of the x^*-weights associated with (M_2, M_3) and (M_3, M_2), would decrease by exactly one unit after shrinking. Then, the LHS would decrease by $x^*(M_2 : M_3)$ ($1.6 + 0.9 = 2.5$ units) while the RHS would decrease by only 2 units. The slack is now negative ($0.2 - 0.5 = -0.3$) and we would obtain a K-C inequality violated by 0.3, i.e. after shrinking we would obtain a new K-C inequality $F'(x) \geq 8$ which is not satisfied by x^*, as $F'(x^*) = 7.7$.

This shrinking procedure can be repeated as long as $K \geq 3$ remains. For example, we can also shrink nodes M_4 and M_5 and decrease the slack by $1.5 + 0.8 - 2 = 0.3$ units more. After these two shrinking steps, we obtain a K-C inequality $F'(x) \geq 6$ violated by x^* ($F'(x^*) = 5.4$).

The previous procedure is not as simple if one of the nodes to be shrunk is M_0 (or M_K). For example, shrinking nodes M_5 and M_K ($K = 6$) in figure 16, the internal arcs in $(M_0 : M_5)$ —with coefficient 5 in the K-C inequality with $K=6$— would become arcs in the set $(M_0 : M_{K'})$ with coefficient $K'-2=3$ in the new K-C configuration with $K'=5$. Then, the LHS would decrease by not only $x^*(M_5 : M_K)$ but $x^*(M_5 : M_K) + x^*(M_0 : M_5)$.

Hence, from the skeleton graph we can conclude with little computational effort if there is, or not, a K-C inequality violated by x^*.

Once the K-C inequalities have been checked, we consider again the initial skeleton graph (figure 16b) in order to find violated K-C$_{02}$ inequalities. Associated with a given K-C configuration, we have two possible K-C$_{02}$ inequalities, depending on whether we assign, respectively, coefficients 0 and 2 to the arcs (M_0, M_1) and (M_1, M_0), or viceversa.

Given that the sets $M_0 \cup M_K$, M_1, $\cdots$, M_{K-1} are not incident with edges, they satisfy $x^*(A^+(M_i)) = x^*(A^-(M_i))$. Then, the differences $x^*(M_i, M_{i+1}) - x^*(M_{i+1}, M_i)$ have the same value for each $i = 0, 2, \ldots, K$. The direction of such 'difference' determines the direction of the arc with coefficient 0. For example, in the skeleton graph of figure 16b, the difference 0.7 corresponds to the direction from M_0 to M_{K-1}. Then, in the K-C$_{02}$ inequality we will assign coefficient 0 to the arc (M_0, M_1) and coefficient 2 to the arc (M_1, M_0).

Then, the LHS of the corresponding K-C$_{02}$ inequality can be easily computed as the sum of all the x^*-weights shown in figure 16b, minus $x^*(M_0 : M_K)$ (because the coefficient for links in $(M_0 : M_K)$ is now $K-1$, instead of $K-2$), minus the 'difference'. If $F(x^*) < 2(K-1)$, the K-C$_{02}$ inequality is violated. In the example in figure 16b, $F(x^*) = 10.6 - 0.2 - 0.7 = 9.7 < 10$ and, then, the K-C$_{02}$ inequality is violated.

Notice that, if the 'difference' is zero, i.e. if $x^*(M_i, M_{i+1}) = x^*(M_{i+1}, M_i)$, then the LHS of the K-C$_{02}$ inequality is less than or equal to the LHS of the standard K-C inequality. Then, we do not consider the former inequality.

As for standard K-C inequalities, by shrinking some consecutive nodes M_i and M_{i+1} with $x^*(M_i, M_{i+1}) + x^*(M_{i+1}, M_i) > 2$ into a single node, we obtain a smaller K-C$_{02}$ configuration with a K-C$_{02}$ inequality having smaller slack.

6.2 Honeycomb and Honeycomb$_{02}$ separation

As for the K-C inequalities, the separation procedures for the Honeycomb and Honeycomb$_{02}$ inequalities with $L = 1$ are based on those developed for the undirected GRP. For each R-set, the two first phases are similar to those for the undirected case, and the direction of the arcs will be ignored.

In phase I (see figure 17a), given a R-set, the vertices that are adjacent with arcs with positive x^*-weight to vertices in different R-sets, are the seeds to form the sets $B_1^1, B_1^2, \ldots, B_1^\gamma$. The remaining vertices in the considered R-set are associated with one of these seeds by trying

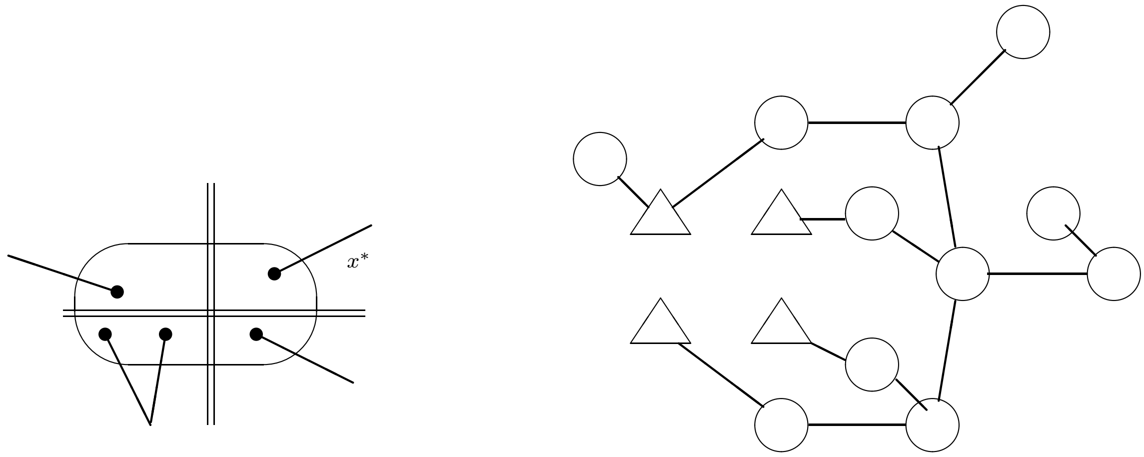

Figure 17: Phases I and II for Honeycomb separation

to get the x^*-weight among the different B_1^i to be as small as possible. Finally, all the B_1^i having an odd number of R-odd vertices are joined to form a single set (which obviously has an even number of R-odd vertices).

Phase II (see figure 17b) consists of finding the (undirected) maximum x^*-weight tree spanning the B_1^i (represented with triangles) and the $p-1$ remaining R-sets (represented with circles). Then, we iteratively shrink each R-set with degree 1 on the tree into its unique adjacent node.

At this point, we have a former Honeycomb configuration and its associated x^*-weighted configuration mixed graph. In the same way as for the K-C inequalities, each non-external arc is removed and its x^*-weight is added to the corresponding external arcs to obtain the skeleton graph shown in figure 18.

Then, the LHS of the corresponding Honeycomb inequality can be computed as the sum of all the x^*-weights shown in figure 18, minus twice the sum of the x^*-weights associated to the links among the B_1^i nodes. If $F(x^*) < 2(K-1)$, the Honeycomb inequality is violated. In the example in figure 18, we have $F(x^*) = 15 \geq 14$ (assuming that all the links among the B_1^i nodes have zero x^*-weight). Then, the Honeycomb inequality is not violated and its associated slack is $+1$.

The shrinking procedure described earlier for the K-C inequalities can also be applied here. In the skeleton graph in figure 18 we could shrink nodes B_3^0 and B_5^0 and nodes B_7^0 and B_8^0. Nevertheless, in this case the slack will only decrease from $+1$ above to $+1 - 0.3 - 0.5 = +0.2$ and therefore, even shrinking nodes, we would not obtain a violated Honeycomb inequality from the Honeycomb configuration corresponding to figure 18.

Once the standard Honeycomb inequalities have been checked, we consider again the skeleton graph in figure 18 in order to find violated Honeycomb$_{02}$ inequalities. Associated with a given Honeycomb configuration, several Honeycomb$_{02}$ inequalities can be defined, depending on the node type $\mathcal{O}$ or $\mathcal{I}$ assigned to each B_1^i. Due to the equation associated with each node B_j^0, the differences between the x^*-weights associated with the opposite arcs in the skeleton graph define the graph shown in figure 19. The direction of such differences on the nodes B_1^i determines their node type, $\mathcal{O}$ or $\mathcal{I}$ (see figure 19). A coefficient 0 will be assigned to the arcs leaving the nodes B_1^i of type $\mathcal{O}$ and coefficient 2 to their opposite arcs. Then, the LHS of the corresponding Honeycomb$_{02}$ inequality can be computed as:

- the sum of all the x^*-weights shown in figure 18,

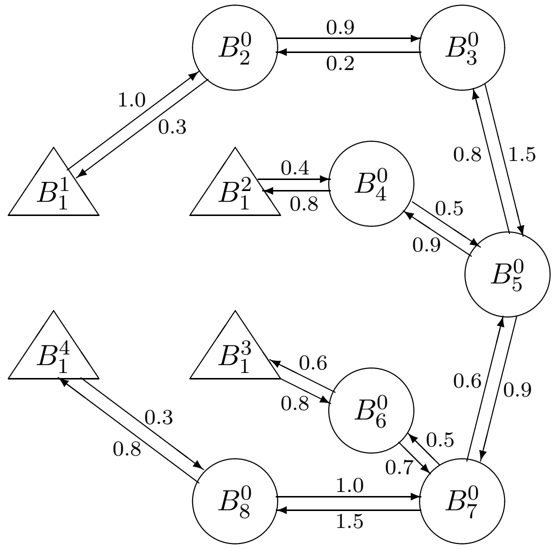

Figure 18: Skeleton graph for Honeycomb separation

- minus twice the sum of the x^*-weights among nodes B_1^i of the same type,
- minus (once) the sum of the x^*-weights among nodes B_1^i of different type,
- minus the sum of the 'differences' associated to nodes of type $\mathcal{O}$.

In the example in figure 18 (again it is assumed that the x^*-weights associated to links joining different nodes B_1^i are zero), we have $F(x^*) = 15 - 0.7 - 0.2 = 14.1 > 14$ and therefore the Honeycomb$_{02}$ inequality is not violated and its associated slack is $+0.1$.

As before, by shrinking some adjacent nodes B_j^0 with x^*-weight greater than 2 into a single node, we obtain a smaller Honeycomb$_{02}$ configuration with a Honeycomb$_{02}$ inequality with less slack. In the example in figure 18, we can shrink nodes B_3^0 and B_5^0 and nodes B_7^0 and B_8^0, with x^*-weight $0.8 + 1.5 = 2.3$ and $1.0 + 1.5 = 2.5$, to obtain a Honeycomb$_{02}$ inequality $F'(x) \geq 10$ violated by x^* ($F'(x^*) = 9.3$).

6.3 The Overall Algorithm

In each iteration of the cutting-plane algorithm, the separation procedures are invoked in the following ordering:

1. R-odd cut and connectivity separation heuristics.
2. Exact connectivity separation if the heuristics failed.
3. Exact R-odd cut separation if the heuristics failed.
4. Exact balanced-set separation.
5. If the number of violated inequalities detected so far is ≤ 10, for each R-set, try K-C and K-C$_{02}$ separation heuristic.
6. If the number of violated inequalities detected so far is ≤ 10, for each R-set, try Honeycomb

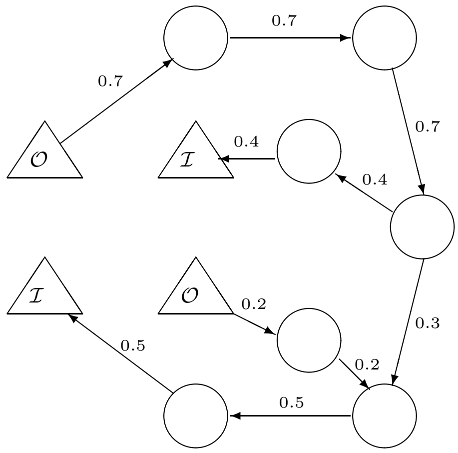

Figure 19: Differences associated with the opposite arcs in the skeleton

and Honeycomb$_{02}$ separation heuristic (that can also find K-Cs).
7. If no violated inequalities have been detected so far, try heuristics for K-C, K-C$_{02}$, Honeycomb and Honeycomb$_{02}$ inequalities by iteratively merging two adjacent R-sets.

When a large number of inequalities have been added, the LPs can become rather large, causing speed and memory problems. Hence, every time a violated inequality is found, it is added to the LP and it is also stored in a cut pool. Every 10 iterations, we rebuild the LP with all the inequalities in the pool having a slack less than 0.1.

When the cutting-plane algorithm does not find more violated inequalities and the LP relaxation is still not integral, we invoke branch-and-bound. The IP to be solved is formed by all the inequalities stored in the pool. If the IP solution is a semitour for the MGRP, we have obtained an optimal MGRP solution. Otherwise, the procedure terminates with a tight lower bound, but no feasible MGRP solution.

6.4 The instances

In Romero (1997), 100 instances for the Mixed Rural Postman Problem were randomly generated with the following features: $3 \leq p \leq 12$, $20 \leq |V| \leq 100$, $15 \leq |E| \leq 220$, $5 \leq |E_R| \leq 150$, $50 \leq |A| \leq 350$, $5 \leq |A_R| \leq 200$. Link costs were randomly generated in the range $1-20$. The procedure described in Corberán, Romero and Sanchis (2002) is capable of finding an optimal solution in 71 out of these 100 MRPP instances and this number increases up to 93 after invoking branch and bound. Given that our cutting plane algorithm (with K-C and Honeycomb separation procedures) solves all these instances up to optimality, we decided to generate harder MGRP instances. These have been grouped into 3 sets:

- 25 instances from the Albaida graph. This is an undirected graph with 116 vertices and 174 edges representing the real street network of the town of Albaida (Valencia, Spain). From this graph we have generated several MGRP instances in the following way. Each edge is selected as 'required' with probability P_1 and is transformed into an arc with probability P_2. If there is more than one strongly connected component in the resulting graph, some arcs are transformed into edges, until a strongly connected graph is obtained. All the vertices in the graph are considered as 'required'. We generated 25 instances, called albaij, by combining the values of $0.1, 0.3, 0.5, 0, 7$ and 0.9 for P_1 and for P_2, where i refers to P_1 and j refers to P_2.

- 25 instances from the Madrigueras graph. This is an undirected graph with 196 vertices and 316 edges representing the real street network of the town of Madrigueras (Albacete, Spain). Another 25 instances, madrij, were generated as above.

- 31 instances from the Aldaya graph. This is a mixed graph with 214 vertices, 224 edges and 127 arcs representing the real street network of the town of Aldaya (Valencia, Spain). The first instance in this group, called *aldaya* in tables, is based on a real garbage collection problem solved in this city. From the original mixed graph, we have generated other MGRP instances in the following way. Each edge is selected as 'required' with probability P_1 and each arc is selected as 'required' with probability P_2. All the vertices of the graph are considered as 'required'. Thirty MGRP instances were generated by combining different values for P_1 and for P_2: aldaij, where i refers to P_1 and j refers to P_2. Instance alda00 is a pure GTSP instance on a mixed graph (all its vertices are required while there are no required links), whilst alda99 is, in practice, a MCPP instance, because it has 2 R-connected components only.

6.5 Computational Results

The algorithm has been coded in *C* and run on a Personal Computer with a 400 MHz Pentium II processor, using CPLEX as LP solver. As stated before, all the 100 MRPP instances from Romero (1997) were solved to optimality with our algorithm, 25 of them after invoking branch and bound. Tables 1, 2 and 3 show the results obtained for the other three sets of instances.

For each instance, we present the number of R-connected components (column p), the number of inequalities found by the separation procedures for each inequality class: connectivity (*conn*), R-odd cut (*odd*), balanced set (*bal*), K-C, K-C$_{02}$, Honeycomb (*HC*) and Honeycomb$_{02}$ (*HC*$_{02}$), the lower bound obtained with the pure cutting-plane procedure (column *LB*), the lower bound obtained after invoking the branch and bound (column *B&B*) and the CPU time in seconds (column *time*). An entry marked with an asterisk means that an optimal solution is reached.

When the branch and bound ends in a non-feasible MGRP solution, it is possible to apply the separation procedures to this integer solution to define a new Integer Program and so on. If this iterative procedure produces an optimal MGRP solution, its cost is shown in column *Opt*.

The algorithm produces an optimal solution on 57 out of the 81 instances shown in tables 1, 2 and 3. Most of the solved instances are those corresponding to greater values for the probabilities P_1 (P_1 and P_2 for the Aldaya instances), which have a medium or large number of required links and hence a smaller value for p. On the other hand, small values for P_1 (P_1 and P_2 for the Aldaya instances) produce instances with a large number of R-sets, most with only one vertex, and it is known that the difficulty of the GRP instances increases with the number of R-sets. Furthermore, fewer K-C and Honeycomb violated inequalities are found for instances with few required links. Nevertheless, for the 15 unsolved instances with known optimal solution, the lower bound obtained (column *B&B*) is on average 0.20% from the optimal value (column *Opt*).

7 Conclusions

In this paper, a large family of facet-inducing inequalities for the polyhedron associated with the Mixed General Routing Problem has been described: the Honeycomb inequalities. The MGRP is an important routing problem as it contains most of the best known routing problems with a single vehicle as special cases and the above inequalities enlarge the known facial description of its polyhedron. In addition, new separation procedures for the known K-C and K-C_{02} inequalities, and for the above Honeycomb and Honeycomb$_{02}$ inequalities have been devised. Finally, a cutting-plane algorithm for the MGRP performing these separation procedures has been presented. The computational results obtained over several sets of instances prove that the K-C, K-C_{02}, Honeycomb and Honeycomb$_{02}$ inequalities are computationally useful to solve MGRP instances of medium size. As far as we know, this is the best algorithm for the MGRP published up to now.

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	p	conn	odd	bal	KC	KC_{02}	HC	HC_{02}	LB	B&B	Time	Opt
alba11	99	153	45	1	36	8	1	1	9302.5	9367	38	
alba13	94	81	60	2	41	1	2	3	10744^*		15	
alba15	103	59	56	-	-	-	-	-	11322.5	11331	4	11332
alba17	97	54	44	-	-	-	-	-	10617	10727	4	10795
alba19	97	43	70	-	12	3	2	2	11351	11401	18	11410
alba31	65	77	125	5	19	11	6	7	9866	9870^*	12	
alba33	70	53	96	6	8	5	1	1	11307	11315^*	12	
alba35	64	35	73	2	10	4	8	1	11435^*		8	
alba37	70	28	60	3	-	1	1	-	11742^*		3	
alba39	64	26	83	3	13	1	1	-	12766^*		5	
alba51	27	27	102	8	13	14	4	3	10931^*		7	
alba53	30	11	60	4	6	-	-	-	12480^*		3	
alba55	34	14	128	6	28	10	1	-	15558^*		21	
alba57	29	13	68	8	5	1	-	-	14893^*		3	
alba59	31	5	73	13	-	6	-	-	15848^*		7	
alba71	10	7	232	30	-	-	-	-	12566^*		32	
alba73	18	5	87	7	-	2	-	-	16647^*		4	
alba75	7	1	70	7	-	-	-	-	14887^*		4	
alba77	12	3	81	13	-	-	-	-	17427^*		7	
alba79	11	-	56	3	-	-	-	-	15501^*		3	
alba91	2	-	158	12	-	-	-	-	14497^*		24	
alba93	4	-	200	18	-	-	-	-	15680^*		21	
alba95	1	-	92	12	-	-	-	-	19032^*		12	
alba97	3	-	62	5	-	-	-	-	19338^*		3	
alba99	3	-	94	15	-	-	-	-	20026^*		16	

Table 1: Computational results for the Albaida-based MGRP instances.

	p	conn	odd	bal	KC	KC_{02}	HC	HC_{02}	LB	B&B	Time	Opt
madr11	163	221	119	2	29	1	-	-	14547.7	14720	259	
madr13	160	194	116	2	28	2	2	1	15221.0	15335	56	
madr15	168	143	74	-	21	1	8	1	17957.4	18100	59	18115
madr17	164	99	98	2	31	3	1	-	17845	17865	19	17875
madr19	156	103	98	2	2	-	1	1	18127.5	18170	14	18185
madr31	103	134	281	8	86	21	8	1	16566.1	16620	119	16725
madr33	106	89	147	6	1	1	1	-	18927.5	18940*	16	
madr35	108	52	129	8	1	2	-	1	23495*		12	
madr37	99	45	112	5	7	7	2	-	23415*		9	
madr39	101	49	88	4	2	-	1	-	23322.5	23345	7	23375
madr51	45	50	249	21	17	28	8	7	18445	18445	29	
madr53	52	36	149	9	16	6	1	-	20490*		9	
madr55	43	12	141	16	3	8	1	1	26730	26775	9	26815
madr57	44	10	114	9	-	-	-	-	35795*		6	
madr59	50	25	116	5	1	3	-	-	25007.5	25015	5	
madr71	10	7	200	24	3	25	1	1	22540	22540*	19	
madr73	11	4	285	41	-	17	-	-	25215*		42	
madr75	11	3	158	24	-	-	-	-	36310*		17	
madr77	7	4	144	17	2	6	-	-	30170*		10	
madr79	11	1	130	14	-	-	-	-	41440*		7	
madr91	2	-	206	21	-	-	-	-	25380*		17	
madr93	4	-	439	95	-	-	-	-	29835*		87	
madr95	3	1	335	77	-	-	-	-	37095*		55	
madr97	3	-	235	54	-	-	-	-	44750*		39	
madr99	2	-	183	34	-	-	-	-	38340*		22	

Table 2: Computational results for the Madrigueras-based MGRP instances.

	p	conn	odd	bal	KC	KC_{02}	HC	HC_{02}	LB	B&B	Time	Opt
aldaya	84	115	239	5	46	21	8	6	25579	25655	131	25703
alda00	214	317	28	-	-	-	-	-	20274.1	20437	119	20503
alda11	185	255	98	1	20	4	1	-	21059.5	21126	112	21209
alda13	157	215	136	3	14	2	-	-	22217.7	22382	122	22430
alda15	128	186	173	4	32	7	10	4	24933.2	25033	151	25053
alda17	100	149	116	4	13	4	-	-	26512.5	26613	24	
alda19	102	147	143	4	12	3	-	-	27291.6	27373*	62	
alda22	140	180	252	8	51	5	11	2	23730.3	23837	203	
alda31	130	151	225	7	36	9	9	-	23219	23312*	77	
alda33	107	125	693	11	42	18	2	-	25424.2	25526	390	
alda35	88	100	545	9	89	18	16	4	25985.6	26013*	214	
alda37	67	82	239	13	26	13	9	5	29393*		29	
alda39	50	63	275	12	20	9	2	2	30498	30591	30	30594
alda44	79	79	229	9	17	3	-	-	26537*		19	
alda51	95	88	550	25	37	26	4	2	25547	25565*	172	
alda53	72	59	199	10	25	5	1	-	26310	26355	45	
alda55	36	40	290	9	4	10	-	-	29876.3	29909*	27	
alda57	30	37	158	8	3	7	-	-	31325*		7	
alda59	30	34	172	11	3	9	-	-	31671*		12	
alda66	30	32	300	19	7	18	-	1	31865	31942*	41	
alda71	64	51	934	43	40	24	4	-	26539	26566*	183	
alda73	38	26	450	20	20	7	7	-	30018	30036*	50	
alda75	21	14	396	30	1	16	1	3	32338*		32	
alda77	7	6	212	10	-	1	-	-	34811*		9	
alda79	8	9	170	9	5	5	-	-	34230*		5	
alda88	3	1	264	34	-	-	-	-	38119*		28	
alda91	35	19	236	30	10	30	6	1	33119*		37	
alda93	16	8	217	26	-	10	-	1	33000*		22	
alda95	6	3	218	18	-	5	-	-	36677*		15	
alda97	5	3	329	27	-	1	-	-	39493*		24	
alda99	2	1	283	18	-	-	-	-	39359*		16	

Table 3: Computational results for the Aldaya-based MGRP instances.

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