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Additional Information

Dependency pairs for proving termination properties of conditional term rewriting systems[☆]

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Dedicated to the memory of Bernhard Gramlich

Abstract

The notion of *operational termination* provides a logic-based definition of termination of computational systems as the absence of *infinite inferences* in the computational logic describing the *operational semantics* of the system. For *Conditional Term Rewriting Systems* we show that operational termination is characterized as the conjunction of *two* termination properties. One of them is traditionally called *termination* and corresponds to the absence of infinite sequences of rewriting steps (a *horizontal* dimension). The other property, that we call *V-termination*, concerns the absence of infinitely many attempts to launch the *subsidiary processes* that are required to perform a *single* rewriting step (a *vertical* dimension). We introduce appropriate notions of *dependency pairs* to characterize termination, V-termination, and operational termination of Conditional Term Rewriting Systems. This can be used to obtain a powerful and more expressive framework for proving termination properties of Conditional Term Rewriting Systems.

Keywords: Conditional term rewriting, dependency pairs, program analysis, operational termination

1. Introduction

Conditional Term Rewriting Systems (CTRSs [6, 11, 24]) extend Term Rewriting Systems (TRSs [5, 36, 41]) by adding a (possibly empty) *conditional* part c to each rewrite rule $\ell \rightarrow r$, thus obtaining a *conditional rewrite rule* $\ell \rightarrow r \Leftarrow c$. The addition of such conditional parts c substantially increases the expressiveness of programming languages that use them (e.g., ASF+SDF [8], CafeOBJ [15], ELAN [7], Haskell [23], OBJ [19], or Maude [9]) and often clarifies the purpose of

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the rules to make programs more readable and self-explanatory. For instance, in functional programs, the use of *guards* and *local definitions* (by means of `where` clauses) is customary.

Example 1. *The following Haskell program implements the well-known quick-sort algorithm [36, Section 1]:*

```
split x [] = ([], [])
split x (y:ys)
  | x <= y = (xs,y:zs)
  | otherwise = (y:xs,zs)
  where (xs,zs) = split x ys
```

```
qsort [] = []
qsort (x:xs) = qsort ys ++ (x:qsort zs)
  where (ys,zs) = split x xs
```

This program can be understood as a CTRS (borrowing from [36, Section 1]; we have added rules to compare natural numbers in Peano's notation (with `leq`), and for implementing Haskell's appending operator `++` for lists with `app`):

$$\begin{aligned}
\text{leq}(0, x) &\rightarrow \text{true} & (1) \\
\text{leq}(s(x), 0) &\rightarrow \text{false} & (2) \\
\text{leq}(s(x), s(y)) &\rightarrow \text{leq}(x, y) & (3) \\
\text{app}(\text{nil}, xs) &\rightarrow xs & (4) \\
\text{app}(\text{cons}(x, xs), ys) &\rightarrow \text{cons}(x, \text{app}(xs, ys)) & (5) \\
\text{split}(x, \text{nil}) &\rightarrow \text{pair}(\text{nil}, \text{nil}) & (6) \\
\text{split}(x, \text{cons}(y, ys)) &\rightarrow \text{pair}(xs, \text{cons}(y, zs)) & (7) \\
&\Leftarrow \text{leq}(x, y) \rightarrow \text{true}, \text{split}(x, ys) \rightarrow \text{pair}(xs, zs) \\
\text{split}(x, \text{cons}(y, ys)) &\rightarrow \text{pair}(\text{cons}(y, xs), zs) & (8) \\
&\Leftarrow \text{leq}(x, y) \rightarrow \text{false}, \text{split}(x, ys) \rightarrow \text{pair}(xs, zs) \\
\text{qsort}(\text{nil}) &\rightarrow \text{nil} & (9) \\
\text{qsort}(\text{cons}(x, xs)) &\rightarrow \text{app}(\text{qsort}(ys), \text{cons}(x, \text{qsort}(zs))) & (10) \\
&\Leftarrow \text{split}(x, xs) \rightarrow \text{pair}(ys, zs)
\end{aligned}$$

Note the following:

1. a guard b in the Haskell program (e.g., $x \leq y$ and `otherwise`, which here means that the condition $x \leq y$ does not hold) is translated as a boolean test $b \rightarrow^* \text{true}$ or $b \rightarrow^* \text{false}$. The intended meaning is that the boolean expression b is evaluated by rewriting (in zero or more steps, denoted as \rightarrow^*) and then the outcome is checked to see whether it is `true` or `false`, respectively.
2. `where` clauses defining pattern matching conditions $p = e$ for an expression e whose value is matched against a pattern p are translated as rewriting conditions $e \rightarrow^* p$. The intended meaning is that e will be evaluated

<p>(Refl) $\frac{}{x \rightarrow^* x}$</p> <p>(Tran) $\frac{x \rightarrow z \quad z \rightarrow^* y}{x \rightarrow^* y}$</p>	<p>(Cong) $\frac{x_i \rightarrow y_i}{f(x_1, \dots, x_i, \dots, x_k) \rightarrow f(x_1, \dots, y_i, \dots, x_k)}$ for all $f \in \mathcal{F}$ and $1 \leq i \leq k = \text{arity}(f)$</p> <p>(Repl) $\frac{s_1 \rightarrow^* t_1 \quad \dots \quad s_n \rightarrow^* t_n}{\ell \rightarrow r}$ for $\ell \rightarrow r \Leftarrow s_1 \rightarrow t_1 \cdots s_n \rightarrow t_n \in \mathcal{R}$</p>
--	--

Figure 1: Inference rules for conditional rewriting ($x, y, z, x_1, \dots, x_k, y_1, \dots, y_k$ are variables)

and the outcome matched against p . In this way, variables in p become instantiated to expressions which are then used in the right-hand side of the rule to return the final result of the computation. Thus, part of such a computation is accomplished in the conditional part of the rules.

The example illustrates two practical uses of conditional rules when defining functions:

1. Testing *boolean conditions* before applying a rule, as in (7) and (8).
2. Local *reductions* of specific expressions followed by *matching* against a pattern in order to obtain pieces of information which can be used to build the outcome as in rules (7), (8), and (10).

Although several *transformations* have been envisaged to *remove* the conditional part of the rules, thus yielding an ‘equivalent’ TRS (see [31, 35, 37, 39] and the references therein), programmers still find conditional rules valuable when writing programs in the aforementioned languages.

1.1. Termination, V-termination, and operational termination of CTRSs

The semantics of rewriting-based computational systems is often described by means of the transitions induced by the *rewriting steps*. The *one-step rewriting relation* $\rightarrow_{\mathcal{R}}$ on terms induced by a CTRS \mathcal{R} is the basis to describe any accomplished *evaluation* or *transformation* of expressions. In this setting, the *absence of infinite rewrite sequences* $t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{R}} \dots$ arises as a natural definition of *terminating behavior* for CTRSs. However, computations with CTRSs with rules $\ell \rightarrow r \Leftarrow s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$ (i.e., the conditional part of a rule consists of a sequence of pairs $s_i \rightarrow t_i$, for $1 \leq i \leq n$) are defined by means of an *Inference System* where each rewriting step $s \rightarrow_{\mathcal{R}} t$ requires a *proof*. Figure 1 shows the inference system for conditional rewriting.¹ This proof also affects the *termination behavior* of \mathcal{R} , representing a different *dimension* of it.

¹All rules in this system are *schematic* in that each inference rule $\frac{B_1 \cdots B_n}{A}$ can actually be used under any *instance* $\frac{\sigma(B_1) \cdots \sigma(B_n)}{\sigma(A)}$ of the rule by a substitution σ . For instance, (Repl) actually establishes that, for every rule $\ell \rightarrow r \Leftarrow s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$ in the CTRS \mathcal{R} , every instance $\sigma(\ell)$ by a substitution σ rewrites into $\sigma(r)$ provided that, for each $s_i \rightarrow t_i$, with $1 \leq i \leq n$, the reachability condition $\sigma(s_i) \rightarrow^* \sigma(t_i)$ can be *proved*.

Assume that we have an *interpreter* for a logic \mathcal{L} , i.e., an *inference machine* that, given a theory \mathcal{S} and a goal formula φ , tries to incrementally build a proof tree for φ . Intuitively, we call \mathcal{S} *terminating* if for any φ the interpreter either finds a proof in finite time, or fails in all possible attempts also in finite time. The notion of *operational termination* captures this idea, meaning that, given an initial goal, an interpreter will either succeed in finite time in producing a closed proof tree, or will fail in finite time, not being able to close or extend further any of the possible proof trees, after exhaustively searching all such proof trees [26]. Besides implying termination in the usual TRS sense, operational termination also captures a ‘*vertical*’ *dimension* of the termination behavior which is missing in the usual ‘without infinite reduction sequences’ definition of termination (the ‘horizontal’ dimension).

Example 2. *The following CTRS \mathcal{R} :*

$$g(a) \rightarrow c(b) \quad (11)$$

$$b \rightarrow f(a) \quad (12)$$

$$f(x) \rightarrow x \leftarrow g(x) \rightarrow c(y) \quad (13)$$

is terminating in the usual sense: there is no infinite rewrite sequence. This is easily proved in this particular case because the underlying TRS \mathcal{R}_u :

$$g(a) \rightarrow c(b)$$

$$b \rightarrow f(a)$$

$$f(x) \rightarrow x$$

which is obtained from \mathcal{R} by removing the conditional part of the rules (see [6, page 324] and [36, Definition 7.1.2], for instance), is terminating.

However, \mathcal{R} is not operationally terminating: we have an infinite proof tree for the inference system for CTRSs in Figure 1 (substitutions provide the necessary match of rules against the different goals; the labels indicating the applied rules have been conveniently shortened):

$$\frac{\frac{\frac{g(a) \rightarrow c(b)}{(Rp)} \quad \frac{\frac{b \rightarrow f(a)}{(Rp)} \quad \frac{c(b) \rightarrow c(f(a))}{(C)}}{(C)} \quad \frac{\frac{\frac{f(a) \rightarrow a}{\vdots} \quad c(f(a)) \rightarrow c(a)}{(C)} \quad c(a) \rightarrow^* c(b)}{(T)} \quad c(f(a)) \rightarrow^* c(b)}{(T)}}{c(b) \rightarrow^* c(b)} \quad \frac{g(a) \rightarrow c(b)}{(Rp)} \quad c(b) \rightarrow^* c(b)}{(T)}}{g(a) \rightarrow^* c(b)} \quad \frac{g(a) \rightarrow^* c(b)}{(T)} \quad f(a) \rightarrow a}{(Rp)}$$

Note that the rightmost uppermost goal $c(a) \rightarrow^ c(b)$ remains open due to the left-to-right evaluation strategy for the goals, which first takes over the infinite branch on its left. This infinite proof tree concerns a single rewriting step $f(a) \rightarrow a$. However such a step is indeed possible; the use of (Refl) instead of (Tran) proves $c(b) \rightarrow^* c(b)$ and closes the tree by cutting its infinite development:*

$$\frac{\frac{\frac{\frac{}{\mathbf{g}(\mathbf{a}) \rightarrow \mathbf{c}(\mathbf{b})}^{(Rp)}}{\mathbf{g}(\mathbf{a}) \rightarrow^* \mathbf{c}(\mathbf{b})}^{(T)}}{\mathbf{f}(\mathbf{a}) \rightarrow \mathbf{a}}^{(Rp)}}{\mathbf{c}(\mathbf{b}) \rightarrow^* \mathbf{c}(\mathbf{b})}^{(Rf)}$$

Available tools for proving operational termination of CTRSs \mathcal{R} (e.g., AProVE [18] or VMTL [38]) rely on *transformations* \mathcal{U} that prove operational termination of \mathcal{R} as termination of the TRS $\mathcal{U}(\mathcal{R})$. However, this transformational approach has substantial limitations.

Example 3. *The following CTRS \mathcal{R} [36, Example 7.2.51]*

$$\mathbf{h}(\mathbf{d}) \rightarrow \mathbf{c}(\mathbf{a}) \quad (14)$$

$$\mathbf{h}(\mathbf{d}) \rightarrow \mathbf{c}(\mathbf{b}) \quad (15)$$

$$\mathbf{f}(\mathbf{k}(\mathbf{a}), \mathbf{k}(\mathbf{b}), x) \rightarrow \mathbf{f}(x, x, x) \quad (16)$$

$$\mathbf{g}(x) \rightarrow \mathbf{k}(y) \Leftarrow \mathbf{h}(x) \rightarrow \mathbf{d}, \mathbf{h}(x) \rightarrow \mathbf{c}(y) \quad (17)$$

is operationally terminating² but $\mathcal{U}(\mathcal{R})$ (where \mathcal{U} is the transformation in [36, Definition 7.2.48]) is not terminating: $\mathcal{U}(\mathcal{R})$ consists of rules (14), (15) and (16) together with the following rules that replace the conditional rule (17)

$$\mathbf{g}(x) \rightarrow \mathbf{U}_1(\mathbf{h}(x), x) \quad (18)$$

$$\mathbf{U}_1(\mathbf{d}, x) \rightarrow \mathbf{U}_2(\mathbf{h}(x), x) \quad (19)$$

$$\mathbf{U}_2(\mathbf{c}(y), x) \rightarrow \mathbf{k}(y) \quad (20)$$

with \mathbf{U}_1 and \mathbf{U}_2 new function symbols. Ohlebusch displays the following cycle:

$$\begin{aligned} \mathbf{f}(\mathbf{k}(\mathbf{a}), \mathbf{k}(\mathbf{b}), \mathbf{U}_2(\mathbf{h}(\mathbf{d}), \mathbf{d})) &\xrightarrow{\mathcal{U}(\mathcal{R})} \mathbf{f}(\mathbf{U}_2(\mathbf{h}(\mathbf{d}), \mathbf{d}), \mathbf{U}_2(\mathbf{h}(\mathbf{d}), \mathbf{d}), \mathbf{U}_2(\mathbf{h}(\mathbf{d}), \mathbf{d})) \\ &\xrightarrow{\mathcal{U}(\mathcal{R})^+} \mathbf{f}(\mathbf{U}_2(\mathbf{c}(\mathbf{a}), \mathbf{d}), \mathbf{U}_2(\mathbf{c}(\mathbf{b}), \mathbf{d}), \mathbf{U}_2(\mathbf{h}(\mathbf{d}), \mathbf{d})) \\ &\xrightarrow{\mathcal{U}(\mathcal{R})^+} \mathbf{f}(\mathbf{k}(\mathbf{a}), \mathbf{k}(\mathbf{b}), \mathbf{U}_2(\mathbf{h}(\mathbf{d}), \mathbf{d})) \end{aligned}$$

In this paper we define the notion of V -termination, which captures the *vertical* dimension of the termination behavior of CTRSs. We provide a *uniform* definition of termination and V -termination of CTRSs as the absence of specific kinds of infinite proof trees. We prove that operational termination is just the *conjunction* of termination and V -termination. We use these results to develop a methodology to *prove or disprove* termination, V -termination, and operational termination of CTRSs. In the TRS setting an interesting approach to develop methods for proving termination of (variants of) rewriting is the extension or generalization of the *Dependency Pair (DP) approach* for TRSs [1]: the rules $\ell \rightarrow r$ that are able to produce infinite sequences are those whose right-hand side r contains (possibly recursive) *function calls* which are represented as new

²Ohlebusch proves \mathcal{R} *quasi-reductive*, which implies its *quasi-decreasingness*, which in turn implies its operational termination [26].

rules $u \rightarrow v$, called *dependency pairs* and collected in a new TRS $\text{DP}(\mathcal{R})$. Pairs in $\text{DP}(\mathcal{R})$ determine *dependency chains* whose finiteness characterizes termination of \mathcal{R} . In this paper we generalize the DP approach to all aforementioned termination properties of CTRSs.

1.2. Our contributions

The contributions of this paper (after Section 2, which introduces some preliminary notions and definitions) can be summarized as follows:

1. In Section 3 we introduce the notion of V -termination as the absence of a specific kind of infinite proof trees. We prove that operational termination is the *conjunction* of termination and V -termination. We also introduce a generic notion of *minimality* of terms and a number of generic results that are the basis for the developments in the subsequent sections.
2. We provide a complete *characterization of termination of CTRSs* using dependency pairs (Section 4). Although our main goal is the development of methods for proving and disproving *operational termination* of CTRSs, a first consideration of the problem of proving termination of CTRSs is useful because: (i) the adaptation of our methodology to characterize termination using dependency pairs is simpler, and (ii) the dependency pairs introduced to analyze termination of CTRSs are also used to analyze operational termination of CTRSs. Our analysis and results apply to arbitrary CTRSs, without any restriction on the shape of the rules $\ell \rightarrow r \leftarrow c$ beyond a general requirement for ℓ not being a variable, something that is fulfilled in most cases.
3. We provide a complete *characterization of V -termination of CTRSs* using dependency pairs (Section 5). As in the previous case (termination of CTRSs) the obtained dependency pairs are also used to prove operational termination of CTRSs as discussed in the next item.
4. Operational termination of CTRSs can be proved by using the previously introduced dependency pairs for proving termination and V -termination of CTRSs. In Section 6 we show that this can be more efficiently done by considering smaller sets of dependency pairs.
5. Section 7 introduces a new kind of chains, called O -chains, which provide a simpler characterization of operational termination of deterministic 3-CTRSs. We also show how to use *O -chains of dependency pairs* to prove and disprove termination, V -termination and operational termination of CTRSs using the different kinds of dependency pairs introduced above.

Section 8 summarizes in more detail our contributions, discusses related work, and briefly describes the practical use of the ideas developed in this paper in the recent version of the termination tool MU-TERM [3]. Section 9 discusses future work and concludes.

This paper is an extended and completely revised version of [28, Sections 3 and 4]. In particular, we have invested a substantial effort in the *foundational* aspects of the paper. There are many new contributions with respect to [28].

The characterization of operational termination in terms of different kinds of proof trees in Section 3 is completely new. Sections 4 and 5 about termination and V -termination of CTRSs are also completely new. We have simplified the definition of the 2D DPs given in [28]. And the corresponding notion of chain has been slightly changed to make it more precise. We also provide full proofs of all results.

This paper is dedicated *in memoriam* to Bernhard Gramlich, who made important contributions, in particular, to the analysis of computations with conditional rewrite systems. He passed away in June 2014, during the preparation of a first version of this paper, where we pay attention to some of his contributions.

2. Preliminaries

The material in this section follows [36]. A binary relation R on a set A is *terminating* if there is no infinite sequence $a_1 R a_2 R a_3 \cdots$. Given relations $R, S \subseteq A \times A$, we let $R \circ S = \{(a, b) \in A \times A \mid \exists c \in A, a R c \wedge c S b\}$.

2.1. Signatures, Terms, and Positions

Throughout the paper, \mathcal{X} denotes a countable set of variables and \mathcal{F} denotes a signature, i.e., a set of function symbols $\{f, g, \dots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \rightarrow \mathbb{N}$. The set of terms built from \mathcal{F} and \mathcal{X} is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. $Var(t)$ is the set of variables occurring in a term t .

Terms are viewed as labelled trees in the usual way. Positions p, q, \dots are represented by sequences of positive natural numbers used to address subterms of t . We denote the empty sequence by ϵ . Given positions p, q , we denote their concatenation as $p.q$. Positions are ordered by the standard prefix ordering: $p \leq q$ if $\exists q'$ such that $q = p.q'$. If p is a position, and Q is a set of positions, then $p.Q = \{p.q \mid q \in Q\}$. The set of positions of a term t is $Pos(t)$. Positions of nonvariable symbols in t are denoted as $Pos_{\mathcal{F}}(t)$. The subterm at position p of t is denoted as $t|_p$, and $t[s]_p$ is t with $t|_p$ replaced by s .

We write $s \supseteq t$, read *t is a subterm of s* , if $t = s|_p$ for some $p \in Pos(s)$ and $s \triangleright t$ if $s \supseteq t$ and $s \neq t$. We write $s \not\supseteq t$ and $s \not\triangleright t$ for the negation of the corresponding properties. The symbol labeling the root of t is denoted as $root(t)$. A *context* is a term $C \in \mathcal{T}(\mathcal{F} \cup \{\square\}, \mathcal{X})$ with a ‘hole’ \square (a fresh constant symbol). We write $C[\]_p$ to denote that there is a (usually single) hole \square at position p of C . Generally, we write $C[\]$ to denote an arbitrary context and make the position of the hole explicit only if necessary. $C[\]_{\epsilon} = \square$ is called the *empty context*.

2.2. Substitutions, renamings, and unifiers

A substitution is a mapping $\sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{X})$. The ‘identity’ substitution $x \mapsto x$ for all $x \in \mathcal{X}$ is denoted ϵ . The set $Dom(\sigma) = \{x \in \mathcal{X} \mid \sigma(x) \neq x\}$ is called the *domain* of σ .

Remark 4. We do not impose that the domain of the substitutions be finite. This is usual practice in the dependency pair approach, where a single substitution is used to instantiate an infinite number of variables coming from renamed versions of the dependency pairs (see below). When substitutions with finite domain are assumed, we explicitly call them finite substitutions.

A *renaming* is a bijective substitution ρ such that $\rho(x) \in \mathcal{X}$ for all $x \in \mathcal{X}$. A finite substitution σ such that $\sigma(s) = \sigma(t)$ for two terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is called a *unifier* of s and t ; we also say that s and t *unify* (with substitution σ). If two terms s and t unify, then there is a unique *most general unifier* σ (up to renaming of variables) such that for every other unifier τ , there is a finite substitution θ such that $\theta \circ \sigma = \tau$.

2.3. Conditional Rewrite Systems

An (*oriented*) CTRS \mathcal{R} is a pair $\mathcal{R} = (\mathcal{F}, R)$ where \mathcal{F} is a signature and R a set of rules $\ell \rightarrow r \Leftarrow s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$. In this paper we assume that $\ell \notin \mathcal{X}$. As usual, ℓ and r are called the left- and right-hand sides of the rule, and the sequence $s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$ (often abbreviated to c) is the *conditional part* of the rule. *Labeled* rules are written $\alpha : \ell \rightarrow r \Leftarrow c$, where α is a *label*, which is often used by itself to refer to the rule.

Rewrite rules $\ell \rightarrow r \Leftarrow c$ are classified according to the distribution of variables among ℓ , r , and c , as follows: *type 1*, if $\text{Var}(r) \cup \text{Var}(c) \subseteq \text{Var}(\ell)$; *type 2*, if $\text{Var}(r) \subseteq \text{Var}(\ell)$; *type 3*, if $\text{Var}(r) \subseteq \text{Var}(\ell) \cup \text{Var}(c)$; and *type 4*, if no restriction is given. A rule of type n is often called an n -rule. A rewrite rule α is a *proper* n -rule if for all $m < n$, α is *not* an m -rule. An n -CTRS contains only m -rules for $m \leq n$. A TRS is a 1-CTRS whose rules have no conditional part; we write them $\ell \rightarrow r$.

A 3-CTRS \mathcal{R} is called *deterministic* if for each rule $\ell \rightarrow r \Leftarrow s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$ in \mathcal{R} and each $1 \leq i \leq n$, we have $\text{Var}(s_i) \subseteq \text{Var}(\ell) \cup \bigcup_{j=1}^{i-1} \text{Var}(t_j)$. Given $\mathcal{R} = (\mathcal{F}, R)$, we consider \mathcal{F} as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$ (called *constructors*) and symbols $f \in \mathcal{D}$ (called *defined functions*), where $\mathcal{D} = \{\text{root}(\ell) \mid (\ell \rightarrow r \Leftarrow c) \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$. If necessary we may write $\mathcal{C}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ to make explicit the CTRS \mathcal{R} which is used to establish the partition of \mathcal{F} into constructor and defined symbols. Terms $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $\text{root}(t) \in \mathcal{D}$ are called *defined* terms. $\text{Pos}_{\mathcal{D}}(t)$ is the set of positions p of subterms $t|_p$ such that $\text{root}(t|_p) \in \mathcal{D}$.

2.4. Proof trees

Given an atom A (in the CTRS logic) of the form $s \rightarrow t$ or $s \rightarrow^* t$ for terms s and t , we write $\text{pred}(A)$ to refer to its *predicate* symbol \rightarrow or \rightarrow^* , respectively. We also write $\text{left}(A)$ to denote term s of A . A finite proof tree T (for the inference system in Figure 1) is either: (i) an *open goal*, simply denoted as G , where G is of the form $s \rightarrow t$ or $s \rightarrow^* t$ for terms s, t ; then, we denote $\text{root}(T) = G$; otherwise, (ii) is a *derivation tree* with root G , denoted as

$$\frac{T_1 \quad \dots \quad T_n}{G}(\rho)$$

where G is as above, T_1, \dots, T_n are finite proof trees (for $n \geq 0$), and $\rho : \frac{B_1 \dots B_n}{A}$ is an inference rule such that $G = \sigma(A)$, $\text{root}(T_1) = \sigma(B_1), \dots, \text{root}(T_n) = \sigma(B_n)$ for some substitution σ . We write $\text{root}(T) = G$. We say that a finite proof tree T is *closed* if it contains no open goals.

A finite proof tree T is a *proper prefix* of a finite proof tree T' if there are one or more open goals G_1, \dots, G_n in T such that T' is obtained from T by replacing each G_i by a derivation tree T_i with root G_i . We denote this as $T \subset T'$. An *infinite proof tree* T is an infinite increasing chain of finite proof trees, that is, a sequence $\{T_i\}_{i \in \mathbb{N}}$ such that for all i , $T_i \subset T_{i+1}$. Note that, for all $i \in \mathbb{N}$, $\text{root}(T_i) = \text{root}(T_{i+1})$; then, we write $\text{root}(T) = \text{root}(T_0)$. We consider two infinite trees $\{T_i\}_{i \in \mathbb{N}}$ and $\{T'_i\}_{i \in \mathbb{N}}$ *equivalent*, written $\{T_i\}_{i \in \mathbb{N}} \equiv \{T'_i\}_{i \in \mathbb{N}}$, iff $(\forall i)(\exists j) T_i \subset T'_j$ and $(\forall i)(\exists j) T'_i \subset T_j$. This captures the intuitive idea that both chains have the same infinite limit.

Remark 5. *There can be different (equivalent) ways to represent the same infinite proof tree T as an infinite increasing chain $\{T_i\}_{i \in \mathbb{N}}$ of finite proof trees T_i . This is due to the possibility of replacing one of more open goals in T_i by proof trees to obtain T_{i+1} . For instance, consider the following sequence³:*

1. T_0 is a goal G_0 .
2. T_1 is $\frac{S_1 \quad G_1 \quad O_1}{G_0}$ where S_1 is a closed proof tree and G_1 and O_1 are open goals.
3. T_2 is $\frac{S_1 \quad S'_1 \quad O_1}{G_0}$ where S'_1 is a closed proof tree with root G_1 .
4. T_3 is $\frac{S_1 \quad S'_1 \quad \frac{S_2 \quad G_2 \quad O_2}{O_1}}{G_0}$ where S_2 is a closed proof tree and G_2 and O_2 are open goals.
5. T_4 is $\frac{S_1 \quad S'_1 \quad \frac{S_2 \quad S'_2 \quad O_2}{O_1}}{G_0}$ where S'_2 is a closed proof tree with root G_2 .
6. T_5 is $\frac{S_1 \quad S'_1 \quad \frac{S_2 \quad S'_2 \quad \frac{S_3 \quad G_3 \quad O_3}{O_2}}{O_1}}{G_0}$ where S_3 is a closed proof tree and G_3 and O_3 are open goals.
7. ...

Note that, for all $i \geq 0$, $T_i \subset T_{i+1}$ as required. Thus, $\{T_i\}_{i \in \mathbb{N}}$ can be used to represent an infinite proof tree T . However, $\{T'_i\}_{i \in \mathbb{N}}$ (where $T'_i = T_{2i}$ for all $i \in \mathbb{N}$) also represents T because $\{T_i\}_{i \in \mathbb{N}} \equiv \{T'_i\}_{i \in \mathbb{N}}$.

A finite proof tree T is *well-formed* if it is either an open goal, or a closed proof tree, or a derivation tree

$$\frac{T_1 \quad \dots \quad T_n}{G}(\rho)$$

where there is i , $1 \leq i \leq n$, such that T_1, \dots, T_{i-1} are closed, T_i is a well-formed but not closed finite proof tree, and T_{i+1}, \dots, T_n are open goals. An infinite proof tree is *well-formed* if it is an increasing chain of well-formed finite proof

³This example has been suggested by one of the referees of the paper.

$$\begin{array}{c}
\vdots \\
\frac{\vec{T}_n \quad \vec{G}_n \quad \vec{O}_n}{\quad} \\
\vdots \\
\frac{\vec{T}_1 \quad \vec{G}_1 \quad \vec{O}_1}{\quad} \\
\vec{G}_0
\end{array}$$

Figure 2: Structure of an infinite well-formed proof tree T

trees. Intuitively, well-formed (finite or infinite) proof trees are the proof trees that an interpreter would incrementally build when trying to solve one condition at a time from left to right.

Remark 6. *Infinite well-formed proof trees T have a single infinite branch as shown in Figure 2 where for all $i \geq 1$, \vec{T}_i are sequences of closed proof trees and \vec{O}_i are sequences of open goals. For all $i \geq 0$, goal G_i is the root of the infinite subtree immediately above it. Formally, T is the limit of the sequence $\{S_i\}_{i \in \mathbb{N}}$, where $S_0 = G_0$ and for all $i \geq 0$, S_{i+1} is obtained from S_i by replacing the open goal G_i by $\frac{\vec{T}_{i+1} \quad G_{i+1} \quad \vec{O}_{i+1}}{G_i}$. Note that, for all $i \geq 0$, $S_i \subset S_{i+1}$, as required.*

Definition 7. *Let T be an infinite well-formed proof tree as in Figure 2. We call the infinite sequence $(G_i)_{i \in \mathbb{N}}$ the spine of T , denoted $\text{spine}(T)$.*

2.5. Conditional rewriting

We write $s \rightarrow_{\mathcal{R}} t$ (resp. $s \rightarrow_{\mathcal{R}}^* t$) iff there is a finite well-formed proof tree for $s \rightarrow t$ (resp. $s \rightarrow^* t$) using \mathcal{R} . We often drop \mathcal{R} from $\rightarrow_{\mathcal{R}}$ or $\rightarrow_{\mathcal{R}}^*$ if no confusion arises. Note that $s \rightarrow_{\mathcal{R}} t$ if and only if there is $p \in \text{Pos}(s)$, $\ell \rightarrow r \leftarrow c \in \mathcal{R}$ and a substitution σ such that $\sigma(u) \rightarrow_{\mathcal{R}}^* \sigma(v)$ for all $u \rightarrow v \in c$, $s|_p = \sigma(\ell)$ and $t = s[\sigma(r)]_p$. We can make this explicit by writing $s \xrightarrow{p}_{\mathcal{R}} t$. We also write $s \xrightarrow{p}_{\mathcal{R}}^* t$ if there is $q > p$ such that $s \xrightarrow{q}_{\mathcal{R}} t$. It is easy to prove that $s \rightarrow_{\mathcal{R}}^* t$ holds if and only if there is a sequence s_1, \dots, s_n of terms for some $n \geq 1$ such that $s = s_1$, $t = s_n$ and for all i , $1 \leq i < n$, $s_i \rightarrow_{\mathcal{R}} s_{i+1}$; in particular, we write $s \xrightarrow{\epsilon}_{\mathcal{R}}^* t$ iff $s \rightarrow^* t$ and for all $i \geq 0$, $s_i \xrightarrow{\epsilon}_{\mathcal{R}} s_{i+1}$ holds.

The following definitions introduce some basic *combinators* for proof trees which are used to connect the usual rewriting notation based on sequences and the corresponding proof trees. First we consider the use of the *congruence* rule.

Definition 8. *Let \mathcal{R} be a CTRS, s, t be terms, T be a proof tree with $\text{root}(T) = s \rightarrow t$, and $C[\]$ be a context.*

1. *If T is a finite proof tree, then $C[T]$ is the finite proof tree*

$$\frac{\begin{array}{c} T \\ \vdots \\ \vdots \end{array}^{(C)}}{C[s] \rightarrow C[t]}^{(C)}$$

2. If T is an infinite proof tree $\{T_i\}_{i \in \mathbb{N}}$ for finite proof trees T_i , $i \geq 0$, then $C[T]$ is the infinite proof tree $\{C[T_i]\}_{i \in \mathbb{N}}$.

The following definition concerns the repeated use of rule $(Tran)$. Note that we do not try to cover all possible uses of the rule, but only those which are useful in our development (see also Proposition 10 below).

Definition 9. For all $i \geq 0$, let t_i be terms and, for all $i \geq 1$, let T_i be finite proof trees with $\text{root}(T_i) = t_{i-1} \rightarrow t_i$. We write the infinite sequence of such trees as T_1, T_2, \dots . Let $n \in \mathbb{N}$ and u be a term. Then

1. $Tr(T_1, \dots, T_n)$ is given by:

$$\frac{\frac{\frac{T_1}{\frac{\frac{\frac{T_2}{\frac{\frac{T_n}{t_n \rightarrow^* t_n} (Rf)}{t_n \rightarrow^* t_n} (Tr)}{\vdots} (Tr)}{t_2 \rightarrow^* t_n} (Tr)}{t_1 \rightarrow^* t_n} (Tr)}{t_0 \rightarrow^* t_n} (Tr)}{t_0 \rightarrow^* t_n} (Tr)}$$

2. $Tr_u(T_1, \dots, T_n)$ is given by:

$$\frac{\frac{\frac{T_1}{\frac{\frac{\frac{T_2}{\frac{\frac{T_n}{t_n \rightarrow^* u} (Tr)}{t_2 \rightarrow^* u} (Tr)}{\vdots} (Tr)}{t_1 \rightarrow^* u} (Tr)}{t_0 \rightarrow^* u} (Tr)}{t_0 \rightarrow^* u} (Tr)}$$

If S is an infinite proof tree $\{S_i\}_{i \in \mathbb{N}}$ with $\text{root}(S) = t_{n-1} \rightarrow t_n$, then $Tr_u(T_1, \dots, T_{n-1}, S)$ is the infinite proof tree $\{Tr_u(T_1, \dots, T_{n-1}, S_i)\}_{i \in \mathbb{N}}$.

3. If T is a finite proof tree with $\text{root}(T) = t_n \rightarrow^* u$, then $Tr_T(T_1, \dots, T_n)$ is given by:

$$\frac{\frac{\frac{T_1}{\frac{\frac{\frac{T_2}{\frac{\frac{T_n}{T} (Tr)}{t_2 \rightarrow^* u} (Tr)}{\vdots} (Tr)}{t_1 \rightarrow^* u} (Tr)}{t_0 \rightarrow^* u} (Tr)}{t_0 \rightarrow^* u} (Tr)}$$

If S is an infinite proof tree $\{S_i\}_{i \in \mathbb{N}}$ with $\text{root}(S) = t_n \rightarrow^* u$, then $Tr_S(T_1, \dots, T_n)$ is the infinite proof tree $\{Tr_{S_i}(T_1, \dots, T_n)\}_{i \in \mathbb{N}}$.

4. $Tr_u^\infty(T_1, T_2, \dots)$ is given as the limit of the infinite sequence $\{S_i\}_{i \in \mathbb{N}}$, where S_0 is

$$\frac{T_1}{t_0 \rightarrow^* u} (Tr)$$

and for all $i > 0$, S_{i+1} is obtained from S_i by replacing the open goal $t_i \rightarrow^* u$ by the derivation tree

$$\frac{T_{i+1} \quad t_{i+1} \rightarrow^* u}{t_i \rightarrow^* u} \text{ (Tr)}$$

Note that, for all $i \geq 0$, $S_i \subset S_{i+1}$, as required.

The following result is obvious from the definitions above and is tacitly used in the sequel; in this result, we write *proof tree* to indistinctly refer to a finite or infinite proof tree.

Proposition 10. *Let \mathcal{R} be a CTRS, s, t be terms and $C[\]$ be a context.*

1. *If T is a closed (resp. infinite, well-formed) proof tree, then $C[T]$ is closed (resp. infinite, well-formed).*
2. *For all $i \geq 1$, let s_i, t_i be terms such that $t_i = s_{i+1}$ and T_i be finite proof trees with $\text{root}(T_i) = s_i \rightarrow t_i$. Let $n \in \mathbb{N}$ and u be a term. Then*
 - (a) *If for all i , $1 \leq i \leq n$, T_i is a closed proof tree, then $\text{Tr}(T_1, \dots, T_n)$ is closed.*
 - (b) *If for all i , $1 \leq i < n$, T_i is a closed proof tree and T_n is a well-formed proof tree, then $\text{Tr}_u(T_1, \dots, T_n)$ is a well-formed proof tree.*
 - (c) *If for all i , $1 \leq i \leq n$, T_i is a closed proof tree and T is a well-formed proof tree with $\text{root}(T) = t_n \rightarrow^* u$, then $\text{Tr}_T(T_1, \dots, T_n)$ is a well-formed proof tree.*
 - (d) *If for all $i \geq 1$, T_i is a closed proof tree, then $\text{Tr}_u^\infty(T_1, T_2, \dots)$ is a well-formed proof tree.*

3. Operational termination of conditional rewriting revisited

Let \mathcal{R} be a CTRS. A term t such that there is no infinite well-formed proof tree T with $\text{left}(\text{root}(T)) = t$ is called *operationally \mathcal{R} -terminating* (or just *operationally terminating*). Actually, we have the following.

Proposition 11. *Let \mathcal{R} be a CTRS. A term t is operationally terminating if and only if there is no infinite well-formed proof tree T with $\text{root}(T) = t \rightarrow^* u$ for any term u .*

PROOF. The *only if* part is obvious. The *if* part follows by contradiction. Assume that t is operationally nonterminating but no goal $t \rightarrow^* u$ has an infinite well-formed tree for any term u . Then, there is an infinite well-formed proof tree T with $\text{root}(T) = t \rightarrow v$ for some term v . Then, $T^\infty = \text{Tr}_u(T)$ is an infinite well-formed proof tree with $\text{root}(T^\infty) = t \rightarrow^* u$, leading to a contradiction. \square

A CTRS \mathcal{R} is called *operationally terminating* if every term is operationally terminating. Termination can be seen as a *horizontal dimension* of operational termination that concerns the absence of infinite *sequences* of rewriting steps: a term t is said to be *\mathcal{R} -terminating* (or just *terminating* if no confusion arises)

if there is no infinite rewrite sequence $t = t_1 \rightarrow t_2 \rightarrow \dots$ starting from t . And \mathcal{R} is said to be *terminating* if every term is terminating. Actually, termination of terms can also be characterized in a *proof-theoretic* way, as follows.

Proposition 12. *Let \mathcal{R} be a CTRS. A term t is terminating if and only if there is no infinite well-formed proof tree T such that $\text{left}(\text{root}(T)) = t$ and for all $G \in \text{spine}(T)$, $\text{pred}(G) = \rightarrow^*$.*

PROOF. By contradiction. For the *if* part, if t is not terminating, then there is an infinite rewrite sequence $t = s_1 \rightarrow_{\mathcal{R}} s_2 \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} s_n \rightarrow_{\mathcal{R}} \dots$. For all $i \geq 1$, let T_i be the closed proof tree of $s_i \rightarrow s_{i+1}$. Let u be an arbitrary term. Then, $T = \text{Tr}_u^\infty(T_1, T_2, \dots)$ is an infinite well-formed proof tree with $\text{root}(T) = t \rightarrow^* u$ and for all $G \in \text{spine}(T)$, $\text{pred}(G) = \rightarrow^*$, leading to a contradiction. For the *only if* part, if there is an infinite well-formed proof tree T with $\text{left}(\text{root}(T)) = t$ and for all $G \in \text{spine}(T)$, $\text{pred}(G) = \rightarrow^*$, then, the only rule that can generate such a tree is (Tran), i.e., T is as follows:

$$\frac{\frac{\frac{\vdots}{T_n} \quad \frac{\vdots}{G_n}}{\text{(Tran)}}}{\vdots} \quad \frac{\vdots}{G_1} \quad \frac{\vdots}{G_0} \quad \text{(Tran)}$$

where for all $i \geq 1$, T_i is a closed proof tree with $\text{root}(T_i) = s_i \rightarrow s_{i+1}$. Thus, there is an infinite rewrite sequence $t = s_1 \rightarrow_{\mathcal{R}} s_2 \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} s_n \rightarrow_{\mathcal{R}} \dots$ for some terms s_1, \dots, s_n, \dots that contradicts termination of t . \square

We define now another termination property, called *V-termination*, which expresses a *vertical* dimension of operational termination.

Definition 13 (V-termination). *Let \mathcal{R} be a CTRS. A term t is said to be V-terminating iff there is no infinite well-formed proof tree T such that $\text{left}(\text{root}(T)) = t$ and $\text{spine}(T)$ contains infinitely many goals G satisfying $\text{pred}(G) = \rightarrow$. We say that \mathcal{R} is V-terminating iff every term is V-terminating.*

We *characterize* operational termination of CTRSs as the conjunction of termination and V-termination. This formalizes the idea of termination and V-termination being *the two dimensions* of operational termination.

Theorem 14. *A CTRS \mathcal{R} is operationally terminating if and only if it is terminating and V-terminating.*

PROOF. If \mathcal{R} is operationally terminating, then there is no infinite well-formed proof tree. Then, \mathcal{R} is terminating (by Proposition 12) and V-terminating.

For the *if* part, we reason by contradiction. If \mathcal{R} is terminating and V-terminating but operationally nonterminating, then there is an infinite well-formed proof tree T . By Proposition 12, $\text{spine}(T)$ contains an infinite number of goals G such that $\text{pred}(G) = \rightarrow$. This contradicts V-termination of \mathcal{R} . \square

Let HT , VT , and OT denote, respectively, the properties of termination, V -termination, and operational termination. Theorem 14 states the equivalence

$$((\forall t) OT(t)) \Leftrightarrow ((\forall t) HT(t) \wedge VT(t))$$

However, the equivalence

$$(\forall t) (OT(t) \Leftrightarrow HT(t) \wedge VT(t))$$

does *not* follow from Theorem 14.⁴ In fact, the latter equivalence does not hold in general (although $(\forall t) (OT(t) \Rightarrow HT(t) \wedge VT(t))$ obviously holds).

Example 15. Consider the following CTRS \mathcal{R}

$$a \rightarrow b \Leftarrow c \rightarrow d \quad (21)$$

$$c \rightarrow c \quad (22)$$

Here, a is terminating and V -terminating but operationally nonterminating.

In order to obtain a characterization of operational termination of *terms*, we have to consider an additional termination property

Definition 16. Let \mathcal{R} be a CTRS and t be a term. We say that t is VH -terminating, written $VHT(t)$, iff there is no infinite well-formed proof tree T with $\text{left}(\text{root}(T)) = t$ and $\text{spine}(T) = (G_i)_{i \in \mathbb{N}}$ such that there is $n \in \mathbb{N}$ satisfying that for all $i \geq n$, $\text{pred}(G_i) = \rightarrow^*$.

Note that, if we let $n = 0$ in Definition 16, we obtain $HT(t)$ (according to Proposition 12). Thus, this property is *stronger* than termination, i.e., $(\forall t) VHT(t) \Rightarrow HT(t)$. For instance, a in Example 15 is not VH -terminating.

Proposition 17. Let \mathcal{R} be a CTRS and t be a term. Then,

$$OT(t) \Leftrightarrow VHT(t) \wedge VT(t)$$

PROOF. The *only if* part is obvious. For the *if* part, we reason by contradiction. If $OT(t)$ does not hold, then there is an infinite well-formed proof tree T with $\text{left}(\text{root}(T)) = t$. Let $\text{spine}(T) = (G_i)_{i \in \mathbb{N}}$. Since t is V -terminating, there is n such that, for all $i \geq n$, $\text{pred}(G_i) = \rightarrow^*$. This contradicts $VHT(t)$. \square

Proposition 17 tells us that we need the stronger property $VHT(t)$ (together with $VT(t)$) to characterize operational termination of a *term* t (i.e., $OT(t)$). However, the following result shows that there is no point in defining a new VH -termination property for CTRSs (namely $(\forall t) VHT(t)$) beyond termination:

Proposition 18. Let \mathcal{R} be a CTRS. Then, $((\forall t) VHT(t)) \Leftrightarrow ((\forall t) HT(t))$.

⁴This observation and the example are due to a reviewer of the paper.

PROOF. We have already noticed that for all terms t , $VHT(t) \Rightarrow HT(t)$ holds. Hence, the implication $((\forall t) VHT(t)) \Rightarrow ((\forall t) HT(t))$ is immediate. We prove that $((\forall t) HT(t)) \Rightarrow ((\forall t) VHT(t))$ by contradiction. If there is a term s such that $VHT(s)$ does not hold, then there is an infinite proof tree T such that $left(root(T)) = s$ and for $spine(T) = (G_i)_{i \in \mathbb{N}}$ there is $n \geq 0$ such that for all $i \geq n$, $pred(G_i) = \rightarrow^*$. Let T' be the infinite well-formed proof tree which is obtained as the subtree of T with root G_n . Note that $spine(T') = (G_{n+i})_{i \in \mathbb{N}}$ and for all $G \in spine(T')$, $pred(G) = \rightarrow^*$. Let $u = left(G_n)$. By Proposition 12, $HT(u)$ does *not* hold, contradicting that $((\forall t) HT(t))$ holds. \square

Now, Theorem 14 can be seen as a consequence of Propositions 17 and 18:

$$\begin{array}{lll}
\mathcal{R} \text{ is operationally terminating} & \Leftrightarrow & \\
(\forall t) OT(t) & \Leftrightarrow & \text{(by definition)} \\
(\forall t) (VHT(t) \wedge VT(t)) & \Leftrightarrow & \text{(by Proposition 17)} \\
((\forall t) VHT(t)) \wedge ((\forall t) VT(t)) & \Leftrightarrow & \\
((\forall t) HT(t)) \wedge ((\forall t) VT(t)) & \Leftrightarrow & \text{(by Proposition 18)} \\
\mathcal{R} \text{ is terminating and } V\text{-terminating} & \Leftrightarrow & \text{(by definition)}
\end{array}$$

Thus, in the following we focus on termination rather than VH -termination.

Notation 19. *In the following, we adopt a uniform notation to designate terms and CTRSs with different (terminating, V -terminating, and operationally terminating) properties by using labels H, V, O to speak of H -, V -, and O -terminating terms and CTRSs, respectively. By choosing H to designate terminating terms, we stress that termination is the horizontal dimension of operational termination. We also let $\Lambda = \{H, V, O\}$ to collect all these termination behaviors.*

3.1. Minimality

As remarked in the introduction, the dependency pair approach is the basis of most automatic tools for proving termination properties of (variants of) TRSs. Our *methodology* for the adaptation of the Dependency Pair approach for TRSs to other variants of rewriting or other extensions of rewrite systems makes explicit previous developments in [1, 17, 21, 22] and involves:⁵

1. The definition of a notion of *minimal non- λ -terminating term* (or just λ -minimal, for $\lambda \in \Lambda$) which can be used to investigate λ -termination by restricting the attention to a smaller subset of non- λ -terminating terms.
2. The investigation of the *structure* of rewrite sequences starting from such λ -minimal terms.
3. The definition of a notion of *dependency pair*, which is able to capture all infinite sequences starting from λ -minimal terms.
4. The definition of an abstract notion of *λ -chain*, which can be used to *characterize* λ -termination as the absence of infinite λ -chains.

⁵This methodology has been successfully applied to, for instance, *context-sensitive rewriting*, as in [2, 20], *Order-Sorted TRSs*, as in [27], and *$A \vee C$ -rewriting*, as in [4].

In this section we therefore prove some technical results which are used in the rest of the paper to prove our results about using dependency pairs to prove termination properties of CTRSs. These results are parametric on $\lambda \in \Lambda$.

For non- λ -terminating terms we introduce the following notion and then prove some essential facts that will be used later.

Definition 20 (Minimality). *Let \mathcal{R} be a CTRS, $\lambda \in \Lambda$ and t be a non- λ -terminating term. We say that t is λ -minimal if every strict subterm s of t is λ -terminating. Let $\mathcal{T}_{\lambda-\infty}$ be the set of minimal non- λ -terminating terms.*

The following lemma shows that non- λ -terminating terms always contain a λ -minimal term.

Lemma 21. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a CTRS, $\lambda \in \Lambda$ and $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If s is non- λ -terminating, then there is a subterm t of s ($s \triangleright t$) such that $t \in \mathcal{T}_{\lambda-\infty}$.*

PROOF. By structural induction. If s is a constant symbol, take $t = s$. If $s = f(s_1, \dots, s_k)$, we assume that $s \notin \mathcal{T}_{\lambda-\infty}$ (otherwise, again let $t = s$). Then, there is a strict subterm s' of s ($s \triangleright s'$) which is non- λ -terminating. By the induction hypothesis, there is $t \in \mathcal{T}_{\lambda-\infty}$ such that $s' \triangleright t$, i.e., $s \triangleright t$. \square

Corollary 22. *Let $\lambda \in \Lambda$. A CTRS \mathcal{R} is λ -terminating iff $\mathcal{T}_{\lambda-\infty} = \emptyset$.*

Subterms and reducts of λ -terminating terms are λ -terminating as well.

Lemma 23. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a CTRS, $\lambda \in \Lambda$, and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If s is λ -terminating, then:*

1. *If $s \triangleright t$, then t is λ -terminating.*
2. *If $s \rightarrow_{\mathcal{R}}^* t$, then t is λ -terminating.*

PROOF. If $\lambda = H$, then the proofs are analogous to the well-known proofs for terminating terms in the TRS setting [21, 22]. Assume $\lambda \in \{V, O\}$.

1. We reason by contradiction. Assume that $s = C[t]$ for some context $C[\]$. If t is non- λ -terminating, then there is an infinite well-formed proof tree T (of the appropriate shape, depending on λ) as in Figure 2, where $G_0 = \text{root}(T)$. We consider two cases:
 - (a) If $G_0 = t \rightarrow u$ for some term u , then $C[T]$ is an infinite well-formed proof tree with $\text{root}(C[T]) = s \rightarrow C[u]$. If $\lambda = O$, this already contradicts OT -termination of s . If $\lambda = V$, then by our assumptions on T , $\text{spine}(T)$ contains an infinite number of goals G_i satisfying $\text{pred}(G_i) = \rightarrow$. Hence $C[T]$ is an infinite well-formed proof tree and $\text{spine}(C[T])$ contains an infinite number of goals G_i satisfying $\text{pred}(G_i) = \rightarrow$. This contradicts V -termination of s .
 - (b) If $G_0 = t \rightarrow^* u$ for some term u , then the first inference rule that applies to G_0 in T is (Tran). We consider two cases:

- i. There are goals $G \in \text{spine}(T)$ such that $\text{pred}(G) = \rightarrow$. Then, there is $m > 0$ such that, for all i , $1 \leq i < m$ only the *transitivity rule* (Tran) has been applied, i.e., \vec{T}_i contains a single closed proof tree T_i with $\text{root}(T_i) = u_{i-1} \rightarrow u_i$ for some term u_i ; \vec{O}_i is empty and $G_i = u_i \rightarrow^* u$. Then, \vec{T}_m is empty, $G_m = u_{m-1} \rightarrow u_m$ is the root of an infinite well-formed subtree T_m^∞ of T , and \vec{O}_m consists of a single open goal $u_m \rightarrow^* u$. Thus, we can build the following infinite well-formed proof tree T'

$$\frac{\frac{\frac{C[T_m^\infty] \quad C[u_m] \rightarrow^* u}{\vdots}}{T'_1} \quad \frac{G'_1}{C[t] \rightarrow^* u}}{G'_1} \text{(Tran)}}{C[t] \rightarrow^* u} \text{(Tran)}$$

(where $T'_i = C[T_i]$ and $G'_i = C[u_i] \rightarrow u$ for all i , $1 \leq i < m$) which already contradicts λ -termination of s if $\lambda = O$. Furthermore, if $\lambda = V$, since T' still contains an infinite number of goals $G \in \text{spine}(T')$ satisfying $\text{pred}(G) = \rightarrow$, it also contradicts λ -termination of s .

- ii. For all $G \in \text{spine}(T)$ we have $\text{pred}(G) = \rightarrow^*$. Note that, in this case, we can only have $\lambda = O$. Then for all $i \geq 1$, \vec{T}_i consists of a closed proof tree T_i with $\text{root}(T_i) = u_{i-1} \rightarrow u_i$ for some term u_i ; \vec{O}_i is empty and $G_i = u_i \rightarrow^* u$. Thus, for all $i \geq 1$, we can replace each T_i by $T'_i = C[T_i]$. Thus, we obtain:

$$\frac{\frac{\frac{\frac{T'_n \quad \frac{G'_n}{\vdots} \text{(Tran)}}{G'_n} \text{(Tran)}}{\vdots} \text{(Tran)}}{T'_1} \quad \frac{G'_1}{C[t] \rightarrow^* u}}{G'_1} \text{(Tran)}}{C[t] \rightarrow^* u} \text{(Tran)}$$

where, for all $i \geq 1$, $G'_i = C[u_i] \rightarrow^* u$ which contradicts O -termination of s .

2. Since $s \rightarrow^* t$, there are terms t_0, \dots, t_n such that, for all i , $0 \leq i \leq n$, $t_i \rightarrow_{\mathcal{R}} t_{i+1}$ has a *closed* proof tree T_{i+1} and we let $t_0 = s$ and $t_n = t$. Thus, there is a finite well-formed proof tree $T = \text{Tr}(T_1, \dots, T_n)$ for $s \rightarrow_{\mathcal{R}}^* t$. Assume that t is not λ -terminating (for $\lambda \in \{V, O\}$). Then, there is an infinite well-formed proof tree T' of the appropriate shape (depending on λ). We consider two cases:

- (a) If $\text{root}(T') = t \rightarrow u$ for some term u , we let T'' be

$$\frac{T' \quad u \rightarrow^* u'}{t \rightarrow^* u'} \text{(Tran)}$$

for some term u' to obtain $\text{Tr}_{T''}(T_1, \dots, T_n)$, an infinite well-formed proof tree with root $s \rightarrow^* u'$, which contradicts λ -termination of s as T' is a subtree of the obtained tree.

- (b) If $\text{root}(T') = t \rightarrow^* u$ for some term u . Again, the infinite well-formed proof tree $\text{Tr}_{T'}(T_1, \dots, T_n)$ with root $s \rightarrow^* u$ contradicts λ -termination of s .

□

Non- λ -terminating inner reducts of λ -minimal terms are λ -minimal as well.

Lemma 24. *Let \mathcal{R} be a CTRS, $\lambda \in \Lambda$, and $t \in \mathcal{T}_{\lambda-\infty}$. If $t \xrightarrow{\epsilon}^* u$ and u is non- λ -terminating, then $u \in \mathcal{T}_{\lambda-\infty}$.*

PROOF. All rewritings below the root of t are issued on λ -terminating terms that remain λ -terminating by Lemma 23. Then, strict subterms of u are all λ -terminating. Since u is non- λ -terminating, $u \in \mathcal{T}_{\lambda-\infty}$. □

Given a signature \mathcal{F} and $f \in \mathcal{F}$, we let $f^\#$ (often just capitalized, e.g., F) be a fresh symbol associated to f . Let $\mathcal{F}^\# = \{f^\# \mid f \in \mathcal{F}\}$ (assume $\mathcal{F} \cap \mathcal{F}^\# = \emptyset$). For $t = f(t_1, \dots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write $t^\#$ to denote the *marked* term $f^\#(t_1, \dots, t_k)$. And given a marked term $t = f^\#(t_1, \dots, t_k)$, where $t_1, \dots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write t^\flat to denote the term $f(t_1, \dots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.

Proposition 25. *Let \mathcal{R} be a CTRS and $\lambda \in \{H, V, O\}$. If $t \in \mathcal{T}_{\lambda-\infty}$, then $t^\#$ is λ -terminating.*

PROOF. (Sketch) By contradiction. If $t^\#$ is not λ -terminating, then there is an infinite well-formed proof tree T with $\text{left}(\text{root}(T)) = t^\#$. Let $\text{spine}(T) = (G_i)_{i \in \mathbb{N}}$. We consider the three possible cases for λ :

- $\lambda = H$. By Proposition 12, we may assume that $\text{root}(T) = t^\# \rightarrow^* u$ for some term u and for all $i \in \mathbb{N}$, $G_i = s_i \rightarrow^* u$ for some terms s_i (with $t^\# = s_0$). Since no rule in \mathcal{R} can remove the mark from (the root of) $t^\#$, this means that there is a strict subterm t_i of $t^\#$ (and hence of t !) starting an infinite rewrite sequence. This contradicts minimality of t .
- $\lambda = V$. There are infinitely many $G \in \text{spine}(T)$ with $\text{pred}(G) = \rightarrow$. The first of such goals corresponds to the application of a rule $\alpha : \ell \rightarrow r \Leftarrow c$ with c nonempty to a reduct u of $t^\#$, i.e., $t^\# \rightarrow_{\mathcal{R}}^* u$ so that T becomes infinite because of attempts of evaluation for some of the conditions in c . Since the root symbol $f^\#$ of $t^\#$ is marked, we can assume that $t^\# = f^\#(t_1, \dots, t_k)$, $u = f^\#(u_1, \dots, u_k)$, and for all $i, 1 \leq i \leq k$, $t_i \rightarrow^* u_i$. Thus, the reduction with α is attempted on u_j for some $j, 1 \leq j \leq k$. Hence, t_j (a proper subterm of t) is non- V -terminating, contradicting minimality of t .
- If $\lambda = O$, then the only missing case in the previous two points is that $t^\#$ reduces to a term u (i.e., $t^\# \rightarrow^* u$) to which a rule $\alpha : \ell \rightarrow r \Leftarrow c$ with c nonempty is attempted but now one of the left-hand sides v of a condition $v \rightarrow w \in c$ starts an infinite rewriting sequence (after instantiation with the matching substitution). Thus, there is $n \in \mathbb{N}$ such that for all $i > n$, $\text{pred}(G_i) = \rightarrow^*$ but $\text{pred}(G_n) = \rightarrow$. This case, however, leads to a contradiction with t being O -minimal as in the case $\lambda = V$.

□

Notation 26. In the following, rather than talking of H -terminating or H -minimal terms, we will use terminating and minimal terms as these notions already exist in the literature. Similarly, rather than $\mathcal{T}_{H-\infty}$, we use the usual notation \mathcal{T}_∞ for TRSs (see [21], for instance) to denote the set of H -minimal terms (minimal nonterminating terms in [21]).

In general, the sets of minimal nonterminating, non- V -terminating and operationally nonterminating terms are *not* related by inclusion.

Example 27. Term \mathbf{a} in Example 15 is minimal operationally nonterminating, but it is terminating and V -terminating. Thus, $\mathcal{T}_{O-\infty} \not\subseteq \mathcal{T}_\infty \cup \mathcal{T}_{V-\infty}$. As for

$$\mathbf{f}(\mathbf{a}) \rightarrow \mathbf{f}(\mathbf{a}) \quad (23)$$

$$\mathbf{a} \rightarrow \mathbf{b} \Leftarrow \mathbf{a} \rightarrow \mathbf{b} \quad (24)$$

$\mathbf{f}(\mathbf{a})$ is not terminating and, since \mathbf{a} is terminating, $\mathbf{f}(\mathbf{a}) \in \mathcal{T}_\infty$. However, $\mathbf{f}(\mathbf{a}) \notin \mathcal{T}_{O-\infty}$ (because $\mathbf{a} \in \mathcal{T}_{O-\infty}$). Therefore, $\mathcal{T}_\infty \not\subseteq \mathcal{T}_{O-\infty}$. Actually, for this example, $\mathcal{T}_{O-\infty} = \{\mathbf{a}\}$ and $\mathcal{T}_\infty = \{\mathbf{f}(\mathbf{a})\}$, i.e., $\mathcal{T}_\infty \cap \mathcal{T}_{O-\infty} = \emptyset$. Now, with

$$\mathbf{a} \rightarrow \mathbf{a} \quad (25)$$

$$\mathbf{f}(x) \rightarrow \mathbf{b} \Leftarrow \mathbf{f}(x) \rightarrow \mathbf{b} \quad (26)$$

we have $\mathbf{f}(\mathbf{a}) \in \mathcal{T}_{V-\infty}$, but $\mathbf{f}(\mathbf{a}) \notin \mathcal{T}_{O-\infty}$ (and $\mathbf{f}(\mathbf{a}) \notin \mathcal{T}_\infty$) because \mathbf{a} is nonterminating. Therefore, $\mathcal{T}_{V-\infty} \not\subseteq \mathcal{T}_{O-\infty}$.

In the following we investigate practical approaches to prove termination, V -termination and operational termination of CTRSs using dependency pairs. Since many definitions and results apply to arbitrary CTRSs, we do not impose any specific restriction on CTRSs when formulating them. Instead, when a specific restriction is required for some result (typically type 3 and determinism) we will make it explicit in its statement.

4. Dependency pairs for proving termination of CTRSs

In this section we consider the analysis of *termination* of CTRSs following the *methodology* summarized in the first paragraphs of Section 3.1.

4.1. Infinite rewrite sequences starting from minimal terms

Our first result in this section is Proposition 29. It establishes that every infinite rewrite sequence starting from a minimal term $t = f(t_1, \dots, t_k)$ first rewrites t_1, \dots, t_k to obtain $t' = f(t'_1, \dots, t'_k)$. Then, a rule $\alpha : f(\ell_1, \dots, \ell_k) \rightarrow r \Leftarrow c$ applies at the root of t' . Proposition 29 below also tells us that, either:

1. the right-hand side r of α depends on other rules in \mathcal{R} , i.e., there are defined symbols in r , or

2. α is *not* a 2-rule, i.e., r contains an *extra variable* $x \in \mathcal{V}ar(r) - \mathcal{V}ar(\ell)$.

Definition 28. Let \mathcal{R} be a CTRS. The set of defined subterms of a term t is:

$$\mathcal{DSubterm}(\mathcal{R}, t) = \{t|_p \mid p \in \mathcal{Pos}_{\mathcal{D}}(t)\}.$$

Proposition 29. Let \mathcal{R} be a CTRS and $t \in \mathcal{T}_{\infty}$. There exist $\alpha : \ell \rightarrow r \Leftarrow c \in \mathcal{R}$, a substitution σ , and $u \in \mathcal{T}_{\infty}$ such that $t \xrightarrow{\epsilon}^* \sigma(\ell) \xrightarrow{\epsilon} \sigma(r) \supseteq u$, and either

1. there is $s \in \mathcal{DSubterm}(\mathcal{R}, r)$, $\ell \not\triangleright s$, such that $u = \sigma(s)$, or
2. there is $x \in \mathcal{V}ar(r) - \mathcal{V}ar(\ell)$ such that $\sigma(x) \supseteq u$.

PROOF. Consider an infinite rewrite sequence starting from t . By definition of \mathcal{T}_{∞} , all proper subterms of t are terminating. Therefore, t has an inner reduction to an instance $\sigma(\ell)$ of the left-hand side of a rule $\alpha : \ell \rightarrow r \Leftarrow s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$ of \mathcal{R} : $t \xrightarrow{\epsilon}^* \sigma(\ell) \xrightarrow{\epsilon} \sigma(r)$ with $\sigma(s_i) \rightarrow^* \sigma(t_i)$ for all i , $1 \leq i \leq n$, and $\sigma(r)$ is nonterminating. Thus, we can write $t = f(t_1, \dots, t_k)$ and $\ell = f(\ell_1, \dots, \ell_k)$ for some k -ary defined symbol f , and $t_i \rightarrow^* \sigma(\ell_i)$ for all i , $1 \leq i \leq k$. Since all t_i are terminating, by Lemma 23 (with $\lambda = H$), $\sigma(\ell_i)$ and all its subterms are also terminating. Hence, $\sigma(\ell)$ is also minimal: $\sigma(\ell) \in \mathcal{T}_{\infty}$. By Lemma 23, $\sigma(y)$ is terminating for all $y \in \mathcal{V}ar(\ell)$. Since $\sigma(r)$ is nonterminating, by Lemma 21, it contains a subterm $u \in \mathcal{T}_{\infty}$: $\sigma(r) \supseteq u$, i.e., there is a position $p \in \mathcal{Pos}(\sigma(r))$ such that $\sigma(r)|_p = u$. We have two cases:

1. If $p \in \mathcal{Pos}_{\mathcal{F}}(r)$, then, since $u \in \mathcal{T}_{\infty}$, $root(u) \in \mathcal{D}$ and there is a defined subterm s of r , $s \in \mathcal{DSubterm}(\mathcal{R}, r)$, such that $u = \sigma(s)$. Note that s cannot be a strict subterm of ℓ , i.e., $\ell \not\triangleright s$. Otherwise, $u = \sigma(s)$ would be a subterm of $\sigma(\ell)$, thus failing to be minimal.
2. If $p \notin \mathcal{Pos}_{\mathcal{F}}(r)$, then there is $x \in \mathcal{V}ar(r)$ such that $\sigma(x) \supseteq u$. Since $\sigma(y)$ is terminating for all $y \in \mathcal{V}ar(\ell)$, it follows that $x \in \mathcal{V}ar(r) - \mathcal{V}ar(\ell)$.

□

The proof of case 1 in Proposition 29 is analogous to [21, Lemma 1] (for TRSs). Case 2 is specifically due to the use of conditional rules (with extra variables in the right-hand side). The following definition collects the rules that are used in the first and second cases of Proposition 29, respectively.

Definition 30. Let \mathcal{R} be a CTRS. Let $\mathcal{DRules}(\mathcal{R})$ be the set of rules in \mathcal{R} which depend on other defined symbols in \mathcal{R} :

$$\mathcal{DRules}(\mathcal{R}) = \{\ell \rightarrow r \Leftarrow c \in \mathcal{R} \mid r \notin \mathcal{T}(\mathcal{C}, \mathcal{X})\}$$

The set of rules in \mathcal{R} whose right-hand sides contain extra variables is:

$$\mathcal{ERules}(\mathcal{R}) = \{\ell \rightarrow r \Leftarrow c \in \mathcal{R} \mid \mathcal{V}ar(r) - \mathcal{V}ar(\ell) \neq \emptyset\}$$

Note that, if \mathcal{R} is a 2-CTRS, then $\mathcal{ERules}(\mathcal{R}) = \emptyset$.

Example 31. For \mathcal{R} in Example 2, $\mathcal{D} = \{\mathbf{b}, \mathbf{f}, \mathbf{g}\}$, $\mathcal{C} = \{\mathbf{a}, \mathbf{c}\}$, and $DRules(\mathcal{R}) = \{(11), (12)\}$, i.e.,

$$\begin{array}{l} \mathbf{g}(\mathbf{a}) \rightarrow \mathbf{c}(\mathbf{b}) \\ \mathbf{b} \rightarrow \mathbf{f}(\mathbf{a}) \end{array}$$

However, (13) $\notin DRules(\mathcal{R})$ because (13) is

$$\mathbf{f}(x) \rightarrow x \Leftarrow \mathbf{g}(x) \rightarrow \mathbf{c}(y)$$

with right-hand side $x \in \mathcal{T}(\mathcal{C}, \mathcal{X})$. Since \mathcal{R} is a 2-CTRS, $ERules(\mathcal{R}) = \emptyset$.

For \mathcal{R} in Example 3, $\mathcal{D} = \{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$, $\mathcal{C} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{k}\}$, $DRules(\mathcal{R}) = \{(16)\}$ and $ERules(\mathcal{R}) = \{(17)\}$.

As a corollary of Proposition 29, we have the following result.

Theorem 32. Let \mathcal{R} be a CTRS. For all $t \in \mathcal{T}_\infty$, there is an infinite sequence

$$t = t_0 \xrightarrow{>\epsilon^*} \sigma_1(l_1) \xrightarrow{\epsilon} \sigma_1(r_1) \supseteq t_1 \xrightarrow{>\epsilon^*} \sigma_2(l_2) \xrightarrow{\epsilon} \sigma_2(r_2) \supseteq t_2 \xrightarrow{>\epsilon^*} \dots$$

where, for all $i \geq 1$, $(\alpha_i : l_i \rightarrow r_i \Leftarrow c_i) \in \mathcal{R}$, σ_i are substitutions such that $\sigma_i(x)$ is terminating for all $x \in \text{Var}(l_i)$, and terms $t_i \in \mathcal{T}_\infty$ such that either:

1. $\alpha_i \in DRules(\mathcal{R})$ and there is a defined subterm s_i of r_i , $r_i \supseteq s_i$, $l_i \not\triangleright s_i$, such that $t_i = \sigma_i(s_i)$, or
2. $\alpha_i \in ERules(\mathcal{R})$ and there is $x_i \in \text{Var}(r_i) - \text{Var}(l_i)$ such that $\sigma_i(x_i) \supseteq t_i$.

Remark 33. Item 2 in Proposition 29 concerns proper 3- and 4-rules only. According to the proof of Proposition 29, this is because instances $\sigma(x)$ of variables x that occur in the left-hand side ℓ of the rule α in Proposition 29 are terminating. This cannot be ensured for variables $x \in \text{Var}(r) - \text{Var}(\ell)$ and this is the reason for including item 2 in Proposition 29. In the following section we investigate conditions to avoid this.

4.2. Preservation of terminating substitutions

In the following, given a CTRS \mathcal{R} , a substitution σ , and $V \subseteq \mathcal{X}$, we say that σ is *terminating over V* if $\sigma(x)$ is terminating for all $x \in V$.

Definition 34 (Preserving terminating substitutions). Let \mathcal{R} be a CTRS and $\alpha : \ell \rightarrow r \Leftarrow c \in \mathcal{R}$. Let σ be a substitution terminating over $\text{Var}(\ell)$. We say that α *preserves termination of σ* iff σ is terminating over $\text{Var}(r)$ whenever $\sigma(s) \rightarrow_{\mathcal{R}}^* \sigma(t)$ for all $s \rightarrow t \in c$. We say that \mathcal{R} *preserves terminating substitutions* if for all substitutions σ and $\alpha : \ell \rightarrow r \Leftarrow c \in \mathcal{R}$, if σ is terminating over $\text{Var}(\ell)$, then α preserves termination of σ .

Rules $\ell \rightarrow r \Leftarrow c$ that preserve terminating substitutions ensure that, if the conditions $s \rightarrow t \in c$ are all satisfied by a substitution σ whose bindings $\sigma(x)$ are terminating for all $x \in \text{Var}(\ell)$, then any other binding $\sigma(y)$ for a (possibly new) variable $y \in \text{Var}(r)$ is terminating as well. Considering Definition 34, the following refinements of Proposition 29 and Theorem 32 are obvious:

Proposition 35. *Let \mathcal{R} be a CTRS which preserves terminating substitutions. Then, for all $t \in \mathcal{T}_\infty$, there exist $\ell \rightarrow r \leftarrow c \in DRules(\mathcal{R}, t)$, a substitution σ , and a term $u \in \mathcal{T}_\infty$ such that $t \xrightarrow{>\epsilon^*} \sigma(\ell) \xrightarrow{\epsilon} \sigma(r) \supseteq u$, and there is $s \in \mathcal{DSubterm}(\mathcal{R}, r)$, $\ell \not\triangleright s$, such that $u = \sigma(s)$.*

PROOF. By Proposition 29, infinite rewrite sequences starting from minimal nonterminating terms can proceed in two precise ways. Either:

1. there is $s \in \mathcal{DSubterm}(\mathcal{R}, r)$, $\ell \not\triangleright s$, such that $u = \sigma(s)$, or
2. there is $x \in \mathcal{Var}(r) - \mathcal{Var}(\ell)$ such that $\sigma(x) \supseteq u$.

Condition 2, though, does not apply as \mathcal{R} preserves terminating substitutions (i.e., $\sigma(x)$ is terminating) and u is nonterminating. \square

Corollary 36. *Let \mathcal{R} be a CTRS which preserves terminating substitutions. For all $t \in \mathcal{T}_\infty$, there is an infinite sequence*

$$t = t_0 \xrightarrow{>\epsilon^*} \sigma_1(l_1) \xrightarrow{\epsilon} \sigma_1(r_1) \supseteq t_1 \xrightarrow{>\epsilon^*} \sigma_2(l_2) \xrightarrow{\epsilon} \sigma_2(r_2) \supseteq t_2 \xrightarrow{>\epsilon^*} \dots$$

where, for all $i \geq 1$, $(l_i \rightarrow r_i \leftarrow c_i) \in DRules(\mathcal{R})$, σ_i is a substitution such that $\sigma_i(x)$ is terminating for all $x \in \mathcal{Var}(l_i)$, and $t_i \in \mathcal{T}_\infty$ is such that $t_i = \sigma_i(s_i)$ for some $s_i \in \mathcal{DSubterm}(\mathcal{R}, r_i)$, $l_i \not\triangleright s_i$.

The following fact easily follows from Definition 34.

Proposition 37. *Every 2-CTRS preserves terminating substitutions.*

Theorem 32 and Corollary 36 are the basis for our next results, which provide several new methods for proving termination of CTRSs.

4.3. Dependency pairs for termination of CTRSs

In the following, given a CTRS \mathcal{R} , we obtain a new CTRS \mathcal{R}' . Each rule in \mathcal{R}' is obtained from a rule $\ell \rightarrow r \leftarrow c \in \mathcal{R}$ by marking the left-hand sides ℓ and also using subterms v in the right-hand side r . The obtained rules are of the form $\ell^\# \rightarrow v' \leftarrow c$ where $v' = v$ if $v \in \mathcal{X}$ and $v' = v^\#$ otherwise. For historical reasons, we usually call them *pairs*.⁶ In our definitions we assume that the signatures \mathcal{F}' of such new CTRSs \mathcal{R}' are obtained by just extracting all function symbols (with the corresponding arity) from the obtained rules⁷ $\ell^\# \rightarrow v' \leftarrow c$. The following set of *horizontal* dependency pairs corresponds to rules issuing root steps in infinite minimal sequences (see Theorem 32, item 1):

⁶The first presentation of the DP approach for TRSs [1] clearly distinguished between rewrite rules and dependency pairs (even using $\langle u, v \rangle$ rather than $u \rightarrow v$ for the latter), although syntactically they are isomorphic structures. This difference became less relevant in other presentations like [21] and [16, 17]. The original terminology, however, is useful as it stresses the different *role* of pairs and rewrite rules (from \mathcal{R}) in the DP approach.

⁷See [40] for an interesting discussion about the formal use of this assumption in termination analysis in the TRS setting. See also [36, Section 8.7] and the references therein for related considerations for CTRSs.

Definition 38. Let \mathcal{R} be a CTRS. The CTRS $DP_H(\mathcal{R})$ of horizontal dependency pairs is:

$$DP_H(\mathcal{R}) = \{\ell^\sharp \rightarrow v^\sharp \Leftarrow c \mid \ell \rightarrow r \Leftarrow c \in DRules(\mathcal{R}), v \in DSubterm(\mathcal{R}, r), \ell \not\prec v\}$$

Example 39. For \mathcal{R} in Example 1, $DP_H(\mathcal{R})$ consists of the rules:

$$LEQ(s(x), s(y)) \rightarrow LEQ(x, y) \quad (27)$$

$$APP(\text{cons}(x, xs), ys) \rightarrow APP(xs, ys) \quad (28)$$

$$\begin{aligned} QSORT(\text{cons}(x, xs)) &\rightarrow APP(\text{qsort}(ys), \text{cons}(x, \text{qsort}(zs))) & (29) \\ &\Leftarrow \text{split}(x, xs) \rightarrow \text{pair}(ys, zs) \end{aligned}$$

$$QSORT(\text{cons}(x, xs)) \rightarrow QSORT(ys) \Leftarrow \text{split}(x, xs) \rightarrow \text{pair}(ys, zs) \quad (30)$$

$$QSORT(\text{cons}(x, xs)) \rightarrow QSORT(zs) \Leftarrow \text{split}(x, xs) \rightarrow \text{pair}(ys, zs) \quad (31)$$

Example 40. For \mathcal{R} in Example 2, $DP_H(\mathcal{R})$ is:

$$G(a) \rightarrow B \quad (32)$$

$$B \rightarrow F(a) \quad (33)$$

For \mathcal{R} in Example 3, $DP_H(\mathcal{R}) = \{F(k(a), k(b), x) \rightarrow F(x, x, x)\}$.

The following set of *collapsing* dependency pairs corresponds to rules extracting minimal terms from *substitution bindings* in infinite minimal sequences (see Theorem 32, item 2):

Definition 41. Let \mathcal{R} be a CTRS. The CTRS $DP_C(\mathcal{R})$ of collapsing dependency pairs is:

$$DP_C(\mathcal{R}) = \{\ell^\sharp \rightarrow x \Leftarrow c \mid \ell \rightarrow r \Leftarrow c \in ERules(\mathcal{R}), x \in \text{Var}(r) - \text{Var}(\ell)\}$$

Example 42. For \mathcal{R} in Example 1, $DP_C(\mathcal{R})$ consists of the rules:

$$\begin{aligned} \text{SPLIT}(x, \text{cons}(y, ys)) &\rightarrow xs & (34) \\ &\Leftarrow \text{less}(x, y) \rightarrow \text{true}, \text{split}(x, ys) \rightarrow \text{pair}(xs, zs) \end{aligned}$$

$$\begin{aligned} \text{SPLIT}(x, \text{cons}(y, ys)) &\rightarrow zs & (35) \\ &\Leftarrow \text{less}(x, y) \rightarrow \text{true}, \text{split}(x, ys) \rightarrow \text{pair}(xs, zs) \end{aligned}$$

$$\begin{aligned} \text{SPLIT}(x, \text{cons}(y, ys)) &\rightarrow xs & (36) \\ &\Leftarrow \text{less}(x, y) \rightarrow \text{false}, \text{split}(x, ys) \rightarrow \text{pair}(xs, zs) \end{aligned}$$

$$\begin{aligned} \text{SPLIT}(x, \text{cons}(y, ys)) &\rightarrow zs & (37) \\ &\Leftarrow \text{less}(x, y) \rightarrow \text{false}, \text{split}(x, ys) \rightarrow \text{pair}(xs, zs) \end{aligned}$$

$$QSORT(\text{cons}(x, xs)) \rightarrow ys \Leftarrow \text{split}(x, xs) \rightarrow \text{pair}(ys, zs) \quad (38)$$

$$QSORT(\text{cons}(x, xs)) \rightarrow zs \Leftarrow \text{split}(x, xs) \rightarrow \text{pair}(ys, zs) \quad (39)$$

For \mathcal{R} in Example 2, $DP_C(\mathcal{R}) = \emptyset$. For \mathcal{R} in Example 3,

$$DP_C(\mathcal{R}) = \{G(x) \rightarrow y \Leftarrow h(x) \rightarrow d, h(x) \rightarrow c(y)\}.$$

Definition 43. In the following, $DP_{HC}(\mathcal{R})$ is:

$$DP_{HC}(\mathcal{R}) = DP_H(\mathcal{R}) \cup DP_C(\mathcal{R}).$$

4.4. Characterization of termination of CTRSs

An essential achievement of the DP approach is the *characterization* of termination of a TRS \mathcal{R} as the absence of infinite *chains of dependency pairs* in $\text{DP}(\mathcal{R})$, the set of dependency pairs associated to \mathcal{R} [1]. In this TRS setting, given a subset $\mathcal{P} \subseteq \text{DP}(\mathcal{R})$ of dependency pairs, a $(\mathcal{P}, \mathcal{R})$ -chain is a (possibly infinite) sequence $u_1 \rightarrow v_1, u_2 \rightarrow v_2, \dots$ of (renamed versions of) rules $u_i \rightarrow v_i \in \mathcal{P}$ together with a substitution σ such that $\sigma(v_i) \rightarrow_{\mathcal{R}}^* \sigma(u_{i+1})$ for all $i \geq 1$. Note the different role of rules in \mathcal{P} and \mathcal{R} : rules in \mathcal{P} (called *pairs*) are the *components* of the chain; rules in \mathcal{R} are used to *connect* consecutive pairs by rewriting.

In the following we generalize this approach to prove termination of CTRSs. First, we introduce a suitable notion of *H-chain* that can be used with our dependency pairs in Section 4.3 to model the infinite sequences of minimal terms whose structure is described by Theorem 32 above. Our H-chains are sequences of renamed variants of rules $\mathcal{P} \subseteq \text{DP}_{HC}(\mathcal{R})$, which we also call *pairs*. Such pairs are *connected* by means of rewriting steps issued not only with \mathcal{R} , but also using a TRS \mathcal{Q} which is used to *extract* and *mark* (minimal) subterms (Theorem 32).

Remark 44. *In the remainder of the paper, unless stated otherwise, \mathcal{P} , \mathcal{Q} , and \mathcal{R} are CTRSs having possibly different signatures.*

In the following definition, given a TRS \mathcal{Q} , we write $\mathcal{Q}_{\triangleright} = \{\ell \rightarrow r \mid \ell \rightarrow r \in \mathcal{Q}, \ell \triangleright r\}$ and $\mathcal{Q}_{\sharp} = \mathcal{Q} - \mathcal{Q}_{\triangleright}$.

Definition 45. *Let \mathcal{P} and \mathcal{R} be CTRSs and \mathcal{Q} be a TRS. A $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -H-chain is a finite or infinite sequence of (renamed) rules $u_i \rightarrow v_i \leftarrow c_i \in \mathcal{P}$, which are viewed as “conditional dependency pairs”, together with a substitution σ satisfying that, for all $i \geq 1$ and all $s \rightarrow t \in c_i$, $\sigma(s) \rightarrow_{\mathcal{R}}^* \sigma(t)$. Also, for all $i \geq 1$, there is a term w_i such that:*

1. *If $v_i \notin \mathcal{X} - \text{Var}(u_i)$, then $\sigma(v_i) = w_i \rightarrow_{\mathcal{R}}^* \sigma(u_{i+1})$.*
2. *If $v_i \in \mathcal{X} - \text{Var}(u_i)$, then $\sigma(v_i) \xrightarrow{\epsilon}_{\mathcal{Q}_{\triangleright}}^* \circ \xrightarrow{\epsilon}_{\mathcal{Q}_{\sharp}} w_i \rightarrow_{\mathcal{R}}^* \sigma(u_{i+1})$.*

A $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -H-chain is called minimal if for all $i \geq 1$, w_i is \mathcal{R} -terminating.

As usual, in Definition 45 we assume that for all $i, j \geq 1$, $i \neq j$, $\text{Var}(u_i \rightarrow v_i \leftarrow c_i) \cap \text{Var}(u_j \rightarrow v_j \leftarrow c_j) = \emptyset$ (renaming substitutions are used if necessary).

In the following, given a signature \mathcal{F} , we write

$$\begin{aligned} \mathcal{E}mb(\mathcal{F}) &= \{f(x_1, \dots, x_k) \rightarrow x_i \mid f \in \mathcal{F}, k = ar(f), 1 \leq i \leq k\} \\ \mathcal{M}^{\sharp}(\mathcal{F}) &= \{f(x_1, \dots, x_k) \rightarrow f^{\sharp}(x_1, \dots, x_k) \mid f \in \mathcal{F}, k = ar(f)\} \end{aligned}$$

where x_1, \dots, x_k are fresh variables. Here, $\mathcal{E}mb(\mathcal{F})$ is used to *extract* from $\sigma(x)$ (for a *collapsing* rule $\ell \rightarrow x \leftarrow c \in \mathcal{P}$) any minimal nonterminating subterm u which may be inside. Clearly, if $s \triangleright t$, then $s \xrightarrow{\epsilon}_{\mathcal{E}mb(\mathcal{F})}^* t$. And we then use $\mathcal{M}^{\sharp}(\mathcal{F})$ to *mark* the root (defined) symbol f of u (i.e., $f = \text{root}(u)$) so that u^{\sharp} can be eventually rewritten into the left-hand side u of a pair in a chain.

Definition 46. For signatures \mathcal{F} and \mathcal{F}' , we let

$$\mathcal{EM}^\sharp(\mathcal{F}, \mathcal{F}') = \mathcal{Emb}(\mathcal{F}) \cup \mathcal{M}^\sharp(\mathcal{F}').$$

We now provide a new characterization of termination of CTRSs.

Theorem 47 (Termination). Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a CTRS.

1. If there is no infinite minimal $(\text{DP}_{HC}(\mathcal{R}), \mathcal{EM}^\sharp(\mathcal{F}, \mathcal{D}), \mathcal{R})$ -H-chain, then \mathcal{R} is terminating.
2. If there is an infinite $(\text{DP}_{HC}(\mathcal{R}), \mathcal{EM}^\sharp(\mathcal{F}, \mathcal{D}), \mathcal{R})$ -H-chain, then \mathcal{R} is not terminating.

PROOF.

1. By contradiction. If \mathcal{R} is nonterminating, then there is a minimal term t starting an infinite minimal sequence. By Theorem 32, this sequence has the following form:

$$t = t_0 \xrightarrow{\geq \epsilon}^* \sigma_1(\ell_1) \xrightarrow{\epsilon} \sigma_1(r_1) \triangleright t_1 \xrightarrow{\geq \epsilon}^* \sigma_2(\ell_2) \xrightarrow{\epsilon} \sigma_2(r_2) \triangleright t_2 \xrightarrow{\geq \epsilon}^* \dots$$

where, for all $i \geq 1$, $(\alpha_i : \ell_i \rightarrow r_i \leftarrow c_i) \in \mathcal{R}$, σ_i are substitutions such that $\sigma_i(x)$ is terminating for all $x \in \mathcal{V}ar(\ell_i)$, and terms $t_i \in \mathcal{T}_\infty$ are minimal nonterminating terms such that either

- (a) $\alpha_i \in \text{DRules}(\mathcal{R})$ and there is a defined subterm s_i of r_i , $r_i \triangleright s_i$, $\ell_i \not\triangleright s_i$, such that $t_i = \sigma_i(s_i)$. In this case, $\ell_i^\sharp \rightarrow s_i^\sharp \leftarrow c_i \in \text{DP}_H(\mathcal{R})$ and $\sigma(s_i^\sharp) = \sigma(s_i)^\sharp = t_i^\sharp \xrightarrow{*}_{\mathcal{R}} \sigma(\ell_{i+1}^\sharp)$.
- (b) $\alpha_i \in \text{ERules}(\mathcal{R})$ and there is $x_i \in \mathcal{V}ar(r_i) - \mathcal{V}ar(\ell_i)$ such that $\sigma_i(x_i) \triangleright t_i$. In this case, $\ell_i^\sharp \rightarrow x_i \leftarrow c_i \in \text{DP}_C(\mathcal{R})$ and $\sigma(x_i) \xrightarrow{\epsilon}_{\mathcal{EM}(\mathcal{F})}^* t_i$. Since t_i is minimal, $\text{root}(t_i) \in \mathcal{D}$ and we have $t_i \xrightarrow{\epsilon}_{\mathcal{M}(\mathcal{D})}^* t_i^\sharp \xrightarrow{*}_{\mathcal{R}} \sigma(\ell_{i+1}^\sharp)$.

Note that, in both cases, by Proposition 25, t_i^\sharp is \mathcal{R} -terminating. Thus, we obtain an infinite minimal $(\text{DP}_{HC}(\mathcal{R}), \mathcal{EM}^\sharp(\mathcal{F}, \mathcal{D}), \mathcal{R})$ -H-chain.

2. Let $(u_i \rightarrow v_i \leftarrow c_i)_{i \geq 1}$ be an infinite $(\text{DP}_{HC}(\mathcal{R}), \mathcal{EM}^\sharp(\mathcal{F}, \mathcal{D}), \mathcal{R})$ -H-chain. If we remove all *marks* from any term in this chain and restore the contexts which were removed from the rules of \mathcal{R} to obtain the pairs in $\text{DP}_{HC}(\mathcal{R})$, we obtain an infinite rewrite sequence witnessing nontermination of \mathcal{R} .

□

Example 48. Consider the CTRS \mathcal{R} in Example 2. With $\text{DP}_H(\mathcal{R})$ as in Example 40 and $\text{DP}_C(\mathcal{R}) = \emptyset$ (Example 42), the longest $(\text{DP}_{HC}(\mathcal{R}), \mathcal{EM}^\sharp(\mathcal{F}, \mathcal{D}), \mathcal{R})$ -H-chain consists of (32) followed by (33). Thus, \mathcal{R} is terminating.

Example 49. Consider the following 3-CTRS \mathcal{R} , which is obtained from the 2-CTRS in Example 2 by a small change in rule (13) to yield (42):

$$\mathbf{g(a)} \rightarrow \mathbf{c(b)} \tag{40}$$

$$\mathbf{b} \rightarrow \mathbf{f(a)} \tag{41}$$

$$\mathbf{f(x)} \rightarrow \mathbf{y \leftarrow g(x) \rightarrow c(y)} \tag{42}$$

Note that $DRules(\mathcal{R}) = \{(40), (41)\}$. Again, $DP_H(\mathcal{R})$ is as in Example 40, but $DP_C(\mathcal{R})$ consists now of a single rule

$$F(x) \rightarrow y \Leftarrow g(x) \rightarrow c(y) \quad (43)$$

With $\mathcal{E}mb(\mathcal{F})$ given by

$$c(x) \rightarrow x \quad (44)$$

$$f(x) \rightarrow x \quad (45)$$

$$g(x) \rightarrow x \quad (46)$$

and $\mathcal{M}^\sharp(\mathcal{D})$ as follows:

$$b \rightarrow B \quad (47)$$

$$f(x) \rightarrow F(x) \quad (48)$$

$$g(x) \rightarrow g(x) \quad (49)$$

there is an infinite $(DP_{HC}(\mathcal{R}), \mathcal{E}M^\sharp(\mathcal{F}, \mathcal{D}), \mathcal{R})$ -H-chain

$$B \rightarrow_{(33)} F(a) \rightarrow_{(43)} b \rightarrow_{(47)} B \rightarrow_{(33)} F(a) \rightarrow_{(43)} b \rightarrow_{(47)} \dots$$

witnessing that \mathcal{R} is not terminating.

The following *simpler* characterization of termination for CTRSs that preserve terminating substitutions is obtained from the proof of Theorem 47 by using Corollary 36 instead of Theorem 32.

Corollary 50. *A CTRS \mathcal{R} which preserves terminating substitutions is terminating if and only if there is no infinite minimal $(DP_H(\mathcal{R}), \emptyset, \mathcal{R})$ -H-chain.*

Corollary 51 (Termination of 2-CTRSs). *A 2-CTRS \mathcal{R} is terminating if and only if there is no infinite minimal $(DP_H(\mathcal{R}), \emptyset, \mathcal{R})$ -H-chain.*

4.5. Termination of which (type of) CTRSs?

Term Rewriting Systems having rules $\ell \rightarrow r$ with *extra variables* (i.e., variables $y \in \mathcal{V}ar(r)$ such that $y \notin \mathcal{V}ar(\ell)$) are *not* terminating: since $r = r[y]$, we can define a *matching* substitution σ to obtain an infinite rewrite sequence:

$$\ell \rightarrow r[\ell] \rightarrow r[r[\ell]] \rightarrow \dots$$

as follows: $\sigma(x) = x$ for all $x \in \mathcal{X} - \{y\}$ and $\sigma(y) = \ell$.

The notion of *type* of a CTRS (see Section 2.3) provides a more precise characterization of where *extra variables* are allowed. In this setting, *proper* 4-CTRSs are intrinsically *nonterminating* if the proper 4-rule $\ell \rightarrow r \Leftarrow c$ is not unsatisfiable. Indeed, if there is a substitution σ such that $\sigma(s) \rightarrow_{\mathcal{R}}^* \sigma(t)$ holds for all $s \rightarrow t \in c$ but α is a *proper* 4-rule, then there is a variable $y \in \mathcal{V}ar(r)$ such

that $y \notin \mathcal{V}ar(\ell) \cup \mathcal{V}ar(c)$ and $r = r[y]$. Thus, we just need to write $\sigma(y) = \sigma(\ell)$ to obtain an infinite rewrite sequence:

$$\sigma(\ell) \rightarrow \sigma(r[y]) = \sigma(r)[\sigma(\ell)] \rightarrow \sigma(r)[\sigma(r)[\sigma(\ell)]] \rightarrow \dots$$

Our treatment of termination applies to CTRS of *arbitrary* type. However, the previous observation shows that 3-CTRSs are a more realistic target when proofs of termination are attempted. The following example shows, however, that *determinism* (which is an important mechanism to control the role of extra variables in CTRSs) does *not* play an essential role to achieve termination.

Example 52. Consider the following nondeterministic 3-CTRS \mathcal{R} :

$$\mathbf{a} \rightarrow x \Leftarrow x \rightarrow \mathbf{b} \quad (50)$$

Since $\text{DP}_H(\mathcal{R}) = \emptyset$, $\text{DP}_C(\mathcal{R}) = \{\mathbf{A} \rightarrow x \Leftarrow x \rightarrow \mathbf{b}\}$, $\mathcal{E}mb(\mathcal{F}) = \emptyset$ and $\mathcal{M}^\sharp(\mathcal{D}) = \{\mathbf{a} \rightarrow \mathbf{A}\}$, there is no infinite $(\text{DP}_{HC}(\mathcal{R}), \mathcal{E}M^\sharp(\mathcal{F}, \mathcal{D}), \mathcal{R})$ -H-chain and \mathcal{R} is terminating. However, for the nondeterministic 3-CTRS \mathcal{R}' :

$$\mathbf{a} \rightarrow x \Leftarrow x \rightarrow \mathbf{a} \quad (51)$$

we have $\text{DP}_H(\mathcal{R}') = \emptyset$, $\text{DP}_C(\mathcal{R}') = \{\mathbf{A} \rightarrow x \Leftarrow x \rightarrow \mathbf{a}\}$, $\mathcal{E}mb(\mathcal{F}) = \emptyset$ and $\mathcal{M}^\sharp(\mathcal{D}) = \{\mathbf{a} \rightarrow \mathbf{A}\}$. There is now an infinite $(\text{DP}_{HC}(\mathcal{R}'), \mathcal{E}M^\sharp(\mathcal{F}, \mathcal{D}), \mathcal{R}')$ -H-chain with substitution $\sigma(x) = \mathbf{a}$:

$$\mathbf{A} \rightarrow \mathbf{a} \rightarrow \mathbf{A} \rightarrow \mathbf{a} \rightarrow \dots$$

Therefore, \mathcal{R}' is nonterminating.

In the following, we will investigate *operational termination* of CTRSs. Since nontermination implies operational nontermination, we cannot expect that 4-CTRSs (without useless rules) are operationally terminating. What about *nondeterministic 3-CTRSs*? Note that \mathcal{R} in Example 52 (which is terminating) is *not* operationally terminating: there is an infinite well-formed proof tree

$$\frac{\frac{\frac{\vdots}{\mathbf{a} \rightarrow \mathbf{a}} \quad \mathbf{a} \rightarrow^* \mathbf{b}}{\mathbf{a} \rightarrow^* \mathbf{b}}}{\mathbf{a} \rightarrow \mathbf{a}}}$$

3-CTRSs \mathcal{R} with a proper nondeterministic 3-rule $\ell \rightarrow r \Leftarrow \bigwedge_{i=1}^n s_i \rightarrow t_i$ without unsatisfiable conditions will be operationally nonterminating: there is i , $1 \leq i \leq n$ and $y \in \mathcal{V}ar(s_i)$ such that $y \notin \mathcal{V}ar(\ell) \cup \bigcup_{j=1}^{i-1} \mathcal{V}ar(t_j)$. If there is a substitution σ such that $\sigma(s_j) \rightarrow^* \sigma(t_j)$ holds for all j , $1 \leq j < i$, then it is not difficult to see that an infinite well-formed proof tree like the previous one is possible.

Remark 53. In the analysis of *V-termination* and *operational termination* that we develop in the following sections we focus on deterministic 3-CTRSs, although our analysis of operational nontermination applies to arbitrary CTRSs.

5. Dependency pairs for proving V -termination of CTRSs

In this section we consider the analysis of V -termination of deterministic 3-CTRSs. Again, we follow the *methodology* already summarized in Section 3.1.

5.1. Infinite computations starting from minimal non- V -terminating terms

Proposition 54 below establishes that, for $t \in \mathcal{T}_{V-\infty}$, there is a precise way for an infinite computation to proceed. Roughly speaking, a ‘true’ conditional rule $\ell \rightarrow r \Leftarrow \bigwedge_{i=1}^n s_i \rightarrow t_i$ with $n > 0$ must be used to try a *failed root-step* on a reduct of t . The reason of this ‘failure’ is that the evaluation of one of the conditions s_i turns into an infinite *climbing* in the proof tree. Then, there is a V -minimal subterm which is an instance of a nonvariable subterm of one of the *left-hand sides* s_i of a condition $s_i \rightarrow t_i$ (the infinite computation continues through the *vertical dimension*). The set of ‘proper’ conditional rules of \mathcal{R} is:

$$CRules(\mathcal{R}) = \{\ell \rightarrow r \Leftarrow c \in \mathcal{R} \mid c \text{ is not empty}\}.$$

Obviously, $CRules(\mathcal{R}) = \emptyset$ if and only if \mathcal{R} is a TRS.

Proposition 54. *Let \mathcal{R} be a deterministic 3-CTRS and $t \in \mathcal{T}_{V-\infty}$. There exist $\alpha : \ell \rightarrow r \Leftarrow \bigwedge_{j=1}^n s_j \rightarrow t_j \in CRules(\mathcal{R})$ and a substitution σ such that*

$$t \xrightarrow{>\epsilon^*} \sigma(\ell_1) \xrightarrow{\epsilon} \sigma(r_1) \supseteq t_1 \xrightarrow{>\epsilon^*} \dots \xrightarrow{>\epsilon^*} \sigma(\ell_m) \xrightarrow{\epsilon} \sigma(r_m) \supseteq t_m \xrightarrow{>\epsilon^*} \sigma(\ell) \quad (52)$$

for rules $\ell_k \rightarrow r_k \Leftarrow c_k \in DRules(\mathcal{R})$, where

1. for all k , $1 \leq k \leq m$,
 - (a) for all $s \rightarrow t \in c_k$, $\sigma(s) \rightarrow^* \sigma(t)$ and
 - (b) $t_k = \sigma(v_k) \in \mathcal{T}_{V-\infty}$ for some $v_k \in \mathcal{DSubterm}(\mathcal{R}, r_k)$
2. there is i , $1 \leq i \leq n$ such that for all j , $1 \leq j < i$,
 - (a) $\sigma(s_j)$ is V -terminating,
 - (b) $\sigma(s_j) \rightarrow^* \sigma(t_j)$, and
 - (c) there is $v \in \mathcal{DSubterm}(\mathcal{R}, s_i)$ such that $\ell \not\vdash v$ and $\sigma(v) \in \mathcal{T}_{V-\infty}$.

PROOF. Since t is non- V -terminating, there is an infinite proof tree T :

$$\frac{\frac{\frac{\frac{T_m^\infty}{\vdots} \text{ (C)}}{C[\sigma(\ell)] \rightarrow C[\sigma(r)] \text{ (C)}}{C[\sigma(r)] \rightarrow^* u} \text{ (Tran)}}{\vdots} \text{ (Tran)}}{T_2 \quad \frac{u_2 \rightarrow^* u}{u_1 \rightarrow^* u} \text{ (Tran)}}{T_1 \quad \frac{u_1 \rightarrow^* u}{t \rightarrow^* u} \text{ (Tran)}} \text{ (Tran)}$$

for some term u , where

1. for all k , $1 \leq k < m$, T_k is a closed proof tree with $root(T_k) = u_{k-1} \rightarrow u_k$ for some terms u_k (being $u_0 = t$),
2. $C[\]$ is a context, and
3. T_m^∞ is an infinite well-formed proof tree

$$\frac{T_{m,1} \cdots T_{m,i-1} T_{m,i}^\infty G_{i+1} \cdots G_n}{\sigma(\ell) \rightarrow \sigma(r)}_{(Rp)}$$

where:

- (a) for all j , $1 \leq j < i$, $T_{m,j}$ are closed proof trees with $root(T_{m,j}) = \sigma(s_j) \rightarrow^* \sigma(t_j)$,
- (b) for all j , $i < j \leq n$, $G_j = \sigma(s_j) \rightarrow^* \sigma(t_j)$ are open goals, and
- (c) $T_{m,i}^\infty$ is an infinite well-formed proof tree with $root(T_{m,i}^\infty) = \sigma(s_i) \rightarrow^* \sigma(t_i)$ and $spine(T_{m,i}^\infty)$ having infinitely many goals G with $pred(G) = \rightarrow$.

The existence of T (which is $Tru(T_1, \dots, T_{m-1}, C[T_m^\infty])$ for short) implies that of a sequence $t \rightarrow^* C[\sigma(\ell)]$ for some rule $\alpha : \ell \rightarrow r \leftarrow \bigwedge_{j=1}^n s_j \rightarrow t_j \in CRules(\mathcal{R})$ such that for all j , $1 \leq j < i$, $\sigma(s_j) \rightarrow^* \sigma(t_j)$. Assume that the length N of this sequence is the shortest possible. We prove by induction on N that a sequence of the form (52) exists.

1. If $N = 0$, then $t = C[\sigma(\ell)]$. Since $\sigma(\ell)$ is obviously non- V -terminating and $t \in \mathcal{T}_{V-\infty}$, we must have $t = \sigma(\ell)$ (i.e., the context $C[\]$ is empty). Therefore, we have a sequence like (52) as required with $m = 0$. Since $root(T_{m,i}^\infty) = \sigma(s_i) \rightarrow^* \sigma(t_i)$, we have that $\sigma(s_i)$ is not V -terminating. We can assume (by a minimality argument on i) that for all j , $1 \leq j < i$, $\sigma(s_j)$ is V -terminating, and $\sigma(s_j) \rightarrow^* \sigma(t_j)$. Clearly, $\alpha \in CRules(\mathcal{R})$. And by Lemma 23, for all j , $1 \leq j < i$, $\sigma(t_j)$ and all its subterms are V -terminating. Since \mathcal{R} is deterministic, $\sigma(x)$ is V -terminating for all $x \in \mathcal{Var}(s_i)$. Since $\sigma(s_i)$ is V -nonterminating, by Lemma 21 it contains a subterm $u \in \mathcal{T}_{V-\infty} : \sigma(s_i) \triangleright u$, i.e., there is a position $p \in \mathcal{Pos}(\sigma(s_i))$ such that $\sigma(s_i)|_p = u$. The case $p \notin \mathcal{Pos}_{\mathcal{F}}(s_i)$ is not possible; otherwise there is $x \in \mathcal{Var}(s_i)$ such that $\sigma(x) \triangleright u$. Since $\sigma(y)$ is V -terminating for all $y \in \mathcal{Var}(s_i)$ we get a contradiction. Thus, $p \in \mathcal{Pos}_{\mathcal{F}}(s_i)$. Then there is a subterm v of s_i , $s_i \triangleright v$, such that $u = \sigma(v)$. Since $root(v) = root(u) \in \mathcal{D}$ (by minimality of u), v is a defined term: $v \in \mathcal{DSubterm}(\mathcal{R}, s_i)$. Note that v cannot be a strict subterm of ℓ , i.e., $\ell \not\triangleright v$. Otherwise, $u = \sigma(v)$ would be a subterm of $\sigma(\ell)$, thus failing to be minimal.
2. If $N > 0$, then we have $t \xrightarrow{p} t' \rightarrow^* C[\sigma(\ell)]$. Note that t' is non- V -terminating. We consider two cases according to p :
 - (a) If $p > \epsilon$, then, by minimality of t , the reduction step is issued on a V -terminating term $t|_p$ by means of a rule $\ell' \rightarrow r' \leftarrow c'$, i.e., $t' = t[\sigma(r')]_p$. By Lemma 23, $\sigma(r')$ is V -terminating and, since t' non- V -terminating and all its immediate subterms are V -terminating, t' is also minimal. By the induction hypothesis, there is a sequence

$$t' \xrightarrow{>\epsilon}^* \sigma(\ell_1) \xrightarrow{\epsilon} \sigma(r_1) \triangleright t_1 \xrightarrow{>\epsilon}^* \dots \xrightarrow{>\epsilon}^* \sigma(\ell_m) \xrightarrow{\epsilon} \sigma(r_m) \triangleright t_m \xrightarrow{>\epsilon}^* \sigma(\ell)$$

and hence, we also have

$$t \xrightarrow{\epsilon}^* \sigma(\ell_1) \xrightarrow{\epsilon} \sigma(r_1) \triangleright t_1 \xrightarrow{\epsilon}^* \dots \xrightarrow{\epsilon}^* \sigma(\ell_m) \xrightarrow{\epsilon} \sigma(r_m) \triangleright t_m \xrightarrow{\epsilon}^* \sigma(\ell)$$

- (b) If $p = \epsilon$, then $t = \sigma(\ell')$ for some $\ell' \rightarrow r' \Leftarrow c' \in \mathcal{R}$ and for all $u' \rightarrow v' \in c'$, $\sigma(u') \rightarrow^* \sigma(v')$. Without loss of generality we can assume that, for all $u' \rightarrow v' \in c'$, $\sigma(u')$ is V -terminating (otherwise, we would be in the base case $N = 0$ by using $\ell' \rightarrow r' \Leftarrow c'$ instead of $\ell \rightarrow r \Leftarrow c$). By Lemma 23, $\sigma(v')$ is V -terminating for all $u' \rightarrow v' \in c'$ as well. Note that $\sigma(r')$ is non- V -terminating. By Lemma 21, $\sigma(r')$ contains a subterm $u \in \mathcal{T}_{V-\infty}$: $\sigma(r') \triangleright u$, i.e., there is a position $q \in \mathcal{Pos}(\sigma(r'))$ such that $\sigma(r')|_q = u$. The case $q \notin \mathcal{Pos}_{\mathcal{F}}(r)$ is not possible: otherwise there is $x \in \mathcal{Var}(r')$ such that $\sigma(x) \triangleright u$, i.e., $\sigma(x)$ is non- V -terminating (by Lemma 23). Since $\sigma(y)$ is terminating for all $y \in \mathcal{Var}(\ell')$, it follows that $x \in \mathcal{Var}(r') - \mathcal{Var}(\ell')$. Since \mathcal{R} is a deterministic 3-CTRS $x \in \mathcal{Var}(v')$ for some $u' \rightarrow v' \in c'$. However, since $\sigma(v')$ is V -terminating, $\sigma(x)$ is V -terminating as well, thus leading to a contradiction. Thus, $q \in \mathcal{Pos}_{\mathcal{F}}(r')$. Then, there is a defined subterm v of r' , $r' \triangleright v$, such that $u = \sigma(v)$. Since $\text{root}(v) = \text{root}(u) \in \mathcal{D}$ (by minimality of u), v is a defined subterm of r' : $v \in \mathcal{DSubterm}(\mathcal{R}, r')$. Note that $r' \notin \mathcal{T}(\mathcal{C}, \mathcal{X})$, i.e., $\alpha \in \mathcal{DRules}(\mathcal{R}, t)$. Note also that v cannot be a strict subterm of ℓ , i.e., $\ell \not\triangleright v$. Otherwise, $u = \sigma(v)$ would be a subterm of $\sigma(\ell)$, thus failing to be minimal.

Therefore, we have $t = \sigma(\ell') \xrightarrow{\epsilon} \sigma(r') = t' \triangleright u$, where $u \in \mathcal{T}_{V-\infty}$. That is, $t \xrightarrow{\epsilon} \sigma(r') = D[u] \rightarrow^* D[E[\sigma(\ell)]] = C[\sigma(\ell)]$ for some contexts $D[\]$ and $E[\]$. Since $u \rightarrow^* E[\sigma(\ell)]$ is shorter, by the induction hypothesis,

$$u \xrightarrow{\epsilon}^* \sigma(\ell_2) \xrightarrow{\epsilon} \sigma(r_2) \triangleright t_2 \xrightarrow{\epsilon}^* \dots \xrightarrow{\epsilon}^* \sigma(\ell_m) \xrightarrow{\epsilon} \sigma(r_m) \triangleright t_m \xrightarrow{\epsilon}^* \sigma(\ell)$$

Therefore, with $\ell' \rightarrow r' \Leftarrow c'$ as $\ell_1 \rightarrow r_1 \Leftarrow c_1$, (52) is obtained. \square

Corollary 55. *Let \mathcal{R} be a deterministic 3-CTRS and $t \in \mathcal{T}_{V-\infty}$. There exist an infinite sequence $(\ell^p \rightarrow r^p \Leftarrow \bigwedge_{j=1}^{n_p} s_j^p \rightarrow t_j^p)_{p \geq 1}$ of (renamed) rules in $\mathcal{CRules}(\mathcal{R})$ and a substitution σ such that $t \xrightarrow{\epsilon}^* \sigma(\ell^1)$ and for all $p \geq 1$,*

1. *there is $m_p \geq 0$ and rules $\ell_k^p \rightarrow r_k^p \Leftarrow c_k^p \in \mathcal{DRules}(\mathcal{R})$ for $1 \leq k \leq m_p$ such that*
 - (a) *for all k , $1 \leq k \leq m_p$ and $s \rightarrow t \in c_k^p$ we have $\sigma(s) \rightarrow^* \sigma(t)$ and there is $v_k^p \in \mathcal{DSubterm}(\mathcal{R}, r_k^p)$ such that $\ell_k^p \not\triangleright v_k^p$ and $\sigma(v_k^p) \in \mathcal{T}_{V-\infty}$.*
 - (b) *$\sigma(v_{m_p}^p) \xrightarrow{\epsilon}^* \sigma(\ell^{p+1})$ and for all j , $1 \leq k < m_p$, $\sigma(v_k^p) \xrightarrow{\epsilon}^* \sigma(\ell_{k+1}^p)$.*
2. *there is i_p , $1 \leq i_p \leq n_p$ such that for all $j < i_p$, $\sigma(s_j^p) \rightarrow^* \sigma(t_j^p)$, and there is $v^p \in \mathcal{DSubterm}(\mathcal{R}, s_{i_p}^p)$ such that $\ell^p \not\triangleright v^p$, $\sigma(v^p) \in \mathcal{T}_{V-\infty}$, and $\sigma(v^p) \xrightarrow{\epsilon}^* \sigma(\ell_1^{p+1})$.*

5.2. Dependency Pairs for V -termination

Our dependency pairs for proving V -termination of deterministic 3-CTRSs are organized into two blocks. The *vertical* block $\text{DP}_V(\mathcal{R})$ contains pairs for shifting the infinite computation to the conditions of the rules as shown in Proposition 54. Each rule in $\text{DP}_V(\mathcal{R})$ is obtained from a rule $\ell \rightarrow r \Leftarrow c \in \mathcal{R}$ by marking the left-hand sides ℓ and using defined subterms v in the left-hand side s of a condition $s \rightarrow t \in c$. The obtained rules are of the form $\ell^\# \rightarrow v^\# \Leftarrow c'$ for some initial subsequence c' of conditions in c .

Definition 56. *Let \mathcal{R} be a CTRS. The CTRS $\text{DP}_V(\mathcal{R})$ of vertical dependency pairs is:*

$$\text{DP}_V(\mathcal{R}) = \left\{ \ell^\# \rightarrow v^\# \Leftarrow \bigwedge_{j=1}^{k-1} s_j \rightarrow t_j \mid \ell \rightarrow r \Leftarrow \bigwedge_{i=1}^n s_i \rightarrow t_i \in \text{CRules}(\mathcal{R}), \right. \\ \left. 1 \leq k \leq n, v \in \mathcal{D}\text{Subterm}(\mathcal{R}, s_k), \ell \not\# v \right\}.$$

The *horizontal* block contains those pairs that correspond to rules issuing root steps that are required to *connect* pairs in $\text{DP}_V(\mathcal{R})$ (also according to Proposition 54). Such pairs are captured by $\text{DP}_H(\mathcal{R})$ in Definition 38.

Example 57. *For \mathcal{R} in Example 1, $\text{DP}_V(\mathcal{R})$ consists of the following rules:*

$$\text{SPLIT}(x, \text{cons}(y, ys)) \rightarrow \text{LEQ}(x, y) \quad (53)$$

$$\text{SPLIT}(x, \text{cons}(y, ys)) \rightarrow \text{SPLIT}(x, ys) \Leftarrow \text{leq}(x, y) \rightarrow \text{true} \quad (54)$$

$$\text{SPLIT}(x, \text{cons}(y, ys)) \rightarrow \text{SPLIT}(x, ys) \Leftarrow \text{leq}(x, y) \rightarrow \text{false} \quad (55)$$

$$\text{QSORT}(\text{cons}(x, xs)) \rightarrow \text{SPLIT}(x, xs) \quad (56)$$

For \mathcal{R} in Examples 2 and 49, $\text{DP}_V(\mathcal{R})$ consists of a single rule:

$$F(x) \rightarrow G(x) \quad (57)$$

For \mathcal{R} in Example 3, $\text{DP}_V(\mathcal{R}) = \{G(x) \rightarrow H(x), G(x) \rightarrow H(x) \Leftarrow h(x) \rightarrow d\}$.

Definition 58. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be CTRSs. A $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ - V -chain is a finite or infinite sequence of (renamed) rules $u_i \rightarrow v_i \Leftarrow c_i \in \mathcal{P}$, together with a substitution σ satisfying that, for all $i \geq 1$,*

1. for all $s \rightarrow t \in c_i$, $\sigma(s) \rightarrow_{\mathcal{R}}^* \sigma(t)$ and
2. $\sigma(v_i) \rightarrow_{\mathcal{R}} \cup \xrightarrow{\mathcal{Q}, \mathcal{R}}^{\epsilon} \sigma(u_{i+1})$, where for all terms s, t , we write $s \xrightarrow{\mathcal{Q}, \mathcal{R}}^{\epsilon} t$ if there is $\ell \rightarrow r \Leftarrow c \in \mathcal{Q}$ and a substitution θ such that $s = \theta(\ell)$, $t = \theta(r)$ and $\theta(u) \rightarrow_{\mathcal{R}}^* \theta(v)$ for all $u \rightarrow v \in c$.

A $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ - V -chain is called *minimal* if for all $i \geq 1$, whenever

$$\sigma(v_i) = w_{i1} \rightarrow_{\mathcal{R}}^* \circ \xrightarrow{\mathcal{Q}, \mathcal{R}}^{\epsilon} w_{i2} \rightarrow_{\mathcal{R}}^* \circ \xrightarrow{\mathcal{Q}, \mathcal{R}}^{\epsilon} \cdots \rightarrow_{\mathcal{R}}^* \circ \xrightarrow{\mathcal{Q}, \mathcal{R}}^{\epsilon} w_{im_i} \rightarrow_{\mathcal{R}}^* \sigma(u_{i+1}),$$

in the chain, then for all j , $1 \leq j \leq m_i$, w_{ij} is V -terminating (w.r.t. \mathcal{R}).

As usual, in Definition 58 we assume that different applications of rules in \mathcal{P} and \mathcal{Q} do not share any variable (renaming substitutions are used if necessary).

We can prove or disprove V -termination of CTRSs as the absence (or existence) of $(\text{DP}_V(\mathcal{R}), \text{DP}_H(\mathcal{R}), \mathcal{R})$ - V -chains. However, a closer examination of the role of $\text{DP}_H(\mathcal{R})$ in such chains shows that we often do not need *all* of them. In the following, we refine the set of pairs that are really necessary to connect pairs $u \rightarrow v \Leftarrow c \in \text{DP}_V(\mathcal{R})$ within a V -chain. In order to achieve this, we adapt the notion of *usable rules* [1] to obtain an overapproximation of the rules that can be applied during a reduction of the right-hand sides of pairs in $\text{DP}_V(\mathcal{R})$.

Definition 59 (Root-Usable rules for CTRSs). *Let \mathcal{R} be a CTRS and t be a term. Let $\text{RULES}_\epsilon(\mathcal{R}, t)$ be the set of rules in \mathcal{R} defining $\text{root}(t)$:*

$$\text{RULES}_\epsilon(\mathcal{R}, t) = \{\ell \rightarrow r \Leftarrow c \in \mathcal{R} \mid \text{root}(\ell) = \text{root}(t)\}$$

Then, the set of root-usable rules of \mathcal{R} for t is:

$$\mathcal{U}_\epsilon(\mathcal{R}, t) = \text{RULES}_\epsilon(\mathcal{R}, t) \cup \bigcup_{\ell \rightarrow r \Leftarrow c \in \text{RULES}_\epsilon(\mathcal{R}, t)} \mathcal{U}_\epsilon(\mathcal{R}^\bullet, r)$$

where $\mathcal{R}^\bullet = \mathcal{R} - \text{RULES}_\epsilon(\mathcal{R}, t)$.

Now, we let

$$\text{DP}_{VH}(\mathcal{R}) = \bigcup_{u \rightarrow v \Leftarrow c \in \text{DP}_V(\mathcal{R})} \mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v)$$

Since $\mathcal{U}_\epsilon(\mathcal{R}, t) \subseteq \mathcal{R}$ for all terms t , we have $\text{DP}_{VH}(\mathcal{R}) \subseteq \text{DP}_H(\mathcal{R})$.

Example 60. *For \mathcal{R} in Example 1, $\text{DP}_H(\mathcal{R})$ contains five rules (see Example 39), but $\text{DP}_{VH}(\mathcal{R})$ consists of the single rule (27), i.e.,*

$$\text{LEQ}(s(x), s(y)) \rightarrow \text{LEQ}(x, y)$$

because this is the only rule in $\mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v)$ when v is one of the right-hand sides of the rules in $\text{DP}_V(\mathcal{R})$ (see Example 57). Actually, for the right-hand side v of pair (53), we have $\mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v) = \{(27)\}$, whereas for the right-hand sides v of pairs (28)-(31), $\mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v) = \emptyset$.

Example 61. *For \mathcal{R} in Example 2 and $\text{DP}_H(\mathcal{R})$ in Example 40, $\text{DP}_{VH}(\mathcal{R}) = \text{DP}_H(\mathcal{R})$. For \mathcal{R} in Example 3, although $\text{DP}_H(\mathcal{R})$ and $\text{DP}_V(\mathcal{R})$ (see Examples 40 and 57) are not empty, we have $\text{DP}_{VH}(\mathcal{R}) = \emptyset$.*

The property that motivates Definition 59 is Proposition 63 below, showing that only pairs in $\text{DP}_{VH}(\mathcal{R}) \subseteq \text{DP}_H(\mathcal{R})$ are used in any $(\text{DP}_V(\mathcal{R}), \text{DP}_H(\mathcal{R}), \mathcal{R})$ - V -chain. The proof relies on the following auxiliary result.

Proposition 62. *Let \mathcal{R} be a CTRS, $u \rightarrow v \Leftarrow c \in \text{DP}_V(\mathcal{R})$, and σ be a substitution such that*

$$\sigma(v) = s_1 \rightarrow_{\mathcal{R}}^* s'_1 \xrightarrow{\epsilon}_{\pi_1} s_2 \rightarrow_{\mathcal{R}}^* \cdots \rightarrow_{\mathcal{R}}^* s'_n \xrightarrow{\epsilon}_{\pi_n} s_{n+1}$$

where, for all i , $1 \leq i \leq n$, $s'_i = \sigma(u_i)$ and $s_{i+1} = \sigma(v_i)$ for some $\pi_i : u_i \rightarrow v_i \Leftarrow c_i \in \text{DP}_H(\mathcal{R})$. Then, for all i , $1 \leq i \leq n$, we have $\pi_i \in \mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v)$.

PROOF. We prove by induction on n that, for all i , $1 \leq i \leq n$, $\pi_i \in \mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v)$. If $n = 0$, this is vacuously true. If $n > 0$, then we write the sequence as follows:

$$s_1 \xrightarrow{\mathcal{R}}^* s'_1 \xrightarrow{\epsilon}_{\text{DP}_H(\mathcal{R})} s_2 \xrightarrow{\mathcal{R}}^* \cdots \xrightarrow{\mathcal{R}}^* s'_{n-1} \xrightarrow{\epsilon}_{\text{DP}_H(\mathcal{R})} s_n \xrightarrow{\mathcal{R}}^* s'_n \xrightarrow{\epsilon}_{\text{DP}_H(\mathcal{R})} s_{n+1}$$

By the induction hypothesis, $\pi_{n-1} \in \mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v)$. Let \mathcal{F} be the signature of \mathcal{R} . Since $\text{root}(v_{n-1}) \notin \mathcal{F}$ (due to the marking procedure for defining the dependency pairs), rewritings with \mathcal{R} do not change the root of s_n and we have $\text{root}(v_{n-1}) = \text{root}(s_n) = \text{root}(s'_n) = \text{root}(u_n)$. Hence, $\pi_n \in \text{RULES}_\epsilon(\text{DP}_H(\mathcal{R}), v_{n-1}) \subseteq \mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v)$, as desired. \square

Proposition 63. *Let \mathcal{R} be a CTRS. Every (minimal) $(\text{DP}_V(\mathcal{R}), \text{DP}_H(\mathcal{R}), \mathcal{R})$ -V-chain is a (minimal) $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain and vice versa.*

PROOF. Since $\text{DP}_{VH}(\mathcal{R}) \subseteq \text{DP}_H(\mathcal{R})$, every $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain is a $(\text{DP}_V(\mathcal{R}), \text{DP}_H(\mathcal{R}), \mathcal{R})$ -O-chain. On the other hand, if $u \rightarrow v \leftarrow c, u' \rightarrow v' \leftarrow c' \in \text{DP}_V(\mathcal{R})$ are such that $\sigma(v)(\rightarrow_{\mathcal{R}} \cup \xrightarrow{\epsilon}_{\text{DP}_H(\mathcal{R}), \mathcal{R}})^* \sigma(u')$, by Proposition 62 the rules in $\text{DP}_H(\mathcal{R})$ that are used belong to $\mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v)$. Since, for all $u \rightarrow v \leftarrow c \in \text{DP}_V(\mathcal{R})$, $\mathcal{U}_\epsilon(\text{DP}_H(\mathcal{R}), v) \subseteq \text{DP}_{VH}(\mathcal{R})$, the conclusion follows. With regard to minimality, since it depends on \mathcal{R} and no change on \mathcal{R} is made, it is preserved as well. \square

5.3. Proving V-termination using dependency pairs

Our next main result (Theorem 65) shows how to use $\text{DP}_V(\mathcal{R})$ and $\text{DP}_{VH}(\mathcal{R})$ to prove V-termination of a deterministic 3-CTRS \mathcal{R} . First, we need the following auxiliary result.

Lemma 64. *Let \mathcal{R} be a CTRS. Let $u \rightarrow v \leftarrow \bigwedge_{j=1}^m s_j \rightarrow t_j, u' \rightarrow v' \leftarrow c$ be a $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain for some substitution σ . Then, there are rules $\ell \rightarrow r \leftarrow \bigwedge_{j=1}^n s_j \rightarrow t_j \in \mathcal{R}$ for some $n > m$, and $\ell' \rightarrow r' \leftarrow c' \in \mathcal{R}$ such that $u = \ell^\sharp$, $s_{m+1} = C[v^b]$ for some context $C[\]$, $u' = (\ell')^\sharp$, and*

$$\frac{T_1 \quad \cdots \quad T_m \quad T_{m+1} \quad O_{m+2} \quad \cdots \quad O_n}{\sigma(\ell) \rightarrow \sigma(r)}$$

is a finite well-formed proof tree T where:

1. for all j , $1 \leq j \leq m$, T_j are closed proof trees with $\text{root}(T_j) = \sigma(s_j) \rightarrow^* \sigma(t_j)$; for all j , $m+2 \leq j \leq n$, $O_j = \sigma(s_j) \rightarrow^* \sigma(t_j)$ is an open goal, and
2. $T_{m+1} = \text{Tr}_{\sigma(t_{m+1})}(\vec{P}^0, V^1, \vec{P}^1, \dots, V^\kappa, \vec{P}^\kappa, E[G'])$ is a well-formed proof tree for some $\kappa \geq 0$ where $\text{root}(T_{m+1}) = \sigma(s_{m+1}) \rightarrow^* \sigma(t_{m+1})$ and
 - (a) for all $1 \leq j \leq \kappa$, V^j are closed proof trees,
 - (b) for all $0 \leq j \leq \kappa$, \vec{P}^j are sequences of α_j closed proof trees (for some $\alpha_j \geq 0$),
 - (c) $E[\]$ is a context and $G' = \sigma(\ell') \rightarrow \sigma(r')$.

PROOF. Since $u \rightarrow v \Leftarrow \bigwedge_{j=1}^m s_j \rightarrow t_j, u' \rightarrow v' \Leftarrow c$ is a $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain with substitution σ , both $u \rightarrow v \Leftarrow \bigwedge_{j=1}^m s_j \rightarrow t_j$ and $u' \rightarrow v' \Leftarrow c$ belong to $\text{DP}_V(\mathcal{R})$ and there is $\kappa \geq 0$ such that

$$\sigma(s_j) \rightarrow_{\mathcal{R}}^* \sigma(t_j) \text{ for all } j, 1 \leq j \leq m, \text{ and} \quad (58)$$

$$\sigma(s_j^k) \rightarrow_{\mathcal{R}}^* \sigma(t_j^k) \text{ for all } j, 1 \leq j \leq n_k \quad (59)$$

where $u^k \rightarrow v^k \Leftarrow \bigwedge_{j=1}^{n_k} s_j^k \rightarrow t_j^k \in \text{DP}_{VH}(\mathcal{R})$ for $k, 1 \leq k \leq \kappa$ and,

$$\sigma(v) \rightarrow_{\mathcal{R}}^* \sigma(u^1) \quad (60)$$

$$\xrightarrow{\epsilon}_{\text{DP}_{VH}(\mathcal{R})} \sigma(v^1) \quad (61)$$

$$\rightarrow_{\mathcal{R}}^* \cdots \quad (62)$$

$$\rightarrow_{\mathcal{R}}^* \sigma(u^\kappa) \quad (63)$$

$$\xrightarrow{\epsilon}_{\text{DP}_{VH}(\mathcal{R})} \sigma(v^\kappa) \quad (64)$$

$$\rightarrow_{\mathcal{R}}^* \sigma(u') \quad (65)$$

If $\kappa = 0$, we simply have $\sigma(v) \rightarrow_{\mathcal{R}}^* \sigma(u')$. By Definition 56, there is $\ell \rightarrow r \Leftarrow \bigwedge_{j=1}^n s_j \rightarrow t_j \in \mathcal{R}$ with $n > m$ such that

$$u = \ell_i^\# \quad (66)$$

$$s_{m+1} = C[v^b] \text{ for some context } C[] \quad (67)$$

Furthermore, there is also $\ell' \rightarrow r' \Leftarrow c' \in \mathcal{R}$ such that $u' = (\ell')^\#$. By Definition 38, there are rules $\ell^k \rightarrow r^k \Leftarrow \bigwedge_{j=1}^{n_k} s_j^k \rightarrow t_j^k \in \mathcal{R}$ for $k, 0 \leq k \leq \kappa$ such that

$$u^k = (\ell^k)^\# \quad (68)$$

$$r^k = C^k[(v^k)^b] \text{ for an appropriate context } C^k[] \quad (69)$$

We apply (Repl) to obtain the following tree

$$\frac{T_1 \quad \cdots \quad T_m \quad T_{m+1} \quad O_{m+2} \quad \cdots \quad O_n}{\sigma(\ell) \rightarrow \sigma(r)}$$

where for all $1 \leq j \leq m$, T_j is a closed proof tree for $\sigma(s_j) \rightarrow_{\mathcal{R}}^* \sigma(t_j)$, see (58). For all $m+2 \leq j \leq n$, O_j are open goals for $\sigma(s_j) \rightarrow^* \sigma(t_j)$. And tree T_{m+1} with root $\sigma(s_{m+1}) \rightarrow^* \sigma(t_{m+1})$ is defined as

$$T_{m+1} = \text{Tr}_{\sigma(t_{m+1})}(\vec{P}^0, V^1, \vec{P}^1, \dots, V^\kappa, \vec{P}^\kappa, E[G'])$$

with $G' = \sigma(\ell') \rightarrow \sigma(r')$ and components $\vec{P}^0, V^1, \vec{P}^1, \dots, V^\kappa, \vec{P}^\kappa$ and $E[]$ obtained from (60)-(65) as follows:

1. Since $\sigma(s_{m+1}) = \sigma(C)[\sigma(v^b)]$ the \mathcal{R} -rewrite sequence (60) is:

$$\sigma(v) = q_0^0 \rightarrow_{\mathcal{R}} q_1^0 \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} q_{\alpha_0}^0 = \sigma(u^1)$$

for some $\alpha_0 \geq 0$. Each step $q_j^0 \rightarrow_{\mathcal{R}} q_{j+1}^0$ for $0 \leq j < \alpha_0$ has a closed proof tree Q_j^0 . Since $root(v) = f^\sharp$ does not occur in \mathcal{R} and does not change during the sequence, removing the marks from the roots of q_j^0 and q_{j+1}^0 does not change the proof tree, which remains the same (except that a (Cong) rule for f instead of f^\sharp is used). Denote such a proof tree with root $(q_j^0)^\flat \rightarrow (q_{j+1}^0)^\flat$ as $(Q_j^0)^\flat$. Thus, if we let $D[\] = \sigma(C[\])$, in:

$$\sigma(s_{m+1}) = D[\sigma(v^\flat)] = D[(q_0^0)^\flat] \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} D[(q_{\alpha_0}^0)^\flat] = D[\sigma(u^1)^\flat]$$

each step $D[(q_j^0)^\flat] \rightarrow_{\mathcal{R}} D[(q_{j+1}^0)^\flat]$ has a closed proof tree $P_j^0 = D[(Q_j^0)^\flat]$.

2. For the k -th $DP_{VH}(\mathcal{R})$ -step (labelled (61) for the first, $k = 1$, and (64) for the last one, $k = \kappa$), there is $\ell^k \rightarrow r^k \leftarrow \bigwedge_{j=1}^{n_k} s_j^k \rightarrow t_j^k \in \mathcal{R}$ and a context $C^k[\]$ such that $u^k = (\ell^k)^\sharp$ (see (68)) and $r^k = C^k[(v^k)^\flat]$ (see (69)). Let U^k be the closed proof tree for $\sigma(\ell^k) \rightarrow_{\mathcal{R}} C^k[(v^k)^\flat]$, which corresponds to the step performed with the dependency pair. Then, we have:

$$D[C^1[\cdots C^{k-1}[\sigma(\ell^k)] \cdots]] \rightarrow_{\mathcal{R}} D[C^1[\cdots C^{k-1}[C^k[\sigma((v^k)^\flat)]]] \cdots]]$$

or, if the notation $E_k[\] = D[C^1[\cdots C^{k-1}[C^k[\]]] \cdots]$ is adopted (for $k \geq 0$; if $k = 0$, then $E_0[\] = D[\]$), then the rewriting step $E_{k-1}[\sigma(\ell^k)] \rightarrow_{\mathcal{R}} E_k[\sigma((v^k)^\flat)] = E_{k-1}[C^k[\sigma(v^k)^\flat]]$ has a closed proof tree $V^k = E_{k-1}[(U^k)^\flat]$.

3. Each sequence $\sigma(v^k) \rightarrow_{\mathcal{R}}^* \sigma(u^{k+1})$ with $0 < k < \kappa$ is as follows:

$$\sigma(v^k) = q_0^k \rightarrow_{\mathcal{R}} q_1^k \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} q_{\alpha_k}^k = \sigma(u^{k+1})$$

for some $\alpha_k \geq 0$, where each step $(q_j^k)^\flat \rightarrow_{\mathcal{R}} (q_{j+1}^k)^\flat$ for $0 \leq j < \alpha_k$ has a closed proof tree Q_j^k . And therefore, we have the following sequence:

$$E_k[\sigma(v^k)^\flat] = E_k[(q_0^k)^\flat] \rightarrow_{\mathcal{R}} E_k[(q_1^k)^\flat] \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} E_k[(q_{\alpha_k}^k)^\flat] = E_k[\sigma(u^{k+1})^\flat]$$

For all j , $0 \leq j < \alpha_k$, each step $E_k[(q_j^k)^\flat] \rightarrow_{\mathcal{R}} E_k[(q_{j+1}^k)^\flat]$ has a closed proof tree $P_j^k = E_k[(Q_j^k)^\flat]$. For all k , $0 \leq k < \kappa$, $E_k[(q_{\alpha_k}^k)^\flat] = E_k[\sigma(\ell_i^k)]$.

4. Finally, for $\sigma(v^\kappa) \rightarrow_{\mathcal{R}}^* \sigma(u')$ (see (65)), we have:

$$\sigma(v^\kappa) = q_0^\kappa \rightarrow_{\mathcal{R}} q_1^\kappa \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} q_{\alpha_\kappa}^\kappa = \sigma(u')$$

for some $\alpha_\kappa \geq 0$, where each step $(q_j^\kappa)^\flat \rightarrow_{\mathcal{R}} (q_{j+1}^\kappa)^\flat$ for $0 \leq j < \alpha_\kappa$ has a closed proof tree Q_j^κ . We therefore have the following sequence:

$$E_\kappa[\sigma(v^\kappa)^\flat] = E_\kappa[(q_0^\kappa)^\flat] \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} E_\kappa[(q_{\alpha_\kappa}^\kappa)^\flat] = E_\kappa[\sigma(u')^\flat]$$

For all j , $0 \leq j < \alpha_\kappa$, there is a closed proof tree $P_j^\kappa = E_\kappa[Q_j^\kappa]$ for $E_\kappa[(q_j^\kappa)^\flat] \rightarrow_{\mathcal{R}} E_\kappa[(q_{j+1}^\kappa)^\flat]$. Since $E_\kappa[(q_{\alpha_\kappa}^\kappa)^\flat] = E_\kappa[\sigma(\ell')]$, we let $E[\] = E_\kappa[\]$.

□

Theorem 65 (V-termination). *Let \mathcal{R} be a CTRS.*

1. If there is no infinite minimal $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain and \mathcal{R} is a deterministic 3-CTRS, then \mathcal{R} is V-terminating.
2. If there is an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain, then \mathcal{R} is non-V-terminating.

PROOF.

1. By contradiction. If \mathcal{R} is not V-terminating, then by Corollary 55, there is an infinite sequence $(\ell^p \rightarrow r^p \Leftarrow \bigwedge_{j=1}^{n_p} s_j^p \rightarrow t_j^p)_{p \geq 1}$ of (renamed) rules in $\text{CRules}(\mathcal{R})$ and a substitution σ such that for all $p \geq 1$,
 - (a) there is $m_p \geq 0$ and rules $\ell_k^p \rightarrow r_k^p \Leftarrow c_k^p \in \text{DRules}(\mathcal{R})$ for $1 \leq k \leq m_p$ such that for all k , $1 \leq k \leq m_p$ and $s \rightarrow t \in c_k^p$ we have $\sigma(s) \rightarrow^* \sigma(t)$ and there is $v_k^p \in \text{DSubterm}(\mathcal{R}, r_k^p)$ such that $\ell_k^p \not\vdash v_k^p$ and $\sigma(v_k^p) \in \mathcal{T}_{V-\infty}$.
 - (b) $\sigma(v_{m_p}^p) \xrightarrow{\geq \epsilon}^* \sigma(\ell^{p+1})$ and for all j , $1 \leq k < m_p$, $\sigma(v_k^p) \xrightarrow{\geq \epsilon}^* \sigma(\ell_{k+1}^p)$.
 - (c) there is i_p , $1 \leq i_p \leq n_p$ such that for all $j < i_p$, $\sigma(s_j^p) \rightarrow^* \sigma(t_j^p)$, and there is $v^p \in \text{DSubterm}(\mathcal{R}, s_{i_p}^p)$ such that $\ell^p \not\vdash v^p$, $\sigma(v^p) \in \mathcal{T}_{V-\infty}$, and $\sigma(v^p) \xrightarrow{\geq \epsilon}^* \sigma(\ell_1^{p+1})$.

We have that, for all $p \geq 1$, if we let $L^p = (\ell^p)^\sharp$, $V^p = (v^p)^\sharp$, and C^p be $\bigwedge_{j=1}^{i_p-1} s_j^p \rightarrow t_j^p$, then $L^p \rightarrow V^p \Leftarrow C^p \in \text{DP}_V(\mathcal{R})$. Similarly, if, for all $p \geq 1$, and k , $1 \leq k \leq m_p$, we let $L_k^p = (\ell_k^p)^\sharp$ and $V_k^p = (v_k^p)^\sharp$, then $L_k^p \rightarrow V_k^p \Leftarrow c_k^p \in \text{DP}_H(\mathcal{R})$.

Now, for all $p \geq 1$, we have that, for all $s \rightarrow t \in C^p$, $\sigma(s) \rightarrow_{\mathcal{R}}^* \sigma(t)$, and $\sigma(V^p) (\rightarrow_{\mathcal{R}} \cup \xrightarrow{\epsilon}_{\text{DP}_H(\mathcal{R}), \mathcal{R}})^* \sigma(L^{p+1})$. Therefore, $A : (L^p \rightarrow V^p \Leftarrow C^p)_{p \geq 1}$ is a $(\text{DP}_V(\mathcal{R}), \text{DP}_H(\mathcal{R}), \mathcal{R})$ -V-chain. By Proposition 63, it is a $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain as well. Furthermore, since for all $p \geq 1$, $\sigma(v^p) \in \mathcal{T}_{V-\infty}$ (item 1c), and for all k , $1 \leq k \leq m_p$, $\sigma(v_k^p) \in \mathcal{T}_{V-\infty}$ (item 1a), we have that $\sigma(V^p)$ and $\sigma(V_k^p)$ are V-terminating. Thus, A is a minimal $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain, leading to a contradiction.

2. If $(u_i \rightarrow v_i \Leftarrow \bigwedge_{j=1}^{m_i} s_{ij} \rightarrow t_{ij})_{i \geq 1}$ is an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain with substitution σ , then, by using Lemma 64, for each $i \geq 1$ there is a rule $\ell_i \rightarrow r_i \Leftarrow \bigwedge_{j=1}^{n_i} s_j^i \rightarrow t_j^i \in \mathcal{R}$, and m_i , $1 \leq m_i < n_i$ with a well-formed proof tree U_i as follows:

$$\frac{\vec{T}_i \quad \text{Tr}_{\sigma(t_{m_i+1}^i)}(\vec{S}_i, E_i[G_i]) \quad \vec{O}_i}{\sigma(\ell_i) \rightarrow \sigma(r_i)}$$

where \vec{T}_i is a sequence of m_i closed proof trees with roots $\sigma(s_j^i) \rightarrow^* \sigma(t_j^i)$ for $1 \leq j \leq m_i$, \vec{O}_i are open goals $\sigma(s_j^i) \rightarrow^* \sigma(t_j^i)$ for $m_i + 2 \leq j \leq n_i$, \vec{S}_i is a sequence of closed proof trees, $E_i[\]$ is a context, and $G_i = \sigma(\ell_{i+1}) \rightarrow \sigma(r_{i+1})$. We define a sequence $(W_i)_{i \in \mathbb{N}}$ of proof trees as follows: $W_0 = U_1$ and for all $i > 0$, W_i is obtained from W_{i-1} by replacing G_i by U_{i+1} . Note that, for all $i \in \mathbb{N}$, $W_i \subset W_{i+1}$. Thus, we obtain an infinite well-formed proof tree $W^\infty = (W_i)_{i \in \mathbb{N}}$ with $\text{spine}(W^\infty)$ containing an infinite number of goals G with $\text{pred}(G) = \rightarrow$ thus proving \mathcal{R} non-V-terminating. \square

6. Dependency pairs for proving operational termination of CTRSs

By Theorem 14, a CTRS \mathcal{R} is operationally terminating if and only if it is terminating and V -terminating. We could use the results in Sections 4 and 5 to prove operational termination using dependency pairs $\text{DP}_H(\mathcal{R})$, $\text{DP}_C(\mathcal{R})$, and $\text{DP}_V(\mathcal{R})$. However, when considering *deterministic 3-CTRSs*, a new, simpler, characterization of operational termination in terms of $\text{DP}_H(\mathcal{R})$ and $\text{DP}_V(\mathcal{R})$ *only* is obtained. In this section we develop such a simpler characterization.

6.1. Infinite computations starting from minimal operationally nonterminating terms

Proposition 66 below establishes that, given $t \in \mathcal{T}_{O-\infty}$, there is a precise way for an infinite computation to proceed: a rule $\ell \rightarrow r \Leftarrow \bigwedge_{i=1}^n s_i \rightarrow t_i$ must be used to try a *root-step* on an inner reduct of t ; then, the computation continues on a minimal operationally nonterminating subterm which is either: (1) an instance of a nonvariable subterm of r , or (2) an instance of a nonvariable subterm of s_i for some $s_i \rightarrow t_i$, $1 \leq i \leq n$.

Proposition 66. *Let \mathcal{R} be a deterministic 3-CTRS and $t \in \mathcal{T}_{O-\infty}$. There exist $\alpha : \ell \rightarrow r \Leftarrow \bigwedge_{i=1}^n s_i \rightarrow t_i \in \mathcal{R}$ and a substitution σ such that $t \xrightarrow{\epsilon}^* \sigma(\ell)$, and there is a term v such that $\ell \not\vdash v$, $\sigma(v) \in \mathcal{T}_{O-\infty}$ and either:*

1. $\alpha \in \text{CRules}(\mathcal{R})$, $\exists i$, $1 \leq i \leq n$ such that for all j , $1 \leq j < i$, $\sigma(s_j)$ is operationally terminating, $\sigma(s_j) \rightarrow^* \sigma(t_j)$, and $v \in \mathcal{DSubterm}(\mathcal{R}, s_i)$, or
2. $\alpha \in \text{DRules}(\mathcal{R})$, for all $1 \leq i \leq n$, $\sigma(s_i)$ is operationally terminating, $\sigma(s_i) \rightarrow^* \sigma(t_i)$, and $v \in \mathcal{DSubterm}(\mathcal{R}, r)$.

PROOF. By definition of $\mathcal{T}_{O-\infty}$, all proper subterms of t are operationally terminating and cannot start any infinite well-formed proof tree. Therefore, t has an inner reduction to an instance $\sigma(\ell)$ of the left-hand side of a rule $\alpha : \ell \rightarrow r \Leftarrow \bigwedge_{i=1}^n s_i \rightarrow t_i \in \mathcal{R}$: $t \xrightarrow{\epsilon}^* \sigma(\ell)$ and $\sigma(\ell)$ is not operationally terminating. By Lemma 24, $\sigma(\ell) \in \mathcal{T}_{O-\infty}$. Thus, since $\ell \notin \mathcal{X}$, by Lemma 23, for all $x \in \text{Var}(\ell)$, $\sigma(x)$ is operationally terminating. We consider two cases:

1. There is i , $1 \leq i \leq n$ such that for all j , $1 \leq j < i$, $\sigma(s_j)$ is operationally terminating, $\sigma(s_j) \rightarrow^* \sigma(t_j)$, and $\sigma(s_i)$ is operationally nonterminating. Clearly, $\alpha \in \text{CRules}(\mathcal{R})$. And by Lemma 23, for all j , $1 \leq j < i$, $\sigma(t_j)$ is operationally terminating. By determinism of \mathcal{R} , $\sigma(x)$ is operationally terminating for all $x \in \text{Var}(s_i)$. Since $\sigma(s_i)$ is operationally nonterminating, by Lemma 21, it contains a subterm $u \in \mathcal{T}_{O-\infty}$: $\sigma(s_i) \supseteq u$, i.e., there is a position $p \in \text{Pos}(\sigma(s_i))$ such that $\sigma(s_i)|_p = u$. The case $p \notin \text{Pos}_{\mathcal{F}}(s_i)$ is not possible; otherwise there is $x \in \text{Var}(s_i)$ such that $\sigma(x) \supseteq u$. Since $\sigma(y)$ is operationally terminating for all $y \in \text{Var}(s_i)$ we get a contradiction. Thus, $p \in \text{Pos}_{\mathcal{F}}(s_i)$. Then there is a subterm v of s_i , $s_i \supseteq v$, such that $u = \sigma(v)$. Since $\text{root}(v) = \text{root}(u) \in \mathcal{D}$ (by O -minimality of u), v is a defined term: $v \in \mathcal{DSubterm}(\mathcal{R}, s_i)$. Note that s cannot be a strict subterm of ℓ , i.e., $\ell \not\vdash v$. Otherwise, $u = \sigma(v)$ would be a subterm of $\sigma(\ell)$, thus failing to be O -minimal.

2. For all i , $1 \leq i \leq n$, $\sigma(s_i)$ is operationally terminating. By Lemma 23, $\sigma(t_i)$ is operationally terminating too. Since t is operationally nonterminating, the rewriting step $\sigma(\ell) \xrightarrow{\epsilon} \sigma(r)$ must be performed, with $\sigma(r)$ operationally nonterminating. Thus, $\sigma(s_i) \rightarrow^* \sigma(t_i)$ holds for all i , $1 \leq i \leq n$. By Lemma 21, $\sigma(r)$ contains a subterm $u \in \mathcal{T}_{O-\infty}$: $\sigma(r) \triangleright u$, i.e., there is a position $p \in \text{Pos}(\sigma(r))$ such that $\sigma(r)|_p = u$. The case $p \notin \text{Pos}_{\mathcal{F}}(r)$ is not possible: otherwise there is $x \in \text{Var}(r)$ such that $\sigma(x) \triangleright u$, i.e., $\sigma(x)$ is operationally nonterminating (by Lemma 23). Since $\sigma(y)$ is operationally terminating for all $y \in \text{Var}(\ell)$, it follows that $x \in \text{Var}(r) - \text{Var}(\ell)$. Since \mathcal{R} is a deterministic 3-CTRS $x \in \text{Var}(t_i)$ for some i , $1 \leq i \leq n$. Since $\sigma(t_i)$ is operationally terminating for all i , $1 \leq i \leq n$, $\sigma(x)$ is operationally terminating as well. Thus, $p \in \text{Pos}_{\mathcal{F}}(r)$. Then, there is a subterm v of r , such that $u = \sigma(v)$. Since $\text{root}(v) = \text{root}(u) \in \mathcal{D}$ (by O -minimality of u), $v \in \mathcal{D}\text{Subterm}(\mathcal{R}, r)$. Note that $r \notin \mathcal{T}(\mathcal{C}, \mathcal{X})$, i.e., $\alpha \in \mathcal{D}\text{Rules}(\mathcal{R}, t)$. Also, v cannot be a strict subterm of ℓ , i.e., $\ell \not\triangleright v$. Otherwise, $u = \sigma(v)$ would be a subterm of $\sigma(\ell)$, thus failing to be O -minimal. □

As a simple corollary of Proposition 66, infinite computations starting from minimal operational nonterminating terms can be visualized as paths over $\mathbb{N} \times \mathbb{N}$, where each bidimensional point (x_i, y_i) is labeled with a rule α_i .

Theorem 67. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a deterministic 3-CTRS and $t \in \mathcal{T}_{O-\infty}$. There is a substitution σ and an infinite sequence $\{(x_i, y_i, \alpha_i)\}_{i \in \mathbb{N}}$ of triples $(x_i, y_i, \alpha_i) \in \mathbb{N} \times \mathbb{N} \times R$, where α_i is $\ell_i \rightarrow r_i \leftarrow \bigwedge_{j=1}^{n_i} s_j^i \rightarrow t_j^i$, such that, for all $i \geq 0$, $x_{i+1} + y_{i+1} = x_i + y_i + 1$ and*

1. $x_0 = y_0 = 0$ and $t \xrightarrow{\epsilon}^* \sigma(\ell_0)$.
2. For all $i \geq 0$, $\sigma(\ell_i) \in \mathcal{T}_{O-\infty}$; furthermore, there is a term v_i such that $\ell_i \not\triangleright v_i$, $\sigma(v_i) \in \mathcal{T}_{O-\infty}$, $\sigma(v_i) \xrightarrow{\epsilon}^* \sigma(\ell_{i+1})$, and one of the following holds:
 - (a) $x_{i+1} = x_i + 1$, $v_i \in \mathcal{D}\text{Subterm}(\mathcal{R}, r_i)$ and $\alpha_i \in \mathcal{D}\text{Rules}(\mathcal{R})$.
 - (b) $y_{i+1} = y_i + 1$ and there is j , $1 \leq j \leq n_i$ s.t. $v_i \in \mathcal{D}\text{Subterm}(\mathcal{R}, s_j^i)$ and $\alpha_i \in \mathcal{C}\text{Rules}(\mathcal{R})$.

Example 68. *For \mathcal{R} in Example 49, Figure 3 shows the representation of the computation starting from $f(\mathbf{a}) \in \mathcal{T}_{O-\infty}$ according to Theorem 67, which corresponds to the following proof tree:*

$$\begin{array}{c}
\vdots \\
\frac{f(\mathbf{a}) \rightarrow \mathbf{b}}{c(\mathbf{f}(\mathbf{a})) \rightarrow c(\mathbf{b})}^{(C)} \quad \frac{c(\mathbf{b}) \rightarrow^* c(\mathbf{b})}{c(\mathbf{f}(\mathbf{a})) \rightarrow^* c(\mathbf{b})}^{(T)} \\
\frac{\frac{b \rightarrow f(\mathbf{a})}{c(\mathbf{b}) \rightarrow c(\mathbf{f}(\mathbf{a}))}^{(Rp)} \quad \frac{c(\mathbf{f}(\mathbf{a})) \rightarrow c(\mathbf{b})}{c(\mathbf{f}(\mathbf{a})) \rightarrow^* c(\mathbf{b})}^{(C)} \quad \frac{c(\mathbf{b}) \rightarrow^* c(\mathbf{b})}{c(\mathbf{f}(\mathbf{a})) \rightarrow^* c(\mathbf{b})}^{(T)}}{c(\mathbf{b}) \rightarrow^* c(\mathbf{b})}^{(T)} \\
\frac{g(\mathbf{a}) \rightarrow c(\mathbf{b})}{g(\mathbf{a}) \rightarrow^* c(\mathbf{b})}^{(Rp)} \quad \frac{c(\mathbf{b}) \rightarrow^* c(\mathbf{b})}{g(\mathbf{a}) \rightarrow^* c(\mathbf{b})}^{(T)} \\
\frac{g(\mathbf{a}) \rightarrow^* c(\mathbf{b})}{f(\mathbf{a}) \rightarrow \mathbf{b}}^{(Rp)}
\end{array}$$

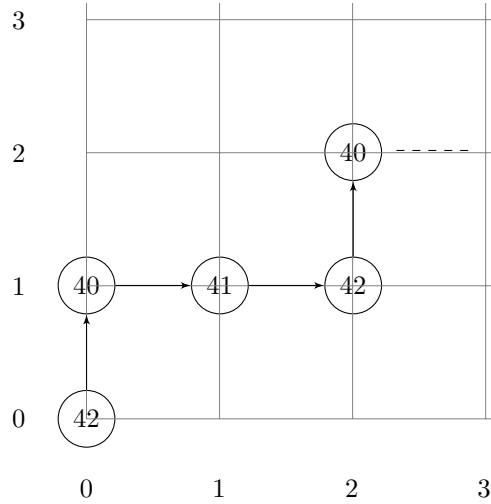


Figure 3: Computations starting with $f(\mathbf{a})$ for \mathcal{R} in Example 49

Remark 69. *The minimal sequence $f(\mathbf{a}) \rightarrow_{(42)} \mathbf{b} \rightarrow_{(41)} f(\mathbf{a}) \rightarrow_{(42)} \mathbf{b} \rightarrow \dots$ is also possible for \mathcal{R} in Example 49. This is because $\sigma(\mathbf{g}(x)) \rightarrow^* \sigma(\mathbf{c}(y))$ for rule (42) is satisfied without any reduction on \mathbf{b} if $\sigma(x) = \mathbf{a}$ and $\sigma(y) = \mathbf{b}$. The implicit assumption in the computation model of Proposition 66 is that only reachability conditions $\sigma(s_i) \rightarrow^* \sigma(t_i)$ that are free of any infinite computation are important to decide the application of a rule. This makes sense in practice.*

The following result shows that an infinite computation starting from a minimal operationally nonterminating term can either:

1. start an infinite (*horizontal*) *rewrite* sequence (possibly as part of the evaluation of one of the conditions of a rule) involving an infinite number of rules in $DRules(\mathcal{R})$ (nontermination), or else
2. climb infinitely many ‘*vertical*’ steps over the conditions in the rules, like in an *infinite stair*. In the last case, those vertical steps can be preceded by a *finite* number of *horizontal* steps using rules in $DRules(\mathcal{R})$, like in the horizontal two-step-segments using rules (40) and (41) in Figure 3 (non-*V*-termination).

Corollary 70. *Let \mathcal{R} be a deterministic 3-CTRS and $t \in \mathcal{T}_{O-\infty}$. Then, the sequence $\{(x_i, y_i, \alpha_i)\}_{i \geq 0}$ associated to t according to Theorem 67 satisfies one of the following conditions. Either:*

1. *There is $k \geq 0$ and an infinite ‘horizontal’ sequence $\{(x_i, y_k, \alpha_i)\}_{i \geq k}$ such that for all $i \geq k$, $x_{i+1} = x_i + 1$ and $\alpha_i \in DRules(\mathcal{R})$, or*
2. *For each $i \in \mathbb{N}$ such that $y_i > 0$ and $y_i = y_{i-1} + 1$, there is $k_i > i$ such that $y_{k_i-1} = y_i$, $y_{k_i} = y_i + 1$, $\alpha_{k_i-1} \in CRules(\mathcal{R})$, and for all j , $i \leq j < k_i - 1$, $\alpha_j \in DRules(\mathcal{R})$.*

7. Proving termination, V -termination, and operational termination of CTRSs using 2D DPs

We often collectively call $\text{DP}_H(\mathcal{R})$ and $\text{DP}_V(\mathcal{R})$ the *2D Dependency Pairs* of a CTRS \mathcal{R} . We can use them to prove termination, V -termination, and operational termination of deterministic 3-CTRSs and also to disprove operational termination of CTRSs (of any type). For this purpose, we use the following notion of *O-chain*.

Definition 71. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be CTRSs. A $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -O-chain is a finite or infinite sequence of (renamed) rules $u_i \rightarrow v_i \leftarrow c_i \in \mathcal{P}$, together with a substitution σ satisfying that, for all $i \geq 1$,*

1. *for all $s \rightarrow t \in c_i$, $\sigma(s) \rightarrow_{\mathcal{R}}^* \sigma(t)$ and*
2. *$\sigma(v_i) \rightarrow_{\mathcal{R}} \cup \xrightarrow{\mathcal{Q}, \mathcal{R}}^* \sigma(u_{i+1})$, where for all terms s, t , we write $s \xrightarrow{\mathcal{Q}, \mathcal{R}} t$ if there is $\ell \rightarrow r \leftarrow c \in \mathcal{Q}$ and a substitution θ such that $s = \theta(\ell)$, $t = \theta(r)$ and $\theta(u) \rightarrow_{\mathcal{R}}^* \theta(v)$ for all $u \rightarrow v \in c$.*

A $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -O-chain is called *minimal* if for all $i \geq 1$, whenever

$$\sigma(v_i) = w_{i1} \rightarrow_{\mathcal{R}}^* \circ \xrightarrow{\mathcal{Q}, \mathcal{R}} w_{i2} \rightarrow_{\mathcal{R}}^* \circ \xrightarrow{\mathcal{Q}, \mathcal{R}} \cdots \rightarrow_{\mathcal{R}}^* \circ \xrightarrow{\mathcal{Q}, \mathcal{R}} w_{im_i} \rightarrow_{\mathcal{R}}^* \sigma(u_{i+1}),$$

in the chain, then for all j , $1 \leq j \leq m_i$, w_{ij} is \mathcal{R} -operationally terminating.

First, we show that termination of CTRSs \mathcal{R} that preserve terminating substitutions can be proved by using O-chains involving pairs in $\text{DP}_H(\mathcal{R})$ only.

Theorem 72 (Termination II). *Let \mathcal{R} be a CTRS.*

1. *If there is no infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain and \mathcal{R} preserves terminating substitutions, then \mathcal{R} is terminating.*
2. *If there is an infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain, then \mathcal{R} is nonterminating.*

PROOF.

1. Every $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -H-chain is also a $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain. Thus, if there is no infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain, then there is no infinite minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -H-chain. Since \mathcal{R} preserves terminating substitutions, by Corollary 50, \mathcal{R} is terminating.
2. If there is an infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain Γ , then there is a substitution σ and an infinite sequence $(u_i \rightarrow v_i \leftarrow c_i)_{i \geq 1}$ of pairs in $\text{DP}_H(\mathcal{R})$ such that for all $s \rightarrow t \in c_i$, $\sigma(s) \rightarrow_{\mathcal{R}}^* \sigma(t)$ and $\sigma(v_i) \rightarrow_{\mathcal{R}}^* \sigma(u_{i+1})$. Since for all $u \rightarrow v \leftarrow c \in \text{DP}_H(\mathcal{R})$, we have that $v \notin \mathcal{X}$, we have $v \notin \mathcal{X} - \text{Var}(u_i) \subseteq \mathcal{X}$. Thus, Γ is an infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -H-chain as well. By Theorem 47, \mathcal{R} is not terminating. □

Requiring the absence of infinite *minimal* $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chains (only) in Theorem 72(1) does *not* imply the absence of infinite minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -H-chains (as required to use Corollary 50 in the proof): since termination does not imply operational termination, there can be minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -H-chains which are *not* minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chains.

Example 73. Consider the first nonterminating CTRS \mathcal{R} in Example 27. We have $\text{DP}_H(\mathcal{R}) = \{F(\mathbf{a}) \rightarrow F(\mathbf{a}), F(\mathbf{a}) \rightarrow \mathbf{A}\}$. Note that \mathbf{a} is an irreducible term, hence terminating; however, it is not operationally terminating due to rule (24). There is an infinite minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -H-chain (where $F(\mathbf{a})$ is obviously \mathcal{R} -terminating):

$$F(\mathbf{a}) \rightarrow_{\text{DP}_H(\mathcal{R})} F(\mathbf{a}) \rightarrow_{\text{DP}_H(\mathcal{R})} \cdots \rightarrow_{\text{DP}_H(\mathcal{R})} F(\mathbf{a}) \rightarrow_{\text{DP}_H(\mathcal{R})} \cdots$$

However, it is not a minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain because $F(\mathbf{a})$ is not operationally terminating. There is no infinite minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain (but \mathcal{R} is nonterminating). Thus, focusing on minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chains in Theorem 72 could lead to wrongly concluding termination of \mathcal{R} .

V-termination of CTRSs can also be investigated by using O-chains.

Theorem 74 (V-termination II). Let \mathcal{R} be CTRS.

1. If there is no infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain and \mathcal{R} is a deterministic 3-CTRS, then \mathcal{R} is V-terminating.
2. If there is an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain, then \mathcal{R} is non-V-terminating.

PROOF.

1. For all CTRSs \mathcal{P} , \mathcal{Q} , and \mathcal{R} , every $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -O-chain is a $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -V-chain and viceversa. Thus, the absence of infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chains implies the absence of infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chains, in particular the absence of infinite minimal $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chains. By Theorem 65, \mathcal{R} is V-terminating.
2. If there is an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain, then there is an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain. By Theorem 65, \mathcal{R} is non-V-terminating.

□

Unfortunately, we cannot restrict our attention to proving the absence of infinite *minimal* $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chains in Theorem 74(1): since V-termination does *not* imply operational termination, there can be minimal $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -V-chains that are not minimal $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -O-chains.

Remark 75. Theorems 72 and 74 allow us to prove and disprove termination and V-termination of CTRSs using the same dependency pairs and chains that are used to prove and disprove operational termination of CTRSs.

7.1. Characterizing operational termination of CTRSs using 2D DPs

We now establish the conditions for proving and disproving operational termination of CTRSs using 2D Dependency Pairs.

Theorem 76 (Operational termination). Let \mathcal{R} be a CTRS.

1. If \mathcal{R} is a deterministic 3-CTRS, there is no infinite minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain, and there is no infinite minimal $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain, then \mathcal{R} is operationally terminating.
2. If there is an infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain or an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain, then \mathcal{R} is operationally nonterminating.

PROOF.

1. By contradiction. If \mathcal{R} is not operationally terminating, then there is $t \in \mathcal{T}_{O-\infty}$ (Lemma 21). By Corollary 70, there is a computation whose bidimensional representation according to Theorem 67 satisfies one of the following conditions:

- (a) There is $k > 0$, a rule $\ell_k \rightarrow r_k \Leftarrow c_k$, and an infinite ‘horizontal’ sequence $\{(x_i, y_k, \alpha_i)\}_{i \geq k}$ such that for all $i \geq k$, $x_{i+1} = x_i + 1$, $\alpha_i : \ell_i \rightarrow r_i \Leftarrow c_i \in \text{DRules}(\mathcal{R})$, and there is $v_i \in \mathcal{DSubterm}(\mathcal{R}, r_i)$ such that v_i is not a proper subterm of ℓ_i , $\sigma(v_i) \in \mathcal{T}_{O-\infty}$, and $\sigma(v_i) \xrightarrow{\geq \epsilon}^* \sigma(\ell_{i+1})$. Then, for all $i \geq k$, $\ell_i^\# \rightarrow v_i^\# \Leftarrow c_i \in \text{DP}_H(\mathcal{R})$, $\sigma(v_i^\#)$ is operationally \mathcal{R} -terminating, and $\sigma(v_i^\#) \xrightarrow{\geq \epsilon}^* \sigma(\ell_{i+1}^\#)$. Thus, there is an infinite minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain which contradicts our initial assumption.
- (b) For each $i \in \mathbb{N}$ such that $y_i > 0$ and $y_i = y_{i-1} + 1$, there is $\nu_i > i$ such that $y_{\nu_i} = y_i + 1$, $\alpha_{\nu_i-1} \in \text{CRules}(\mathcal{R})$ and for all j , $i \leq j < \nu_i - 1$, $\alpha_j \in \text{DRules}(\mathcal{R})$. Therefore, by Theorem 67, we have:

- i. Since $y_i = y_{i-1} + 1$, we have $\ell_{i-1}^\# \rightarrow v_{i-1}^\# \Leftarrow \bigwedge_{j=1}^{k-1} s_j^{i-1} \rightarrow t_j^{i-1} \in \text{DP}_V(\mathcal{R})$ for some $1 \leq k \leq n_{i-1}$.

- ii. there is a rule $\alpha_{\nu_i-1} : \ell_{\nu_i-1} \rightarrow r_{\nu_i-1} \Leftarrow \bigwedge_{j=1}^{n_{\nu_i-1}} s_j^{\nu_i-1} \rightarrow t_j^{\nu_i-1}$ and a term $v_{\nu_i-1} \in \mathcal{DSubterm}(\mathcal{R}, s_k^{\nu_i-1})$ for some k , $1 \leq k \leq n_{\nu_i-1}$, such that $\sigma(v_{\nu_i-1}) \in \mathcal{T}_{O-\infty}$, and $\sigma(v_{\nu_i-1}) \xrightarrow{\geq \epsilon}^* \sigma(\ell_{\nu_i})$. Therefore, $\ell_{\nu_i-1}^\# \rightarrow v_{\nu_i-1}^\# \Leftarrow \bigwedge_{j=1}^{k-1} s_j^{\nu_i-1} \rightarrow t_j^{\nu_i-1} \in \text{DP}_V(\mathcal{R})$, $\sigma(v_{\nu_i-1}^\#)$ is operationally \mathcal{R} -terminating, and $\sigma(v_{\nu_i-1}^\#) = \sigma(v_{\nu_i-1}^\#)$ also is.

- iii. for all j , $i \leq j < \nu_i - 1$, and $\alpha_j : \ell_j \rightarrow r_j \Leftarrow c_j \in \text{DRules}(\mathcal{R})$, there is $v_j \in \mathcal{DSubterm}(\mathcal{R}, r_j)$ such that $\sigma(v_j) \in \mathcal{T}_{O-\infty}$, $\sigma(v_j^\#)$ is operationally \mathcal{R} -terminating, and $\ell_j^\# \rightarrow v_j^\# \Leftarrow c_j \in \text{DP}_H(\mathcal{R})$ is such that $\sigma(c_j)$ holds and $\sigma(v_j^\#) \rightarrow_{\mathcal{R}}^* \sigma(\ell_{j+1}^\#)$. Furthermore, $\sigma(v_{i-1}^\#) \rightarrow_{\mathcal{R}}^* \sigma(\ell_i^\#)$.

We repeat this to obtain an infinite minimal $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain, which leads to a contradiction.

2. If there is an infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain, then, by Theorem 72, \mathcal{R} is not terminating and, therefore, it is not operationally terminating. If there is an infinite $(\text{DP}_H(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain, then, by Theorem 74, \mathcal{R} is not V -terminating and, therefore, it is not operationally terminating.

□

Example 77. For \mathcal{R} in Example 2, and $\text{DP}_V(\mathcal{R})$ and $\text{DP}_{VH}(\mathcal{R})$ as given in Examples 57 and 61, there is an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain:

$$\mathbf{F}(\mathbf{a}) \rightarrow_{\text{DP}_{VH}(\mathcal{R})} \mathbf{G}(\mathbf{a}) \rightarrow_{\text{DP}_{VH}(\mathcal{R})} \mathbf{B} \rightarrow_{\text{DP}_{VH}(\mathcal{R})} \mathbf{F}(\mathbf{a}) \rightarrow_{\text{DP}_V(\mathcal{R})} \dots$$

witnessing operational nontermination of \mathcal{R} . For \mathcal{R} in Example 49, we have a similar proof of operational nontermination.

The following result establishes that we can always move rules from \mathcal{Q} to \mathcal{P} without losing infinite (minimal) O-chains.

Proposition 78. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be CTRSs. For every infinite (minimal) $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -O-chain there is an infinite (minimal) $(\mathcal{P} \cup \mathcal{Q}, \emptyset, \mathcal{R})$ -O-chain.

PROOF. Let Γ be an infinite $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ -O-chain with substitution σ . Then, there is an infinite number of renamings $u_i \rightarrow v_i \leftarrow c_i$ of rules in \mathcal{P} such that, for all $i \geq 1$, $\sigma(s) \rightarrow_{\mathcal{R}}^* \sigma(t)$ for all $s \rightarrow t \in c_i$, $\sigma(v_i) (\rightarrow_{\mathcal{R}} \cup \xrightarrow{\mathcal{Q}, \mathcal{R}})^* \sigma(u_{i+1})$. If Γ is minimal, then $\sigma(v_i)$ is operationally terminating. Assume that each connection between $\sigma(v_i)$ and $\sigma(u_{i+1})$ involves $q_i \geq 0$ steps with $\xrightarrow{\mathcal{Q}, \mathcal{R}}$ using variants $u'_1 \rightarrow v'_1 \leftarrow c'_1, \dots, u'_{q_i} \rightarrow v'_{q_i} \leftarrow c'_{q_i}$ of rules in \mathcal{Q} . Note that each $\xrightarrow{\mathcal{Q}, \mathcal{R}}$ -step with a rule $u' \rightarrow v' \leftarrow c' \in \mathcal{Q}$ implies that $\sigma(s') \rightarrow_{\mathcal{R}}^* \sigma(t')$ for all $s' \rightarrow t' \in c'$. Then, we can build the infinite $(\mathcal{P} \cup \mathcal{Q}, \emptyset, \mathcal{R})$ -O-chain Γ'

$$u_1 \rightarrow v_1 \leftarrow c_1, u'_1 \rightarrow v'_1 \leftarrow c'_1, \dots, u'_{q_1} \rightarrow v'_{q_1} \leftarrow c'_{q_1}, \dots, u_2 \rightarrow v_2 \leftarrow c_2, \dots$$

with the same substitution σ . If Γ is minimal, for all j , $1 \leq j \leq q_1$, $\sigma(v'_j)$ is operationally terminating, i.e., Γ' is minimal as well. □

In general, the converse does not hold. For instance, with $\mathcal{Q} = \{\mathbf{a} \rightarrow \mathbf{a}\}$ we have a trivial infinite $(\mathcal{Q}, \emptyset, \mathcal{R})$ -O-chain. But there is no $(\emptyset, \mathcal{Q}, \mathcal{R})$ -O-chain!

8. Contributions and related work

Termination of CTRSs was defined in [6, Definition 4.7(i)] as the absence of infinite rewrite sequences. Further research on the topic led to the understanding that such a traditional view of termination based on rewrite sequences did not provide a good account of the termination behavior of rewriting with CTRSs because the role of the conditions was neglected. Ohlebusch [36, Section 7.2] provides a good account of the development of the area until the end of the nineties. However, [36] presented five different notions of CTRS termination, some of them actually wrong, in the sense that any reasonable interpreter will loop evaluating CTRSs declared terminating under the given notion, so considerable uncertainty about the right notion of CTRS termination remained. The notion of *operational termination* was proposed in [26] for general logics [32] and used in [14] to study and characterize the termination of *MEL Rewrite Theories*

(based on the Membership Equational Logic (MEL) in [33]) and of CTRSs in [26], where the notion of *quasi-decreasingness* (see [36, Definition 7.2.39]) and of operational termination of a CTRS were proved equivalent.

In this setting, the contributions of this paper can be summarized as follows. First, we have revisited operational termination of CTRSs to establish connections with existing termination properties of CTRSs:

1. We describe termination of CTRSs in a *proof-theoretic* style as the absence of a specific class of infinite well-formed proof trees of the CTRS logic.
2. We have defined a new termination property of CTRSs, which we call *V-termination*, and proved that operational termination of CTRSs is characterized as the *conjunction* of termination and *V-termination*.
3. We have defined different notions of *minimal non- λ -terminating terms* (for $\lambda \in \{H, V, O\}$, corresponding to the three aforementioned termination properties of CTRSs) and proved a number of auxiliary results which are the basis for the development of the different notions of dependency pairs that are introduced to characterize termination, *V-termination*, and operational termination of CTRSs.

Thanks to the above-mentioned results, we have also been able to develop a *unified methodology* (described in the first paragraphs of Section 3.1) to analyze termination, *V-termination*, and operational termination using the notions of minimal term and minimal computation, in agreement with previous experiences developing frameworks for proving termination of rewriting [1, 17, 21] and variants of rewriting like context-sensitive rewriting (as in [2, 20]), order-sorted rewriting (as in [27]), and equational rewriting with *AVC*-theories (as in [4]).

Regarding termination, *V-termination* and operational termination of CTRSs:

1. Our characterization of termination of CTRSs using dependency pairs $\text{DP}_H(\mathcal{R})$ and $\text{DP}_C(\mathcal{R})$ is novel in the literature, valid for arbitrary CTRSs, and draws interesting connections with other termination analyses where collapsing dependency pairs also play an essential role (see, e.g., [2, 20]).
2. The analysis of termination of 2-CTRSs can also be accomplished as termination of the *underlying TRS* (i.e., the TRS \mathcal{R}_u which is obtained by just dropping the conditional part of the rules), see [36, Lemma 7.1.2] for instance. However, in contrast to our Corollary 51, the analysis of termination of 2-CTRSs \mathcal{R} as termination of the underlying TRS \mathcal{R}_u provides a *sufficient* condition only; it may fail in those cases where taking into account the conditions of the rules is essential to prove termination.

Example 79. *The one rule 1-CTRS \mathcal{R}*

$$a \rightarrow a \Leftarrow a \rightarrow b \tag{70}$$

is terminating but $\mathcal{R}_u = \{a \rightarrow a\}$ is not. With $\text{DP}_H(\mathcal{R}) = \{A \rightarrow A \Leftarrow a \rightarrow b\}$ we see that there is no infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain (due to the insatisfiability of the conditional part of the pair). Thus, \mathcal{R} is terminating, by Theorem 72.

3. We provide a complete characterization of V -termination of deterministic CTRSs using dependency pairs $\text{DP}_V(\mathcal{R})$ and $\text{DP}_{VH}(\mathcal{R})$ (which is a subset of the dependency pairs in $\text{DP}_H(\mathcal{R})$). Since, as far as we know, this property has never been formulated before, our developments are a new contribution to the analysis of termination properties of CTRSs.
4. Thanks to our results in the first part of the paper, proofs of termination and V -termination of CTRSs can be immediately used to conclude operational termination. However, we provide a simpler analysis of *operational termination* of CTRSs on the basis of $\text{DP}_H(\mathcal{R})$, $\text{DP}_V(\mathcal{R})$ and $\text{DP}_{VH}(\mathcal{R})$, where $\text{DP}_C(\mathcal{R})$ becomes unnecessary.
5. Our definitions highlight the flexibility of our approach: with the same notion of O-chain we can prove not only operational termination of CTRSs, but also termination and V -termination of CTRSs. We believe that this is a good basis for the development of a unified *dependency pair framework* for CTRSs which can be used to prove or disprove all these properties.

8.1. Related work

8.1.1. Nakamura et al.'s Conditional Dependency Pairs

The *Conditional Dependency Pairs* (CDPs) by Nakamura et al. [34] apply to a restricted subclass of *1-CTRSs*: the condition c in the 1-rules ($\ell \rightarrow r \Leftarrow c$) considered in [34] is a *term* rather than a sequence $s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$. An instance $\sigma(c)$ of condition c is satisfied if and only if $\sigma(c) \rightarrow^* \text{true}$. For the 1-CTRSs considered in [34], our proposal generates a *subset* of the pairs considered in [34, Definition 3.1], i.e., $\text{DP}_H(\mathcal{R}) \cup \text{DP}_V(\mathcal{R}) \subseteq \text{CDP}(\mathcal{R})$. Often, the inclusion is strict due to our more restrictive generation of pairs: we avoid using defined subterms v in the right-hand side r of a rule $\ell \rightarrow r \Leftarrow c$ (or in the left-hand side s of a condition $s \rightarrow t \in c$) which are *strict subterms* of the left-hand side⁸, ℓ .

Example 80. Consider the following 1-CTRS with a conditional rule à la Nakamura et al.:

$$\text{f}(\text{false}) \rightarrow \text{true} \tag{71}$$

$$\text{f}(\text{f}(x)) \rightarrow \text{false} \Leftarrow \text{f}(x) \rightarrow \text{true} \tag{72}$$

Here, $\text{CDP}(\mathcal{R}) = \{\text{F}(\text{f}(x)) \rightarrow \text{F}(x)\}$. However, we would not include it as part of (in this case) $\text{DP}_V(\mathcal{R})$. This is because $\text{f}(x)$ in the condition of the conditional rule is a strict subterm of the left-hand side $\text{f}(\text{f}(x))$ of the rule. Such defined subterms are ruled out by our definition of $\text{DP}_V(\mathcal{R})$ regarding the generation of a dependency pair. Actually, $\text{DP}_H(\mathcal{R}) = \text{DP}_V(\mathcal{R}) = \emptyset$.

Their notion of chain ([34, Definition 3.2]) is also different from our Definition 71 (no component \mathcal{Q} is used in [34]). The following results, involving chains of a simpler type (like those in [34], where pairs are connected by rewritings with \mathcal{R} only), also characterize operational termination of CTRSs.

⁸This observation is originally due to Dershowitz [10] and exploited by Hirokawa and Middeldorp to refine the definition of DPs for TRSs [21, 22].

Theorem 81 (Operational termination II). *Let \mathcal{R} be a CTRS.*

1. *If there is no infinite (minimal) $(\text{DP}_H(\mathcal{R}) \cup \text{DP}_V(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain and \mathcal{R} is a deterministic 3-CTRS, then \mathcal{R} is operationally terminating.*
2. *If there is an infinite $(\text{DP}_H(\mathcal{R}) \cup \text{DP}_V(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain, then \mathcal{R} is operationally nonterminating.*

PROOF.

1. By contradiction. If \mathcal{R} is not operationally terminating, Theorem 76 ensures that (i) there is an infinite minimal $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain or (ii) there is an infinite minimal $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain. In both cases (with (ii) using Proposition 78 and since $\text{DP}_{VH}(\mathcal{R}) \subseteq \text{DP}_H(\mathcal{R})$), there is an infinite minimal $(\text{DP}_H(\mathcal{R}) \cup \text{DP}_V(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain.
2. By contradiction. If \mathcal{R} is operationally terminating, Theorem 76 ensures that there is no infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain and there is no infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain. Assume that there is an infinite $(\text{DP}_H(\mathcal{R}) \cup \text{DP}_V(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain $\Gamma : (u_i \rightarrow v_i \leftarrow c_i)_{i \geq 1}$ with substitution σ . Since there is no infinite $(\text{DP}_H(\mathcal{R}), \emptyset, \mathcal{R})$ -O-chain, Γ contains an infinite number of occurrences of pairs in $\text{DP}_V(\mathcal{R})$ and Γ can be seen as a (nonminimal) $(\text{DP}_V(\mathcal{R}), \text{DP}_H(\mathcal{R}), \mathcal{R})$ -V-chain. By Proposition 63, Γ is an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -V-chain. Since every V-chain is also an O-chain (but minimality is not guaranteed to be preserved!), Γ is an infinite $(\text{DP}_V(\mathcal{R}), \text{DP}_{VH}(\mathcal{R}), \mathcal{R})$ -O-chain. This leads to a contradiction. □

Theorem 81 provides an alternative characterization of operational termination of CTRSs, which is already more general than the one in [34] because it applies to deterministic 3-CTRSs. However, these results are *less powerful* than those in Section 7 in that there is no explicit distinction between the two dimensions of operational termination; thus, no analysis of termination or V-termination is possible with these results (or within [34]).

8.1.2. Transformation techniques

As remarked in the introduction, existing tools for proving termination of deterministic 3-CTRSs currently use transformation techniques. Except for [30] (see Section 8.2 below), we are not aware of any implementation of *direct* CTRS termination methods. The transformation which is typically used for this purpose is \mathcal{U} in [36, Definition 7.2.48]. This transformation is not complete, however (see Example 3, where \mathcal{R} is operationally terminating but $\mathcal{U}(\mathcal{R})$ is not). Thus, disproving operational termination is *not possible* with this transformation, in sharp contrast to our approach. Furthermore, when $\mathcal{U}(\mathcal{R})$ is terminating, tools may fail to find a proof. This is often due to the loss of information introduced by transformations, and also to the presence of new symbols and rules that prevent the search process from finding a proof.

Schernhammer and Gramlich investigated a variant of the transformation \mathcal{U} originally proposed in [13], where symbols are given *replacement restrictions*

by means of a *replacement map* μ which associates a subset $\mu(f)$ of reducible arguments to each function symbol f in the signature [25]. In this variant of the transformation \mathcal{U} , only the new symbols U introduced by the transformation are given replacement restrictions; in particular, $\mu(U) = \{1\}$ for all such symbols, whilst $\mu(f) = \{1, \dots, ar(f)\}$ for all symbols $f \in \mathcal{F}$ in the signature \mathcal{F} of \mathcal{R} (i.e., the *original* symbols get no replacement restriction). They proved that this variant $\mathcal{U}_\mu(\mathcal{R})$ of the transformation is *complete* for proving operational termination *of terms of the original signature*, i.e., if a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is terminating with respect to the *context-sensitive rewrite relation* $\rightarrow_{\mathcal{U}_\mu(\mathcal{R}), \mu}$ induced by the replacement map μ and the TRS $\mathcal{U}_\mu(\mathcal{R})$, then t is operationally \mathcal{R} -terminating. Still, there can be terms in the signature of $\mathcal{U}_\mu(\mathcal{R})$ which are *not* terminating even when \mathcal{R} is operationally terminating. Thus, \mathcal{U}_μ is still *unable* to prove operational nontermination of \mathcal{R} as nontermination of $\mathcal{U}_\mu(\mathcal{R})$.

8.2. Practical use of our results

Our results are the basis of the implementation described in [30] as part of the tool MU-TERM. We have participated in the 2014 and 2015 editions of the *International Termination Competition* where we were able to obtain the first position among the participating tools of the *TRS Conditional* subcategory, see

http://nfa.imn.htwk-leipzig.de/termcomp/show_job_results/5382

With our tool, we were able to outperform for the CTRS class other termination tools like AProVE [18] and VMTL [38] that rely on the use of the aforementioned sound transformations. Moreover, these tools are *not* able to disprove operational termination of CTRSs due to the incompleteness of the transformations.

9. Conclusion

To the best of our knowledge this is the first correct and complete characterization of both *termination* and *operational termination* of CTRSs which is based on the notion of dependency pair. We have proposed the new notion of V -termination of CTRSs and showed that, together with termination, it is one of the *dimensions* of operational termination of CTRSs. The corresponding notions of minimal non- V -terminating and operationally nonterminating term and the properties explored here are also new in the literature. Our *bidimensional* approach to the problem of proving operational termination of CTRSs is useful to simplify the analysis of operational termination and also to prove already known termination properties like nontermination of 3-CTRSs and termination of (a subclass of) 3-CTRSs.

The theoretical notions presented in this paper are the basis for the implementation of our techniques for automatically proving operational termination of CTRSs that have been developed in [30], and incorporated in the latest version of the tool MU-TERM [3]. This makes these techniques available to tools like MTT [12], which use MU-TERM as a backend for achieving proofs of operational termination of more general theories like membership equational programs or

order-sorted rewrite theories (see [29] for a recent account of the computational problems arising when computing with these theories and some envisaged solutions). Direct termination methods for these wider logics will require extending the techniques presented here to the case of order-sorted conditional rewrite theories with types and subtypes, and where rewriting is context-sensitive and can take place modulo axioms B . This is a subject for future work. Also, a deeper investigation about the role of V -termination in the practical use of strategies for conditional rewriting is another interesting subject of future work.

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