

Closed subsets of compact-like topological spaces

SERHII BARDYLA^a AND ALEX RAVSKY^b

^a Institute of Mathematics, University of Vienna, Austria. (sbardyla@yahoo.com)

^b Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, Nat. Acad. Sciences of Ukraine, Lviv, Ukraine. (alexander.ravsky@uni-wuerzburg.de)

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ABSTRACT

We investigate closed subsets (subsemigroups, resp.) of compact-like topological spaces (semigroups, resp.). We show that each Hausdorff topological space is a closed subspace of some Hausdorff ω -bounded pracomact topological space and describe open dense subspaces of countably pracomact topological spaces. We construct a pseudocompact topological semigroup which contains the bicyclic monoid as a closed subsemigroup. This example provides an affirmative answer to a question posed by Banach, Dimitrova, and Gutik in [4]. Also, we show that the semigroup of $\omega \times \omega$ -matrix units cannot be embedded into a Hausdorff topological semigroup whose space is weakly H -closed.

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1. PRELIMINARIES

In this paper all topological spaces are assumed to be Hausdorff. By ω we denote the first infinite cardinal. For ordinals α, β put $\alpha \leq \beta$, ($\alpha < \beta$, resp.) iff $\alpha \subset \beta$ ($\alpha \subset \beta$ and $\alpha \neq \beta$, resp.). By $[\alpha, \beta]$ ((α, β) , $(\alpha, \beta]$, (α, β) , resp.) we denote the set of all ordinals γ such that $\alpha \leq \gamma \leq \beta$ ($\alpha \leq \gamma < \beta$, $\alpha < \gamma \leq \beta$, $\alpha < \gamma < \beta$, resp.). The cardinality of a set X is denoted by $|X|$.

For a subset A of a topological space X by \overline{A} we denote the closure of the set A in X .

A family \mathcal{F} of subsets of a set X is called a *filter* if it satisfies the following conditions:

- (1) $\emptyset \notin \mathcal{F}$;
- (2) If $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$;
- (3) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

A family \mathcal{B} is called a *base* of a filter \mathcal{F} if for each element $A \in \mathcal{F}$ there exists an element $B \in \mathcal{B}$ such that $B \subset A$. A filter on a topological space X is called an ω -*filter* if it has a countable base. A filter \mathcal{F} is called *free* if $\bigcap \mathcal{F} = \emptyset$. A filter on a topological space X is called *open* if it has a base which consists of open subsets. A point x is called an *accumulation point* (θ -*accumulation point*, resp.) of a filter \mathcal{F} if for each open neighborhood U of x and for each $F \in \mathcal{F}$ the set $U \cap F$ ($\overline{U} \cap F$, resp.) is non-empty. A topological space X is said to be

- *compact*, if each filter has an accumulation point;
- *sequentially compact*, if each sequence $\{x_n\}_{n \in \omega}$ of points of X has a convergent subsequence;
- ω -*bounded*, if each countable subset of X has compact closure;
- *totally countably compact*, if each sequence of X contains a subsequence with compact closure;
- *countably compact*, if each infinite subset $A \subseteq X$ has an accumulation point;
- ω -*bounded precompact*, if there exists a dense subset D of X such that each countable subset of the set D has compact closure in X ;
- *totally countably precompact*, if there exists a dense subset D of X such that each sequence of points of the set D has a subsequence with compact closure in X ;
- *countably precompact*, if there exists a dense subset D of X such that every infinite subset $A \subseteq D$ has an accumulation point in X ;
- *pseudocompact*, if X is Tychonoff and each real-valued function on X is bounded;
- *H-closed*, if each filter on X has a θ -accumulation point;
- *feebly ω -bounded*, if for each sequence $\{U_n\}_{n \in \omega}$ of non-empty open subsets of X there is a compact subset K of X such that $K \cap U_n \neq \emptyset$ for each $n \in \omega$;
- *totally feebly compact*, if for each sequence $\{U_n\}_{n \in \omega}$ of non-empty open subsets of X there is a compact subset K of X such that $K \cap U_n \neq \emptyset$ for infinitely many $n \in \omega$;
- *selectively feebly compact*, if for each sequence $\{U_n\}_{n \in \omega}$ of non-empty open subsets of X , for each $n \in \omega$ we can choose a point $x_n \in U_n$ such that the sequence $\{x_n : n \in \omega\}$ has an accumulation point.
- *feebly compact*, if each open ω -filter on X has an accumulation point.

The interplay between some of the above properties is shown in the diagram at page 3 in [13].

Remark 1.1. H-closed topological spaces have few different equivalent definitions. For a topological space X the following conditions are equivalent:

- X is H-closed;
- if X is a subspace of a Hausdorff topological space Y , then X is closed in Y ;
- each open filter on X has an accumulation point;
- for each open cover $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$ of X there exists a finite subset $B \subset A$ such that $\cup_{\alpha \in B} \overline{F_\alpha} = X$.

H-closed topological spaces in terms of θ -accumulation points were investigated in [7, 16, 17, 19, 21, 22, 28, 29]. Also observe that each H-closed space is feebly compact.

In this paper we investigate closed subsets (subsemigroups, resp.) of compact-like topological spaces (semigroups, resp.). We prove that each topological space can be embedded as a closed subspace into an H-closed topological space. However, the semigroup of $\omega \times \omega$ -matrix units cannot be embedded into a topological semigroup which is a weakly H-closed topological space. We show that each topological space is a closed subspace of some ω -bounded precompact topological space and describe open dense subspaces of countably precompact topological spaces. Also, we construct a pseudocompact topological semigroup which contains the bicyclic monoid as a closed subsemigroup, providing a positive solution of [4, Problem 7.2].

2. CLOSED SUBSPACES OF COMPACT-LIKE TOPOLOGICAL SPACES

The productivity of compact-like properties is a known topic in general topology. According to Tychonoff's theorem, a (Tychonoff) product of a family of compact spaces is compact, On the other hand, there are two countably compact spaces whose product is not feebly compact (see [10], the paragraph before Theorem 3.10.16). The product of a countable family of sequentially compact spaces is sequentially compact [10, Theorem 3.10.35]. But already the Cantor cube $D^{\mathfrak{c}}$ is not sequentially compact (see [10], the paragraph after Example 3.10.38). On the other hand some compact-like properties are also preserved by products, see [27, § 3-4] (especially Theorem 3.3, Proposition 3.4, Example 3.15, Theorem 4.7, and Example 4.15), [26, § 5], and [13, Sec. 2.3].

Proposition 2.1. *A product of any family of feebly ω -bounded spaces is feebly ω -bounded.*

Proof. Let $X = \prod\{X_\alpha : \alpha \in A\}$ be a product of a family of feebly ω -bounded spaces and let $\{U_n\}_{n \in \omega}$ be a family of non-empty open subsets of the space X . For each $n \in \omega$ let V_n be a basic open set in X which is contained in U_n . For each $n \in \omega$ and $\alpha \in A$ let $V_{n,\alpha} = \pi_\alpha(V_n)$ where by π_α we denote the projection on X_α . For each $\alpha \in A$ there exists a compact subset K_α of X_α , intersecting each $V_{n,\alpha}$. Then the set $K = \prod\{K_\alpha : \alpha \in A\}$ is a compact subset of X intersecting each $V_n \subset U_n$. □

A non-productive compact-like properties still can be preserved by products with more strong compact-like spaces. For instance, a product of a countably compact space and a countably compact k -space or a sequentially compact space is countably compact, and a product of a pseudocompact space and a pseudocompact k -space or a sequentially compact Tychonoff space is pseudocompact (see [10, Sec. 3.10]).

Proposition 2.2. *A product $X \times Y$ of a countably pracomact space X and a totally countably pracomact space Y is countably pracomact.*

Proof. Let D be a dense subset of X such that each infinite subset of D has an accumulation point in X and F be a dense subset of Y such that each sequence of points of the set F has a subsequence contained in a compact set. Then $D \times F$ is a dense subset of $X \times Y$. So to prove that the space $X \times Y$ is countably pracomact it suffices to show that each sequence $\{(x_n, y_n)\}_{n \in \omega}$ of points of $D \times F$ has an accumulation point. Taking a subsequence, if needed, we can assume that a sequence $\{y_n\}_{n \in \omega}$ is contained in a compact set K . Let $x \in X$ be an accumulation point of a sequence $\{x_n\}_{n \in \omega}$ and $\mathcal{B}(x)$ be the family of neighborhoods of the point x . For each $U \in \mathcal{B}(x)$ put $Y_U = \overline{\{y_n \mid x_n \in U\}}$. Then $\{Y_U \mid U \in \mathcal{B}(x)\}$ is a centered family of closed subsets of a compact space K , so there exists a point $y \in \bigcap \{Y_U \mid U \in \mathcal{B}(x)\}$. Clearly, (x, y) is an accumulation point of the sequence $\{(x_n, y_n)\}_{n \in \omega}$. \square

Proposition 2.3. *A product $X \times Y$ of a selectively feebly compact space X and a totally feebly compact space Y is selectively feebly compact.*

Proof. Let $\{U_n\}_{n \in \omega}$ be a sequence of open subsets of $X \times Y$. For each $n \in \omega$ pick a non-empty open subsets U_n^1 of X and U_n^2 of Y such that $U_n^1 \times U_n^2 \subset U_n$. Taking a subsequence, if needed, we can assume that there exists a compact subset K of the space Y intersecting each set U_n^2 , $n \in \omega$. Since X is selectively feebly compact, for each $n \in \omega$ we can choose a point $x_n \in U_n^1$ such that a sequence $\{x_n\}_{n \in \omega}$ has an accumulation point $x \in X$. For each $n \in \omega$ pick a point $y_n \in U_n^2 \cap K$. Then $(x_n, y_n) \in U_n^1 \times U_n^2 \subset U_n$. Let $\mathcal{B}(x)$ be the family of neighborhoods of the point x in X . For each $U \in \mathcal{B}(x)$ put $Y_U = \overline{\{y_n \mid x_n \in U\}}$. Then $\{Y_U \mid U \in \mathcal{B}(x)\}$ is a centered family of closed subsets of the compact space K , so there exists a point $y \in \bigcap \{Y_U \mid U \in \mathcal{B}(x)\}$. Clearly, (x, y) is an accumulation point of the sequence $\{(x_n, y_n)\}_{n \in \omega}$. \square

An *extension* of a space X is a Hausdorff space Y containing X as a dense subspace. Extensions of topological spaces were investigated in [8, 18, 23, 24, 25]. A class \mathcal{C} of spaces is called *extension closed* provided each extension of each space of \mathcal{C} belongs to \mathcal{C} . If Y is a space, a class \mathcal{C} of spaces is *Y -productive* provided $X \times Y \in \mathcal{C}$ for each space $X \in \mathcal{C}$. It is well-known or easy to check that each of the following classes of spaces is extension closed: countably pracomact, ω -bounded pracomact, totally countably pracomact, feebly compact, selectively feebly compact, and feebly ω -bounded. Since $[0, \omega_1)$ endowed with the order topology is ω -bounded and sequentially compact, each

of these classes is $[0, \omega_1)$ -productive by Proposition 2.2, [13, Proposition 3], [13, Proposition 2], [9, Lemma 4.2], Proposition 2.3, and Proposition 2.1, respectively.

Next we introduce a construction which helps us to construct a pseudo-compact topological semigroup which contains the bicyclic monoid as a closed subsemigroup providing a positive answer to [4, Problem 7.2].

Let X and Y be topological spaces such that there exists a continuous injection $f : X \rightarrow Y$. Then by $E_Y^f(X)$ we denote the subset $[0, \omega_1] \times Y \setminus \{(\omega_1, y) \mid y \in Y \setminus f(X)\}$ of a product $[0, \omega_1] \times Y$ endowed with a topology τ which is defined as follows. A subset $U \subset E_Y^f(X)$ is open if it satisfies the following conditions:

- for each $\alpha < \omega_1$, if $(\alpha, y) \in U$ then there exist $\beta < \alpha$ and an open neighborhood V_y of y in Y such that $(\beta, \alpha] \times V_y \subset U$;
- if $(\omega_1, f(x)) \in U$ then there exist $\alpha < \omega_1$, an open neighborhood $V_{f(x)}$ of $f(x)$ in Y and an open neighborhood W_x of x in X , such that $f(W_x) \subset V_{f(x)}$ and $(\alpha, \omega_1) \times V_{f(x)} \cup \{\omega_1\} \times f(W_x) \subset U$.

Remark that $\{\omega_1\} \times f(X)$ is a closed subset of $E_Y^f(X)$ homeomorphic to X .

Proposition 2.4. *Let X be a topological space which admits a continuous injection f into a space Y and \mathcal{C} be any extension closed, $[0, \omega_1)$ -productive class of spaces. If $Y \in \mathcal{C}$ then $E_Y^f(X) \in \mathcal{C}$.*

Proof. Let $Y \in \mathcal{C}$. Since \mathcal{C} is $[0, \omega_1)$ -productive, $[0, \omega_1) \times Y \in \mathcal{C}$. A space $E_Y^f(X)$ is an extension of the space $[0, \omega_1) \times Y$ providing that $E_Y^f(X) \in \mathcal{C}$. \square

If a space X is a subspace of a topological space Y and id is the identity embedding of X into Y , then by $E_Y(X)$ we denote the space $E_Y^{id}(X)$. It is easy to see that $E_Y(X)$ is a subspace of a product $[0, \omega_1] \times Y$ which implies that if Y is Tychonoff then so is $E_Y(X)$.

Proposition 2.5. *Let X be a subspace of a pseudocompact space Y . Then $E_Y(X)$ is pseudocompact and contains a closed copy of X .*

Proof. The argument above implies that $E_Y(X)$ is Tychonoff. Fix any continuous real valued function f on $E_Y(X)$. Observe that the dense subspace $[0, \omega_1) \times Y$ of $E_Y(X)$ is pseudocompact. Then the restriction of f on the subset $[0, \omega_1) \times Y$ is bounded, i.e., there exist reals a, b such that $f([0, \omega_1) \times Y) \subset [a, b]$. Then $f^{-1}[a, b]$ is closed in $E_Y(X)$ and contains the dense subset $[0, \omega_1) \times Y$ witnessing that $f^{-1}[a, b] = E_Y(X)$. Hence the space $E_Y(X)$ is pseudocompact. \square

Embeddings into countable compact and ω -bounded topological spaces were investigated in [2, 3].

A family \mathcal{A} of countable subsets of a set X is called *almost disjoint* if for each $A, B \in \mathcal{A}$ the set $A \cap B$ is finite. Given a property P , an almost disjoint family \mathcal{A} is called *P -maximal* if each element of \mathcal{A} has the property P and for

each countable subset $F \subset X$ which has the property P there exists $A \in \mathcal{A}$ such that the set $A \cap F$ is infinite.

Let \mathcal{F} be a family of closed subsets of a topological space X . The topological space X is called

- \mathcal{F} -regular, if for any set $F \in \mathcal{F}$ and point $x \in X \setminus F$ there exist disjoint open sets $U, V \subset X$ such that $F \subset U$ and $x \in V$;
- \mathcal{F} -normal, if for any disjoint sets $A, B \in \mathcal{F}$ there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.

Given a topological space X , by \mathcal{D}_ω we denote the family of countable closed discrete subsets of X . We say that a subset A of X satisfies a property \mathcal{D}_ω iff $A \in \mathcal{D}_\omega$.

Theorem 2.6. *Each \mathcal{D}_ω -regular topological space X can be embedded as an open dense subset into a Hausdorff countably pracomact topological space.*

Proof. By Zorn's Lemma, there exists a \mathcal{D}_ω -maximal almost disjoint family \mathcal{A} in X . Let $Y = X \cup \mathcal{A}$. We endow Y with the topology τ defined as follows. A subset $U \subset Y$ belongs to τ iff it satisfies the following conditions:

- if $x \in U \cap X$, then there exists an open neighborhood V of x in X such that $V \subset U$;
- if $A \in U \cap \mathcal{A}$, then there exists a cofinite subset $A' \subset A$ and an open set V in X such that $A' \subset V \subset U$.

Observe that X is an open dense subset of Y and \mathcal{A} is a discrete and closed subspace of Y . Since X is \mathcal{D}_ω -regular for each distinct points $x \in X$ and $y \in Y$ there exist disjoint open neighborhoods U_x and U_y in Y . By Proposition 2.1 from [2], each \mathcal{D}_ω -regular topological space is \mathcal{D}_ω -normal. Fix any distinct $A, B \in \mathcal{A}$. Put $A' = A \setminus (A \cap B)$ and $B' = B \setminus (A \cap B)$. By the \mathcal{D}_ω -normality of X , there exist disjoint open neighborhoods $U_{A'}$ and $U_{B'}$ of A' and B' , respectively. Then the sets $U_A = \{A\} \cup U_{A'}$ and $U_B = \{B\} \cup U_{B'}$ are disjoint open neighborhoods of A and B , respectively, in Y . Hence the space Y is Hausdorff.

Observe that the maximality of the family \mathcal{A} implies that there exists no countable discrete subset $D \subset X$ which is closed in Y . Hence each infinite subset A in X has an accumulation point in Y , that is, Y is countably pracomact. \square

However, there exists a Hausdorff topological space which cannot be embedded as a dense open subset into any Hausdorff countably pracomact topological space.

Example 2.7. Let τ be the usual topology on the real line \mathbb{R} and $C = \{A \subset \mathbb{R} : |\mathbb{R} \setminus A| \leq \omega\}$. By τ^* we denote the topology on \mathbb{R} which is generated by the subbase $\tau \cup C$. Obviously, the space $\mathbb{R}^* = (\mathbb{R}, \tau^*)$ is Hausdorff. We claim that \mathbb{R}^* cannot be embedded as a dense open subset into any Hausdorff countably pracomact topological space. Assuming the contrary, let X be a Hausdorff countably pracomact topological space which contains \mathbb{R}^* as a dense open

subspace. Since X is countably pracomact there exists a dense subset Y of X such that each infinite subset of Y has an accumulation point in X . Since \mathbb{R}^* is open and dense in X the set $Z = \mathbb{R}^* \cap Y$ is dense in X . Moreover, it is dense in (\mathbb{R}, τ) . Fix any point $z \in Z$ and a sequence $\{z_n\}_{n \in \omega}$ of distinct points of $Z \setminus \{z\}$ converging to z in (\mathbb{R}, τ) . Since Z is dense in (\mathbb{R}, τ) such a sequence exists. Observe that $\{z_n\}_{n \in \omega}$ is closed and discrete in \mathbb{R}^* . So, its accumulation point x belongs to $X \setminus \mathbb{R}^*$. Note that for each open neighborhood U of z in \mathbb{R}^* all but finitely many z_n belongs to the closure of U . Hence $x \in \overline{U}$ for each open neighborhood U of z which contradicts to the Hausdorffness of X .

Theorem 2.8. *Each topological space can be embedded as a closed subset into a Hausdorff ω -bounded pracomact topological space.*

Proof. Let X be a topological space. By X_d we denote the set X endowed with a discrete topology. Let X^* be the one point compactification of the space X_d . The unique non-isolated point of X^* is denoted by ∞ . Put $Y = [0, \omega_1] \times X^* \setminus \{(\omega_1, \infty)\}$. We endow Y with a topology τ defined as follows. A subset U is open in (Y, τ) if it satisfies the following conditions:

- if $(\alpha, \infty) \in U$, then there exist $\beta < \alpha$ and a cofinite subset A of X^* which contains ∞ such that $(\beta, \alpha] \times A \subset U$;
- if $(\omega_1, x) \in U$, then there exist $\alpha < \omega_1$ and an open (in X) neighborhood V of x such that $(\alpha, \omega_1] \times V \subset U$.

It is easy to check that the space (Y, τ) is Hausdorff. Observe that the subset $[0, \omega_1] \times X^* \subset Y$ is open, dense and ω -bounded. Hence Y is ω -bounded pracomact. Finally, note that the subset $\{\omega_1\} \times X \subset Y$ is closed and homeomorphic to X . □

Next we introduce a construction which helps us to prove that any space can be embedded as a closed subspace into an H-closed topological space.

Denote the subspace $\{1 - 1/n \mid n \in \mathbb{N}\} \cup \{1\}$ of the real line by J . Let X be a dense open subset of a topological space Y . By Z we denote the set $(J \times Y) \setminus \{(t, y) \mid y \in Y \setminus X \text{ and } t > 0\}$. By $H_Y(X)$ we denote the set Z endowed with a topology defined as follows. A subset $U \subset Z$ is open in $H_Y(X)$ if it satisfies the following conditions:

- for each $x \in X$ if $(t, x) \in U$, then there exist open neighborhoods V_t of t in J and V_x of x in X such that $V_t \times V_x \subset U$;
- for each $y \in Y \setminus X$ if $(0, y) \in U$, then there exists an open neighborhood V_y of y in Y such that $\{0\} \times (V_y \setminus X) \cup (J \setminus \{1\}) \times (V_y \cap X) \subset U$.

Obviously, the space $H_Y(X)$ is Hausdorff and the subset $\{(1, x) \mid x \in X\} \subset H_Y(X)$ is closed and homeomorphic to X .

Proposition 2.9. *If Y is an H-closed topological space, then $H_Y(X)$ is H-closed.*

Proof. Fix an arbitrary filter \mathcal{F} on $H_Y(X)$. One of the following three cases holds:

- (1) there exists $t \in J \setminus \{1\}$ such that for each $F \in \mathcal{F}$ there exists $y \in Y$ such that $(t, y) \in F$;
- (2) for each $F \in \mathcal{F}$ there exists $x \in X$ such that $(1, x) \in F$;
- (3) for every $t \in J$ there exists $F \in \mathcal{F}$ such that $(t, y) \notin F$ for each $y \in Y$.

Consider case (1). For each $F \in \mathcal{F}$ put $F_t = F \cap (\{t\} \times X \cup \{0\} \times (Y \setminus X))$. Clearly, a family $\mathcal{F}_t = \{F_t \mid F \in \mathcal{F}\}$ is a filter on $\{t\} \times X \cup \{0\} \times (Y \setminus X)$. Observe that for each $t \in J \setminus \{1\}$ the subspace $\{t\} \times X \cup \{0\} \times (Y \setminus X)$ is homeomorphic to Y and hence is H-closed. Then there exists a θ -accumulation point $z \in \{t\} \times X \cup \{0\} \times (Y \setminus X)$ of the filter \mathcal{F}_t . Obviously, z is a θ -accumulation point of the filter \mathcal{F} .

Consider case (2). For each $F \in \mathcal{F}$ put $F_0 = \{(0, x) \mid (1, x) \in F\}$. Clearly, the family $\mathcal{F}_0 = \{F_0 \mid F \in \mathcal{F}\}$ is a filter on the H-closed space $\{0\} \times Y$. Hence there exists $y \in Y$ such that $(0, y)$ is a θ -accumulation point of the filter \mathcal{F}_0 . If $y \in X$, then $(1, y)$ is a θ -accumulation point of the filter \mathcal{F} . If $y \in Y \setminus X$, then we claim that $(0, y)$ is a θ -accumulation point of the filter \mathcal{F} . Indeed, let U be any open neighborhood of the point $(y, 0)$. There exists an open neighborhood V_y of y in Y such that $V = \{0\} \times (V_y \setminus X) \cup (J \setminus \{1\}) \times (V_y \cap X) \subset U$. Since $(0, y)$ is a θ -accumulation point of the filter \mathcal{F}_0 , $\overline{V} \cap F_0 \neq \emptyset$ for each $F_0 \in \mathcal{F}_0$. Fix any $F \in \mathcal{F}$ and $(0, z) \in \overline{V} \cap F_0$. The definition of the topology on $H_Y(X)$ yields that the set $\{(t, z) \mid t \in J \setminus \{1\}\}$ is contained in \overline{V} . Then $(1, z) \in \overline{\{(t, z) \mid t \in J \setminus \{1\}\}} \subset \overline{V}$. Hence for each $F \in \mathcal{F}$ the set $\overline{V} \cap F$ is non-empty providing that $(0, y)$ is a θ -accumulation point of the filter \mathcal{F} .

Consider case (3). For each $F \in \mathcal{F}$ denote

$$F^* = \{(0, x) \mid \text{there exists } t \in I \text{ such that } (t, x) \in F\}.$$

Let $(0, y)$ be a θ -accumulation point of the filter $\mathcal{F}^* = \{F^* \mid F \in \mathcal{F}\}$.

If $y \in X$, then we claim that $(1, y)$ is a θ -accumulation point of the filter \mathcal{F} . Indeed fix any $F \in \mathcal{F}$ and an open neighborhood V of $(1, y)$. Then there exist a positive integer n and an open neighborhood U of y in X such that $\{t \in J \mid t > 1 - 1/n\} \times U \subset V$. By the assumption, there exist sets $F_0, \dots, F_n \in \mathcal{F}$ such that $F_i \cap \{(1 - 1/i, x) \mid x \in Y\} = \emptyset$ for every $i \leq n$. Then the set $H = \bigcap_{i \leq n} F_i \cap F$ belongs to \mathcal{F} and for each $(t, x) \in H$, $t > 1 - 1/n$. Since $(0, y)$ is a θ -accumulation point of the filter \mathcal{F}^* the set $\{0\} \times U \cap H^*$ is non-empty. Fix any $(0, x) \in \{0\} \times U \cap H^*$. Then there exists $k > n$ such that $(1 - 1/k, x) \in H \subset F$. The definition of the topology of $H_Y(X)$ implies that $(1 - 1/k, x) \in \overline{V} \cap H \subset \overline{V} \cap F$ which implies that $(1, y)$ is a θ -accumulation point of the filter \mathcal{F} .

If $y \in Y \setminus X$, then even more simple arguments show that $(0, y)$ is a θ -accumulation point of the filter \mathcal{F} .

Hence the space $H_Y(X)$ is H-closed. □

Theorem 2.10. *For any Hausdorff topological space X there exists an H-closed space Z which contains X as a closed subspace.*

Proof. For each Hausdorff topological space X there exists an H-closed space Y which contains X as a dense open subspace (see [10, Problem 3.12.6]). By

Proposition 2.9, the space $H_Y(X)$ is H-closed. It remains to note that the set $\{(1, x) \mid x \in X\} \subset H_Y(X)$ is closed and homeomorphic to X . \square

3. APPLICATIONS FOR TOPOLOGICAL SEMIGROUPS

A set endowed with an associative binary operation is called a *semigroup*. A semigroup S is called an *inverse semigroup*, if for each element $a \in S$ there exists a unique element $a^{-1} \in S$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. The map which associates every element of an inverse semigroup to its inverse is called an *inversion*.

A topological (inverse) semigroup is a Hausdorff topological space endowed with a continuous semigroup operation (and a continuous inversion, resp.). In this case the topology of the space is called (*inverse*, resp.) *semigroup topology*. A semitopological semigroup is a Hausdorff topological space endowed with a separately continuous semigroup operation. In this case the topology of the space is called *shift-continuous*.

Let X be a non-empty set. By \mathcal{B}_X we denote the set $X \times X \cup \{0\}$ where $0 \notin X \times X$ endowed with the following semigroup operation:

$$(a, b) \cdot (c, d) = \begin{cases} (a, d), & \text{if } b = c; \\ 0, & \text{if } b \neq c, \end{cases}$$

and $(a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0$, for each $a, b, c, d \in X$.

The semigroup \mathcal{B}_X is called the *semigroup of $X \times X$ -matrix units*. Observe that semigroups \mathcal{B}_X and \mathcal{B}_Y are isomorphic iff $|X| = |Y|$.

If a set X is infinite then the semigroup of $X \times X$ -matrix units cannot be embedded into a compact topological semigroup (see [11, Theorem 3]). In [12, Theorem 5] this result was generalized for countably compact topological semigroups. Moreover, in [6, Theorem 4.4] it was shown that for an infinite set X the semigroup \mathcal{B}_X cannot be embedded densely into a feebly compact topological semigroup.

A bicyclic monoid $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by two elements p and q subject to the condition $pq = 1$. The bicyclic monoid is isomorphic to the set $\omega \times \omega$ endowed with the following semigroup operation:

$$(a, b) \cdot (c, d) = \begin{cases} (a + c - b, d), & \text{if } b \leq c; \\ (a, d + b - c), & \text{if } b > c. \end{cases}$$

Neither stable nor Γ -compact topological semigroups can contain a copy of the bicyclic monoid (see [1, 15]). In [14] it was proved that the bicyclic monoid does not embed into a countably compact topological inverse semigroup. Also, a topological semigroup with a feebly compact square cannot contain the bicyclic monoid [4]. On the other hand, in [4, Theorem 6.1] it was proved that there exists a Tychonoff countably precompact topological semigroup S densely containing the bicyclic monoid. Moreover, under Martin's Axiom the semigroup S is countably compact (see [4, Theorem 6.6 and Corollary 6.7]). However, it is still unknown whether there exists under ZFC a countably

compact topological semigroup containing the bicyclic monoid (see [4, Problem 7.1]). Also, in [4] the following problem was posed:

Problem 3.1 ([4, Problem 7.2]). *Is there a pseudocompact topological semigroup S that contains a closed copy of the bicyclic monoid?*

Embeddings of semigroups which are generalizations of the bicyclic monoid into compact-like topological semigroups were investigated in [5, 6]. Namely, in [6] it was proved that for each cardinal $\lambda > 1$ a polycyclic monoid \mathcal{P}_λ does not embed as a dense subsemigroup into a feebly compact topological semigroup. In [5] embeddings of graph inverse semigroups into CLP-compact topological semigroups were described.

Observe that the space $[0, \omega_1]$ endowed with a semigroup operation of taking minimum becomes a topological semilattice and therefore a topological inverse semigroup.

Lemma 3.2. *Let X and Y be semitopological (topological, topological inverse, resp.) semigroups such that there exists a continuous injective homomorphism $f : X \rightarrow Y$. Then $E_Y^f(X)$ is a semitopological (topological, topological inverse, resp.) semigroup with respect to the semigroup operation inherited from a direct product of semigroups (ω_1, \min) and Y .*

Proof. We prove this lemma for the case of topological semigroups X and Y . Proofs in other cases are similar. Fix any elements $(\alpha, x), (\beta, y)$ of $E_Y^f(X)$. Also, assume that $\beta \leq \alpha$. In the other case the proof will be similar. Fix any open neighborhood U of $(\beta, xy) = (\alpha, x) \cdot (\beta, y)$. There are three cases to consider:

- (1) $\beta \leq \alpha < \omega_1$;
- (2) $\beta < \alpha = \omega_1$;
- (3) $\alpha = \beta = \omega_1$.

In case (1) there exist $\gamma < \beta$ and an open neighborhood V_{xy} of xy in Y such that $(\gamma, \beta] \times V_{xy} \subset U$. Since Y is a topological semigroup there exist open neighborhoods V_x and V_y of x and y , respectively, such that $V_x \cdot V_y \subset V_{xy}$. Put $U_{(\alpha, x)} = (\gamma, \alpha] \times V_x$ and $U_{(\beta, y)} = (\gamma, \beta] \times V_y$. It is easy to check that $U_{(\alpha, x)} \cdot U_{(\beta, y)} \subset (\gamma, \beta] \times V_{xy} \subset U$.

Consider case (2). Similarly as in case (1) there exist an ordinal $\gamma < \beta$ and open neighborhoods V_x, V_y and V_{xy} of x, y and xy , respectively, such that $(\gamma, \beta] \times V_{xy} \subset U$ and $V_x \cdot V_y \subset V_{xy}$. Since the map f is continuous there exists an open neighborhood $V_{f^{-1}(x)}$ of $f^{-1}(x)$ in X such that $f(V_{f^{-1}(x)}) \subset V_x$. Put $U_{(\omega_1, x)} = (\beta, \omega_1) \times V_x \cup \{\omega_1\} \times f(V_{f^{-1}(x)})$ and $U_{(\beta, y)} = (\gamma, \beta] \times V_y$. It is easy to check that $U_{(\omega_1, x)} \cdot U_{(\beta, y)} \subset (\gamma, \beta] \times V_{xy} \subset U$.

Consider case (3). There exist ordinal $\gamma < \omega_1$, an open neighborhood V_{xy} of xy in Y and an open neighborhood $W_{f^{-1}(xy)}$ of $f^{-1}(xy)$ in X such that $(\gamma, \omega_1) \times V_{xy} \cup \{\omega_1\} \times f(W_{f^{-1}(xy)}) \subset U$.

Since Y is a topological semigroup there exist open (in Y) neighborhoods V_x and V_y of x and y , respectively, such that $V_x \cdot V_y \subset V_{xy}$. Since the map f is continuous and X is a topological semigroup there exist open (in X)

neighborhoods $W_{f^{-1}(x)}$ and $W_{f^{-1}(y)}$ of $f^{-1}(x)$ and $f^{-1}(y)$, respectively, such that $W_{f^{-1}(x)} \cdot W_{f^{-1}(y)} \subset W_{f^{-1}(xy)}$, $f(W_{f^{-1}(x)}) \subset V_x$ and $f(W_{f^{-1}(y)}) \subset V_y$. Put $U_{(\omega_1, x)} = (\gamma, \omega_1) \times V_x \cup \{\omega_1\} \times f(W_{f^{-1}(x)})$ and $U_{(\omega_1, y)} = (\gamma, \omega_1) \times V_y \cup \{\omega_1\} \times f(W_{f^{-1}(y)})$. It is easy to check that $U_{(\omega_1, x)} \cdot U_{(\omega_1, y)} \subset U$.

Hence the semigroup operation in $E_Y^f(X, \tau_X)$ is continuous. \square

Remark 3.3. The subsemigroup $\{(\omega_1, f(x)) \mid x \in X\} \subset E_Y^f(X)$ is closed and topologically isomorphic to X .

Proposition 2.4, Lemma 3.2 and Remark 3.3 imply the following:

Proposition 3.4. *Let X be a (semi)topological semigroup which admits a continuous injective homomorphism f into a (semi)topological semigroup Y and \mathcal{C} be any $[0, \omega_1]$ -productive, extension closed class of spaces. If $Y \in \mathcal{C}$ then the (semi)topological semigroup $E_Y^f(X) \in \mathcal{C}$ and contains a closed copy of a (semi)topological semigroup X .*

Proposition 2.5, Lemma 3.2 and Remark 3.3 imply the following:

Proposition 3.5. *Let X be a subsemigroup of a pseudocompact (semi)topological semigroup Y . Then the (semi)topological semigroup $E_Y(X)$ is pseudocompact and contains a closed copy of the (semi)topological semigroup X .*

By [4, Theorem 6.1], there exists a Tychonoff countably pracomact (and hence pseudocompact) topological semigroup S containing densely the bicyclic monoid. Hence Proposition 3.5 implies the following corollary which gives an affirmative answer to Problem 3.1.

Corollary 3.6. *There exists a pseudocompact topological semigroup which contains a closed copy of the bicyclic monoid.*

Further we will need the following definitions. A subset A of a topological space is called θ -closed if for each element $x \in X \setminus A$ there exists an open neighborhood U of x such that $\overline{U} \cap A = \emptyset$. Observe that if a topological space X is regular then each closed subset A of X is θ -closed. A topological space X is called *weakly H-closed* if each ω -filter \mathcal{F} has a θ -accumulation point in X . Generalizations of H-closed spaces were investigated by Osipov in [19, 20]. Obviously, for a topological space X the following implications hold: X is H-closed $\Rightarrow X$ is weakly H-closed $\Rightarrow X$ is feebly compact. Neither of the above implications can be inverted. Indeed, an arbitrary pseudocompact but not countably compact space will be an example of feebly compact space which is not weakly H-closed. The space $[0, \omega_1]$ with an order topology is an example of weakly H-closed but not H-closed space.

The following theorem shows that Theorem 2.10 cannot be generalized for topological semigroups.

Theorem 3.7. *The semigroup \mathcal{B}_ω of $\omega \times \omega$ -matrix units does not embed into a weakly H-closed topological semigroup.*

Proof. Suppose to the contrary that \mathcal{B}_ω is a subsemigroup of a weakly H-closed topological semigroup S . By $E(\mathcal{B}_\omega)$ we denote the semilattice of idempotents of \mathcal{B}_ω . Observe that $E(\mathcal{B}_\omega) = \{(n, n) \mid n \in \omega\} \cup \{0\}$ and $a \cdot b = 0$ for each distinct elements $a, b \in E(\mathcal{B}_\omega)$. Let \mathcal{F} be an arbitrary free ω -filter on the set $\{(n, n) \mid n \in \omega\}$. Since S is weakly H-closed, there exists a θ -accumulation point $s \in S$ of the filter \mathcal{F} . Fix any open neighborhood U_p of the point $p = s \cdot s$. The continuity of the semigroup operation in S yields an open neighborhood V_s of s such that $\overline{V_s} \cdot \overline{V_s} \subset \overline{U_p}$. Since s is a θ -accumulation point of the filter \mathcal{F} there exist distinct elements $(n, n), (m, m) \in \overline{V_s} \cap E(\mathcal{B}_\omega)$. Hence $0 = (n, n) \cdot (m, m) \in \overline{V_s} \cdot \overline{V_s} \subset \overline{U_p}$ which implies that $0 \in \overline{U_p}$ for each open neighborhood U_p of p witnessing that $p = 0$.

We claim that 0 is a θ -accumulation point of the filter \mathcal{F} . Indeed, fix any open neighborhood U of 0 . Since $s \cdot s = 0$ and S is a topological semigroup there exists an open neighborhood V_s of s such that $\overline{V_s} \cdot \overline{V_s} \subset U$. Observe that $(n, n) = (n, n) \cdot (n, n) \in \overline{V_s} \cdot \overline{V_s} \subset U$ for each $(n, n) \in \overline{V_s}$. Hence 0 is a θ -accumulation point of the filter \mathcal{F} . Since the filter \mathcal{F} was selected arbitrarily we have that 0 is a θ -accumulation point of any free ω -filter on the set $\{(n, n) \mid n \in \omega\}$.

Thus for each open neighborhood U of 0 the set $A_U = \{(n \mid (n, n) \notin \overline{U})\}$ is finite, because if there exists an open neighborhood U of 0 such that the set A_U is infinite, then 0 is not a θ -accumulation point of the ω -filter \mathcal{F} which has a base consisting of cofinite subsets of A_U .

Let \mathcal{F} be an arbitrary free ω -filter on the set $\{(1, n) \mid n \in \omega\}$. Since S is weakly H-closed there exists a θ -accumulation point $s \in S$ of the filter \mathcal{F} .

We claim that $s \cdot 0 = 0$. Indeed, fix any open neighborhood W of $s \cdot 0$. The continuity of the semigroup operation in S yields open neighborhoods V_s of s and V_0 of 0 such that $\overline{V_s} \cdot \overline{V_0} \subset W$. Since the set $A_{V_0} = \{(n \mid (n, n) \notin \overline{V_0})\}$ is finite and s is a θ -accumulation point of the filter \mathcal{F} there exist distinct $n, m \in \omega$ such that $(1, n) \in \overline{V_s}$ and $(m, m) \in \overline{V_0}$. Then $0 = (1, n) \cdot (m, m) \in \overline{V_s} \cdot \overline{V_0} \subset W$. Hence $0 \in W$ for each open neighborhood W of $s \cdot 0$ witnessing that $s \cdot 0 = 0$.

Fix an arbitrary open neighborhood U of 0 . Since $s \cdot 0 = 0$ and S is a topological semigroup, there exist open neighborhoods V_s of s and V_0 of 0 such that $\overline{V_s} \cdot \overline{V_0} \subset U$. Recall that the set $\{n \mid (n, n) \notin \overline{V_0}\}$ is finite. Then $(1, n) = (1, n) \cdot (n, n) \in \overline{V_s} \cdot \overline{V_0} \subset U$ for all but finitely many elements $(1, n) \in \overline{V_s}$. Hence 0 is a θ -accumulation point of the ω -filter \mathcal{F} . Since the filter \mathcal{F} was selected arbitrarily, 0 is a θ -accumulation point of any free ω -filter on the set $\{(1, n) \mid n \in \omega\}$. As a consequence, for each open neighborhood U of 0 the set $B_U = \{n \mid (1, n) \notin \overline{U}\}$ is finite.

Similarly it can be shown that for each open neighborhood U of 0 the set $C_U = \{n \mid (n, 1) \notin \overline{U}\}$ is finite.

Fix an open neighborhood U of 0 such that $(1, 1) \notin \overline{U}$. Since $0 = 0 \cdot 0$ the continuity of the semigroup operation implies that there exists an open neighborhood V of 0 such that $\overline{V} \cdot \overline{V} \subset U$. The finiteness of the sets B_V

and C_V implies that there exists $n \in \omega$ such that $\{(1, n), (n, 1)\} \subset \overline{V}$. Hence $(1, 1) = (1, n) \cdot (n, 1) \in \overline{V} \cdot \overline{V} \subset \overline{U}$, which contradicts to the choice of U . \square

Corollary 3.8. *The semigroup of $\omega \times \omega$ -matrix units does not embed into a topological semigroup S whose space is H -closed.*

However, we have the following questions:

Problem 3.9. *Does there exist a feebly compact topological semigroup S which contains the semigroup of $\omega \times \omega$ -matrix units?*

Problem 3.10. *Does there exist a topological semigroup S which cannot be embedded into a feebly compact topological semigroup T ?*

We remark that these questions were posed at the Lviv Topological Algebra Seminar a few years ago.

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REFERENCES

- [1] L. Anderson, R. Hunter, and R. Koch, Some results on stability in semigroups, *Trans. Amer. Math. Soc.* 117 (1965), 521–529.
- [2] T. Banakh, S. Bardyla and A. Ravsky, Embedding topological spaces into Hausdorff κ -bounded spaces, preprint, [arXiv:1906.00185v3](https://arxiv.org/abs/1906.00185v3).
- [3] T. Banakh, S. Bardyla and A. Ravsky, Embeddings into countably compact Hausdorff spaces, preprint, [arXiv:1906.04541](https://arxiv.org/abs/1906.04541).
- [4] T. Banakh, S. Dimitrova and O. Gutik, Embedding the bicyclic semigroup into countably compact topological semigroups, *Topology Appl.* 157, no. 18 (2010), 2803–2814.
- [5] S. Bardyla, Embedding of graph inverse semigroups into CLP-compact topological semigroups, *Topology Appl.* 272 (2020), 107058.
- [6] S. Bardyla and O. Gutik, On a semitopological polycyclic monoid, *Algebra Discr. Math.* 21, no. 2 (2016), 163–183.
- [7] D. Dikranjan and E. Giuli, $S(n)$ - θ -closed spaces, *Topology Appl.* 28 (1988), 59–74.
- [8] D. Dikranjan and J. Pelant, Categories of topological spaces with sufficiently many sequentially closed spaces, *Cahiers de Topologie et Geometrie Differentielle Categoriqes* 38, no. 4 (1997), 277–300.
- [9] A. Dow, J. R. Porter, R.M. Stephenson, Jr. and R. G. Woods, Spaces whose pseudo-compact subspaces are closed subsets, *Appl. Gen. Topol.* 5, no. 2 (2004), 243–264.
- [10] R. Engelking, *General Topology*, 2nd ed., Heldermann, Berlin, 1989.
- [11] O. Gutik and K. Pavlyk, Topological semigroups of matrix units, *Algebra Discr. Math.* 3 (2005), 1–17.
- [12] O. Gutik, K. Pavlyk and A. Reiter, Topological semigroups of matrix units and countably compact Brandt λ^0 -extensions, *Mat. Stud.* 32, no. 2 (2009), 115–131.
- [13] O. Gutik and A. Ravsky, On old and new classes of feebly compact spaces, *Visn. Lviv. Univ. Ser. Mech. Math.* 85 (2018), 48–59.
- [14] O. Gutik and D. Repovs, On countably compact 0-simple topological inverse semigroups, *Semigroup Forum.* 75, no. 2 (2007), 464–469.

- [15] J. Hildebrandt and R. Koch, Swelling actions of Γ -compact semigroups, *Semigroup Forum* 33, no. 1 (1986), 65–85.
- [16] J. E. Joseph, On H-closed spaces, *Proc. Amer. Math. Soc.* 55, no. 1 (1976), 223–226.
- [17] J. E. Joseph, More characterizations of H-closed spaces, *Proc. Amer. Math. Soc.* 63, no. 1 (1977), 160–164.
- [18] D. D. Mooney, Spaces with unique Hausdorff extensions, *Topology Appl.* 61, no. 1 (1995), 241–256.
- [19] A. Osipov, Nearly H-closed spaces, *Journal of Mathematical Sciences* 155, no. 4 (2008), 626–633.
- [20] A. Osipov, Weakly H-closed spaces, *Proceedings of the Steklov Institute of Mathematics* 10, no. 1 (2004), S15–S17.
- [21] J. Porter, On locally H-closed spaces, *Proc. London Math. Soc.* 20, no. 3 (1970), 193–204.
- [22] J. Porter and J. Thomas, On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.* 138 (1969), 159–170.
- [23] J. Porter and C. Votaw, H-closed extensions I, *Gen. Topology Appl.* 3 (1973), 211–224.
- [24] J. Porter and C. Votaw, H-closed extensions II, *Trans. Amer. Math. Soc.* 202 (1975), 193–208.
- [25] J. Porter and R. Woods, *Extensions and Absolues of Hausdorff Spaces*, Springer, Berlin, 1988, 856 pp.
- [26] R. M. Stephenson, Jr, Initially κ -compact and related compact spaces, in: K. Kunen, J. E. Vaughan (eds.), *Handbook of Set-Theoretic Topology*, Elsevier, 1984, pp. 603–632.
- [27] J.E. Vaughan, Countably compact and sequentially compact spaces, in: K. Kunen, J. E. Vaughan (eds.), *Handbook of Set-Theoretic Topology*, Elsevier, 1984, 569–602.
- [28] N. Velichko, H-Closed Topological Spaces, *American Mathematical Society* 78, no. 2 (1968), 103–118.
- [29] J. Vermeer, Closed subspaces of H-closed spaces, *Pacific Journal of Mathematics* 118, no. 1 (1985), 229–247.