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The depth and the attracting centre for a continuous map on a fuzzy metric interval

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Abstract

Let I be a fuzzy metric interval and f be a continuous map from I to I. Denote by $R(f)$, $\Omega(f)$ and $\omega(x,f)$ the set of recurrent points of f, the set of non-wandering points of f and the set of ω- limit points of x under f, respectively. Write $\omega(f)=\cup_{x\in I}\omega(x,f)$, $\omega^{n+1}(f)=\cup_{x\in\omega^n(f)}\omega(x,f)$ and $\Omega_{n+1}(f)=\Omega(f|_{\Omega_n(f)})$ for any positive integer n. In this paper, we show that $\Omega_2(f)=\overline{R(f)}$ and the depth of f is at most 2, and $\omega^3(f)=\omega^2(f)$ and the depth of the attracting centre of f is at most 2.

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1. Introduction

By extending the notion of Menger space to the fuzzy setting, Kramosil and Michalek [7] obtained the notion of fuzzy metric space with the help of continuous t-norms. In order to obtain a Hausdorff topology in fuzzy metric

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spaces, George and Veeramani [1] modified the notion given by Kramosil and Michalek in a slight but appealing way. Recently many authors studied several properties of the Hausdorff fuzzy metric spaces (see [6, 8, 11]) and introduced and investigated the different types of fuzzy contractive maps and obtained a lot of fixed point theorems (see [3, 4, 9, 10, 13, 14, 15, 16]). Until now, there are little of works that investigates some properties of discrete dynamical systems on fuzzy metric spaces. In this paper, we introduce the notion of fuzzy metric interval and study the depth and the attracting centre for a continuous map on a fuzzy metric interval.

The rest of this paper is organized as follows. In Section 2 we give some definitions and notations. In Section 3 we study the depth for a continuous map on a fuzzy metric interval. In Section 4 we study the depth of the attracting centre for a continuous map on a fuzzy metric interval.

2. Preliminaries

Throughout the paper, let $\mathbf{N}$ be the set of all positive integers and $\mathbf{N}^\star = \mathbf{N} \cup \{0\}$. Firstly, we recall the basic definitions and the properties about fuzzy metric spaces.

Definition 2.1 (see [12])**.** We say that a continuous map $\xi : [0,1]^2 \longrightarrow [0,1]$ is a continuous t-norm if for any $a, b, c, d \in [0,1]$, the following conditions hold:
(1) $\xi(a,b) = \xi(b,a)$.
(2) $\xi(a,b) \leq \xi(c,d)$ for $a \leq c$ and $b \leq d$.
(3) $\xi(\xi(a,b),c) = \xi(a,\xi(b,c))$.
(4) $\xi(a,0) = 0$ and $\xi(a,1) = a$.

For $a, b \in [0,1]$, we will use the notation $a * b$ instead of $\xi(a,b)$. For example, $\xi(a,b) = \min\{a,b\}$, $\xi(a,b) = ab$ and $\xi(a,b) = \max\{a+b-1, 0\}$ are the most commonly used t-norms.

In the present paper, we also use the following definition of the fuzzy metric space.

Definition 2.2 (see [1])**.** We say that a triple $(X, M, *)$ is a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and M is a map defined on $X^2 \times (0,+\infty)$ into $[0,1]$ and for any $x, y, z \in X$ and $s, t \in (0,+\infty)$, the following conditions hold:
(1) $M(x,y,t) > 0$.
(2) $M(x,y,t) = 1$ (for any $t > 0$) $\Longleftrightarrow x = y$.
(3) $M(x,y,t) = M(y,x,t)$.
(4) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$.
(5) $M_{xy} : (0,+\infty) \longrightarrow [0,1]$ is a continuous mapping (where $M_{xy}(t) = M(x,y,t)$).

Remark 2.3. (1) M_{xy} is a non-decreasing function on $(0,\infty)$ for all $x, y \in X$ (see [2]). (2) M is a continuous function on $X \times X \times (0,+\infty)$ (see [11]).

If $(X, M, *)$ is a fuzzy metric space, then we will say that $(M, *)$, or simply M, is a fuzzy metric on X. In [1], George and Veeramani showed that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x,\varepsilon,t) : x \in X, 0 < \varepsilon < 1, t > 0\}$, where $B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1-\varepsilon\}$ for all $x \in X, \varepsilon \in (0,1)$ and $t > 0$, and (X,τ_M) is a Hausdorff space.

Definition 2.4 (see [5])**.** Let $(X,M,*)$ be a fuzzy metric space. We say that a sequence of points $x_n \in X$ converges to x (denoted by $x_n \longrightarrow x$) $\Longleftrightarrow$ $\lim_{n\longrightarrow+\infty} M(x_n,x,t) = 1$ (for any $t > 0$), i.e. for each $\delta \in (0,1)$ and $t > 0$, there exists $N \in \mathbf{N}$ such that $M(x_n,x,t) > 1-\delta$ for all $n \geq N$.

Definition 2.5. (1) We say that a fuzzy metric space $(X,M,*)$ is compact if (X,τ_M) is compact. We say that a subset A of X is compact if A as a fuzzy metric subspace is compact.

(2) We say that a fuzzy metric space $(X,M,*)$ is connected if (X,τ_M) is connected. We say that a subset A of X is connected if A as a fuzzy metric subspace is connected.

By [6] we know that: (1) $(X,M,*)$ is a compact fuzzy metric space $\Longleftrightarrow$ each sequence of points in X has a convergent subsequence. (2) If $(X,M,*)$ is a compact fuzzy metric space and A is a subset of X, then A is compact $\Longleftrightarrow$ A is closed. (3) If $(X,M,*)$ is a compact fuzzy metric space, then for any $x,y \in X$ with $x \neq y$, there exist $B(x,\varepsilon_1,t_1)$ and $B(y,\varepsilon_2,t_2)$ with $\varepsilon_1,\varepsilon_1 \in (0,1)$ and $t_1,t_2 \in (0,+\infty)$ such that $B(x,\varepsilon_1,t_1) \cap B(y,\varepsilon_2,t_2) = \varnothing$.

Definition 2.6. Let $(X,M,*)$ be a compact fuzzy metric space and $a,b \in X$. We say that X is a fuzzy metric interval with ends a and b if the following conditions hold:

(1) $M(a,x,t) \geq M(a,b,t)$ for any $x \in X$ and $t > 0$.

(2) For any $x,y \in X$ with $x \neq y$, we have $M(a,x,t) < M(a,y,t)$ for any $t > 0$, which is denoted by $x > y$, or $M(a,x,t) > M(a,y,t)$ for any $t > 0$, which is denoted by $x < y$.

(3) For any $x,y \in X$ with $M(a,x,t) \geq M(a,y,t)$ for any $t > 0$, set $\{z \in X : M(a,x,t) \geq M(a,z,t) \geq M(a,y,t)$ for any $t > 0\}$, which is denoted by $[x;y]$, is a connected subset of X.

(4) If $y \in B(x,\varepsilon,t)$ for some $x \in X$ and some $\varepsilon \in (0,1)$ and some $t > 0$, then $[y;x] \subset B(x,\varepsilon,t)$ if $y \leq x$ or $[x;y] \subset B(x,\varepsilon,t)$ if $y \geq x$.

Write $[x;y) = [x;y] - \{y\} \equiv \{z \in X : M(a,x,t) \geq M(a,z,t) > M(a,y,t)$ and $(x;y) = [x;y) - \{x\} \equiv \{z \in X : M(a,x,t) > M(a,z,t) \geq M(a,y,t)$.

Remark 2.7. (1) $[a;b] = X$. (2) Let $x,y \in [a;b]$. If $M(a,x,t) = M(a,y,t)$ for some $t > 0$, then $x = y$.

Example 2.8. Let $I = [a,b]$ be a compact interval of $\mathbf{R}= (-\infty,+\infty)$. Define $s * t = st$ for any $s;t \in [0,1]$, and let $M_d : I \times I \times (0,\infty) \longrightarrow [0,1]$ such that for any $x,y \in I$ and $t > 0$,

$$M_d(x,y,t) = \frac{t}{t+|x-y|}.$$

Then $(I, M_d, *)$ is a fuzzy metric interval. Further, it was proven [1] that the topologies induced by (I, d) with $d(x, y) = |x-y|$ for any $x, y \in I$ and $(I, M_d, *)$ are the same.

Let $(I, M, *)$ be a fuzzy metric interval and $A \subset I$. Denote by $\overline{A}$ the closure of A in (I, τ_M). Let $C^0(I)$ denote the set of all continuous maps on I. For any $f \in C^0(I)$ and $x \in I$, write $f^0(x) = x$ and $f^n = f \circ f^{n-1}$ for any $n \in \mathbf{N}$. We also introduce a number of notations:

$$\begin{aligned}
O(x,f) &= \{f^n(x) : n \in \mathbf{N}^\star\}.\\
\Lambda(x,f) &= \cup_{n=1}^{\infty} f^{-n}(x).\\
Fix(f) &= \{x : f(x) = x\}.\\
P(f) &= \{x : \text{there exists some } n \in \mathbf{N} \text{ such that } f^n(x) = x\}.\\
\omega(x,f) &= \{y : \text{ there exists a sequence of positive integers } k_1 < k_2 < \cdots \\
&\qquad \text{such that } \lim_{n\longrightarrow\infty} M(f^{k_n}(x), y, t) = 1 \text{ (for any } t > 0)\}.\\
R(f) &= \{x : x \in \omega(x,f)\}.\\
U\Gamma(f) &= \{y : \text{ there exist a connected component } J \text{ of } I - \{y\}, \text{ a point } x \\
&\qquad \in \overline{J}, \text{a sequence of points } x_1, x_2, \cdots \in \Lambda(x,f) \cap J \text{ and a} \\
&\qquad \text{sequence of positive integers } k_1 < k_2 < \cdots \text{such that } f^{k_n}(x) \in J \\
&\qquad \text{for any } n \in \mathbf{N} \text{ and } \lim_{n\longrightarrow\infty} M(x_n, y, t) = \lim_{n\longrightarrow\infty} M(f^{k_n}(x), y, t) \\
&\qquad = 1 (\text{for any } t > 0)\}.\\
\Omega(f) &= \{y : \text{there exist a sequence of points } x_1, x_2, \cdots \in I \text{ and a sequence} \\
&\qquad \text{of positive integers } k_1 \le k_2 \le \cdots \text{ such that } \lim_{n\longrightarrow\infty} M(x_n, y, t) = \\
&\qquad \lim_{n\longrightarrow\infty} M(f^{k_n}(x_n), y, t) = 1 \text{ (for any } t > 0)\}.
\end{aligned}$$

$P(f)$, $R(f)$, $U\Gamma(f)$ and $\Omega(f)$ are called the set of periodic points, the set of recurrent points, the set of unilateral γ- limit points and the set of non-wandering points of f, respectively. $O(x, f)$, $\Lambda(x, f)$ and $\omega(x, f)$ are called the orbit of x under f, the reverse orbit of x under f and the set of ω-limit points of x under f, respectively.

Remark 2.9. Let $(I, M, *)$ be a fuzzy metric interval and $f \in C^0(I)$. Then the following statements hold:
(1) $f(\Omega(f)) \subset \Omega(f)$ and $\Omega(f)$ is closed.
(2) $Fix(f) \subset P(f) \subset R(f) \subset \omega(f) \subset \Omega(f)$.
(3) $A\Gamma(f) \subset \omega(f)$ and $R(f) \subset \Omega_n(f)$ for any $\in \mathbf{N}$.

Definition 2.10. Let $(I, M, *)$ be a fuzzy metric interval and $f \in C^0(I)$.
(1) For any $A \subset I$, write $\omega(A) = \cup_{x\in A}\omega(x, f)$ and $\omega^1(f) = \omega(f) = \omega(I)$ and $\omega^{n+1}(f) = \cup_{x\in\omega^n(f)}\omega(x, f)$ for any $n \in \mathbf{N}$. The minimal $m \in \mathbf{N} \cup \{\infty\}$ such that $\omega_m(f) = \omega_{m+1}(f)$ is called the depth of the attracting centre of f.
(2) Write $\Omega_1(f) = \Omega(f)$ and $\Omega_{n+1}(f) = \Omega(f|_{\Omega_n(f)})$ for any $n \in \mathbf{N}$. The minimal $m \in \mathbf{N} \cup \{\infty\}$ such that $\Omega_m(f) = \Omega_{m+1}(f)$ is called the depth of f.

In this paper, we show that if f is a continuous map on a fuzzy metric interval, then $\Omega_2(f) = \overline{R(f)}$ and the depth of f is at most 2, and $\omega^3(f) = \omega^2(f)$ and the depth of the attracting centre of f is at most 2.

3. The depth for a continuous map on a fuzzy metric interval

In this section, we study the depth of a continuous map on a fuzzy metric interval $I = [a, b]$ with $a \neq b$.

Proposition 3.1. *The following two statements hold:*

(1) Let $x_n \in I$ for any $n \in \mathbf{N}$. Then $x_n \longrightarrow x \Longleftrightarrow M(a, x_n, t) \longrightarrow M(a, x, t)$ for any $t > 0$.

(2) For any $x, y \in I$ with $x \leq y$, $[x; y]$ is closed.

(3) For any $x, y, z \in X$ with $x < y < z$, $[x; z] - \{y\}$ is not connected.

Proof. (1) $\Longrightarrow$ is obvious since M is continuous.

$\Longleftarrow$ Assume on the contrary that $x_n \not\longrightarrow x$. Then there exist an open neighbourhood U of x and a sequence of positive integers $k_1 < k_2 < \cdots$ such that $f^{k_n}(x) \in I - U$ for any $n \in \mathbf{N}$. By taking subsequence we let $x_{k_n} \longrightarrow u \in I - U$ since $I - U$ is closed. Then by the above ($\Longrightarrow$) we have $M(a, x_{k_n}, t) \longrightarrow M(a, u, t)$ for any $t > 0$. Thus $M(a, x, t) = M(a, u, t)$. By Remark 2.7 we have $x = u$. This is a contradiction.

(2) Let $x_n \in [x; y]$ and $x_n \longrightarrow u \in I$ since I is compact. Then $M(a, x, t) \geq M(a, x_n, t) \geq M(a, y, t)$ with $M(a, x_n, t) \longrightarrow M(a, u, t) \in [M(a, y, t), M(a, x, t)]$, which implies $u \in [x; y]$.

(3) We claim that $[x; y)$ is an open subset of $[x; z]$. Indeed, for any $w \in [x; y)$, there exists an open neighbourhood $U = B(w, \varepsilon, t_0)$ of w such that $y \notin U$. By Definition 2.6 (4) we see that $U \cap [x; z] \subset [x; y)$, which implies that $[x; y)$ is an open subset of $[x; z]$. In a similar fashion we can also show that $(y; z]$ is an open subset of $[x; z]$. Since $[x; y) \cap (y; z] = \varnothing$ and $[x; z] - \{y\} = [x; y) \cap (y; z]$, from which it follows that $[x; z] - \{y\}$ is not connected. The proof is completed. ☐

Lemma 3.2. *Let $f \in C(I)$. If there exist $x, y \in I$ with $x \leq y$ such that $f(x) \leq x \leq y \leq f(y)$ or $x \leq f(x)$ and $f(y) \leq y$, then $[x; y] \cap Fix(f) \neq \varnothing$.*

Proof. We can assume that $f(x) \neq x$ and $f(y) \neq y$. Define $F : [x; y] \longrightarrow \mathbf{R}$ such that for any $z \in [x; y]$,

$$F(z) = M(a, z, t) - M(a, f(z), t).$$

By Remark 2.3 we see that F is continuous. Since $F(x)F(y) < 0$ and $[x; y]$ is connected, we have $0 \in f([x; y])$ and there exists $p \in [x; y]$ such that $F(p) = 0$. Thus $M(a, p, t) = M(a, f(p), t)$ for $t > 0$. By Remark 2.7 we see $f(p) = p$. The proof is completed. ☐

Corollary 3.3. *Let $f \in C(I)$. If there exist $x \in I$ and $n \in \mathbf{N}$ such that $f^n(x) < x$ and $[f^n(x); x] \cap P(f) = \varnothing$, Then $f^{kn}(x) < f^n(x)$ for any $k \geq 2$. If there exist $x \in I$ and $n \in \mathbf{N}$ such that $f^n(x) > x$ and $[x; f^n(x)] \cap P(f) = \varnothing$, Then $f^{kn}(x) > f^n(x)$ for any $k \geq 2$.*

Proof. Let $f^n(x) < x$. Assume on the contrary that $f^n(x) \leq f^{kn}(x)$ for some $k \geq 2$. Then it follows from Lemma 3.2 that $[f^n(x); x] \cap Fix(f^{(k-1)n}) \neq \varnothing$. This contradicts $[f^n(x); x] \cap P(f) = \varnothing$. For the another case, the proof is similar. The proof is completed. □

By Lemma 3.2 and Corollary 3.3 we obtain the following corollary.

Corollary 3.4. *Let $f \in C(I)$ and $x, y \in I$ with $x < y$ and $[x; y] \cap P(f) = \varnothing$. If $u, v, f^m(u), f^n(v) \in [x; y]$ for some $m, n \in \mathbf{N}$, then $u \in [x; f^m(u))$ and $v \in [x; f^n(v))$, or $u \in (f^m(u); y]$ and $v \in (f^n(v); y]$.*

Theorem 3.5. *Let $f \in C(I)$. Then $\overline{P(f)} = \overline{R(f)}$.*

Proof. By Remark 2.9 it is suffice to show that $\overline{P(f)} \supset \overline{R(f)}$. Let $x \in R(f) - P(f)$. Take an open neighbourhood $U = B(x, \varepsilon, t_0)$ of x. Then there exists a sequence of positive integers $k_1 < k_2 < \cdots$ such that $f^{k_n}(x) \longrightarrow x$ and $f^{k_n}(x) \in U$. Choose $f^{k_m}(x) \in U$. Without loss of generality we may assume that $f^{k_m}(x) < x$. Then by Remark 2.7 we see that there exists $r > m$ such that $f^{k_m}(x) < f^{k_r}(x)$. By Corollary 3.4 we obtain $[f^{k_m}(x); f^{k_r}(x)] \cap P(f) \neq \varnothing$ if $f^{k_r}(x) \geq x$ or $[f^{k_m}(x); x] \cap P(f) \neq \varnothing$ if $f^{k_r}(x) \leq x$. Thus $U \cap P(f) \neq \varnothing$ (Definition 2.6 (4)), which implies $x \in \overline{P(f)}$ and $\overline{R(f)} \subset \overline{P(f)}$. The proof is completed. □

Theorem 3.6. *Let $f \in C(I)$. Then $\Omega(f|\Omega(f)) = \overline{R(f)}$ and the depth of f is at most 2.*

Proof. By Remark 2.9 it is suffice to show that $\Omega(f|\Omega(f)) - R(f) \subset \overline{R(f)}$. Let $x \in \Omega(f|\Omega(f)) - R(f)$. Take an open neighbourhood $U = B(x, \varepsilon, t_0)$ of x. Then there exist a sequence of positive integer $k_1 \leq k_2 \leq \cdots$ and a sequence of points $x_n \in \Omega(f)$ such that $f^{k_n}(x_n) \longrightarrow x$, $x_n \longrightarrow x$ and $f^{k_n}(x_n), x_n \in U$ for any $n \in \mathbf{N}$. Without loss of generality we may assume that $x_n \notin P(f)$ for any $n \in \mathbf{N}$. Choose $x_m, f^{k_m}(x_m) \in U$. Without loss of generality we may assume that $x_m < f^{k_m}(x_m)$. We can choose an open neighbourhood $W = B(x_n, \delta, t_1)$ of x_m such that $W, f^{k_m}(W) \subset U$ and $W \cap f^{k_m}(W) = \varnothing$ since I is a compact Hausdorff space. Note that $x_m \in \Omega(f)$ and $x_m \notin P(f)$. Then there exist a sequence of positive integers $r_1 < r_2 < \cdots$ and a sequence of points $y_n \in I$ such that $f^{r_n}(y_n) \longrightarrow x_m$, $y_n \longrightarrow x_m$ and $f^{r_n}(y_n), y_n \in W$ for any $n \in \mathbf{N}$. By Proposition 3.1 we see that there exists $r_n > k_m$ such that $M(a, f^{r_n}(y_n), t) > M(a, f^{k_m}(y_n), t)$ and $M(a, y_n, t) > M(a, f^{k_m}(y_n), t)$. Thus $f^{r_n}(y_n) < f^{k_m}(y_n)$ and $y_n < f^{k_m}(y_n)$. By Corollary 3.4 we obtain $[y_n; f^{k_m}(y_n)] \cap P(f) \neq \varnothing$ if $y_n \leq f^{r_n}(y_n)$, or $[f^{r_n}(y_n); f^{k_m}(y_n)] \cap P(f) \neq \varnothing$ if $y_n \geq f^{r_n}(y_n)$, which implies $U \cap P(f) \neq \varnothing$ (Definition 2.6 (4)). Thus $x \in \overline{P(f)} = \overline{R(f)}$. The proof is completed. □

4. The attracting centre for a continuous map on a fuzzy metric interval

In this section, we study the attracting centre of a continuous map on a fuzzy metric interval $I = [a; b]$ with $a \neq b$.

Lemma 4.1. *Let $f \in C^0(I)$ and $x, w, y \in I$ with $w \in [x; y]$ and $[x; y] \cap (R(f) \cup \Lambda(w, f)) = \varnothing$. If there exist $x_1, x_2, \cdots \in \Lambda(w, f)$ with $x_n < x$ and $y_1, y_2, \cdots \in \Lambda(w, f)$ with $y < y_n$ for any $n \in \mathbf{N}$ such that $\lim_{n\longrightarrow\infty} M(x_n, x, t) = \lim_{n\longrightarrow\infty} M(y_n, y, t) = 1$ (for any $t > 0$), then $\{x, y\} \cap A\Gamma(f) \neq \varnothing$.*

Proof. We claim that $f^n([x; y]) \cap [x; y] = \varnothing$ for any $n \in \mathbf{N}$. Indeed, if $f^m([x; y]) \cap [x; y] \neq \varnothing$ for some $m \in \mathbf{N}$, then $f^m([x; y]) \not\subset [x; y]$ since $[x; y] \cap P(f) = \varnothing$. Thus there exists $N \in \mathbf{N}$ such that $x_n \in f^m([x; y])$ or $y_n \in f^m([x; y])$ for any $n \geq N$ (Proposition 3.1), which implies that $[x; y] \cap \Lambda(w, f) \neq \varnothing$. This is a contradiction.

Since $x, y \notin R(f)$, we see that there exist $\delta_0 \in (0, 1)$, $t_1 > 0$ and $t_2 > 0$ such that $M(f^n(x), x, t_1) \leq 1 - \delta_0$ and $M(f^n(y), y, t_2) \leq 1 - \delta_0$ for any $n \in \mathbf{N}$. Thus by taking subsequence we can assume that $f^n([x; y]) \cap [x_1; y_1] = \varnothing$ for any $n \in \mathbf{N}$ and choose two sequences of positive integers $\mu_1, \mu_2, \cdots$ and $\lambda_1, \lambda_2, \cdots$ such that

(1) $f^{\mu_n}(x_n) = f^{\lambda_n}(y_n) = w$ for all $n \in \mathbf{N}$.

(2) (i) $f^{\mu_n}([x_n; x]) \supset [x_1; x]$ for any $n \in \mathbf{N}$ or (ii) $f^{\mu_n}([x_n; x]) \supset [y; y_1]$ for any $n \in \mathbf{N}$.

(3) (i) $f^{\lambda_n}([y; y_n]) \supset [x_1; x]$ for any $n \in \mathbf{N}$ or (ii) $f^{\lambda_n}([y; y_n]) \supset [y; y_1]$ for any $n \in \mathbf{N}$.

If (2(i)) holds, then write

$$E_1 = [x_1; x]$$

and for $n \geq 2$, write

$$E_n = E_{n-1} \cap f^{-\mu_1-\cdots-\mu_{n-1}}([x_n; x]).$$

Thus $E_{n+1} \subset E_n$ and $E_n \neq \varnothing$ for all $n \in \mathbf{N}$, which implies that $\cap_{n=1}^{\infty} E_n \neq \varnothing$. Let $u \in \cap_{n=1}^{\infty} E_n$. Then one has $f^{\mu_1+\cdots+\mu_{n-1}}(u) \in [x_n; x]$ and $u \in f^{\mu_n}([x_n; x])$ for all $n \in \mathbf{N}$, from which we see that $x \in A\Gamma(f)$.

If (3(i)) holds, then using arguments similar to the ones developed in the proof of above case, also we can show that $y \in A\Gamma(f)$.

If (2(ii)) and (3(ii)) hold, then $f^{\lambda_n+\mu_n}([y_n; y]) \supset [y_1; y]$ and $f^{\lambda_n+\mu_n}([x; x_n]) \supset [x; x_1]$ for any $n \in \mathbf{N}$. In a similar fashion, also we can show that $x, y \in A\Gamma(f)$. The proof is completed. □

Lemma 4.2. *Let $f \in C^0(I)$. Then $R(f) \cup A\Gamma(f) \subset \omega(R(f) \cup A\Gamma(f))$.*

Proof. If $x \in R(f)$, then $x \in \omega(x, f) \subset \omega(R(f) \cup A\Gamma(f))$. In the following we show that $x \in \omega(R(f) \cup A\Gamma(f))$ if $x \in A\Gamma(f) - R(f)$.

Since $x \in A\Gamma(f) - R(f)$, there exist $\delta_0 \in (0, 1)$ and $t_1 > 0$ such that $M(f^n(x), x, t_1) \leq 1 - \delta_0$ for any $n \in \mathbf{N}$ and without loss of generality we may assume that there exist a sequence of points $x_0, x_1, \cdots, x_n, \cdots \in (x; b]$, and two sequences of positive integers $\lambda_1 \leq \lambda_2 \leq \cdots$ and $\mu_1 < \mu_2 < \cdots$ such that:

(1) $M(x, x_n, t_1) > 1 - \delta_0/2$ for every $n \in \mathbf{N}^\star$.

(2) $f^{\lambda_n}(x_n) = x_0$ and $x_n \in (x; x_{n-1})$ for every $n \in \mathbf{N}$ and $\lim_{n\longrightarrow\infty} M(x_n, x, t) = 1$ (for any $t > 0$).

(3) $f^{\mu_n}(x_0) \in (x; x_0)$ and $f^{\mu_{n+1}}(x_0) \in (x; f^{\mu_n}(x_0))$ for every $n \in \mathbf{N}$ and $\lim_{n\longrightarrow\infty} M(f^{\mu_n}(x_0), x, t) = 1$ (for any $t > 0$).

By Corollary 3.4 we see that there exists a sequence of periodic points $p_1, p_2, \cdots$ such that $p_{n+1} \in (x; p_n)$ for any $n \in \mathbf{N}$ and $\lim_{n\longrightarrow\infty} M(p_n, x, t) = 1$ (for any $t > 0$). Without loss of generality we may assume that

$$x < \cdots < x_{2n} < x_{2n-1} < f^{\mu_n}(x_0) < \cdots < f^{\mu_3}(x_0)$$

$$< x_4 < x_3 < f^{\mu_2}(x_0) < x_2 < x_1 < p_2 < f^{\mu_1}(x_0) < p_1 < x_0.$$

Let E_n be the connected component of $I - \Lambda(f^{\mu_1}(x_0), f)$ containing $f^{n+\mu_1}(x_0)$ and $\overline{E_0} = [u; v]$. Then $f^n([u; v]) \subset \overline{E_n}$.

We claim that $[u; v] \cap (A\Gamma(f) \cup R(f)) \neq \varnothing$. Without loss of generality we may assume that $[u; v] \cap R(f) = \varnothing$. Then $[u; v] \subset [p_2, p_1]$ and $f^n([u; v]) \not\subset [u; v]$ for all $n \in \mathbf{N}$ and the following two statements hold:

(4) (i) $u \in \Lambda(f^{\mu_1}(x_0), f)$, or (ii) there exist $u_1, u_2, \cdots \in \Lambda(f^{\mu_1}(x_0), f)$ with $p_2 < u_1 < u_2 < \cdots < u$ such that $\lim_{n\longrightarrow\infty} M(u_n, u, t) = 1$ (for any $t > 0$).

(5) (i) $v \in \Lambda(f^{\mu_1}(x_0), f)$, or (ii) there exist $v_1, v_2, \cdots \in \Lambda(f^{\mu_1}(x_0), f)$ with $v < \cdots < v_2 < v_1 < p_1$ such that $\lim_{n\longrightarrow\infty} M(v_n, v, t) = 1$ (for any $t > 0$).

Now we show that $u \notin \Lambda(f^{\mu_1}(x_0), f)$. Otherwise, if $u \in \Lambda(f^{\mu_1}(x_0), f)$, then there exists some $n \in \mathbf{N}$ such that $f^n(u) = f^{\mu_1}(x_0) \neq u$. By Lemma 3.2 and Corollary 3.4 we have $f^n(v) > v$ and $v \notin \Lambda(f^{\mu_1}(x_0), f)$ since $[u; v] \cap P(f) = \varnothing$. Thus by (5(ii)) we see that there exists $N \in \mathbf{N}$ such that $v_n \in f^n((u; v))$ for any $n \geq N$, which contradicts the definition of E_0. In a similar fashion we can also show that $v \notin \Lambda(f^{\mu_1}(x_0), f)$. Since $u, v \notin \Lambda(f^{\mu_1}(x_0), f)$, we know that (4(ii)) and (5(ii)) hold. By Lemma 4.1 we see that $\{u, v\} \cap A\Gamma(f) \neq \varnothing$. The claim is proven

It follows from above claim that $x \in \omega(u, f) \cap \omega(v, f)$. The proof is completed. □

Lemma 4.3. *Let $f \in C^0(I)$. Then $\omega(\Omega(f)) \subset R(f) \cup A\Gamma(f)$.*

Proof. We may assume that $y \in \omega(\Omega(f)) - R(f)$. Then there exist $\delta_0 \in (0, 1)$ and $t_1 > 0$ such that $M(f^n(y), y, t_1) \leq 1 - \delta_0$ for any $n \in \mathbf{N}$ and without loss of generality we may assume that there exists some $z \in \Omega(f)$ such that $y \in \omega(z, f)$ and there exist two sequences of positive integers $\mu_1 < \mu_2 < \cdots$ and $\lambda_1 < \lambda_2 < \cdots$, a sequence of points $z_1, z_2, \cdots$ such that

(1) $\lim_{n\longrightarrow\infty} M(f^{\mu_n}(z), y, t) = \lim_{n\longrightarrow\infty} M(f^{\lambda_n}(z_n), z, t) = \lim_{n\longrightarrow\infty} M(z_n, z, t) = 1$ (for any $t > 0$).

(2) $y < \cdots < f^{\mu_n}(z) < \cdots < f^{\mu_2}(z) < f^{\mu_1}(z)$ with $M(f^{\mu_n}(z), y, t_1) > 1 - \delta_0/2$.

For every $n \geq 2$, there exists a point z_i with $s_n = \lambda_i + \mu_n - \mu_{n+4} > 0$ such that

$$f^{\mu_{n+5}}(z) < u_{n+4} = f^{\mu_{n+4}}(z_i) < f^{\mu_{n+3}}(z) < f^{\mu_{n+2}}(z)$$

$$< f^{\mu_{n+1}}(z) < v_n = f^{\lambda_i+\mu_n}(z_i) = f^{s_n}(u_{n+4}) < f^{\mu_{n-1}}(z).$$

If there exist $k_1 < k_2 < \cdots$ such that $f^{s_{k_n}}(y) > y$, then $f^{s_{k_n}}(y) > f^{\mu_1}(z)$ and $f^{s_{k_n}}([y; u_{k_n+4}]) \supset [v_{k_n}; f^{\mu_1}(z)]$, then there exists $w_n \in (y; u_{k_n+4}]$ such that $f^{s_{k_n}}(w_n) = f^{\mu_1}(z)$ for any $n \geq 2$ with $w_n \longrightarrow y$, which implies that $y \in A\Gamma(f)$. In the following we may assume that $f^{s_n}(y) < y$ for any $n \geq 2$.

Since $f^{s_n}([y;u_{n+4}]) \supset [y;v_n]$ for any $n \geq 2$, there exists $w_n \in [y;u_{n+4}]$ such that $f^{s_n}(w_n) = w_{n-1}$ for any $n \geq 2$, $w_1 = f^{\mu_3}(z)$ and $w_n \longrightarrow y$, which implies that $y \in A\Gamma(f)$. The proof is completed. □

Theorem 4.4. *Let $f \in C^0(I)$. Then for any $n \in \mathbf{N}$, $\omega^{n+2}(f) = \omega^2(f) = \omega(\Omega(f)) = \omega(R(f) \cup A\Gamma(f))$ and the depth of the attracting centre of f is at most 2.*

Proof. It follows from Lemma 4.2 and Lemma 4.3 that

$$\begin{aligned} \omega(R(f) \cup (A\Gamma(f))) &\subset \omega^2(f) \subset \omega(\Omega(f)) \\ &\subset R(f) \cup (A\Gamma(f) \\ &\subset \omega(R(f) \cup A\Gamma(f)). \end{aligned}$$

The last implies that

$$\begin{aligned} \omega(R(f) \cup (A\Gamma(f))) &= \omega^2(f) = \omega(\Omega(f)) \\ &= R(f) \cup (A\Gamma(f) \\ &= \omega(R(f) \cup A\Gamma(f)). \end{aligned}$$

Thus we know that for any $n \in \mathbf{N}$, $\omega^{n+2}(f) = \omega^2(f) = \omega(\Omega(f)) = \omega(R(f) \cup A\Gamma(f)) = \omega(R(f) \cup A\Gamma(f))$. The proof is completed. □

5. Conclusion

In this paper, we introduce the notion of fuzzy metric interval, and study the depth and the attracting centre for a continuous map f on a fuzzy metric interval, and show that $\Omega_2(f) = \overline{R(f)}$ and the depth of f is at most 2, and $\omega^3(f) = \omega^2(f)$ and the depth of the attracting centre of f is at most 2.

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