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Factorization of operators through Orlicz spaces

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Abstract We study factorizations of operators between quasi-Banach spaces. We prove the equivalence between certain vector norm inequalities and the factorization of operators through Orlicz spaces. As a consequence, we obtain the Maurey-Rosenthal factorization of operators into L_p-spaces. We give several applications of our results. In particular, we prove a variant of Maurey's Extension Theorem.

Keywords Factorization · Banach function lattice · Banach envelope · Orlicz space.

Mathematics Subject Classification (2000) 46E30 · 47B38 · 46B42.

1 Introduction

This paper is devoted to factorization of operators between quasi-Banach function spaces. We recall that if X, Y are quasi-Banach spaces and $T\colon X \to Y$ is a continuous operator, then T *factors* through a quasi-Banach space Z if there are continuous operators $R\colon X \to Z$ and $S\colon Z \to Y$ so that $T = SR$.

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We are motivated by the fact that factorization theorems play a fundamental role in functional analysis. Curiously, the theorems about factorization of operators with values in L_p-spaces, $0 \le p \le \infty$ found numerous applications in the theory of Banach spaces. For instance, notice that Nikishin's [18] factorization theorem found applications in harmonic analysis (see, e.g., [22, p. 263]).

There are many classical results on the description of concrete factorizations of operators between Banach lattices in terms of vector norm inequalities. The so called Maurey-Rosenthal theorems relate some p-convexity/p-concavity inequalities for operators with values on Banach lattices spaces with their factorizations through weighted L_p-spaces (see, e.g., [22, Proposition III.H.10] and [3,5,6,19]). For instance, if X is an order continuous p-convex Banach lattice on a σ-finite measure space (Ω, Σ, μ) and $T\colon X(\mu) \to E$ is a p-concave operator, then there exist a measurable function $g \ge 0$ a.e. such that the multiplication operator $M_g\colon X(\mu) \to L_p(\mu)$ is bounded and an operator $S\colon L_p(\mu) \to X$ such that T factors through the space $L_p(\mu)$ as follows:

$$\begin{array}{ccc} X(\mu) & \xrightarrow{\quad T \quad} & E. \\ & {\scriptstyle M_g}\searrow \quad \nearrow{\scriptstyle S} & \\ & L_p(\mu) & \end{array}$$

We refer to Pisier's book [20] for more on abstract factorization theorems and their applications to Banach space theory.

The general problem we consider in this paper is to provide the inequalities for the operators that characterize their factorizations and extensions through Orlicz spaces.

We now describe our main results in more detail. The paper consists of five sections. In Section 2, we recall some of the basic facts about the Banach envelope of a quasi-Banach space. It consists of fundamental facts about the duality in Banach function lattices we will need for the proofs of our main theorems. In Section 3 we prove a separation theorem for operators $T\colon X \to E$ which involves the generalized Orlicz spaces X_φ generated by a Banach function lattice X and an increasing, continuous positive function φ on $[0, \infty)$ (an Orlicz function). We then investigate this result further motivated by applications to factorization. In particular we show that under mild assumptions $T\colon X \to E$ can be extended to the Orlicz space $L_\varphi(\nu)$ as follows, where $i\colon X \to L_\varphi(\nu)$ is the continuous inclusion map:

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & E. \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle S} & \\ & L_\varphi(\nu) & \end{array}$$

Recall that a function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an Orlicz function if φ is a convex continuous function at zero such that $\varphi(u) = 0$ if any only if $u = 0$.

Let $(\Omega, \mathcal{S}, \nu)$ be a measure space. The Orlicz space $L_\varphi(\nu)$ is defined to be the space of all $f \in L^0(\nu)$ satisfying

$$\int_\Omega \varphi(\lambda|f(t)|)\, d\nu < \infty$$

for some $\lambda > 0$. It is well known that $L_\varphi(\nu)$ is a Banach lattice equipped with the norm

$$\|f\| = \inf\Big\{\varepsilon > 0;\ \int_\Omega \varphi(f/\varepsilon)\, d\nu \le 1\Big\}.$$

In Section 4 we present results and examples regarding the two notions of (ϕ, φ)-concavity for operators that we introduce, where ϕ and φ are Orlicz functions. Some consequences and applications in form of factorization theorems hold from our general results and are shown in Section 5. We apply our arguments to give a variant of Maurey's Extension Theorem. We prove that if X is a 2-convex quasi-Banach lattice and Y is a closed subspace of X, then every 2-concave operator T from Y to a quasi-Banach space E admits an extension $\widetilde{T}\colon X \to E$ that factors through a Hilbert space. We also show that when X is a p-convex Banach lattice and $\varphi(t) = t^p$ for all $t \ge 0$, we recover the Maurey-Rosenthal Theorem for p-concave operators. We conclude by presenting some applications to the study of the continuous inclusions between quasi-Banach function lattices.

2 Preliminaries

We now introduce notation and conventions and provide some technical observations which we will use later without any references. All the operators considered in this paper are linear.

By definition, a quasi-normed space is a vector space X over $\mathbb{K}$ with a quasi-norm $\|\cdot\|_X$ satisfying:

(i) $\|x\|_X > 0, \qquad x \in X,\ x \ne 0,$
(ii) $\|\lambda x\|_X = |\lambda|\|x\|_X, \qquad \lambda \in \mathbb{K},\ x \in X,$
(iii) $\|x + y\|_X \le C_X(\|x\|_X + \|y\|_X), \qquad x, y \in X$

for some constant C_X independent of x, y. If a quasi-normed space is complete, we say that it is a *quasi-Banach space.*

Let $(X, \|\cdot\|_X)$ be a quasi-Banach space. The Mackey semi-norm $\|\cdot\|_X^c$ on X is the Minkowski functional of the convex hull $\mathrm{conv}(B_X)$ of the unit ball $B_X := \{x \in X;\ \|x\|_X \le 1\}$,

$$\|x\|_X^c = \inf\big\{\lambda > 0;\ x \in \lambda\, \mathrm{conv}(B_X)\big\}, \qquad x \in X.$$

If the topological dual X^* of X separates the points of X, then X^* is a Banach space under the norm

$$\|x^*\|_{X^*} = \sup_{x \in B_X} |x^*(x)|.$$

Since $\|x^*\|_{X^*} = \sup\big\{|x^*(x)|;\, x \in \operatorname{conv}(B_X)\big\}$, it follows

$$\|x\|_X^{\mathrm{c}} = \sup_{\|x^*\|_{X^*} \le 1} |x^*(x)|, \quad x \in X.$$

and so $X^* = (X, \|\cdot\|_X^{\mathrm{c}})^*$ with equality of norms.

The completion of $(X, \|\cdot\|_X^{\mathrm{c}})$ is called the *Banach envelope* of X. The above formulas imply

$$\|\kappa x\|_{X^{**}} = \sup_{\|x^*\|_{X^*} \le 1} |x^*(x)| = \|x\|_X^{\mathrm{c}},$$

where $\kappa\colon X \to X^{**}$ is the canonical embedding defined by $\kappa x(x^*) = x^*(x)$, $x \in X$, $x^* \in X^*$. In particular, it follows that the Banach envelope of X is the closure of the range of κ in X^{**}.

Let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a complete σ-finite measure space, and let $L_0(\mu)$ denote the space of all equivalence classes of extended real-valued Σ-measurable functions on Ω equipped with the topology of convergence in measure on μ-finite sets. As usual if $x, y \in L_0(\mu)$, then $|x| \le |y|$ means that $|x(\omega)| \le |y(\omega)|$ for μ-almost all $\omega \in \Omega$.

A quasi-normed *function lattice* X on a measure space (Ω, μ) is defined to be a quasi-normed space $X \subset L_0(\mu)$ such that there exists a strictly positive $u \in X$ and X is an ideal in $L_0(\mu)$, i.e., if $|x| \le |y|$, where $y \in X$ and $x \in L_0(\mu)$, then $x \in X$ and $\|x\|_X \le \|y\|_X$. If in addition, $(X, \|\cdot\|_X)$ is complete then it is called a quasi-Banach function lattice.

Fix a quasi-normed function lattice X on (Ω, μ). It is said that X is order continuous if for any decreasing sequence $0 \le x_n \in X$, if $x_n \downarrow 0$ a.e., then $\|x_n\|_X \to 0$. We say that X has the *weak Fatou property* whenever if $x_n, x \in X$, $x_n \uparrow x$ a.e., then $\|x_n\|_X \to \|x\|_X$. It is said that X satisfies the *Fatou property* if for any $x \in L_0(\mu)$ and $x_n \in X$ such that $0 \le x_n \le x$ and $x_n \uparrow x$ we have that $x \in X$ and $\|x_n\|_X \to \|x\|_X$.

The Köthe dual space (or associate space) X' of a normed function lattice X on (Ω, μ) is defined to be the collection of all $x \in L_0(\mu)$ such that $\int_\Omega |xy|\, d\mu < \infty$ for every $y \in X$ equipped with the norm

$$\|x\|_{X'} = \sup_{\|y\|_X \le 1} \int_\Omega |xy|\, d\mu\,.$$

X' is a Banach function lattice with the Fatou property. Notice that if X is a Banach function lattice, then $X \subset X''$ with $\|x\|_{X''} \le \|x\|_X$ for every $x \in X$. Moreover, $X = X''$ with equality of the norms if and only if X has the Fatou property (see [13]).

It is well known that a normed function lattice X is order continuous if and only if the map $X' \ni y \mapsto x_y^* \in X^*$ where

$$x_y^*(x) := \int_\Omega xy\, d\mu, \quad x \in X$$

is an order isometrical isomorphism of X' onto X^* (see, e.g., [13]). It is also known that for any $x \ge 0$ there exists $0 \le x^* \in X^*$ such that $\|x^*\|_{X^*} = 1$

and $x^*(x) = \|x\|_X$ (see e.g. [14]). In particular, by the above fact, there exists $y \in X'$ with $\|y\|_{X'} = 1$ and $\|x\|_X = \int_\Omega xy \, d\mu$ provided X is order continuous. We will write X_a for the order continuous part of the normed function lattice X, that is, the biggest order continuous Banach function lattice that is included in X.

Let us observe that if $(X, \|\cdot\|_X)$ is a quasi-normed function lattice on (Ω, μ) such that the topological dual X^* separates the points of X, then $(X, \|\cdot\|_X^{\mathrm{c}})$ is a normed function lattice on (Ω, μ), which is order continuous provided X is order continuous.

Quasi-Banach function spaces on measure spaces are the natural setting for the development of the ideas of this paper. However, they can also be applied in the case of abstract quasi-Banach lattices. This is the case for instance of the spaces $C(K)$, that will be relevant in this paper regarding some examples and applications.

3 Factorization through Orlicz spaces

Let Φ be the set of all increasing and continuous functions $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$. Given a quasi-normed lattice X on (Ω, Σ, μ) and a function $\varphi \in \Phi$, we define a linear subspace of $L_0(\mu)$ by

$$X_\varphi = \big\{ f \in L_0(\mu);\ \exists\, \lambda > 0,\ \varphi(\lambda |f|) \in X \big\}.$$

and the functional $\|\cdot\|_{X_\varphi}\colon X_\varphi \to [0,\infty)$ by

$$\|f\|_{X_\varphi} := \inf \big\{ \lambda > 0;\ \|\varphi(|f|/\lambda)\|_X \le 1 \big\}, \quad f \in X_\varphi.$$

Clearly $\|f\|_{X_\varphi} = 0$ if and only if $f = 0$ and $\|\lambda f\|_{X_\varphi} = |\lambda| \, \|f\|_{X_\varphi}$ for all $\lambda \in \mathbb{R}$, $f \in X_\varphi$. The space X_φ is an ideal in $L_0(\mu)$ on which $\|\cdot\|_{X_\varphi}$ is monotone, i.e., $f \in L_0(\mu)$, $g \in X_\varphi$ and $|f| \le |g|$ a.e. implies $f \in X_\varphi$ and $\|f\|_{X_\varphi} \le \|g\|_{X_\varphi}$.

Notice that there is a large class of functions $\varphi \in \Phi$ for which $\|\cdot\|_{X_\varphi}$ is a quasi-norm on X_φ. To see this let C_X be a constant from the quasi-triangle inequality of a quasi-normed function lattice X. If $\varphi \in \Phi$ is such that there is a constant C such that

$$\varphi(t/C) \le \varphi(t)/2C_X, \quad t > 0,$$

then for all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ we have

$$\varphi\Big(\frac{1}{C}(\alpha s + \beta t)\Big) \le \frac{1}{2C_X}(\varphi(s) + \varphi(t)), \quad s, t > 0.$$

Since $\big\|\varphi\big(|f|/(\|f\|_{X_\varphi} + \varepsilon)\big)\big\|_X \le 1$ for every $\varepsilon > 0$ and all $f \in X_\varphi$, the above inequality yields

$$\|f + g\|_{X_\varphi} \le C\big(\|f\|_{X_\varphi} + \|g\|_{X_\varphi}\big), \quad f, g \in X_\varphi.$$

Thus $\|\cdot\|_{X_\varphi}$ is a quasi-norm on X_φ and it may be verified that $(X_\varphi, \|\cdot\|_{X_\varphi})$ is complete. In particular, X_φ is a Banach function lattice provided φ is a convex function and X is a Banach function lattice. Note that in the case $\varphi(t) = t^p$ for all $t \geq 0$, the same notion makes sense for the case of abstract quasi-Banach lattices and the space X_φ is known as X_p the *p-convexification* of X (see [14] or [19, Ch.2]) equipped with the quasi-norm

$$\|f\|_{X_p} = \||f|^p\|_X^{1/p}.$$

If $X = L_1(\mu)$ and $\varphi \in \Phi$, then X_φ is the *Orlicz space* denoted by $L_\varphi(\mu)$ (for short L_φ).

Let $\varphi \in \Phi$. Throughout the paper a quasi-Banach function lattice X is said to be *φ-admissible* provided that $\|\cdot\|_{X_\varphi}$ is a quasi-norm on X_φ. If in addition the topological dual $(X_\varphi)^*$ separates the points of X_φ, then X is said to be *strongly φ-admissible.*

It is easy to check that if (Ω, Σ, μ) is an atomless measure space and $\varphi \in \Phi$, then $L_1(\mu)$ is φ-admissible if and only if there exists $C > 0$ such that

$$\varphi(t/C) \leq \varphi(t)/2, \quad 0 < t < \mu(\Omega).$$

Let us observe that if an order continuous Banach function lattice X is φ-admissible with $\varphi \in \Delta_2$ (i.e., there exists a constant $C > 0$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t > 0$), then $\|f_n\|_{X_\varphi} \to 0$ if and only if $\|\varphi(|f_n|)\|_X \to 0$. This implies that X_φ is order continuous if and only if X is order continuous.

We will need the following obvious technical result.

Proposition 1 *Let $\varphi \in \Phi$ and let X be a quasi-Banach function lattice on (Ω, μ). If X has the weak Fatou property, then the inequalities $\|f\|_{X_\varphi} \leq 1$ and $\|\varphi(|f|)\|_X \leq 1$ are equivalent.*

The following lemma provides general examples of $\varphi \in \Phi$ and quasi-normed lattices X which are strongly φ-admissible.

Lemma 1 *Let $\varphi \in \Phi$ and let X be a φ-admissible quasi-normed function lattice on (Ω, μ) such that $\chi_\Omega \in X$.*

(i) *If φ is concave then $X \hookrightarrow X_\varphi$.*
(ii) *If φ is convex then $X_\varphi \hookrightarrow X$. As a consequence X is strongly φ-admissible provided the topological dual X^* separates points of X.*

Proof (i). Since φ is a concave function, the set

$$\mathcal{C}_\varphi := \{(a, b);\, a, b \geq 0,\ \forall t \geq 0,\ \varphi(t) \leq at + b\}$$

is non-empty and we have

$$\varphi(t) = \inf\{at + b;\, (a, b) \in \mathcal{C}_\varphi\}, \qquad t \geq 0.$$

Let $C_1 = \|\chi_\Omega\|_X$ and let $C > 0$ be such that $C_X C_1 \varphi\big(1/C_1 C\big) \le 1$. For any $(a, b) \in \mathcal{C}_\varphi$ and $f \in X$, $\|f\| = 1$, we have

$$0 \le \varphi\big(|f|/C\big) \le \frac{a}{C}\,|f| + b \quad \mu - a.e.$$

Hence $\varphi\big(|f|/C\big) \in X$ by the monotonicity of the quasi-norm $\|\cdot\|_X$, and

$$\big\|\varphi\big(|f|/C\big)\big\|_X \le \big\|a(|f|/C)+b\big\|_X \le C_X\Big(\frac{a}{C}\|f\|_X + bC_1\Big) \le C_X C_1\Big(a\frac{1}{C_1 C} + b\Big).$$

Since $(a, b) \in C_\varphi$ is arbitrary,

$$\|\varphi\big(|f|/C\big)\|_X \le C_X C_1 \inf_{(a,b)\in\mathcal{C}_\varphi}\Big(a\frac{1}{C_1 C} + b\Big) = C_X C_1 \varphi\big(1/C_1 C\big) \le 1.$$

This implies $f \in X_\varphi$ with $\|f\|_{X_\varphi} \le C$ and shows that the inclusion map $i\colon X \to X_\varphi$ is bounded with $\|i\| \le C$. The proof of (ii) follows immediately from (i).

In the sequel we will need Ky-Fan's Lemma (see, e.g., [8, p. 190]). For the sake of completeness we state it here.

Lemma 2 *Let E be a Hausdorff topological vector space, and let K be a compact convex subset of E. Let Ψ be a set of functions on K with values in $(-\infty, \infty]$ having the following properties*:

(a) *each $f \in \Psi$ is convex and lower semicontinuous,*
(b) *Ψ is concave, i.e., if $g \in conv(\Psi)$, there is an $f \in \Psi$ with $g(x) \le f(x)$, for every $x \in K$,*
(c) *there is an $r \in \mathbb{R}$ such that each $f \in \Psi$ has a value not greater than r.*

Then there is an $x_0 \in K$ such that $f(x_0) \le r$ for all $f \in \Psi$.

We now state and prove a separation theorem that gives directly an extension theorem for operators.

Theorem 1 *Let $\varphi, \phi \in \Phi$ and X be a quasi-Banach lattice on (Ω, μ) such that X is strongly φ^{-1}-admissible. Suppose that T is an operator from X into a quasi-Banach space E. Suppose $0 < C < \infty$ and Y is a non trivial subset of X. Consider the following conditions*:

(i) For every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and every finite sequence $\{f_k\}_{k=1}^n$ in Y,

$$\sum_{k=1}^n \alpha_k \phi(\|Tf_k\|_E) \le C\,\Big\|\sum_{k=1}^n \alpha_k \varphi(|f_k|)\Big\|_{X_{\varphi^{-1}}}^c.$$

(ii) *There exists a positive functional* x^* *in the closed unit ball of* $(X_{\varphi^{-1}})^*$ *such that*

$$\phi\big(\|Tf\|_E\big) \le Cx^*(\varphi(|f|)), \quad f \in Y.$$

(iii) There exists $0 \le w \in B_{(X_{\varphi^{-1}})'}$ *such that for* $A = supp\, w$ *we have*

$$\phi\big(\|Tf\|_E\big) \le C \int_A \varphi(|f|)w\, d\mu, \quad f \in Y.$$

Then (*i*) *is equivalent to* (*ii*). *If* $X_{\varphi^{-1}}$ *is order continuous then all three conditions are equivalent.*

Proof (i) $\Rightarrow$ (ii). Let us recall the following consequence of the Banach-Alaoglu theorem, which will be used below: if the topological dual of a quasi-normed space E separates the points then the closed unit ball of the dual space E^*, B_{E^*}, is compact for the weak* topology $\sigma(E^*, E)$ on E^*.

Put $K := \{x^* \in B_{(X_{\varphi^{-1}})^*};\, x^* \ge 0\}$. Combining our hypotheses with the basic facts shown at the beginning of Section 2, we deduce that the inequality given in the condition (i) is equivalent to

$$\sum_{k=1}^n \alpha_k \phi(\|Tf_k\|_E) \le C \sup_{x^* \in K} \sum_{k=1}^n \alpha_k x^*(\varphi(|f_k|)). \qquad (*)$$

Let $S_{\ell_1^n}$ denotes be the unit sphere of n-dimensional ℓ_1^n-space. Consider the family

$$\Psi := \Big\{\psi_{\alpha,f};\, 0 \le \alpha = \{\alpha_k\}_{k=1}^n \in S_{\ell_1^n},\, f = \{f_k\}_{k=1}^n \in \prod_{k=1}^n Y,\, n \in \mathbb{N}\Big\}$$

of convex functions $\psi_{\alpha,f} \colon K \to \mathbb{R}$ defined for $0 \le \alpha = \{\alpha_k\} \in S_{\ell_1^n}$, $f = \{f_k\} \in \prod_{k=1}^n Y$ by

$$\psi_{\alpha,f}(x^*) = \sum_{k=1}^n \alpha_k \phi(\|Tf_k\|_E) - C \sum_{k=1}^n \alpha_k x^*(\varphi(|f_k|)), \quad x^* \in K.$$

It may be easily verified that Ψ is a concave family. Our hypotheses yields that the unit ball $B_{(X^{\varphi^{-1}})^*}$ is a compact convex subset for the weak*-topology $w^* = \sigma((X_{\varphi^{-1}})^*, X_{\varphi^{-1}})$. Since the set K is w^*-closed, it is w^*-compact. Clearly, every function in Ψ is continuous for the w^*-topology. Thus the Hahn-Banach theorem together with the inequality $(*)$ imply that for every $\psi \in \Psi$ there exists $x^* \in B_{(X^{\varphi^{-1}})^*}$ such that $\Psi(x^*) \le 0$. Then Ky-Fan's lemma applies and we deduce that there exists $x^* \in K$ such that

$$\phi\big(\|Tf\|\big) \le Cx^*(\varphi(|f|)), \quad f \in Y$$

and this completes the proof of (i).

The implication (ii) $\Rightarrow$ (i) is obvious.

If $X_{\varphi^{-1}}$ is order continuous, then the topological dual $(X_{\varphi^{-1}})^*$ can be isometrically identified with the Köthe dual $(X_{\varphi^{-1}})'$. This completes the proof.

We remark that if X is a quasi-Banach lattice which contains no copy of c_0 and has a weak unit then a standard representation theorem can be applied to represent X as an order continuous quasi-Banach function lattice on a measure space (Ω, Σ, μ) where Ω is a compact Hausdorff space and Σ is the σ-algebra of Borel sets of Ω (for details see [11]). Notice also that no ideal structure for X is needed for the separation argument used in (i) $\Rightarrow$ (ii) of the proof above: the same result remains true for the case $X = C(K)$.

We state now a theorem in which the domain space is an abstract quasi-Banach lattice. The proof is similar to the one given above.

Theorem 2 *Let $0 < p < \infty$ and let X be a quasi-Banach lattice such that the dual $(X_{1/p})^*$ separates the points of $X_{1/p}$. Suppose $\phi \in \Phi$ and T is an operator from X in a quasi-Banach space E. Suppose $0 < C < \infty$ and Y is a non trivial subset of X. Then the following conditions are equivalent*:

(i) For every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and every finite sequence $\{f_k\}_{k=1}^n$ in Y,

$$\sum_{k=1}^{n} \alpha_k \phi(\|Tf_k\|_E) \le C \Big\| \sum_{k=1}^{n} \alpha_k |f_k|^p \Big\|_{X_{1/p}}^c .$$

(ii) *There exists a positive functional x^* in the closed unit ball of $(X_{1/p})^*$ such that*

$$\phi\big(\|Tf\|_E\big) \le C x^*(|f|^p), \quad f \in Y.$$

In what follows if Σ is a σ-algebra of subsets of Ω and $A \in \Sigma$, then $\Sigma_A = \{F \cap A;\, F \in \Sigma\}$ denotes a σ-algebra of subsets of A.

The following theorem characterizes vector inequalities in terms of factorization under a mild assumption on $\varphi \in \Phi$. Recall that we write $L_\varphi(\nu)_a$ for the order continuous part of the Orlicz space $L_\varphi(\nu)$.

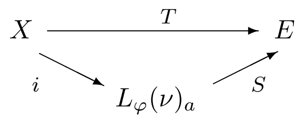
Theorem 3 *Let $\varphi \in \Phi$ and let X be a quasi-Banach lattice on (Ω, Σ, μ) that is strongly φ^{-1}-admissible and such that $X_{\varphi^{-1}}$ is order continuous. Suppose T is an operator from X into a quasi-Banach space E. Consider the following statements*:

(i) There exists a constant $C > 0$ such that for every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and every finite sequence $\{f_k\}_{k=1}^n$ in X,

$$\sum_{k=1}^{n} \alpha_k \varphi(\|Tf_k\|_E) \le C \Big\| \sum_{k=1}^{n} \alpha_i \varphi(|f_k|) \Big\|_{X_{\varphi^{-1}}}^c .$$

(ii) There exists $0 \le w \in B_{(X_{\varphi^{-1}})'}$ such that T admits the following factorization where $L_\varphi(\nu)$ is the Orlicz space on (A, Σ_A, ν) with $A = \operatorname{supp} w$,

$d\nu = wd\mu$ and $i\colon X \to L_\varphi(\nu)_a$ is an operator given by $i(f) = f\chi_A$ for all $f \in X$:



Then (i) implies (ii). If in addition φ is super-multiplicative (i.e., there exists $\widetilde{C} > 0$ such that $\varphi(s)\varphi(t) \le \widetilde{C}\varphi(st)$ for all $s, t > 0$) then (ii) implies (i) too.

Proof (i) $\Rightarrow$ (ii). From Theorem 1 it follows that there is $0 \le w \in B_{X'_{\varphi^{-1}}}$ such that

$$\varphi(\|Tf\|_E) \le C \int_A \varphi(|f|)d\nu, \quad f \in X \tag{$*$}$$

where $A = \operatorname{supp} w$, $d\nu := wd\mu$.

We claim that the map $i\colon X \to L_\varphi(\nu)_a$ and is bounded. To see that fix $f \in X$ with $\|f\| \le 1$. Then $\varphi(|f|) \in X_{\varphi^{-1}}$ with $\|\varphi(|f|)\|_{X_{\varphi^{-1}}} \le 1$ and so

$$\int_A \varphi(|f|)w\, d\mu \le \|\varphi(|f|)\|_{X_{\varphi^{-1}}}\, \|w\|_{(X_{\varphi^{-1}})'} \le 1.$$

Combining with our hypothesis $X_{\varphi^{-1}} = (X_{\varphi^{-1}})_a$ yields $i\colon X \to L_\varphi(\nu)_a$ with $\|i\| \le 1$. Thus the claim follows.

Using the inequality $(*)$ we obtain with $\lambda := \|f\|_{L_\varphi(\nu)} > 0$,

$$\varphi\big(\|Tf\|_E/\lambda\big) \le \int_A \varphi(|f|/\lambda)\, d\nu \le 1, \quad f \in X$$

and so

$$\|Tf\|_E \le \varphi^{-1}(C)\|f\|_{L_\varphi(\nu)}, \quad f \in X.$$

The required result then holds for the extension S of T to $L_\varphi(\nu)_a$, by the density of simple functions in $L_\varphi(\nu)_a$.

(ii) $\Rightarrow$ (i). Let $d\nu = wd\mu$, $w \in B_{(X_{\varphi^{-1}})'}$. Assume that there exists $\widetilde{C} > 0$ such that $\varphi(s)\,\varphi(t) \le \widetilde{C}\varphi(s\,t)$ for all $s, t > 0$. Clearly $\varphi \in \Delta_2$ and so $L_\varphi(\nu)_a = L_\varphi(\nu)$. Hence our hypothesis yields

$$\begin{aligned}\widetilde{C}\int_A \varphi(|f|)\, d\nu &= \int_A \widetilde{C}\varphi\Big(\frac{|f|}{\|f\|_{L_\varphi(\nu)}}\|f\|_{L_\varphi(\nu)}\Big)\, d\nu \\ &\ge \varphi(\|f\|_{L_\varphi(\nu)}) \cdot \int_A \varphi\Big(\frac{|f|}{\|f\|_{L_\varphi(\nu)}}\Big)\, d\nu \\ &= \varphi(\|f\|_{L_\varphi(\nu)}).\end{aligned}$$

Thus, for every $f \in X$,

$$\|Tf\|_E \le \|S(i(f))\| \le \|S\|\|f\|_{L_\varphi(\nu)} \le \|S\|\varphi^{-1}\big(\widetilde{C}\int_A \varphi(|f|)d\nu\big).$$

Therefore,

$$\varphi(\|Tf\|_E)\varphi\big(1/\|S\|\big) \le \widetilde{C}\varphi\big(\|Tf\|_E/\|S\|\big)\big) \le \widetilde{C}^2 \int_A \varphi(|f|)\, d\nu.$$

An application of Theorem 1 for $C = \widetilde{C}^2/\varphi(1/\|S\|)$ gives the result.

Notice that in the above results the Mackey norms as well as the dual (or the Köthe dual) space of $X_{\varphi^{-1}}$ appears. Both notions require delicate investigations when concrete spaces are considered. In the present paper we do not study this problem in general, since we are more interested in showing that our results find applications in some general classes of spaces and operators. Let us show one of them to close this section.

Let (X_0, X_1) is a pair of quasi-Banach function lattices on a measure space (Ω, μ). Denote by $\mathcal{U}$ the class of all functions $\psi\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ that are concave and positively homogeneous of degree 1. The Calderón-Lozanovskii space $\psi(X_0, X_1)$ consists of all $x \in L_0(\mu)$ such that $|x| = \psi(|x_0|, |x_1|)$ for some $x_j \in X_j$, $j = 0, 1$. The space $\psi(X_0, X_1)$ is a quasi-Banach lattice equipped with the quasi-norm (cf. [15,21])

$$\|x\| = \inf\big\{\max\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\};\ |x| = \psi(|x_0|, |x_1|)\big\}.$$

For instance, in the case when $\psi(s,t) = s^{1-\theta}t^\theta$, $\psi(X_0, X_1)$ is the well known Calderón space (see [1]).

The Lozanovskii Köthe duality formula (see [15,21]) states that when (X_0, X_1) is a couple of Banach function lattices on a meaasure space, then

$$\psi(X_0, X_1)' = \widehat{\psi}(X_0', X_1')$$

with equality of norms, where the space on the right hand side of the above equality is equipped with the norm

$$\|x'\| = \inf\big\{\|x_0'\|_{X_0'} + \|x_1'\|_{X_1'};\ |x'| = \widehat{\psi}(|x_0'|, |x_1'|)\big\}.$$

Here, for $\psi \in \mathcal{U}$, the conjugate function $\widehat{\psi}$ is defined by

$$\widehat{\psi}(s,t) := \inf\Big\{\frac{\alpha s + \beta t}{\psi(\alpha, \beta)};\ \alpha, \beta > 0\Big\}$$

for all $s, t \ge 0$. We have $\psi \in \mathcal{U}$ and $\widehat{(\widehat{\psi})} = \psi$ (see [16]).

Notice that if X is a quasi-Banach function lattice on (Ω, μ), $\varphi \in \Phi$ is a concave function and $\psi \in \mathcal{U}$ is defined by $\psi(s,t) = t\varphi(s/t)$ for all $s \ge 0$, $t > 0$ and $\psi(0,0) = 0$, then it is not difficult to check that

$$X_{\varphi^{-1}} = \psi(X, L_\infty)$$

with equality of norms. In particular, the Köthe duality formula given above implies that if X is a Banach function lattice, then we have

$$(X_{\varphi^{-1}})' = \widehat{\psi}(X', L_1)$$

also with equality of norms.

We end this section with the remark that in the case when $\varphi \in \Phi$ is a convex function and X is a quasi-Banach function lattice, then $X_{\varphi^{-1}}$ is not normable in general. We refer to [12], where the Banach envelopes as well as the topological duals of a large class of rearrangement invariant quasi-Banach lattices are described. These results can be applied under some mild assumptions in our setting.

4 Generalized concavity of operators

The main Theorem 1 of Section 2 motivates the study of classes of operators which satisfy inequality (i) of Theorem 1. In the present section we study classes of operators which satisfy the mentioned requirement. In particular, we introduce definitions which generalize the classical concept of (q, p)-concave operators (see [8]).

Let X be a quasi-Banach function lattice and let Y be a subspace of X. Let T be an operator from X to a quasi-Banach space E and let ℓ_φ and ℓ_ϕ be Orlicz sequence lattices on $\mathbb{N}$. T is said to be (ϕ, φ)-*concave* if there is a positive constant C such that, for every finite sequence $\{f_k\}_{k=1}^n$ in X,

$$\big\|\{\|Tf_k\|_E\}_{k=1}^n\big\|_{\ell_\phi^n} \leq C\,\big\|\|\{|f_k|\}_{k=1}^n\|_{\ell_\varphi^n}\big\|_X.$$

Here ℓ_φ^n is $\mathbb{R}^n$ equipped with the quasi-norm $\|\cdot\|_\varphi$. Notice that the definition makes sense also in the case of an abstract quasi-Banach lattice X with $\varphi(t) = t^p$ for all $t \geq 0$ where $1 \leq p < \infty$, since the expression $(\sum |x_i|^p)^{1/p}$ makes sense also in this case due to Krivine´s calculus (see [14]). The notion of (ϕ, φ)-concave operator has been analyzed and has found applications recently in a series of papers (see for instance [4,17] where examples of (ϕ, φ)-concave operators can be found).

We write $K_{\phi,\varphi}(T)$ for the least constant C that works in the inequality above. If $0 < p < \infty$ and $\varphi(t) = t^p$ (resp., $\phi(t) = t^q$ and $\varphi(t) = t^p$ for all $t \geq 0$ with $0 < q \leq p$), we say that T is (ϕ, p)-concave (resp., (q, p)-concave) and write $K_{\phi,p}(T)$ (resp., $K_{q,p}(T)$). As usual, the (q, q)-concave operators are called q-concave, and $K_q(T)$ is used instead of $K_{q,q}(T)$. We say that a quasi-Banach lattice X is q-*concave* if id_X is a q-concave operator and we write $K_q(X)$ instead of $K_q(id_X)$. We refer to [8] for the definitions and more information on relationships between (q, p)-concave and (q, p)-summing operators between Banach spaces.

We introduce now a different notion of concavity for operators, that coincides with the previous one for the case of q-concave operators. Let $\phi, \varphi \in \Phi$. An operator T from a quasi-Banach lattice X to a quasi-Banach space E is said to be *strongly* (ϕ,φ)*-concave* if there is a constant C such that, for any finite sequence $\{f_k\}_{k=1}^n$ in X,

$$\phi^{-1}\Big(\sum_{k=1}^n \alpha_k \phi\big(\|Tf_k\|_E/C\big)\Big) \le \Big\|\varphi^{-1}\Big(\sum_{k=1}^n \alpha_k \varphi(|f_k|)\Big)\Big\|_X.$$

We let $\widetilde{K}_{\phi,\varphi}(T)$ be the least constant C for which the above inequality holds.

Example 1 Let $\phi \in \Phi$ be given by $\phi(t) = t^2$ for $0 \le t \le 1$ and $\phi(t) = t$ for $1 < t < \infty$. Then every 2-concave operator $T\colon X \to E$ from a Banach lattice X into a Banach space E is strongly $(\phi, 2)$-concave.

To show this we first notice that we can and do assume that $K_2(T) = 1$. Fix a finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$. Consider a finite sequence $\{f_1, ..., f_n\}$ of elements of X, and order them in a way that the first $f_1, ..., f_m$ satisfy $\|Tf_k\|_E \le 1$ and $\|Tf_k\|_E > 1$ for $m+1 \le k \le n$. We put

$$C := \sum_{k=1}^m \alpha_k \|Tf_k\|_E^2 + \sum_{k=m+1}^n \alpha_k \|Tf_k\|_E.$$

If $C \le 1$, then we have

$$\begin{aligned}\phi^{-1}\Big(\sum_{k=1}^n \alpha_k \phi(\|Tf_k\|_E)\Big) &= \phi^{-1}\Big(\sum_{k=1}^m \alpha_k \|Tf_k\|_E^2 + \sum_{k=m+1}^n \alpha_k \|Tf_k\|_E\Big)\\ &= \phi^{-1}(C) \le \Big(\sum_{k=1}^n \alpha_k \|Tf_k\|_E^2\Big)^{1/2}.\end{aligned}$$

If $C \ge 1$, then it follows by concavity of the function $t \mapsto t^{1/2}$ that

$$C \le \Big(\sum_{k=1}^n \alpha_k \|Tf_k\|_E\Big) \le \Big(\sum_{k=1}^n \alpha_k \|Tf_k\|_E^2\Big)^{1/2}.$$

Since the operator is 2-concave with $K_2(T) = 1$, we obtain

$$\phi^{-1}\Big(\sum_{k=1}^n \alpha_k \phi(\|Tf_k\|_E)\Big) \le \Big\|\Big(\sum_{k=1}^n \alpha_k |f_k|^2\Big)^{1/2}\Big\|_X.$$

Example 2 Every 1-concave operator $T\colon X \to E$ from a Banach function lattice X into a Banach space E is strongly $(1,\varphi)$-concave where $\varphi(t) = e^t - 1$ for all $t \ge 0$. In fact $\varphi^{-1}(t) = \log(t+1)$, $t \ge 0$ and so, for any $f_1, ..., f_n \in X$ and $\alpha_1, ..., \alpha_n \ge 0$ with $\sum_{k=1}^n \alpha_k = 1$, we have

$$\begin{aligned}\Big\|\varphi^{-1}\Big(\sum_{i=k}^n \alpha_k \varphi(|f_k|)\Big)\Big\|_X &= \Big\|\log\Big(\sum_{k=1}^n (\alpha_k e^{|f_k|} - \alpha_k) + 1\Big)\Big\|_X\\ &\ge \|\sum_{k=1}^n \alpha_k |f_k|\| \ge \frac{1}{K_1(T)} \sum_{k=1}^n \alpha_k \|Tf_k\|\end{aligned}$$

and so the statement follows.

4.1 (φ, ϕ)-concavity for operators on $C(K)$-spaces

The following result shows that, for the case of spaces of continuous functions both notions of (ϕ, φ)-concavity of operators coincides, providing in this case a broad class of examples. As we mentioned in Section 2, these spaces are not Banach function lattices in our sense, but the same arguments that prove Theorem 1 works also in this case. If $\phi, \varphi \in \Phi$ we say that $\varphi \prec \phi$ *near* 0 if both functions are equivalent in a neighborhood of 0.

Theorem 4 *Let ϕ, $\varphi \in \Phi$ be convex functions such that $\varphi \prec \phi$ near* 0 *and $\varphi \in \Delta_2$. Suppose that K is a compact Hausdorff space and that Y is a Banach space. Suppose $T\colon C(K) \to Y$ is a bounded operator. Then the following conditions on T are equivalent*:

(i) *There is a constant C_1 so that for every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and every finite sequence $\{f_k\}_{k=1}^n$ in the unit ball of $C(K)$,*

$$\sum_{k=1}^{n} \alpha_k \phi(\|Tf_k\|_Y) \le C_1 \left\| \sum_{k=1}^{n} \alpha_k \varphi(|f_k|) \right\|_{C(K)}.$$

(ii) *There is a constant C_2 and a probability Borel measure μ on K such that for all f in the unit ball of $C(K)$,*

$$\phi(\|Tf\|_Y) \le C_2 \int_K \varphi(|f|)\, d\mu.$$

(iii) *T is a (ϕ, φ)-concave operator.*

Proof (i) $\Rightarrow$ (ii). Since $(C(K))_{\varphi^{-1}} = C(K)$ with equality of norms, thus by Theorem 1 we can find a positive functional x^* in the unit ball of $C(K)^*$ such that for all f in the unit ball of $C(K)$,

$$\phi(\|Tf\|_Y) \le C_1 x^*(\varphi(|f|)).$$

Combining with the Riesz representation Theorem we obtain the statement. The implication (ii) $\Rightarrow$ (iii) is easy and we omit it. (iii) $\Rightarrow$ (ii) is proved in [17], and (ii) $\Rightarrow$ (i) is obvious.

4.2 The modulus of convexity and the (ϕ, p)-concavity on Banach function spaces

In this subsection we present a new class of (ϕ, p)-concave operators. Suppose X is a quasi-normed lattice. Following the Banach case, we define the modulus of uniform monotonicity by

$$\sigma_X(\varepsilon) = \inf\left\{\|x+y\| - 1;\, x, y \ge 0,\, \|x\| = 1,\, \|y\| \ge \varepsilon\right\}, \quad \varepsilon \ge 0.$$

Clearly, σ_X is non-decreasing with $\sigma_X(0) = 0$. X is said to be uniformly monotone if $\sigma_X(\varepsilon) > 0$ for all $\varepsilon > 0$.

We prove the following result, which is a lattice analogue of the theorem of Kadec, for uniformly convex Banach spaces (see, e.g., [7, p. 125]). We remark that the theorem of Kadec applies only to Banach spaces since there is no equivalent uniformly convex quasi-norm on non-locally convex quasi-normed spaces.

Lemma 3 *If $\sum_{k=1}^{\infty} |x_k|$ is a convergent series in a quasi-normed lattice X, then*

$$\sum_{k=1}^{\infty} \sigma_X(\|x_k\|) < \infty.$$

Proof We may assume, by subtracting the first few terms, that $x_1 \neq 0$. We may also assume that $\| \sum_{k=1}^{n} |x_k| \| \leq 1$ for all $n \geq 0$. Then, it follows from the definition of σ_X that

$$\|x_1\|(1 + \sigma_X(\|x_2\|)) \leq \|x_1 + x_2\|.$$

Since $\|x_1 + x_2\| \leq 1$,

$$\|x_1 + x_2\|\big(1 + \sigma_X(\|x_3\|)\big) \leq \|x_1 + x_2 + x_3\| \leq 1.$$

Repeating, we find that for all $n \geq 2$

$$\|x_1\| \prod_{k=2}^{n} \big(1 + \sigma_X(\|x_k\|)\big) \leq \|x_1 + ... + x_n\| \leq 1.$$

This implies

$$\sum_{k=2}^{n} \sigma_X(\|x_k\|) \leq \|x_1\|^{-1}$$

and the result then follows.

We have the following application of Lemma 3 and the Closed Graph Theorem.

Corollary 1 *Assume that X is uniformly monotone quasi-Banach lattice. Let $\phi \in \Phi$ be an admissible function such that $\phi \prec \sigma_X$ near 0. Then the identity map on X is $(\phi, 1)$-concave. In particular every operator T from X into a quasi-Banach space is $(\varphi, 1)$-summing.*

To provide next example let us recall that the modulus of convexity δ_X of a Banach space X is given by

$$\delta_X(\varepsilon) = \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\|;\ \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\Big\}, \quad 0 \le \varepsilon \le 2.$$

X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \le 2$. In this case, the modulus of convexity is equivalent (on (0, 2]) to a canonic Orlicz function $\tilde{\delta}_X$ (cf. [17, pp.65ff]), and $\ell_2 \hookrightarrow \ell_{\tilde{\delta}_X}$.

Since the modulus of convexity is equivalent at zero to the modulus $\hat{\delta}_X$ (see [2]) given by

$$\hat{\delta}_X(\varepsilon) = \inf\{\max\{\|x+y\|, \|x-y\|\} - 1;\ \|x\| = 1,\ \|y\| = \varepsilon\}, \quad \varepsilon \ge 0,$$

it follows that an uniformly convex Banach lattice X is uniformly monotone and

$$\sigma_X(\varepsilon) \le \hat{\delta}_X(\varepsilon), \quad 0 \le \varepsilon \le 2.$$

The content of the following result is essentially due to Figiel and Pisier [10] (for details we refer to [4]), and provides a relevant example of (ϕ, φ)-concave operator.

Proposition 2 *Let X be a uniformly convex Banach space. Then X is of $\tilde{\delta}_X$-cotype, i.e., there exists $C > 0$ such that for arbitrarily many $x_1, ..., x_n \in X$*

$$\big\|\{\|x_k\|_X\}_{k=1}^n\big\|_{\ell_{\tilde{\delta}_X}} \le C\Big(\int_0^1 \Big\|\sum_{k=1}^n r_k(t)x_k\Big\|_X^2\, dt\Big)^{1/2},$$

where $(r_k)_{k=1}^\infty$ is the sequence of Rademacher functions. Moreover, if X is a uniformly convex Banach lattice, then X is $(\tilde{\delta}_X, 2)$-concave.

5 φ-convexity of quasi-Banach function spaces and factorization of strongly (ϕ, φ)-concave operators

In this section we present some general applications of the notions and results given in the previous ones using our factorization tools. First we recall that a quasi-normed space $(X, \|\cdot\|)$ is *normable* whenever there exists a norm $\|\cdot\|^*$ in X and a constant $C > 0$ such that $C^{-1}\,\|\cdot\|^* \le \|\cdot\| \le C\,\|\cdot\|^*$.

In the sequel, we will make use of the following result.

Proposition 3 *Suppose that $\varphi \in \Phi$ and that a quasi-normed function lattice X is φ^{-1}-admissible. If φ is super-multiplicative (i.e. there exists a constant $C > 0$ such that $\varphi(s)\varphi(t) \le C\varphi(st)$ for every $s, t \ge 0$), then*

$$C\,\|\varphi(|f|)\|_{X_{\varphi^{-1}}} \ge \varphi\big(\|f\|_X\big), \quad f \in X.$$

Proof Observe that $\|g\|_X = 1$ implies $\|\varphi(|g|)\|_{X_{\varphi^{-1}}} = 1$. If $0 \neq f \in X$, then combining with our hypothesis on φ yields,

$$\begin{aligned} C\,\|\varphi(|f|)\|_{X_{\varphi^{-1}}} &= C\left\|\varphi\Big(\|f\|_X \frac{|f|}{\|f\|_X}\Big)\right\|_{X_{\varphi^{-1}}} \geq \varphi\big(\|f\|_X\big)\,\big\|\varphi(|f|/\|f\|_X)\big\|_{X_{\varphi^{-1}}} \\ &= \varphi\big(\|f\|_X\big). \end{aligned}$$

A quasi-normed lattice $X = (X, \|\cdot\|)$ is said to be *p-convex* if there exists a constant $C > 0$ such that for every every finite sequence $\{x_1, ..., x_n\}$ in X,

$$\Big\|\Big(\sum_{k=1}^{n} |x_k|^p\Big)\Big\|_X^{1/p} \leq C\Big(\sum_{k=1}^{n} \|x_k\|^p\Big)^{1/p}.$$

We notice the well-known easily verified fact that X is p-convex if and only if $X_{1/p}$ is normable. This motivates the following definition. Given $\varphi \in \Phi$, a quasi-normed function lattice X is said to be φ-convex if there is a constant $M > 0$ such that for every finite sequence $\{f_1, ..., f_n\}$ in $X_{\varphi^{-1}}$,

$$\|f_1 + ... + f_n\|_{X_{\varphi^{-1}}} \leq M\big(\|f_1\|_{X_{\varphi^{-1}}} + ... + \|f_n\|_{X_{\varphi^{-1}}}\big).$$

Proposition 4 *Let $\varphi \in \Phi$ and X be a quasi-normed function lattice. Then X is φ-convex if and only if $X_{\varphi^{-1}}$ is normable.*

Proof Assume that X is φ-convex with constant M. Let us define

$$\|f\|^*_{X_{\varphi^{-1}}} = \inf\Big\{\sum_{k=1}^{n} \|f_k\|_{X_{\varphi^{-1}}};\, |f| \leq \sum_{k=1}^{n} |f_k|,\ f_k \in X_{\varphi^{-1}}\Big\}, \quad f \in X_{\varphi^{-1}}.$$

Clearly, this expression defines a lattice norm on $X_{\varphi^{-1}}$ and for every $f \in X_{\varphi^{-1}}$, $\|f\|_{X_{\varphi^{-1}}} \leq M\|f\|^*_{X_{\varphi^{-1}}} \leq \|f\|_{X_{\varphi^{-1}}}$, so $X_{\varphi^{-1}}$ is normable. The converse is a consequence of a direct calculation using an equivalent norm to $\|\cdot\|_{X_{\varphi^{-1}}}$.

Remark 1 Notice that for any quasi-normed Orlicz space $X := L_\varphi$ on a measure space (Ω, Σ, μ), we have $X_{\varphi^{-1}} = L_1$ with $\|\cdot\|_{X_{\varphi^{-1}}} = \|\cdot\|_{L_1}$. In particular this implies that X is φ-convex. To see this observe that if $f \in X_{\varphi^{-1}}$, then for any $k > \|f\|_{X_{\varphi^{-1}}}$ we have $c = \|\varphi^{-1}(|f|/k)\|_{L_\varphi} \leq 1$. Hence

$$\int_\Omega \varphi(\varphi^{-1}(|f|/k))d\mu \leq \int_\Omega \varphi\Big(\frac{\varphi^{-1}(|f|/k)}{c}\Big)\,d\mu \leq 1.$$

This implies $f \in L_1(\mu)$ with $\|f\|_{L_1} \leq k$ and so $L_1 \subset X_{\varphi^{-1}}$ with

$$\|f\|_{L_1} \leq \|f\|_{X_{\varphi^{-1}}}, \quad f \in X_{\varphi^{-1}}.$$

The converse inclusion and inequality are obvious.

Example 3 Let (Ω, μ) be an infinite measure space. Then $L_2 := L_2(\mu)$ is ϕ-convex, where ϕ is the function given in Example 1. To see this we first observe that $L_\phi = L_1(\mu) + L_2(\mu)$. Since $\phi^{-1}(t) = t^{1/2}$ for $0 \le t \le 1$ and $\phi^{-1}(t) = t$ for $1 \le t < \infty$, we conclude that for any non zero function $f \in (L_2)_{\phi^{-1}}$ there exists $\lambda > 0$ such that

$$1 \ge \|\phi^{-1}(|f|/\lambda)\|_{L_2} = \Big(\int_\Omega |\phi^{-1}(|f|/\lambda)^2\, d\mu\Big)^{1/2} = \Big(\int_\Omega \varphi(|f|/\lambda)\, d\mu\Big)^{1/2},$$

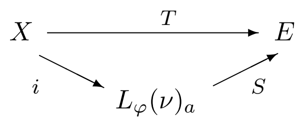
where $\varphi(t) = t$ for $0 \le t \le 1$ and $\varphi(t) = t^2$ for $1 \le t < \infty$. This gives that $(L_2)_{\phi^{-1}} = L_\varphi$. Clearly that $L_\varphi = L_1 \cap L_2$ and so Proposition 4 implies that L_2 is ϕ-convex.

Theorem 5 *Let $\varphi, \phi \in \Phi$ be such that φ is super-multiplicative and $t \mapsto \varphi \circ \phi^{-1}(t)$ is a concave function on $(0, \infty)$. Assume that a quasi-Banach lattice X on (Ω, Σ, μ) is φ-convex and that an operator T from X into a quasi-Banach space E is strongly (ϕ, φ)-concave.*

(i) *There exists a constant $C > 0$ and a positive functional x^* in the closed unit ball of $(X_{\varphi^{-1}})^*$ such that*

$$\phi\big(\|Tf\|_E/\widetilde{K}_{\phi,\varphi}(T)\big) \le C\, x^*(\varphi(|f|)), \quad f \in X.$$

(ii) *If $X_{\varphi^{-1}}$ is order continuous, then there exists $0 \le w \in B_{(X_{\varphi^{-1}})'}$ such that T admits the following factorization where $L_\varphi(\nu)$ is the Orlicz space on (A, Σ_A, ν) with $A = supp\, w$, $d\nu = w d\mu$ and $i\colon X \to L_\varphi(\nu)_a$ is an operator given by $i(f) = f\chi_A$ for all $f \in X$:*



Proof (i). Assume without loss of generality that $\widetilde{K}_{\phi,\varphi}(T) = 1$. We claim that the condition (i) in Theorem 3 is satisfied. Fix a finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and a finite sequence $\{f_k\}_{k=1}^n$ in X. Put $\psi(t) := \varphi \circ \phi^{-1}(t)$ for all $t \ge 0$. Since ψ is concave, our hypothesis on T yields

$$\sum_{k=1}^n \alpha_k \varphi(\|Tf_k\|_E) = \sum_{k=1}^n \alpha_k \psi\big(\phi(\|Tf_k\|_E)\big) \le \psi\Big(\sum_{k=1}^n \alpha_k \phi(\|Tf_k\|_E)\Big)$$
$$\le \varphi\Big(\Big\|\varphi^{-1}\Big(\sum_{k=1}^n \alpha_k \varphi(|f_k|)\Big)\Big\|_X\Big).$$

Since φ is super-multiplicative, we find by Proposition 3 that there exists a constant C_φ such that

$$\varphi\Big(\Big\|\varphi^{-1}\Big(\sum_{k=1}^{n}\alpha_k\varphi(|f_k|)\Big)\Big\|_X\Big) \le C_\varphi\Big\|\sum_{k=1}^{n}\alpha_k\varphi(|f_k|)\Big\|_{X_{\varphi^{-1}}}.$$

Then, using φ-convexity of X, we conclude that there exists a positive constant $\widetilde{C} = \widetilde{C}(\varphi, X)$ such that the Mackey norm on $X_{\varphi^{-1}}$ satisfies $\|\cdot\|^c_{X_{\varphi^{-1}}} \ge \widetilde{C}\,\|\cdot\|_{X_{\varphi^{-1}}}$. Combining this fact with both obtained estimates yields

$$\sum_{k=1}^{n}\alpha_k\varphi(\|Tf_k\|_E) \le C\,\Big\|\sum_{k=1}^{n}\alpha_i\varphi(|f_k|)\Big\|^c_{X_{\varphi^{-1}}},$$

where $C = \widetilde{C}\,C_\varphi$. This proves the claim and so Theorem 1 applies. The statement (ii) follows by Theorem 3.

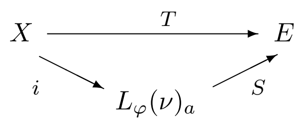
We obtain as direct applications of Theorem 5 the following corollaries. Both of them recover the Maurey-Rosenthal Theorem for $\varphi(t) = t^p$ and $\phi(t) = t^p$, respectively.

Corollary 2 *Let $0 < p < \infty$, $\varphi \in \Phi$ and let T be a strongly (p, φ)-concave operator from a quasi-Banach lattice X on (Ω, Σ, μ) into a quasi-Banach space E. Suppose that $\varphi \in \Phi$ is a super-multiplicative function such that X is φ-convex and $t \mapsto \varphi(t^{1/p})$ is a concave function.*

(i) *There exists a constant $C > 0$ and a positive functional x^* in the closed unit ball of $(X_{\varphi^{-1}})^*$ such that*

$$\varphi\big(\|Tf\|_E/\widetilde{K}_{p,\varphi}\|\big) \le C\,x^*(\varphi(|f|)), \quad f \in X.$$

(ii) *If $X_{\varphi^{-1}}$ is order continuous, then there exists $0 \le w \in B_{(X_{\varphi^{-1}})'}$ such that T admits the following factorization through $L_p(\nu)$ defined on (A, Σ_A, ν), where $A = \operatorname{supp} w$, $d\nu = w d\mu$ and $i\colon X \to L_\varphi(\nu)_a$ is an operator given by $i(f) = f\chi_A$ for all $f \in X$:*

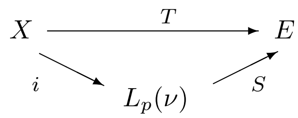


Corollary 3 *Let $0 < p < \infty$, $\varphi \in \Phi$ and let T be a strongly (ϕ, p)-concave operator from a quasi-Banach lattice X on (Ω, Σ, μ) into a quasi-Banach space E. Suppose that X is p-convex and $t \mapsto \phi\big(t^{1/p}\big)$ is a convex function.*

(i) *There exists a positive constant C and a positive functional x^* in the closed unit ball of $(X_{1/p})^*$ such that*

$$\|Tf\|_E \le C\big(x^*(|f|^p)\big)^{1/p}, \quad f \in X.$$

(ii) *If X is order continuous, then there exists $0 \le w \in B_{(X_{1/p})'}$ such that T admits the following factorization through $L_p(\nu)$ defined on (A, Σ_A, ν), where $A = \operatorname{supp} w$, $d\nu = w d\mu$ and $i\colon X \to L_p(\nu)$ is an operator given by $i(f) = f\chi_A$ for all $f \in X$:*



We state the following result that is interesting on its own. Based on Theorem 2 the proof of this result is similar to the proof of Theorem 5.

Theorem 6 *Let $0 < p < \infty$. Let X be a p-convex quasi-Banach lattice and let Y be a closed subspace of X. Suppose that T is a p-concave operator from Y into a quasi-Banach space E. Then there exists a positive functional x^* in the closed unit ball of $(X_{1/p})^*$ such that*

$$\|Tx\|_E \le K_p(T)\big(x^*(|x|^p)\big)^{1/p}, \quad x \in Y.$$

As an application of our results we prove a variant of the classical extension theorem for operators that we state in Theorem 7, that is known as Maurey's Extension Theorem. We refer to [8, p.248]) where several remarkable consequences of this famous result are presented.

Theorem 7 *Let X and Y be Banach spaces such that X has type 2 and Y has cotype 2 and Z is a subspace of X. Then every operator $T\colon Z \to Y$ admits an extension $\widetilde{T}\colon X \to Y$ that factors through a Hilbert space.*

We recall that a Banach space E has type 2, respectively, cotype 2 provided there exists a constant $C > 0$ such that for every finite sequence $\{x_1, ..., x_n\}$ of elements in E,

$$\Big(\int_0^1 \Big\|\sum_{k=1}^n r_k(t)x_k\Big\|_E^2 dt\Big)^{1/2} \le C\Big(\sum_{k=1}^n \|x_k\|_E^2\Big)^{1/2},$$

respectively,

$$\Big(\sum_{k=1}^n \|x_k\|_E^2\Big)^{1/2} \le C\Big(\int_0^1 \Big\|\sum_{k=1}^n r_k(t)x_k\Big\|_E^2 dt\Big)^{1/2}.$$

We can now state the following variant of Maurey's Extension Theorem.

Theorem 8 *Let X be 2-convex quasi-Banach lattice and Y be a closed subspace of X. Then every 2-concave operator T from Y to a quasi-Banach space E admits an extension $\widetilde{T}\colon X \to E$ that factors through a Hilbert space with $\|\widetilde{T}\|_{X\to E} \le K_2(T)$.*

Proof By Theorem 6 there exists a positive functional x^* in the closed unit ball of $(X_{1/2})^*$ such that

$$\|Tx\|_E \le Cx^*(|x|^2)^{1/2}, \quad x \in Y, \qquad (*)$$

where $C = K_2(T)$. Notice that the formula

$$p(x) := x^*(|x|^2)^{1/2}, \quad x \in X$$

defines a semi-norm on X which satisfies

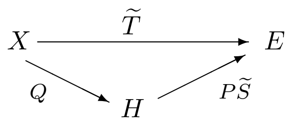
$$p(x+y)^2 + p(x-y)^2 = 2(p(x)^2 + p(y)^2), \quad x, y \in X.$$

Since $p(x) \le \||x|^2\|_{X_{1/2}}^{1/2} = \|x\|_X$, $N := \{x \in Y;\, p(x) = 0\}$ is a closed subspace of X. Combining the above equality we conclude that the quotient space X/N under the norm

$$\|[x]\|_{X/N} = p(x), \quad [x] \in X/N$$

satisfies the parallelogram law. Thus X/N is an inner product space. The norm completion H of X/N is a Hilbert space.

It follows from $(*)$ that $\|Tx\|_E \le C\,\|[x]\|_{Y/N}$ holds for all $[x] \in Y/N$. In particular, this implies that the formula $S([x]) := Tx$ defines a continuous operator from Y/N into E with $\|\widetilde{S}\| \le K_2(T)$. Denote by $\widetilde{S}$ the unique linear continuous extension of S to G, the closure of Y/N in H. Let $P\colon H \to G$ be a norm one continuous projection and let $Q\colon X \to H$ be defined by $Qx = [x]$ for all $x \in X$. Then the operator $\widetilde{T} = P\widetilde{S}Q\colon X \to E$ is an extension of T and we have the required factorization of $\widetilde{T}$ through a Hilbert space H,



To complete the proof it is enough to observe that $\|\widetilde{T}\|_{X\to E} \le K_2(T)$.

We show some applications of our results to study continuous inclusions between quasi-Banach function lattices.

Corollary 4 *Let $\varphi \in \Phi$ and let X be a quasi-Banach function lattice on (Ω, Σ, μ). Assume that $Y \subset X$ is a quasi-Banach function lattice such that Y is φ^{-1}-admissible and $Y_{\varphi^{-1}}$ is order continuous. If there exists a constant $C > 0$ such that for every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and every finite sequence $\{f_k\}_{k=1}^n$ in Y,*

$$\sum_{k=1}^n \alpha_k \varphi(\|f_k\|_X) \le C \Big\| \sum_{k=1}^n \alpha_k \varphi(|f_k|) \Big\|_{Y_{\varphi^{-1}}}^c ,$$

then there exists $0 \le w \in B_{(Y_{\varphi^{-1}})'}$ such that $Y \hookrightarrow L_\varphi(\nu)_a \hookrightarrow X$, where the Orlicz space $L_\varphi(\nu)$ is defined on (Ω, Σ_A, ν) with $d\nu = w d\mu$. If φ is a super-multiplicative function, then the converse is also true, i.e., the shown continuous inclusions imply the above inequality.

The proof is just an application of Theorem 3 to the factorization

$$\begin{array}{ccc} Y & \xrightarrow{\quad i \quad} & X \\ & {}_{i}\searrow \quad \nearrow_{i} & \\ & L_\varphi(\nu)_a & \end{array}$$

Example 4 Let us show an application regarding a generalization of the function ϕ that appeared in Example 1. Let $1 \le p < \infty$ and let (Ω, Σ, μ) be a measure space. Consider the function $\phi_p \in \Phi$ given by $\phi_p(t) = t^p$ for $0 \le t \le 1$ and $\phi_p(t) = t$ for $t \ge 1$. Clearly this function is super-multiplicative with constant 1 and $L_{\phi_p} = L_1 + L_p$ (see for instance [9, Th.16]).

An application of Corollary 4 provides the following characterization of the spaces $L_1 + L_p$. If f_k is a measurable function, define the measurable sets $A_k = \{f_k \le 1\}$ and $B_k = \{f_k > 1\}$. For an order continuous Banach function lattice X over the Lebesgue measure μ, the following assertions are equivalent.

(1) $X = L_{\phi_p}(\nu)$ where $d\nu = w d\mu$ with $0 < w \in (X_{\phi_p^{-1}})'$.

(2) The space X is ϕ_p-convex, and there is a constant $C > 0$ such that for every set of functions $f_1, ..., f_n \in X$ with $f_1, ..., f_m$ with norm less or equal than 1 and $f_{m+1}, ...f_n$ with norm bigger than 1, and positive scalars α_k such that $\sum_{k=1}^n \alpha_k = 1$,

$$\sum_{k=1}^m \alpha_k \|f_i\|_X^p + \sum_{k=m+1}^n \alpha_k \|f_i\|_X \le C \Big\| \sum_{k=1}^n \alpha_k (|f_k|^p \chi_{A_k} + |f_k| \chi_{B_k}) \Big\|_{X_{\phi_p^{-1}}} .$$

Notice that this result extends the one that is known for L_p-spaces, which can be proved using the Maurey-Rosenthal factorization theorem: an order continuous Banach function lattice X on (Ω, μ) is p-convex and p-concave if and only if it is isomorphic to an $L_p(w d\mu)$ space for function $w \in (X_{1/p})'$.

The following result is a consequence of Theorem 5.

Corollary 5 *Let $\varphi, \phi \in \Phi$ be such that φ is super-multiplicative and $t \mapsto \varphi \circ \phi^{-1}(t)$ is a concave function. Assume that X and Y are quasi-Banach function lattices on (Ω, Σ, μ) are such that Y is a φ-convex, $Y_{\varphi^{-1}}$ is order continuous and the inclusion map $i\colon Y \to X$ is strongly (ϕ, φ)-concave. Then Y is continuously embedded into an Orlicz space $L_\varphi(wd\mu)$ on (Ω, Σ_A, ν), where $0 \le w \in (X_{\varphi^{-1}})'$.*

We conclude by showing an application of a direct extension of one of the most famous instances of the Maurey-Rosenthal Theorem for operators from Banach function lattices taking values in L_1 that factors through L_2. Consider a finite measure space (Ω, μ) and an order continuous 2-convex Banach function space X on (Ω, μ) such that $X \subset L_1(\mu)$. Suppose that $\phi \in \Phi$ is such that $t \mapsto (\phi^{-1}(t))^2$ is a concave function on $[0, \infty)$. If there exists a constant $K > 0$ such that for every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and every finite sequence $\{f_k\}_{k=1}^n$ in X,

$$\phi^{-1}\Big(\sum_{k=1}^n \alpha_k \phi\Big(\int_\Omega |f_k|\, d\mu\Big)\Big) \le K \Big\| \Big(\sum_{k=1}^n \alpha_k |f_k|^2\Big)^{1/2} \Big\|_X,$$

then there exists $w \in (X_{1/2})'$ with $w \ge 0$ a.e. such that $X \hookrightarrow L_2(wd\mu)$. The inclusion $L_p(\mu) \hookrightarrow L_2(\mu)$ is a particular case of this result for $2 \le p < \infty$ and $\phi(t) = t^p$ for all $t \ge 0$.

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