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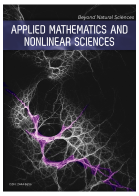


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# Stability analysis of fourth-order iterative method for finding multiple roots of non-linear equations

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**Abstract**

The use of complex dynamics tools in order to deepen the knowledge of qualitative behaviour of iterative methods for solving non-linear equations is a growing area of research in the last few years with fruitful results. Most of the studies dealt with the analysis of iterative schemes for solving non-linear equations with simple roots; however, the case involving multiple roots remains almost unexplored. The main objective of this paper was to discuss the dynamical analysis of the rational map associated with an existing class of iterative procedures for multiple roots. This study was performed for cases of double and triple multiplicities, giving as a conjecture that the wideness of the convergence regions of the multiple roots increases when the multiplicity is higher and also that this family of parametric methods includes some specially fast and stable elements with global convergence.

**Keywords:** Nonlinear equations, iterative methods, multiple roots, stability, strange fixed points, free critical points.
**AMS 2010 codes:** 65H05, 37F10.

## 1 Introduction

With the advancement of computer algebra, finding higher-order multi-point methods, not requiring the computation of second-order derivative for multiple roots, becomes very important and is an interesting task from the practical point of view. These multi-point methods are of great practical importance since they overcome theoretical limits of one-point methods concerning the order and computational efficiency. Further, these multi-point iterative methods are also capable to generate root approximations of high accuracy.

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Here, discuss the dynamical analysis of iterative methods for finding a zero of a continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$ that is, a solution $\xi$ of the nonlinear equation $f(x)=0$. In the case of simple roots, many robust and efficient methods have been proposed with a high convergence order. Here, we focus in the case of a root $\xi$ of multiplicity $m>1$, namely, $f(\xi)=0$, $f^{(k)}(\xi)=0$ for $k=1,\ldots,m-1$ and $f^{(m)}(\xi)\neq 0$. It is well known that the convergence order of iterative methods decreases in the presence of a multiple root. In this sense, modifications in the iterative function can improve the behavior of the method. Newton's method preserves the second order convergence for multiple roots [6], with the modification given by

$$x_{n+1}=x_n-m\frac{f(x_n)}{f'(x_n)}.$$

This is quadratically convergent and is optimal in the sense of Kung and Traub [5]. Existing third and fourth-order methods of finding simple roots to multiple roots have been extended (see, for example, [7]- [9]). However, the number of families of optimal iterative methods for finding multiple roots of nonlinear equations available in the literature, such as [1]- [10], is very much reduced.

Recently, Hueso et al. [4] considered the following fourth-order iterative method

$$\begin{aligned} y_n &= x_n-\frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)},\\ x_{n+1} &= x_n-\left[s_1+s_2h(y_n,x_n)+s_3h(x_n,y_n)+s_4h(y_n,x_n)^2\right]\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1}$$

where $h(x_n,y_n)=\frac{f'(y_n)}{f'(x_n)}$, $\mu=1-\frac{2}{m+2}$, $b=\frac{2m}{m+2}$,

$$\begin{aligned} s_1 = -\frac{\mu^{-1-2m}}{b(b-2m+bm)}\Bigg(\mu^{2+2m}\Bigg(-m^2s_4\mu^{2-2m}-m^2s_3\mu^{-1+m}+\frac{2b(b-m)(b-2m+bm)s_4\mu^{-2m}}{m}\\ +\frac{bm^2(b-2m+bm)s_3\mu^m}{(b-m)^2}\Bigg)+(-2bm+b^2(1+m)+m^2\mu)(s_4\mu^3+\mu^{2m}(-m\mu+s_3\mu^m))\Bigg), \end{aligned}$$

$$\begin{aligned} s_2 = \frac{\mu^{-m}}{bm(-b+m)^2(b-2m+bm)}\Big(2b^5(1+m)s_4+6b^3m^2(3+m)s_4-2b^4m(5+3m)s_4-m^6\mu^{2m}\\ +2bm^4(2s_4+\mu^{2m}(m-s_3\mu^m))+b^2m^3(-2(7+m)s_4+\mu^{2m}(-m+s_3\mu^m+ms_3\mu^m))\Big), \end{aligned}$$

$$s_3=\frac{m^3\mu^{-3m}(-8s_4+8m\mu^{2m}+12m^2\mu^{2m}+6m^3\mu^{2m}+m^4\mu^{2m})}{8(2+m)^3},$$

and $s_4$ is a real free parameter.

Here, our main concern is to discuss the dynamical analysis of the rational map associated with the above mentioned scheme for multiple roots. First, we are going to recall some dynamical concepts of complex dynamics (see [3]) that we use in this work. Given a rational function $R:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the orbit of a point $z_0\in\hat{\mathbb{C}}$ is defined as:

$$\{z_0,R(z_0),R^2(z_0),\ldots,R^n(z_0),\ldots\}.$$

We analysed the phase plane of the map $R$ by classifying the starting points from the asymptotic behaviour of their orbits. $z_f\in\hat{\mathbb{C}}$ is called a fixed point of $R$ if $R(z_f)=z_f$. A periodic point $z$ of period $p>1$ is a point such that $R^p(z)=z$ and $R^k(z)\neq z$, for $k<p$. A preperiodic point is a point $z$ that is not periodic but there exists a $k>0$

such that $R^k(z)$ is periodic. A critical point $z^*$ is a point where the derivative of the rational function vanishes, $R'(z^*)=0$. Moreover, a fixed point $z_f$ is called attractor if $|R'(z_f)|<1$, superattractor if $|R'(z_f)|=0$, repulsor if $|R'(z_f)|>1$ and parabolic if $|R'(z_f)|=1$. So, a superattracting fixed point is also a critical point. The fixed points that are not associated with the roots of the polynomial are called strange fixed points.

On the other hand, the basin of attraction of an attractor $\alpha \in \hat{\mathbb{C}}$ is defined as the set of starting points whose orbits tend to $\alpha$:

$$A(\alpha)=\{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to \alpha, n \to \infty\}.$$

In the Fatou set of the rational function $R$, $\mathscr{F}(R)$ is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complementary set in $\hat{\mathbb{C}}$ is the Julia set, $\mathscr{J}(R)$. That is, the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

By using these tools of complex dynamics, we study the general convergence of family (1) on polynomials with multiple roots of multiplicity 2 and 3. It is known that the roots of a polynomial can be transformed by an affine map with no qualitative changes on the dynamics of the family. So, we can use the polynomials $p(z)=(z-a)^2(z-b)$ and $q(z)=(z-a)^3(z-b)$.

The rest of the paper is organised as follows. In Section 2 the complex dynamics of the family is studied on low-degree polynomials with multiplicity $m=2$, by means of an analysis and the strange fixed points and their stability, the critical points and confirming the results by dynamical planes, showing curious behaviour in terms of stability and convergence to the roots. In Section 3 this study is extended to the case of multiplicity 3, finding the similitude and differences with respect to the case of double roots. Finally, some conclusions and remarks are presented in Section 4.

## 2 Dynamical analysis of polynomials with double roots in the complex plane

In order to study the stability of the family on polynomials with double roots, the operator of the family on $p(z)$ is calculated, obtaining a rational function that depends, not only on $s_4$, but also on parameters $a$ and $b$. In order to eliminate these parameters, the following transformation is usually applied.

Blanchard [3] considered the conjugacy map (a Möbius transformation)

$$h(z)=\frac{z-a}{z-b},$$

with the following properties:

$$i)\ h(\infty)=1,\ \ ii)\ h(a)=0,\ \ (iii)\ h(b)=\infty,$$

and proved that, for quadratic polynomials, Newton's operator is conjugate to the rational map $z^2$. In what follows, we use this transformation in order to avoid the appearance of parameters $a$ and $b$ in the rational functions resulting from applying the fixed point operator of the iterative method on polynomials $p(z)$ and $q(z)$.

Next, we are going to analyze, under the dynamical point of view, the stability and reliability of the members of the proposed family. First, we will study the fixed points of the rational function that are not related to the original roots of the polynomial and then the free critical points, that is, the critical points of the associated rational function different from 0 and $\infty$.

For $p(z)$, the operator associated with family (1) is the rational function $M_p(z,s_4,a,b)$ depending on the parameters $s_4$, $a$ and $b$. On the other hand, operator $M_p(z,s_4,a,b)$ on $p(z)$ is conjugated to operator $O_p(z,s_4)$,

$$O_p(z,s_4)=(h\circ M\circ h^{-1})(z)=\frac{N_1}{D_1}, \tag{2}$$

where

$$N_1 = z^4(112+388z+546z^2+409z^3+196z^4+66z^5+10z^6+z^7+s_4(-2-2z+z^2)^3),$$

$$D_1 = 256+896z+1168z^2+528z^3-8(37+s_4)z^4-4(125+6s_4)z^5-3(87+4s_4)z^6+4(-17+4s_4)z^7 \\ +(-2+6s_4)z^8+(8-6s_4)z^9+(-1+s_4)z^{10}.$$

Let us observe that the parameters $a$ and $b$ have been obviated in $O_p(z,s_4)$.

### 2.1 Fixed and critical points

As we have seen, the fourth-order family of iterative methods (1), applied on the polynomial $p(z)$, after Möbius transformation, gives rise to the rational function (2), depending on parameter $s_4$. It is clear that this rational function has not only 0 and $\infty$ as fixed points but also different strange fixed points (that do not correspond to the roots of $p(z)$), $r_1(s_4)=1$ and the roots of the high-degree polynomial,

$$p_r(t) = 256+1152t+2320t^2+(2736+8s_4)t^3+(2052+24s_4)t^4+(1006+12s_4)t^5+(336-16s_4)t^6 \\ +(72-6s_4)t^7+(4+6s_4)t^8+(2-s_4)t^9,$$

that are denoted by $r_i(s_4), i=2,\ldots,10$. Nevertheless, the complexity of the operator can be lower depending on the value of the parameter, as we can see in the following result.

**Theorem 1.** *The number of strange fixed points of operator $O_p(z,s_4)$ is ten (including $r_1(s_4)=1$, except for $s_4=64$), except in the following cases:*

*(i) For $s_4=370$, there are nine strange fixed points and*

$$O_p(z,s_4) = z^4\frac{-2848-8492z-3894z^2+6329z^3+2416z^4-2154z^5+380z^6+z^7}{256+896z+1168z^2+528z^3-3256z^4-9380z^5-4701z^6+5852z^7+2218z^8-2212z^9+369z^{10}},$$

*(ii) For $s_4=64$, there are nine strange fixed points and*

$$O_p(z,s_4) = -z^4\frac{400+1548z+1770z^2+337z^3-243z^4+75z^5+z^6}{256+1152z+2320z^2+2848z^3+2040z^4+4z^5-1025z^6-69z^7+313z^8-63z^9},$$

*(iii) For $s_4=-368$, there are eight strange fixed points and*

$$O_p(z,s_4) = z^4\frac{3056+9220z+4962z^2-5479z^3-2012z^4+2274z^5-358z^6+z^7}{256+896z+1168z^2+528z^3+2648z^4+8332z^5+4155z^6-5956z^7-2210z^8+2216z^9-369z^{10}},$$

*(iv) For $s_4=1$, there are seven strange fixed points and*

$$O_p(z,s_4) = z^4\frac{13+26z+18z^2+5z^3+z^4}{32+64z+26z^2-25z^3-28z^4-8z^5+2z^6}.$$

As we will see in the following, not only the number but also the stability of the fixed points depend on the parameter of the family. The expression of the differential operator, necessary for analyzing the stability of the fixed points and for obtaining the critical points, is

$$O'_p(z,s_4) = \frac{N_2}{D_2}, \tag{3}$$

where

$$N_2 = z^3(2+z)^3(4+5z+2z^2+z^3)(3584+15072z+25192z^2+19908z^3+5970z^4-2145z^5-3300z^6 \\ -1707z^7-366z^8-23z^9+24z^{10}-z^{11}+s_4(-2-2z+z^2)^2 \\ (-64-104z+84z^2+80z^3-50z^4-39z^5-16z^6+z^7)),$$

and

$$D_2 = (256+896z+1168z^2+528z^3-8(37+s_4)z^4-4(125+6s_4)z^5-3(87+4s_4)z^6+4(-17+4s_4)z^7 +(-2+6s_4)z^8+(8-6s_4)z^9+(-1+s_4)z^{10})^2.$$

As it comes from the double root of the polynomial, it is clear that $z=0$ is always a superattracting fixed point. However, the stability of $z=\infty$ and the rest of fixed points change depending on the values of the parameter $s_4$. In the following results we establish the stability of both fixed points.

**Theorem 2.** *The character of $z=\infty$ depends on $s_4$ as follows:*

*(i) If $|s_4-1|<1$, then $z=\infty$ is an attractor, being superattractive if $s_4=1$.*

*(ii) When $|s_4-1|=1$, $z=\infty$ is a parabolic point.*

*(iii) If $|s_4-1|>1$, then $z=\infty$ is repulsive.*

*Proof.* In order to analyze the stability of $z=\infty$, its inverse relationship with $z=0$ (induced by Möbius transformation) is used to define the associated rational function:

$$Inf(z,s_4) = \frac{1}{O_p(1/z,s_4)} = z\left(s_4(-1+2z(1+z))^3-(1+z)\left(1+z+4z^2\right)\right) \frac{-1+z(10+z(-17+8z(-10+z(-7+2z(1+z)(5+4z)))))}{-1+z(-10+s_4(-1+2z(1+z))^3-z(66+z(196+z(409+2z(273+2z(97+28z))))))}$$

The rest of the proof is straightforward as the stability function of the infinity is $|Inf'(0,s_4)|=|s_4-1|$.

**Theorem 3.** *The character of the strange fixed point $z=1$ is as follows (except for $s_4=64$):*

*(i) If $|s_4-64|>432$, then $z=1$ is an attractor and it cannot be a superattractor.*

*(ii) When $|s_4-64|=432$, $z=1$ is a parabolic point.*

*(iii) If $|s_4-64|<432$, then $z=1$ is a repulsor.*

*Proof.* It can been easily seen that

$$O'_p(1,s_4) = \frac{432}{64-s_4}.$$

So,

$$\left|\frac{432}{64-s_4}\right| \leq 1 \text{ is equivalent to } 432 \leq |64-s_4|.$$

Let us consider $s_4=a+ib$ as an arbitrary complex number. Then,

$$432^2 \leq 64^2-128a+a^2+b^2,$$

that is,

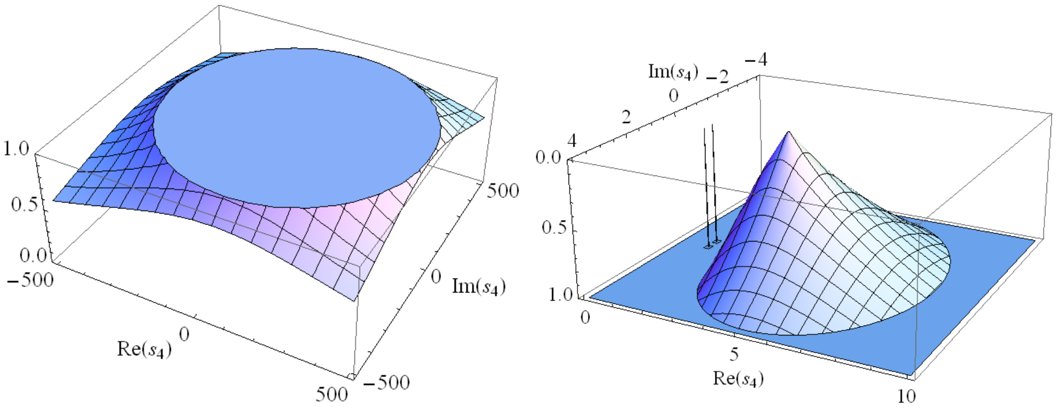
$$(a-64)^2+b^2 \geq 432^2.$$

Therefore,

$$|O'_p(1,s_4)| \leq 1 \text{ iff } |64-s_4| \geq 432.$$

In addition, if $s_4$ is such that $|64-s_4|<432$, then $|O'_p(1,s_4)|>1$ and $z=1$ is a repulsive point, provided $s_4 \neq 64$, for which $z=1$ is not a fixed point.

The analysis of the stability function of the strange fixed points, $|O'_p(r_i(s_4), s_4)|$, where $i = 1, 2, \ldots, 10$ gives us relevant information about their stability and, therefore, on the whole stability of the iterative methods of the class. In case $i = 1$, this function is quite simple and can be studied analytically (Theorem 3) (see the stability function in Figure 1(a)); in other cases, the analytical expression of the strange fixed points is not available (as they are known as a roots of a certain high-degree polynomial), and thus, they are studied graphically. In Figure 1(b), the stability functions of $r_i(s_4)$, for $i = 2, 3, \ldots, 10$, are plotted, presenting a big area in the region $[0, 10] \times [-4, 4]$ of the complex plane where one of them is attracting. However, out of this region and inside the much bigger are of repulsion of $z = 1$, only the roots are attracting fixed points.




(a) $s_1(s_4) = 1$ (b) $s_i(s_4), i = 2, 3, \ldots, 10$

**Fig. 1** Stability functions of the strange fixed points

Regarding the critical points, they are calculated by solving $O'_p(z, s_4) = 0$ and are presented in the following result. The relevance of the knowledge of the free critical points (critical points different from those associated with the roots) is the following known fact: each invariant Fatou component is associated with, at least, one critical point.

**Theorem 4.** *The free critical points of operator $O_p(z, s_4)$ are $cr_1 = -1$, $cr_2 = -2$, $cr_3 = \frac{1}{2}(-1 - i\sqrt{15})$, $cr_4 = \frac{1}{2}(-1 + i\sqrt{15})$, which are preimages of the strange fixed point $z = 1$ (called pre-periodic critical points), and the roots of polynomial*

$$\begin{aligned} p_c(t) = {} & 3584 - 256s_4 + (15072 - 928s_4)t + (25192 - 496s_4)t^2 + (19908 + 1248s_4)t^3 + (5970 + 792s_4)t^4 \\ & + (-2145 - 996s_4)t^5 + (-3300 - 612s_4)t^6 + (-1707 + 156s_4)t^7 + (-366 + 114s_4)t^8 \\ & + (-23 + 25s_4)t^9 + (24 - 20s_4)t^{10} + (-1 + s_4)t^{11}, \end{aligned}$$

*are denoted by $cr_i$, $i = 5, 6, \ldots, 15$. So, the number of free critical points (including the pre-periodic ones) is fifteen, except in the following cases:*

*(i) For $s_4 = 14$, $cr_7 = 0$ and there are only fourteen free critical points.*

*(ii) For $s_4 = 64$, there are only thirteen free critical points.*

*(iii) For $s_4 = 1$, $cr_2 = -2$ is not a free critical point and there are only ten free critical points.*

*(iv) For $s_4 = 0$, $cr_{14} = cr_3$, $cr_{15} = cr_4$ and $cr_1 = -1$ are not free critical point and there are only nine free critical points.*

### 2.2 Dynamical planes

In this section, we show, by means of dynamical planes, the qualitative behaviour of the different elements of the proposed family by using the conclusions obtained in the analysis of the stability of strange fixed points.

The dynamical plane associated with a value of the parameter $s_4$, that is, obtained by iterating an element of family, is generated by using each point of the complex plane as initial estimation (we have used a mesh of $800 \times 800$ points). We paint in blue the points whose orbit converges to infinity, in orange the points converging to zero (with a tolerance of $10^{-3}$), in green, red, etc. the points whose orbit converges to one of the strange fixed points and in black the points whose orbit reaches the maximum number of 80 iterations without converging to any of the fixed points. The colour is brighter for lower number of iterations needed to converge. Moreover, all fixed points appear marked as a white circle in the figures, with a white star if the fixed point is an attractor and with a white square if the point is critical.

Some values of the parameter whose associated iterative method shows stable behavior with convergence only to the "roots" (multiple $z = 0$ or simple $z = \infty$) are presented in Figure 2. Depending on the value of the parameter, only the basin of attraction of $z = 0$ is observed (global convergence to the multiple root, as is shown when we widen the rectangle where Figures 2(a) to 2(c) and 2(g) to 2(l) are plotted), or both $z = 0$ and $z = \infty$ appear (Figures 2(d), 2(e) and 2(f)).

On the other hand, unstable behaviour is found when we choose values of the parameter in the stability region of attracting strange fixed points (Figures 3(a), 3(b), 3(e), 3(f) and 3(i)) or attracting periodic orbits (Figures 3(c), 3(d), 3(g) and 3(h)). The periodic orbits are marked with yellow lines, with yellow circles at the elements of the orbit (Figures 3(d) and 3(h)).

## 3 Complex dynamics on cubic roots

For $q(z) = (z-a)^3(z-b)$, the operator associated with family (1) is the rational function $M_q(z,s_4,a,b)$ depending on parameters $s_4$, $a$ and $b$. On the other hand, by means of Möbius transformation, operator $M_q(z,s_4,a,b)$ on $q(z)$ is conjugated to operator $O_q(z,s_4)$,

$$O_q(z,s_4) = (h \circ M \circ h^{-1})(z) = \frac{N_3}{D_3}, \tag{4}$$

where

$$\begin{aligned} N_3 = {} & z^4(8000s_4(-81-90z+15z^2+20z^3)^3+729(9+5z)^2(8430885+19582398z+19680084z^2+10889802z^3 \\ & +3773466z^4+911250z^5+128700z^6+3750z^7+625z^8)), \end{aligned}$$

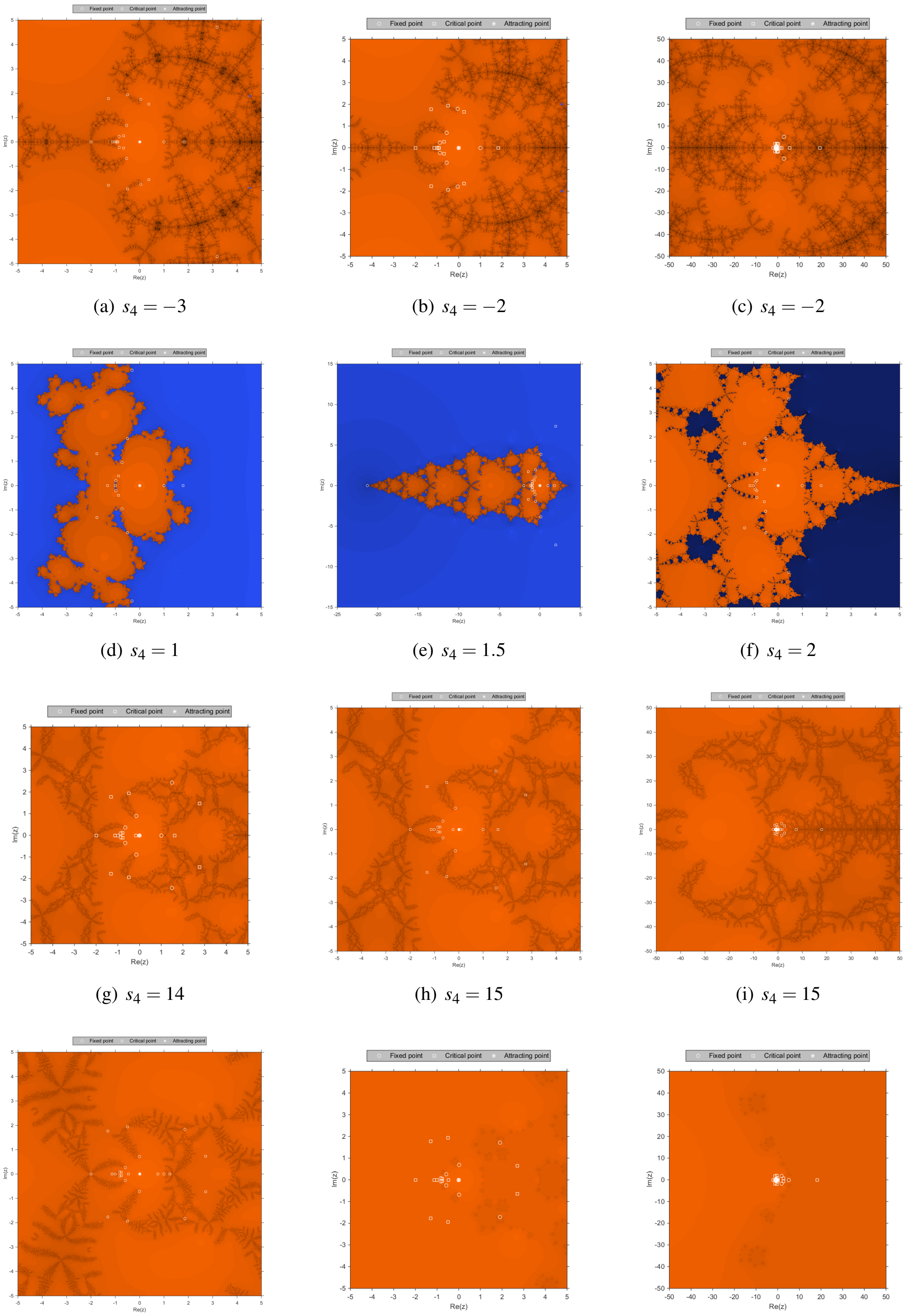
and

$$\begin{aligned} D_3 = {} & 2353579470675+7426850774130z+8915707713357z^2+3315802825188z^3-42515280(81351+100s_4)z^4 \\ & -354294(14946147+40000s_4)z^5-669222(4780377+20000s_4)z^6+5832(-168866289+440000s_4)z^7 \\ & +3645(-24326001+2600000s_4)z^8+12150(3992733+184000s_4)z^9-16875(-1320219+121280s_4)z^{10} \\ & -67500(-50787+11200s_4)z^{11}+56250(-7047+2560s_4)z^{12}+31250(-5103+2048s_4)z^{13}. \end{aligned}$$

Let us observe that the parameters $a$ and $b$ have been obviated in $O_q(z,s_4)$.

### 3.1 Fixed and critical points

As we have seen, the fourth-order family of iterative methods (1), applied on the polynomial $q(z)$, after Möbius transformation, gives rise to the rational function (4), depending on parameter $s_4$. It is clear that this rational function has not only 0 and $\infty$ as fixed points, but also different strange fixed points (that do not correspond



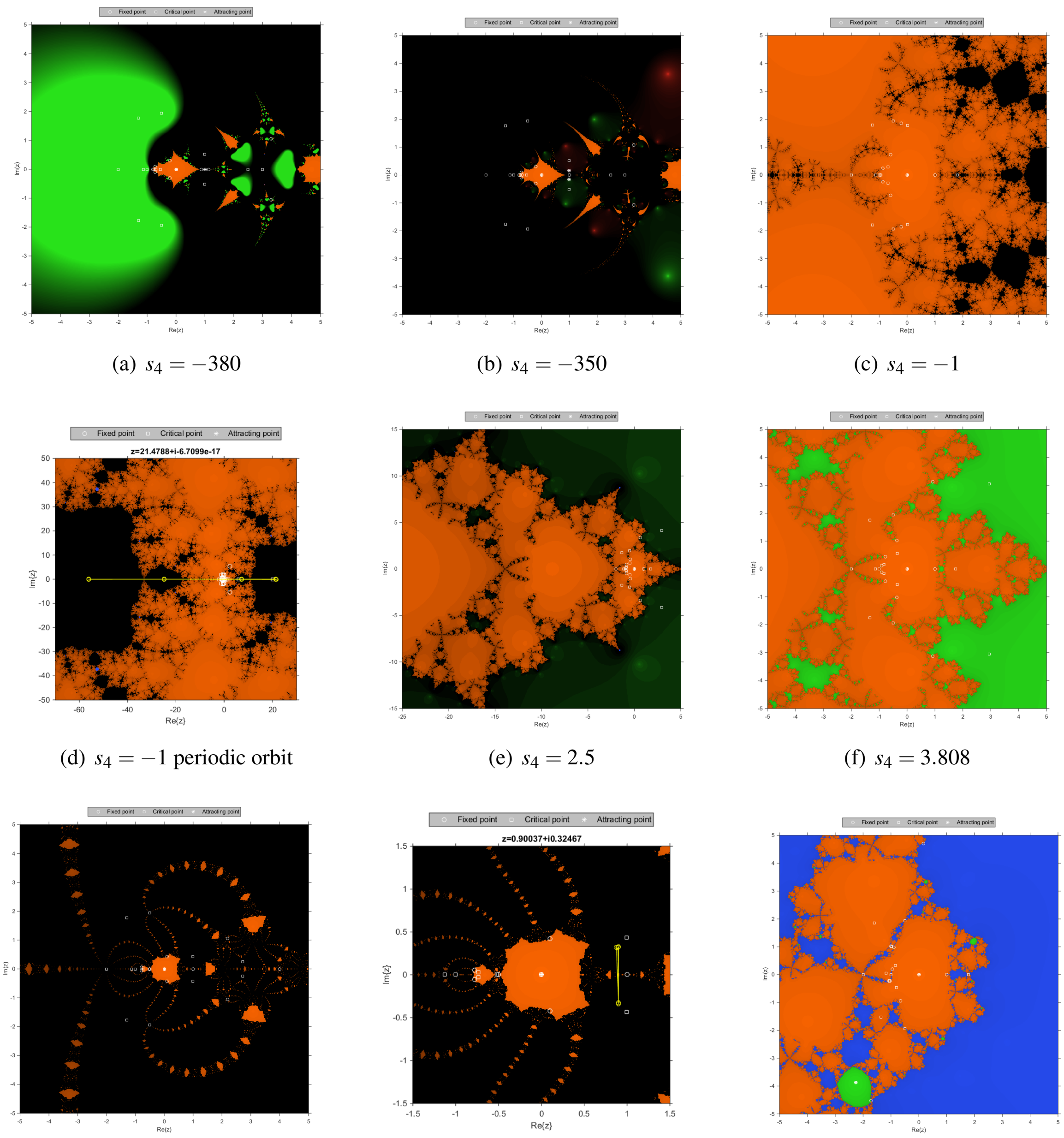

(a) $s_4 = -3$ (b) $s_4 = -2$ (c) $s_4 = -2$

(d) $s_4 = 1$ (e) $s_4 = 1.5$ (f) $s_4 = 2$

(g) $s_4 = 14$ (h) $s_4 = 15$ (i) $s_4 = 15$

(j) $s_4 = 50$ (k) $s_4 = 64$ (l) $s_4 = 64$

**Fig. 2** Dynamical planes with stable behavior




(a) $s_4 = -380$ (b) $s_4 = -350$ (c) $s_4 = -1$

(d) $s_4 = -1$ periodic orbit (e) $s_4 = 2.5$ (f) $s_4 = 3.808$

(g) $s_4 = 370$ (h) $s_4 = 370$ periodic orbit (i) $s_4 = 0.99724 + 0.187187i$

**Fig. 3** Dynamical planes with unstable behavior

to the roots of $q(z)$): $R_1(s_4) = 1$ and the roots of the high-degree polynomial, are denoted by $R_i(s_4), i = 2, \ldots, 13$:

$$\begin{aligned} q_R(t) = {} & 2353579470675 + 9780430244805t + 18696137958162t^2 + (21514105454985 + 4251528000s_4)t^3 \\ & + (16345973527353 + 14171760000s_4)t^4 + (8450100030114 + 13384440000s_4)t^5 \\ & + (2959835140332 - 2566080000s_4)t^6 + (679038108930 - 9477000000s_4)t^7 \\ & + (90517688325 - 2235600000s_4)t^8 + (2871257625 + 2046600000s_4)t^9 \\ & - (123018750 - 756000000s_4)t^{10} + (676603125 - 144000000s_4)t^{11} + (170859375 - 64000000s_4)t^{12}. \end{aligned}$$

Nevertheless, the complexity of the operator can be lower depending on the value of the parameter, as we can see in the following result.

**Theorem 5.** *The number of strange fixed points of operator $O_q(z,s_4)$ is thirteen (including $R_1(s_4)=1$, except for $s_4=442333143/982600$), excluding the following cases:*

*(i) For $s_4=442333143/982600$, the fixed point operator is simplified. In this case, $R_1(s_4)$ is not a fixed point and there are twelve strange fixed points, obtained as roots of a polynomial of this degree.*

*(ii) For $s_4=40095/8$, there are twelve strange fixed points: $z=-\frac{3}{5}$, $z=1$, $z=\frac{3}{25}\left(3\pm 8\sqrt{6}\right)$ and the roots of an 8th-degree polynomial.*

*(iii) For $s_4=-19744461261/4913000$, there are eleven strange fixed points of $O_q(z,s_4)$, calculated as $z=1$, $z=\frac{3}{73}\left(-37\pm 8\sqrt{2}\right)$ and the roots of an 8th-degree polynomial.*

*(iv) For $s_4=2187/1000$, there are nine strange fixed points ($z=1$ and the roots of a polynomial of degree eight).*

*(v) For $s_4=0$, there are again nine strange fixed points ($z=1$ and the roots of a polynomial of degree eight).*

As we will see in the following, not only the number but also the stability of the fixed points depend on the parameter of the family. The expression of the differential operator, necessary for analyzing the stability of the fixed points and for obtaining the critical points, is

$$O'_q(z,s_4)=\frac{N_4}{D_4}, \tag{5}$$

where

$$\begin{aligned} N_4 = {} & 1458z^3(3+z)^4(9+5z)^3(45+6z+5z^2)\left[(16000s_4(81+90z-15z^2-20z^3)^2\right.\\ & (-98415-115911z+130491z^2+93636z^3-51840z^4-48735z^5-14025z^6-1950z^7+125z^8)\\ & -729(9+5z)^2(-20487050550-49465642545z-45092237409z^2-11214605763z^3+10551446127z^4\\ & +10478743686z^5+4604951574z^6+1167748650z^7+94138200z^8-35569125z^9-12853125z^{10}\\ & \left.-1884375z^{11}+109375z^{12}))\right], \end{aligned}$$

and

$$\begin{aligned} D_4 = {} & \left[2353579470675+7426850774130z+8915707713357z^2+3315802825188z^3\right.\\ & -42515280(81351+100s_4)z^4-354294(14946147+40000s_4)z^5-669222(4780377+20000s_4)z^6\\ & +5832(-168866289+440000s_4)z^7+3645(-24326001+2600000s_4)z^8+12150(3992733+184000s_4)z^9\\ & -16875(-1320219+121280s_4)z^{10}-67500(-50787+11200s_4)z^{11}+56250(-7047+2560s_4)z^{12}\\ & \left.+31250(-5103+2048s_4)z^{13}\right]^2. \end{aligned}$$

Regarding the stability of the fixed points, it is clear that 0 is a superattractive fixed point. However, as in case $m=2$, the stability of the other fixed points change depending on the values of the parameter $s_4$. In the following results we establish the stability of $z=\infty$ and the strange fixed point $z=1$, whose proofs are similar to those of Theorem 3, and thus, they are omitted.

**Theorem 6.** *The stability of the fixed point $z=\infty$ is defined as follows:*

*(i) If $|s_4-\frac{5103}{2048}|<\frac{729}{4096}$, then $z=\infty$ is attracting and superattracting for $s_4=\frac{5103}{2048}$.*

*(ii) For $|s_4-\frac{5103}{2048}|=\frac{729}{4096}$, $z=\infty$ is a parabolic point.*

*(iii) If* $|s_4 - \frac{5103}{2048}| > \frac{729}{4096}$*, then* $z = \infty$ *is repulsive.*

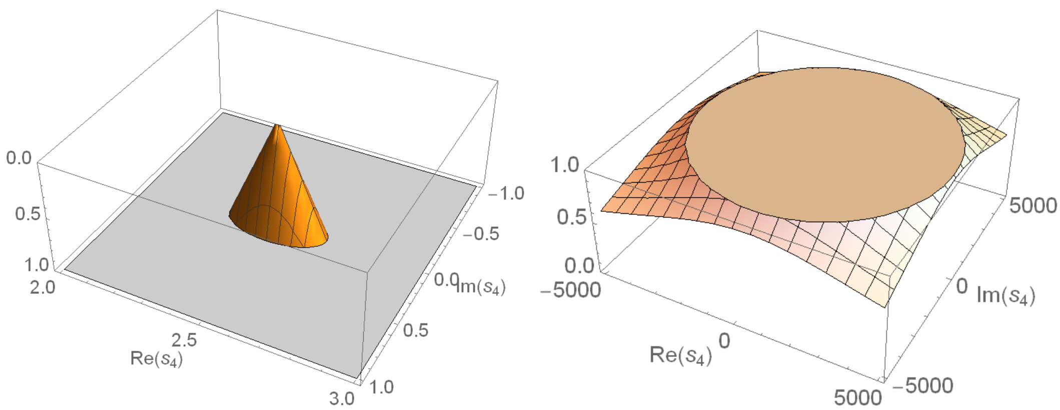
**Theorem 7.** *The character of the strange fixed point* $z = 1$ *is as follows (except for* $s_4 = 442333143/982600$*, as in this case* $z = 1$ *is not a fixed point ):*

*(i) If* $|4913000s_4 - 2211665715| > 21956126976$*, then* $z = 1$ *is an attractor and it cannot be a superattractor.*

*(ii) When* $|4913000s_4 - 2211665715| = 21956126976$*,* $z = 1$ *is a parabolic point.*

*(iii) If* $|4913000s_4 - 2211665715| < 21956126976$*, then* $z = 1$ *is a repulsor.*

Let us remark that area of the complex plane where $z = 1$ is repulsive has been highly incremented with respect to the case in $m = 2$ (see Figure 4). This makes the family even more stable, as the study of the stability of the rest of strange fixed points establishes.




(a) $z = \infty$ (b) $R_1(s_4) = 1$

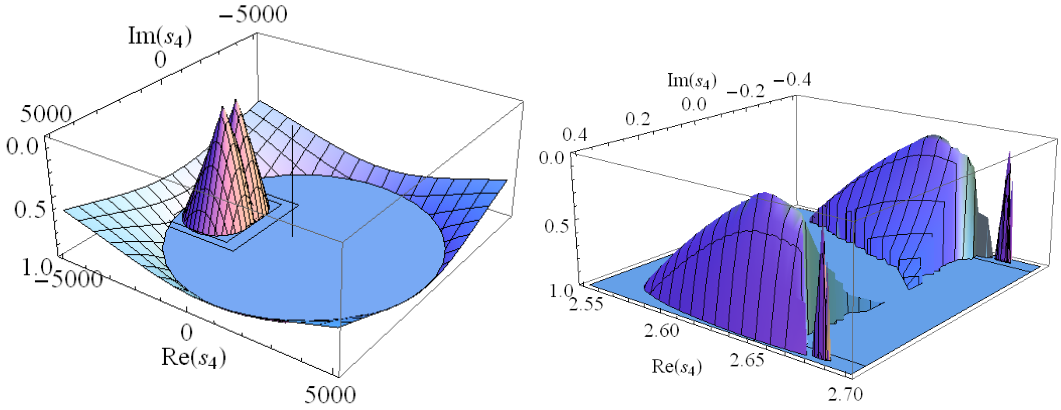
**Fig. 4** Stability function of the fixed points $z = \infty$ and $z = 1$, for $m = 3$

Regarding the rest of strange fixed points, we remark some interesting aspects that have been stated both numerical and graphically:

- Most of them are repulsive, for any complex value of parameter $s_4$. Only four of them can be attractive or superattractive.
- Four of the strange fixed points are superattracting if $s_4 \approx 2.685 \pm 0.378i$. Around these values, their respective stability functions are lower than one and they are attractive (see Figure 5(b)).
- In addition, one of these points is superattracting if $s_4 \approx -2410.18 \pm 331.827i$. A big area surrounding these two values of $s_4$ forms the stability function of the strange fixed points (see also Figure 5(b)).

For the global behaviour of the strange fixed points, Figure 5(a) can be observed. In it, it is clear that, except for values of parameter close to 2.5, the area around the origin is completely stable for the methods, as the strange fixed points are repulsive. The rest of dangerous behaviour is, moreover, very far from the standard values of a parameter in real applications.

For better understanding the behaviour of the elements of the family applied on polynomials with cubic multiplicity, it is necessary to analyse the number of critical points of the associated operator, as a lower number decreases the number of attracting areas different from the roots. In the following result these items are discussed, from the analysis of the equation $O'_q(z, s_4) = 0$.




(a) $R_i(s_4)$, $i = 2,3,\ldots,13$ (b) A detail

**Fig. 5** Stability functions of the strange fixed points $R_i(s_4)$, $i = 2,3,\ldots,13$ for $m = 3$

**Theorem 8.** *The free critical points of rational function $O_q(z,s_4)$ are $CR_1 = -3$, $CR_2 = -9/5$, $CR_3 = \frac{3}{5}(-1-2i\sqrt{6})$, $CR_4 = \frac{3}{5}(-1+2i\sqrt{6})$ and the roots $CR_i, i = 5,6,...,18$, of a 14th-polynomial, except in the following cases:*

*(i) If $s_4 = 187353/1600$, then $CR_6 = 0$ and there are only seventeen free critical points.*

*(ii) For $s_4 = 442333143/982600$, the operator is simplified and there are only sixteen free critical points.*

*(iii) If $s_4 = 2187/1000$, then $CR_i = -3$, $i \in \{1,5,6,7,8\}$ is not a critical point and there are only thirteen free critical points.*

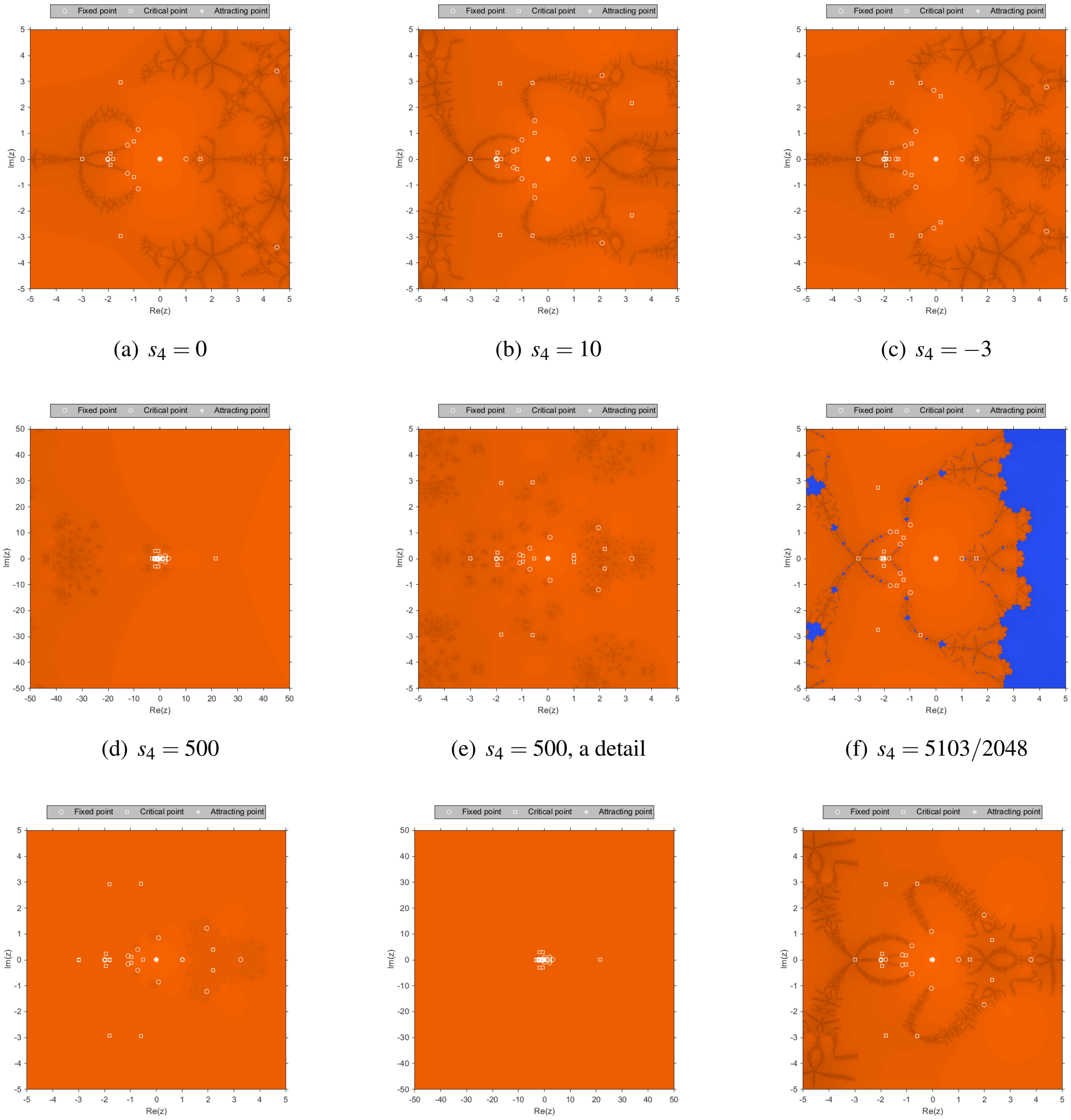
*(iv) For $s_4 = 0$, $CR_6 = CR_7 = CR_2 = -9/5$; on the other hand, $CR_3 = \frac{3}{5}(-1-2i\sqrt{6})$ and $CR_4 = \frac{3}{5}(-1+2i\sqrt{6})$ are not critical points. So, there are only twelve free critical points, $z = -3$, $z = -9/5$ and the roots of a 10th-degree polynomial.*

Let us also remark that the first critical points $CR_1 = -3$, $CR_2 = -9/5$, $CR_3 = \frac{3}{5}(-1-2i\sqrt{6})$ and $CR_4 = \frac{3}{5}(-1+2i\sqrt{6})$ are, for all values of $s_4$, preimages of the fixed point $z = 1$. Therefore, they are preperiodic points and their orbits depend on the stability of the strange fixed points they converge to.

### 3.2 Dynamical planes

In this section, some of the values of $s_4$ that have appeared along the analysis are used to plot the respective dynamical planes and observe the performance of the method. In the studied cases, convergence to the multiple root $z = 0$ or to both $z = 0$ and $z = \infty$ are observed, depending on the value of the parameter. For the first case, see Figures 6(a) to 6(e) and 6(g) to 6(i); regarding the former case, a dynamical plane for a superattracting $z = \infty$ is shown in Figure 6(f). As in the previous section, all the dynamical planes have been obtained by using a mesh of $800 \times 800$ points and a maximum of 80 iterations.

There have been also appeared some values of $s_4$ corresponding to unstable behaviour, under different circumstances, when we have analysed the stability region of attracting strange fixed points (Figures 7(b), 7(c), 7(e) and 7(f), but also some of them correspond to attracting periodic orbits Figures 7(a), 3(d), 3(g) and 3(h)). The periodic orbits are marked with yellow lines, with yellow circles at the elements of the orbit. Let us also remark that in any basin of attracting, including those of periodic orbits, some white square corresponding to critical points appear.




(a) $s_4 = 0$ (b) $s_4 = 10$ (c) $s_4 = -3$

(d) $s_4 = 500$ (e) $s_4 = 500$, a detail (f) $s_4 = 5103/2048$

(g) $s_4 = 442333143/982600$ (h) $s_4 = 442333143/982600$ a detail (i) $s_4 = 187353/1600$

**Fig. 6** Dynamical planes with stable behavior for $m = 3$

## 4 Conclusions

In this article, investigation has been made on the complex plane for class (1) to reveal its dynamical behaviour on polynomials with double and triple roots. The dynamical study of family (1) of iterative methods allows us to select iterative schemes with good stability and reliability properties and detect iterative methods with dangerous numerical behaviour. Indeed, wide regions for parameter $s_4$ have been obtained where the schemes have very good stability properties, mainly for the multiple root. In fact, the simple root does not appear as attracting fixed points for many values of the parameter.

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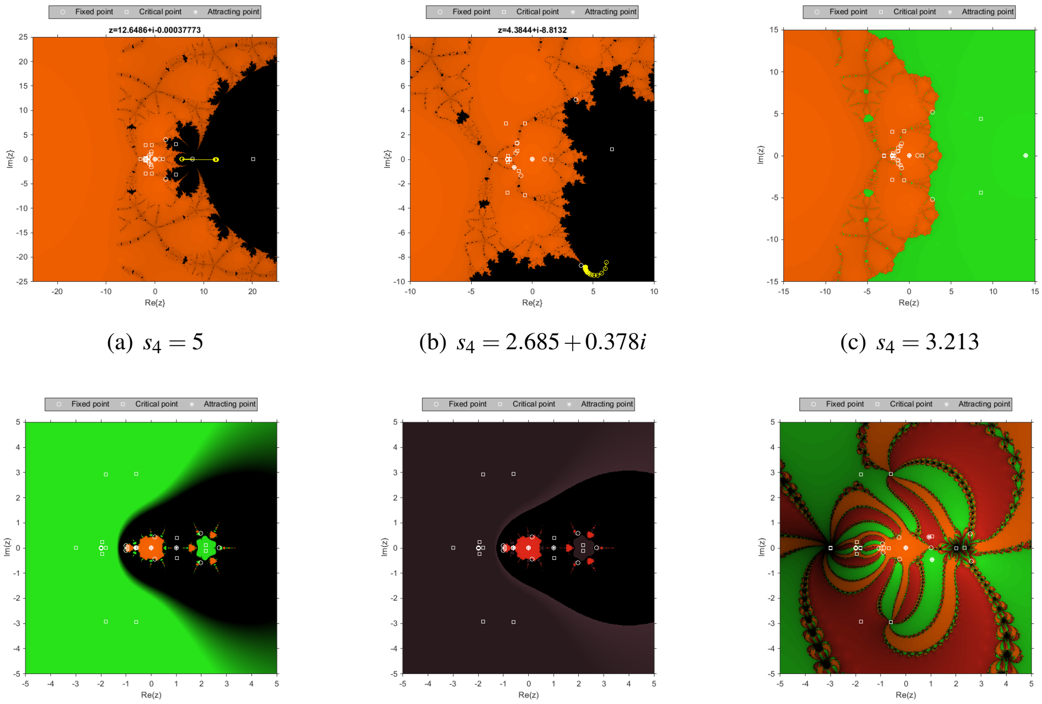


(a) $s_4 = 5$ (b) $s_4 = 2.685 + 0.378i$ (c) $s_4 = 3.213$

(d) $s_4 = 5066$ (e) $s_4 = 40095/8$ (f) $s_4 = -2410.18 + 331.827i$

**Fig. 7** Dynamical planes with unstable behavior

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