Critical damping in nonviscously damped linear systems

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Abstract

In structural dynamics, energy dissipative mechanisms with nonviscous damping are characterized by their dependence on the time-history of the response velocity, mathematically represented by convolution integrals involving hereditary functions. Combination of damping parameters in the dissipative model can lead the system to be overdamped in some (or all) modes. In the domain of the damping parameters, the thresholds between induced oscillatory and non-oscillatory motion are named critical damping surfaces (or critical manifolds, since several parameters can be involved). In this paper the theoretical foundations to determine critical damping surfaces in nonviscously damped systems are established. In addition, a numerical method to obtain critical curves is developed. The approach is based on the transformation of the algebraic equations, which define implicitly the critical curves, into a system of differential equations. The derivations are validated with three numerical methods covering single and multiple degree of freedom systems.

Keywords: critical damping surfaces, nonviscous damping, viscoelastic damping, eigenvalues, hereditary functions, overdamping

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1. Introduction

In this paper, nonviscously damped linear systems are under consideration. Nonviscous (also named viscoelastic) materials have been widely used for vibrating control in mechanical, aerospace, automotive and civil engineering applications. This paper deals precisely with those applications where vibrations are tried to be disappeared, that is, designing of damping devices which are able to avoid oscillatory motion at dynamical systems. In nonviscous models, damping forces are assumed to be dependent on the history of the response velocity via kernel time functions. As far as the motion equations concerned, this fact is represented by convolution integrals involving the velocities of the degrees–of–freedom (dof) and affected by the hereditary kernels. Denoting by $\mathbf{u}(t) \in \mathbb{R}^n$ to the array with the degrees of freedom of the system, this vector verifies the dynamic equilibrium equations which in turn has an integro-differential form

$$\mathbf{M}\ddot{\mathbf{u}} + \int_{-\infty}^{t} \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\mathbf{u}}\,\mathrm{d}\tau + \mathbf{K}\mathbf{u} = \mathfrak{f}(t) \tag{1}$$

where $\mathbf{M}, \mathbf{K} \in \mathbb{R}^{n\times n}$ are the mass and stiffness matrices assumed to be positive definite and positive semidefinite, respectively; $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{n\times n}$ is the nonviscous damping matrix in the time domain, assumed symmetric, which satisfies the necessary conditions of Golla and Hughes [1] for a strictly dissipative behavior. As known, the viscous damping is just a particular case of Eq. (1) with $\boldsymbol{\mathcal{G}}(t) \equiv \mathbf{C}\,\delta(t)$, where $\mathbf{C}$ is the viscous damping matrix and $\delta(t)$ the Dirac's delta function. The time–domain system of motion equations are then reduced to the well known expressions

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathfrak{f}(t) \tag{2}$$

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Considering now the free motion case $\mathfrak{f}(t) \equiv \mathbf{0}$ in Eq. (1), exponential solutions of the type $\mathfrak{u}(t) = \boldsymbol{u}\, e^{st}$ can be checked, with $\boldsymbol{u}$ and s to be found, leading the classical nonlinear eigenvalue problem associated to viscoelastic vibrating structures

$$\left[s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}\right] \boldsymbol{u} \equiv \mathbf{D}(s)\, \boldsymbol{u} = \mathbf{0} \tag{3}$$

where $\mathbf{G}(s) = \mathcal{L}\{\boldsymbol{\mathcal{G}}(t)\} \in \mathbb{C}^{n\times n}$ is the damping matrix in the Laplace domain and $\mathbf{D}(s)$ is the dynamical stiffness matrix or transcendental matrix.

The nature of the response governed by Eqs. (1) is closely related to the eigensolutions of the eigenvalue problem (3). Adhikari [2] derived modal relationships and closed form expressions for the transfer function in the Laplace domain. Due to the nonlinearity, induced by a frequency-dependent damping matrix, to search eigensolutions is in general much more expensive computationally than that of classical viscous damping [3]. A survey of the different viscoelastic models can be found in the references [4, 5]. In this paper, hereditary damping models based on exponential kernels will be considered [6].

Since the present paper is devoted on critical damping and this field has been deeply studied in the bibliography for viscously damped systems, we consider relevant to review the main works related to them. Duffin [7] defined an overdamped system in terms of the quadratic forms of the coefficient matrices. Nicholson [8] obtained eigenvalue bounds for free vibration of damped linear systems. Based on these bounds, sufficient condition for subcritical damping were derived. Müller [9] characterized an underdamped system in similar terms to Duffin's work deriving a sufficient condition expressed as function of the definiteness of the system matrices. Inman and Andry [10] proposed sufficient conditions for underdamped, overdamped and critically damped motions in terms of the definiteness of the system matrices. These conditions are valid for classically damped systems although Inman and Andry shown that they also could work for nonclassical systems. Inman and Orabi [11] and Gray and Andry [12] proposed more efficient method for computing the critical damping condition. However, Barkwell and Lancaster [13] pointed out some defficiencies in the Inman and Andry criterion of ref. [10] presenting a counterexample and they provided some reasons explaining why this criterion had been usually adopted to check criticality in damped systems. Additionally, Barkwell and Lancaster [13] obtained necessary and sufficient conditions for overdamping in gyroscopic vibrating sytems. Bhaskar [14] presented a more complete overdamping condition which somehow corrected that of Inman and Andry [10] giving a generalization to a class of nonconservative systems. Beskos and Boley [15] established conditions for finding critical damping surfaces from the determinant of the system and its derivative. They proved that a critically damped eigenvalue was simultaneously root of the characteristic equation and its derivative, something that can be used to detect critical damping surfaces. In the work [16] the same authors studied conditions for critical damping in continuous systems. Beskou and Beskos [17] presented an approximate method computationally efficient to find critical damping surfaces separating overdamping or partially overdamping regions from those of underdamping for viscously damped systems.

As far as nonviscous systems concerned, research on critical damping has not been as exhaustive as that of viscous damping. Mainly, investigations have been focused on single degree–of–freedom systems on the discussion of the type of response attending at the damping parameters of a single–exponential hereditary kernel. Muravyov and Hutton [18] and Adhikari [19] analyzed the conditions under which single degree–of–freedom nonviscously damped systems by one exponential kernel becomes critically damped. In reference [19] a complete analysis of the roots arising from the third order characteristic polynomial is carried out . Adhikari [20] studied the dynamic response of nonviscously damped oscillators and discussed the effect of the damping parameters on the frequency response function. Müller [21] performed a detailed analysis on the nature of the eigenmotions of a single degree of freedom Zener 3–parameter viscoelastic model. Muravyov [18] obtained closed–form solutions for forced nonviscoulsy damped beams studying the conditions for overdamping or underdamping time response. As known, viscoelastic systems modeled by hereditary exponential functions are characterized by having extra real overdamped modes associated to those kernels.

In reference to these type of modes, the references [22, 23, 24] provide a mathematical characterization and some numerical methods to their evaluation. Lázaro [5] observed that certain recursive method to obtain eigenvalues in proportionally damped viscoelastic systems always converges under a linear rate except just in the critical surfaces where the scheme is underlineal.

In this paper, critical damping surfaces of nonviscously damped linear systems are presented. Critical damping is refereed to the set of damping parameters within the threshold between induced oscillatory and non-oscillatory motion (for all or for some modes). The general procedure to extract these manifolds in the domain of the damping parameters is to eliminate the eigenparameter from a system of two algebraic equations. In addition, since this elimination is not possible for polynomials with order greater than four, a new method to construct critical curves is developed. This methodology is based on to transform the algebraical equations into a system of two ordinary differential equations. The approach is validated with three numerical examples. The two first are devoted on single degree–of–freedom systems with one and two hereditary exponential kernels, respectively. The application of the current approach for multiple degree–of–freedom systems is presented in the third example.

2. Conditions of criticality in terms of determinant of the system

2.1. Theoretical background

In this section, the main results derived by Beskos and Boley [15] in critical viscously damped systems will be extended to nonviscous damping. In order to establish the main assumptions of the research, the type of damping model will be described in its most general form. A general damping matrix based on hereditary Biot's exponential kernels will be considered. Mathematically, this model adopts the following form in time and frequency domain

$$\boldsymbol{\mathcal{G}}(t) = \sum_{k=1}^{N} \mathbf{C}_k \, \mu_k \, e^{-\mu_k t} \ , \quad \mathbf{G}(s) = \mathcal{L}\{\boldsymbol{\mathcal{G}}(t)\} = \sum_{k=1}^{N} \frac{\mu_k}{s+\mu_k} \mathbf{C}_k \tag{4}$$

where $\mu_k > 0$, $1 \leq k \leq N$ represent the nonviscous relaxation coefficients and $\mathbf{C}_k \in \mathbb{R}^{n\times n}$ are the (symmetric) matrices of the limit viscous damping model, defined as

$$\sum_{k=1}^{N} \mathbf{C}_k = \lim_{\mu_1 \dots \mu_N \to \infty} \mathbf{G}(s) \tag{5}$$

Coefficients μ_k control the time and frequency dependence of the damping model while the spatial location and the level of damping are controlled by coefficients within matrices $\mathbf{C}_k$. It is straightforward that the following relationships hold

$$\sum_{k=1}^{N} \mathbf{C}_k = \int_0^{\infty} \boldsymbol{\mathcal{G}}(t)\, dt = \mathbf{G}(0) \tag{6}$$

Henceforth, the damping matrix (and by extension the transcendental matrix) will be assumed to be function of a set of parameters controlling the dissipative behavior, additionally to the frequency dependence. In the most general case, the symmetric damping model presented in (4) depends on $l_{\max} = N + Nn(n+1)/2$ parameters. Indeed, N nonviscous coefficients $\mu_1, \dots, \mu_N$ plus $n(n+1)/2$ possible independent entrees in every symmetric matrix $\mathbf{C}_k$, with $1 \leq k \leq N$. Thus, the complete set of parameters can be listed as

$$\mu_1, \dots, \mu_N, C_{111}, \dots, C_{1nn}, \dots, C_{N11}, \dots, C_{Nnn} \tag{7}$$

where $C_{kij} = C_{kji}$ is the ijth entree of matrix $\mathbf{C}_k$. Real applications depend in general on much less parameters, say $l << l_{\max}$. For the sake of clarity, the set of independent damping parameter will be denoted by $\boldsymbol{\theta} = \{\theta_1, \dots, \theta_l\}$. Consequently, the damping matrix can be written as $\mathbf{G}(s, \boldsymbol{\theta})$, highlighting its dependence on the frequency and, additionally, on the array of damping parameters.

The determinant associated to the nonlinear eigenvalue problem (3) can be denoted by $\mathcal{D}(s,\boldsymbol{\theta}) = \det[\mathbf{D}(s)]$ and the eigenvalues are then the roots of the characteristic equation defined as

$$\mathcal{D}(s,\boldsymbol{\theta}) = \det\left[s^2\mathbf{M} + s\sum_{k=1}^{N}\frac{\mu_k}{s+\mu_k}\mathbf{C}_k + \mathbf{K}\right] = 0 \tag{8}$$

The algebraic structure of the spectrum for this problem can vary depending on the actual value of $\boldsymbol{\theta} \in \mathbb{R}^l$. Thus, if the level of damping induced by matrix $\mathbf{G}(s,\boldsymbol{\theta})$ is light, the set of roots is formed by $2n$ complex eigenvalues with oscillatory nature and r real eigenvalues with non-oscillatory nature and associated to the nonviscous hereditary kernels (usually named as nonviscous eigenvalues). Furthermore, the total number of these real eigenvalues is [3]

$$r = r_1 + \cdots + r_N = \sum_{k=1}^{N}\mathrm{rank}(\mathbf{C}_k) \tag{9}$$

As long as the set of $R = 2n + r$ eigenvalues is formed by n conjugate complex pairs and r nonviscous eigenvalues, it will be said that the system is completely underdamped. As the damping level increases, the real part of eigenvalues (not necessary all) becomes higher (in absolute value) and the imaginary part decreases. For certain value of the damping parameters a conjugate–complex pair could merge into a double real negative root. The set of damping parameters is said then to be on a critical surface, which in turn represents the threshold between underdamping and overdamping. If oscillatory and non-oscillatory modes coexist, then the system is said to be partially overdamped (or under mixed overdamping). Finally, the system is said to be completely overdamped if all modes are overcritically damped. For mixed or complete overdamping, some roots (or maybe all) of Eq. (8) are negative real numbers, say $s = \lambda$, with $\lambda < 0$ so that

$$\mathcal{D}(\lambda,\boldsymbol{\theta}) = \det\left[\lambda^2\mathbf{M} + \lambda\mathbf{G}(\lambda,\boldsymbol{\theta}) + \mathbf{K}\right] = 0 \tag{10}$$

In addition, if an eigenvalue lies exactly on the threshold between underdamped and overdamped motion (critical), another mathematical condition in additon to that of Eq. (10) can be derived. Consider that for a particular combination of the damping parameters, say $\boldsymbol{\theta}_0$, the system lies on a critical surface. Then, there exists a negative real eigenvalue, say $\lambda = -\sigma_0$ so that $\mathcal{D}(-\sigma_0,\boldsymbol{\theta}_0) = 0$. Let us assume that a small perturbation of the damping parameters (from $\boldsymbol{\theta}_0$ to $\boldsymbol{\theta}$) shifts the system to the underdamped region. Under such case, the real eigenvalue $-\sigma_0$ (associated to $\boldsymbol{\theta}_0$) is transformed into the conjugate complex pair, say $-\sigma(\boldsymbol{\theta}) \pm i\,\epsilon(\boldsymbol{\theta})$, verifying $\sigma(\boldsymbol{\theta}_0) = \sigma_0$ and $\epsilon(\boldsymbol{\theta}_0) = 0$. According to the algebra fundamental theorem, the expression of the determinant $\mathcal{D}(\lambda,\boldsymbol{\theta})$ can be written as

$$\mathcal{D}(\lambda,\boldsymbol{\theta}) = \left[(\lambda + \sigma(\boldsymbol{\theta}))^2 + \epsilon^2(\boldsymbol{\theta})\right]\,\mathcal{P}(\lambda,\boldsymbol{\theta}) \tag{11}$$

where $\mathcal{P}(\lambda,\boldsymbol{\theta})$ is a polynomial of order $2R-2$, where $R = 2n + r$ is the total number of eigenvalues. The functions $-\sigma(\boldsymbol{\theta})$ and $\pm\,\epsilon(\boldsymbol{\theta})$ represent the variation of real and imaginary part of both eigenvalues around the critical eigenvalue $-\sigma_0$. Let us calculate the derivative of Eq.(11) respect λ

$$\frac{\partial\mathcal{D}}{\partial\lambda} = 2\left(\lambda + \sigma(\boldsymbol{\theta})\right)\,\mathcal{P}(\lambda,\boldsymbol{\theta}) + \left[(\lambda + \sigma(\boldsymbol{\theta}))^2 + \epsilon^2(\boldsymbol{\theta})\right]\,\frac{\partial\mathcal{P}}{\partial\lambda} \tag{12}$$

Evaluating at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, yields

$$\left.\frac{\partial\mathcal{D}}{\partial\lambda}\right|_{(-\sigma_0,\boldsymbol{\theta}_0)} = 2\left(\lambda + \sigma_0\right)\,\mathcal{P}(-\sigma_0,\boldsymbol{\theta}_0) + \left(\lambda + \sigma_0\right)^2\,\left.\frac{\partial\mathcal{P}}{\partial\lambda}\right|_{(-\sigma_0,\boldsymbol{\theta}_0)} \equiv \left(\lambda + \sigma_0\right)\,\mathcal{Q}(\lambda) \tag{13}$$

where $\mathcal{Q}(\lambda)$ is a polynomial of the same order as $\mathcal{P}(\lambda)$. Eq. (13) shows that the critical eigenvalue $-\sigma_0$ is also root of the equation $\partial\mathcal{D}/\partial\lambda = 0$.

$$\frac{\partial}{\partial\lambda}\mathcal{D}(\lambda,\boldsymbol{\theta}) = \frac{\partial}{\partial\lambda}\det\left[\mathbf{D}(\lambda,\boldsymbol{\theta})\right] = 0 \tag{14}$$

This result is in agreement with the condition derived by Beskos and Boley [15] for viscous systems. Indeed, for each value of λ, Eq. (10) defines a l–dimensional surface in the space where the parameters array $\boldsymbol{\theta}$ can take values. Since for light damping n pairs of conjugate–complex eigenvalues exist initially, we will have as much as n critical surfaces because, as Beskos and Boley [15] point out: "*there are at most as many partial critical damping possibilities as the number of the pairs in* (8) *of roots s with zero imaginary part*". The mathematical principle which characterizes a critical damping surfaces can be extrapolated to nonviscous damping and therefore these ones can be found imposing a minimum among all possible values of λ in Eq. (10), that is $\partial\mathcal{D}/\partial\lambda = 0$. This condition is consistent with the fact that a critical root is real and double under critical condition. Therefore, Eqs. (10) and (14) define a set of critical surfaces resulting after eliminating parameter λ from both equations. This process, although well defined from a theoretical point of view, can only be carried out provided that an analytical closed–form expression of $\mathcal{D}(\lambda, \boldsymbol{\theta})$ is available, something that only occurs for small to moderately sized systems. Analysis of critical damping in non-viscous systems using simultaneously the information provided by Eqs. (10) and (14) has not been carried out yet in the bibliography, up to the author's knowledge. Instead, those available results have been found for single degree–of–freedom systems and just for one hereditary kernel since for this particular problem the roots' nature can be studied using the Cardano's formulas. In them, explicit expressions of the critical curves can be derived from the discussion on the discriminant [18, 19, 20]. In the current work, the two deduced conditions (10) and (14) will be used to validate the available results for single dof. Additionally, a new numerical method to evaluate the critical damping surfaces for multiple dof systems is proposed in the following point.

2.2. Derivation of the proposed numerical method

The numerical determination of critical damping surfaces consists in solving Eqs. (10) and (14) simultaneously for a prefixed range of values within the damping parameters. In general, from a computational point of view, this process becomes highly inefficient. Indeed, for each combination of the prefixed parameters, a system of two nonlinear equations must be solved. The main objective of this paper is to propose a new numerical method suitable to construct critical curves based on two parameters. In other words, the approach allows to draw critical overdamped regions located in two dimensional cross sections of the l–dimensional real critical manifolds. Thus, considering the l–dimensional domain of all possible values of the array $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_l\}$, two of them can be chosen arbitrarily: they will be named as *design parameters*. Let us take, without loss of generality, θ_1 and θ_2 as the design parameters, while the rest will remain fixed, say $\theta_{30}, \ldots, \theta_{l0}$. The challenge is to find critical damping curves in the 2-dimensional domain of the design parameters (θ_1, θ_2). for the sake of clarity in the notation, let us introduce two new symbols, p and q, to refer the new free parameters: $p = \theta_1$ and $q = \theta_2$. Thus, the critical curves are then functions of the form $q = q(p)$. For each value of p, both equations

$$\mathcal{D}(\lambda, p, q) = 0 \quad , \qquad \frac{\partial}{\partial\lambda}\mathcal{D}(\lambda, p, q) = 0 \tag{15}$$

allow to find a pair (λ, q) (or several pairs, since λ is affected by a polynomial). Let us consider a point p_0 for which q_0 and λ_0 are solutions of both equations (15) and let us assume that around $p = p_0$ the functions $q(p)$ and $\lambda(p)$ exist. It is said then that the three numbers (p_0, q_0, λ_0) form a initial point of the proposed approach. The derivatives $\lambda'(p)$= $\mathrm{d}\lambda/\mathrm{d}p$ and $q'(p)$= $\mathrm{d}q/\mathrm{d}p$ can be evaluated just applying the chain rule in Eqs. (15). Indeed,

$$\frac{\partial\mathcal{D}}{\partial\lambda}\,\lambda'(p) + \frac{\partial\mathcal{D}}{\partial q}\,q'(p) + \frac{\partial\mathcal{D}}{\partial p} \quad = \quad 0 \tag{16}$$

$$\frac{\partial^2\mathcal{D}}{\partial\lambda^2}\,\lambda'(p) + \frac{\partial^2\mathcal{D}}{\partial\lambda\partial q}\,q'(p) + \frac{\partial^2\mathcal{D}}{\partial\lambda\partial p} \quad = \quad 0 \tag{17}$$

From Eq. (15), it is verified that $\partial \mathcal{D}/\partial \lambda = 0$. Hence, both $\lambda'(p)$ and $q'(p)$ can be solved, resulting

$$\begin{aligned} q'(p) &= -\frac{\mathcal{D}_{,p}}{\mathcal{D}_{,q}} \\ \lambda'(p) &= \frac{\mathcal{D}_{,p}\,\mathcal{D}_{,\lambda q}}{\mathcal{D}_{,q}\,\mathcal{D}_{,\lambda\lambda}} - \frac{\mathcal{D}_{,\lambda p}}{\mathcal{D}_{,\lambda\lambda}} \end{aligned} \tag{18}$$

where

$$\mathcal{D}_{,p} = \frac{\partial \mathcal{D}}{\partial p} \quad , \; \mathcal{D}_{,q} = \frac{\partial \mathcal{D}}{\partial q} \quad , \; \mathcal{D}_{,\lambda\lambda} = \frac{\partial^2 \mathcal{D}}{\partial \lambda^2} \quad , \; \mathcal{D}_{,\lambda p} = \frac{\partial^2 \mathcal{D}}{\partial \lambda \partial p} \quad , \; \mathcal{D}_{,\lambda q} = \frac{\partial^2 \mathcal{D}}{\partial \lambda \partial q} \tag{19}$$

The equations (18) together with the initial conditions

$$\lambda(p_0) = \lambda_0 \quad , \quad q(p_0) = q_0 \tag{20}$$

form an initial value problem which is well defined provided that the derivatives $\mathcal{D}_{,\lambda\lambda}$ and $\mathcal{D}_{,q}$ do not vanish at the initial point (p_0, q_0, λ_0). The existence of the critical curve beyond the surroundings of the initial point is subordinate to the existence of those derivatives along the curve. Then, critical curves arise as the numerical solution of a system of ordinary differential equations, for which Runge–Kutta based methods can be used. Previously, the method requires solving the system of two algebraical equations (15) and two unkowns, say λ_0, q_0, which in general results in several solutions because of the polynomial form. Real pairs (λ_0, q_0) verifying $\lambda_0 < 0$ and $q_0 \geq 0$ (it is assumed a positive range for damping parameters) will be appropriate solutions lying on a critical curve. Taking derivatives repeatedly respect p in Eq. (17) also can lead to obtain higher order derivatives, allowing to expand in its Taylor series the critical curve around $p = p_0$. This procedure is used to find an approximation of a critical curve in the one–kernel single–dof numerical example.

3. Numerical examples

3.1. Single degree of freedom systems, $N = 1$ exponential kernel

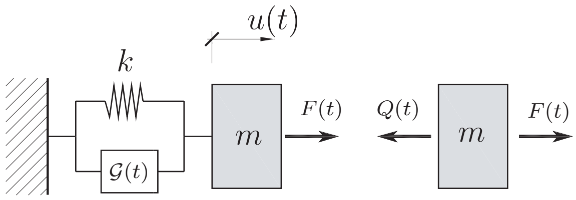

Figure 1: A single degree–of–freedom viscoelastic oscillator

First, a single dof nonviscous system with one hereditary kernel will be considered. The dof represents the displacement of certain mass m attached to the ground by a linear viscoelastic constraint. Fig. 1 shows the schematic configuration *mass–spring–viscoelastic damper* and the corresponding free body diagram. Hence, the internal force is related to the displacement by

$$Q(t) = \int_{-\infty}^{t} \mathcal{G}(t-\tau)\dot{u}(\tau)\mathrm{d}\tau + ku(t) \tag{21}$$

where k is the constant of the linear–elastic spring and $\mathcal{G}(t)$ is the dissipative kernel or damping function with the general form, both in time and frequency domain

$$\mathcal{G}(t) = c\,\mu\,e^{-\mu t} \;, \quad G(s) = \frac{\mu\, c}{s + \mu} \tag{22}$$

where μ and c are respectively is the nonviscous (relaxation) and the viscous coefficients, which verifies at the limit $\lim_{\mu\to\infty} G(s) = c$. This latter relationship represents the particular case of Eq. (5) for single dof systems with one exponential kernel. The free motion equation can be deduced from the dynamic equilibrium for $F(t) \equiv 0$

$$m\ddot{u} + \int_{-\infty}^{t} \mathcal{G}(t-\tau)\dot{u}(\tau)\mathrm{d}\tau + ku(t) = 0 \tag{23}$$

And the associated characteristic equation yields

$$ms^2 + sG(s) + k = ms^2 + s\frac{\mu\, c}{s+\mu} + k = 0 \tag{24}$$

In order to compare the results of the current example with those presented in the bibliography, it is suitable to introduce the following dimensionless variables.

$$x = \frac{s}{\omega_n}\ , \quad \nu = \frac{\omega_n}{\mu}\ , \quad \zeta = \frac{c}{2m\omega_n} \tag{25}$$

where $\omega_n = \sqrt{k/m}$ is he natural frequency of the undamped system, ν and ζ denote the nonviscous parameter and the viscous damping ratio, respectively. After straight operations and multiplying Eq. (24) by the denominator of the hereditary function, the characterisitc polynomial in nondimensional form is obtained as

$$\mathcal{D}(x,\nu,\zeta) = (1+\nu x)(x^2+1) + 2x\zeta = \nu x^3 + x^2 + (\nu + 2\zeta)x + 1 = 0 \tag{26}$$

As known, the three roots are available as function of the coefficients using the Cardano's formulas so that a detailed discussion of the nature of the three roots can be addressed as function of the values of $\zeta > 0$ and $\nu > 0$. This work has been carried out by Adhikari in the references [19, 20] where closed form expressions of the critical curves enclosing the overdamped region were derived. For the sake of our exposition, it is considered of interest to transcript here the Adhikari's results of the critical curves because approximations based on Taylor expansions will also be presented later. Thus, the overdamped region is a 2-dimensional set in the domain of the damping parameters (ζ, ν) which can be defined mathematically as

$$\left\{(\zeta,\nu) \in \mathbb{R}^+ : \zeta_L(\nu) \le \zeta \le \zeta_U(\nu)\right\} \tag{27}$$

where the critical damping curves $\zeta_L(\nu)$, $\zeta_U(\nu)$ are

$$\begin{aligned} \zeta_L(\nu) &= \frac{1}{24\nu}\left[1 - 12\nu^2 + 2\sqrt{1+216\nu^2} + \cos\left(\frac{4\pi+\varphi}{3}\right)\right] \\ \zeta_U(\nu) &= \frac{1}{24\nu}\left[1 - 12\nu^2 + 2\sqrt{1+216\nu^2} + \cos\left(\frac{\varphi}{3}\right)\right] \end{aligned} \tag{28}$$

with

$$\varphi = \arccos\left[-\frac{5832\nu^4 + 540\nu^2 - 1}{(1+216\nu^2)^{3/2}}\right] \tag{29}$$

Let us apply the proposed method to find critical damping curves based on the solution of the system of differential equations (18). According to the theoretical derivations, the critical surfaces arise from eliminating the variable x from the two following equations

$$\begin{aligned} \mathcal{D}(x,\nu,\zeta) &= \nu x^3 + x^2 + (\nu+2\zeta)x + 1 = 0 \\ \frac{\partial \mathcal{D}}{\partial x} &= 3\nu x^2 + 2x + \nu + 2\zeta = 0 \end{aligned} \tag{30}$$

From the second equation the two roots $x_{1,2} = (-1 \pm \sqrt{-6\zeta\nu - 3\nu^2 + 1})/3\nu$ can be determined. Hence, plug them into the first equation and after some simplifications, it yields

$$8\zeta^3\nu + 12\zeta^2\nu^2 - \zeta^2 + 6\zeta\nu^3 - 10\zeta\nu + \nu^4 + 2\nu^2 + 1 = 0 \tag{31}$$

expression which defines the critical surface in implicit form and it matches with the third order polynomial obtained by Adhikari [19].

The proposed numerical method to determine critical curves will be now applied to the current single dof system. The main objective is to find critical curves of the type $\zeta = \zeta(\nu)$, therefore the nonviscous parameter ν is chosen as independent variable ($p \equiv \nu$) while the viscous damping ratio is defined as a dependent variable ($q \equiv \zeta$). As described in Sec. 2.2, to construct the system of differential equations, the partial derivatives of $\mathcal{D}(x, \nu, \zeta)$ and $\mathcal{D}_{,x}(x, \nu, \zeta)$ respect to x, ζ and ν need to be found

$$\begin{aligned} \mathcal{D}_{,\nu} &= x + x^3 \quad , \qquad \mathcal{D}_{,\zeta} = 2x \\ \mathcal{D}_{,x\nu} &= 1 + 3x^2 \quad , \qquad \mathcal{D}_{,x\zeta} = 2 \quad , \qquad \mathcal{D}_{,xx} = 2 + 6\nu x \end{aligned} \tag{32}$$

After some math, the two differential equations are set as

$$\begin{aligned} \zeta'(\nu) &= -\frac{1}{2}\left(1 + x^2\right) \\ x'(\nu) &= -\frac{x^2}{1 + 3\nu x} \end{aligned} \tag{33}$$

Viscous damping ratio	ζ_0	1.00	1.00	1.00
Nonviscous damping factor	ν_0	0.00	0.134884	$-3.06744 \pm 2.32772i$
Dimensionless critical eigenvalue	x_0	-1.00	-3.38298	$0.191488 \pm 0.50885i$
Type of solution		Real	Real	Complex

Table 1: Initial conditions used for the critical damping curves shown in Fig. 2

In order to define properly the initial value problem of the proposed method, the corresponding initial conditions must be added. Taking $\zeta_0 = 1$, Eqs. (30) can be solved obtaining the four pairs of roots with the form (ν, x) shown in Table 1. Only real solutions with $x < 0$, $\nu \geq 0$ are of interest as initial values. The first pair results in the initial values $\zeta_0 = 1$, $\nu_0 = 0$, $x_0 = -1$ of the critical curve $\zeta_L(\nu)$ while the second one $\zeta_0 = 1$, $\nu_0 = 0.134884$, $x_0 = -3.38298$ is used to obtain the curve $\zeta_U(\nu)$. Both curves have been plotted in Fig. 2 achieving a perfect fitting with those of the exact expressions from Eqs. (28).

As stated above, the limit viscous behavior associated to a nonviscous oscillator arises for high values of the relaxation parameter μ or, equivalently, for low values of the nonviscous dimensionless parameter $\nu = \omega_n/\mu$. The plot of Fig. (2) represents the nature of the response for any pair of the parameters (ζ, ν) within $0 \leq \zeta, \nu \leq \infty$. Therefore, it is expected that the oscillatory nature of the associated viscous system is also shown within the parametric domain as particular case. Indeed, the limit $\mu \to \infty$ is depicted as the set defined by the abscissas axis ($\nu = 0$). It is well known that for viscous systems the eigenfrequency has oscillatory nature for $0 \leq \zeta < 1$ while non-oscillatory motion occurs for $\zeta \geq 1$. This latter behavior can be observed along the line ($\nu = 0$) and it has been highlighted with a blue line.

The new perspective proposed in this work allows to interpret critical curves as solutions of certain differential equations. Based on such equations, the Taylor series expansion of the critical curves can also be determined. Although the resulting expressions are approximations, its derivation is considered of interest because it leads to much simpler and more intuitive expressions for both the real critical eigenvalues and parameters.

Encouraged by the fact that the initial point of the critical curve $\zeta_L(\nu)$ is as simple as $\zeta_L(0) = 1$ and also by its regularity and low curvature (information already known since the exact result is available in Fig. 2,

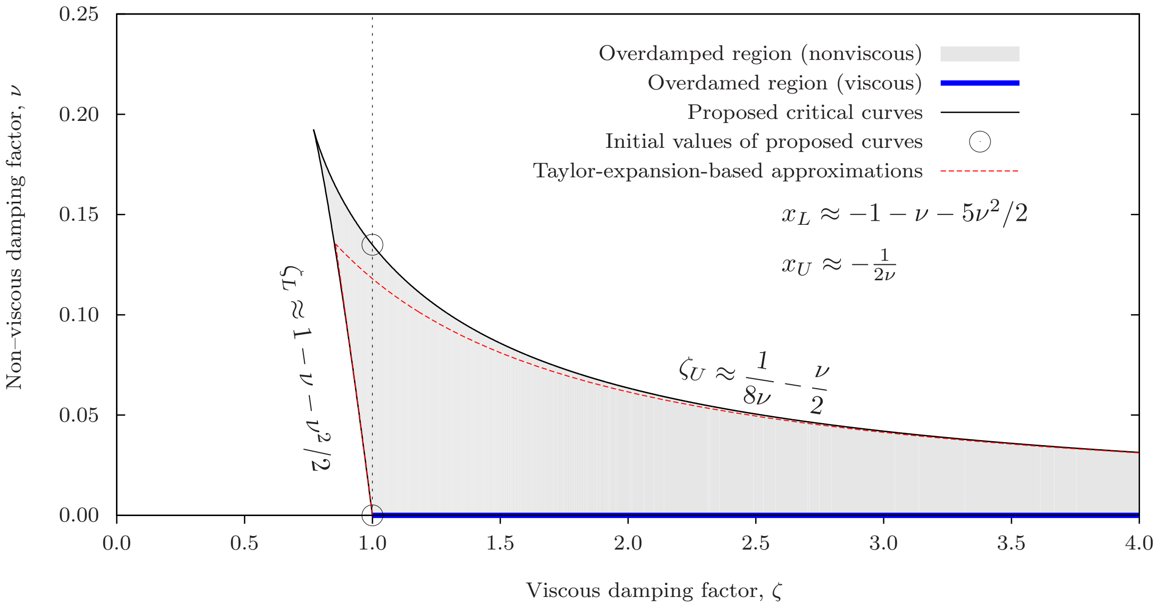

Figure 2: Example 1: single degree–of–freedom with one exponential kernel. Representation of overdamped region and approximated critical curves obtained from Taylor expansion —Eq. (34)— and from the method of polynomial pivots —Eqs.(41), ref. [25]—. Formulas of $x_L(\nu)$ and $x_U(\nu)$ evaluate the approximate critical eigenvalues along the critical curves

we think that the Taylor series expansion around $\nu_0 = 0$ can provide accurate results and in turn simple in form. Indeed, the first derivative can be determined just from Eq. (33) for $x_0 = -1$ and $\nu_0 = 0$, yielding

$$\zeta_L'(0) = -1 \ , \quad x'(0) = -1$$

Now, taking again derivatives respect to ν in Eqs. (32) and after some operations, second derivatives $\zeta_L''(0)$ and $x''(0)$ can be found, so that

$$\zeta_L''(0) = -1 \ , \quad x''(0) = -5$$

Hence, Taylor series expansions up to the second order of the critical damping curve $\zeta_L(\nu)$, and its associated critical eigenvalue $x_L(\nu)$ are then

$$\zeta_L(\nu) \approx 1 - \nu - \nu^2/2 \tag{34}$$
$$x_L(\nu) \approx -1 - \nu - 5\nu^2/2 \tag{35}$$

Similar procedure could be followed to find a Taylor based approximation around upper critical curve ζ_U. However, this function presents higher changes of curvatures and a wider domain of ζ (in fact an infinite range). It is expected that a polynomial based approximation only will perform satisfactory results around the initial point and of course it will not be able to represent the asymptotic behavior. To undertake this approach, a recent result on asymptotic behavior of polynomial roots proposed by Lázaro et al. [25] will be used. Given a kth order polynomial ($k \in \mathbb{N}$)

$$a_0 + a_1 x + \cdots + a_{k-2}x^{k-2} + a_{k-1}x^{k-1} + x^k \tag{36}$$

then the following two numbers (called polynomial pivots)

$$-\frac{a_{k-1}}{2} \pm \sqrt{\left(\frac{a_{k-1}}{2}\right)^2 - a_{k-2}} \tag{37}$$

present the property of lying close to two roots provided that they are not relatively smaller than the rest of the polynomial coefficients. The exact and rigorous mathematical conditions describing this statement are given in form of several theorems in the reference [25]. Let us verify if the so defined pivots of the third order polynomial $\mathcal{D}(x, \nu, \zeta)$ can give us valuable information respect to the nature of the roots. Thus, the closed-form expression of the determinant (of order $k = 3$) expressed in the form (36), is

$$\frac{1}{\nu} + \frac{\nu + 2\zeta}{\nu} x + \frac{1}{\nu} x^2 + x^3 \tag{38}$$

and the coefficients $a_{k-2} = a_1$ and $a_{k-1} = a_2$ are

$$a_1 = \frac{\nu + 2\zeta}{\nu} \quad , \quad a_2 = \frac{1}{\nu} \tag{39}$$

Therefore, from Eqs. (37) the pivots are

$$-\frac{a_2}{2} \pm \sqrt{\left(\frac{a_2}{2}\right)^2 - a_1} = -\frac{1}{2\nu} \pm \sqrt{\frac{1}{4\nu^2} - \frac{2\zeta}{\nu} - 1} \tag{40}$$

It is known that the roots have multiplicity two (double roots) on the critical damping curves. Therefore, if the two pivots are forced to be equal, it will represent a double root and therefore it will lie approximately on a critical curve. To impose that both pivots are equal is equivalent to impose the argument within the square root of Eq. (40) to be zero. From this condition, an approximation of the upper curve $\zeta_U(\nu)$ is found resulting

$$\frac{1}{4\nu^2} - \frac{2\zeta}{\nu} - 1 = 0 \quad \rightarrow \quad \zeta_U(\nu) \approx \frac{1}{8\nu} - \frac{\nu}{2} \tag{41}$$

Returning to the expression of the pivots, the associated (approximate) critically damped eigenvalue results in the unique value (with double multiplicity)

$$x_U(\nu) \approx -\frac{1}{2\nu} \tag{42}$$

obtained from Eq. (40) after deleting the discriminant, as imposed in Eq. (41). According to the results of [25] the higher the pivots (in absolute value) the more accurate the approximation to the polynomial roots. Therefore, since the (double) pivot has the nonviscous parameter ν in the denominator (see Eq. (42)), the lower the nonviscous parameter ν, the better is the prediction of the associated critical curve $\zeta_U(\nu)$, as it can be appreciated in the Fig. 2.

3.2. Single degree of freedom systems, $N = 2$ exponential kernels

In this point, critical damping surfaces for single dof system with $N = 2$ hereditary kernels will be determined. The approach can easily be extrapolated to the general case of N kernels. According to Eq. (4), the damping function is

$$\mathcal{G}(t) = c_1 \, \mu_1 \, e^{-\mu_1 t} + c_2 \, \mu_2 \, e^{-\mu_2 t} \quad , \quad G(s) = \mathcal{L}\{\mathcal{G}(t)\} = \frac{\mu_1 \, c_1}{s + \mu_1} + \frac{\mu_2 \, c_2}{s + \mu_2} \tag{43}$$

And the characteristic equation yields

$$ms^2 + sG(s) + k = ms^2 + s\left(\frac{\mu_1 \, c_1}{s + \mu_1} + \frac{\mu_2 \, c_2}{s + \mu_2}\right) + k = 0 \tag{44}$$

As noticed, the dissipative model has four parameters, c_1, c_2, μ_1, μ_2. The proposed method has been designed to draw critical curves of two parameters. Hence, the other two parameters must be fixed before attempting the numerical solution by the differential equations. For the sake of the representation, let us consider the particular case with $c_1 = c_2 = c$ and let us define the following dimensionless parameters

$$x = \frac{s}{\omega_n} \ , \quad \nu_1 = \frac{\omega_n}{\mu_1} \ , \quad \nu_2 = \frac{\omega_n}{\mu_2} \ , \quad \zeta = \frac{c}{m\omega_n} \tag{45}$$

Where $\omega_n = \sqrt{k/m}$ is the natural frequency of the system, ν_1 and ν_2 represent the nonviscous parameters and ζ denotes the viscous damping ratio.. Using the new dimensionless parameters, the Eq. (44) can be expressed as

$$x^2 + x\,\zeta\left(\frac{1}{1+\nu_1\,x} + \frac{1}{1+\nu_2\,x}\right) + 1 = 0 \tag{46}$$

Note that the limit viscous system emerges when the nonviscous parameters become zero $\nu_1 = \nu_2 = 0$ or, equivalently, when the relaxation coefficients are infinite ($\mu_1 = \mu_2 = \infty$). At that situation, it is clear that the critical damping ratio is $\zeta_{cr} = 1$. Multiplying Eq. (46) by $(1+\nu_1 x)(1+\nu_2 x)$ the characteristic equation can be transformed into a four order polynomial equation

$$\mathcal{D}(x,\nu_1,\nu_2,\zeta) = (1+\nu_1 x)(1+\nu_2 x)(x^2+1) + x\zeta(2+\nu_1 x+\nu_2 x) = 0 \tag{47}$$

This equation together with its derivative respect to x

$$\mathcal{D}_{,x}(x,\nu_1,\nu_2,\zeta) = 2\zeta + 2x + 2x\,\nu_1\nu_2(1+2x^2) + (1+2x\zeta+3x^2)(\nu_1+\nu_2) = 0 \tag{48}$$

allow to find the critical surfaces after eliminating x. Observe that the last equation is a third order polynomial, therefore the Cardano formulas could be used to obtain the three roots. Plugging them into Eq. (47) would lead the exact critical curves. These derivations will not be carried out here explicitly due to the complexity of the resulting expressions. On the other hand, they can easily be programmed in a symbolic software and their results can be compared to those of the proposed approach.

According to the derived method, the critical curves are defined in terms of two parameters, leaving fixed the rest. In the current example two cases will be considered:

Case (a): critical curves in the plane (ν_1,ν_2) (i.e. functions $\nu_2 = f(\nu_1)$ with ζ: fixed)

Case (b): critical curves in the plane (ζ,ν_2) (i.e. functions $\nu_2 = f(\zeta)$ with ν_1: fixed)

Case (a): Critical curves in plane (ν_1,ν_2)

For this case the nonviscous parameter ν_1 is considered as independent variable and the overdamped regions will be plotted in the 2D domain (ν_1,ν_2). Thus, critical curves are functions of the form $\nu_2 = f(\nu_1)$ and, for their representation, the proposed system of two differential equations with two unknowns, $x(\nu_1)$ and $\nu_2(\nu_1)$, must be solved. Such system, derived in Section 2.2 in terms of two functions $\lambda(p)$ and $q(p)$ can be adapted to the actual notation considering $\lambda \equiv x$, $q \equiv \nu_2$ and $p \equiv \nu_1$. Thus, the system can be written in matrix form as

$$\begin{bmatrix} 0 & \mathcal{D}_{,\nu_2} \\ \mathcal{D}_{,xx} & \mathcal{D}_{,x\nu_2} \end{bmatrix} \left\{ \begin{array}{c} x'(\nu_1) \\ \nu_2'(\nu_1) \end{array} \right\} = -\left\{ \begin{array}{c} \mathcal{D}_{,\nu_1} \\ \mathcal{D}_{,x\nu_1} \end{array} \right\} \ , \quad \left\{ \begin{array}{c} x(\nu_{10}) \\ \nu_2(\nu_{10}) \end{array} \right\} = \left\{ \begin{array}{c} x_0 \\ \nu_{20} \end{array} \right\} \tag{49}$$

where

$$\begin{aligned} \mathcal{D}_{,\nu_1} &= x\left[1 + x(x+\zeta+\nu_2+x^2\nu_2)\right] & \qquad \mathcal{D}_{,\nu_2} &= x\left[1 + x(x+\zeta+\nu_1+x^2\nu_1)\right] \\ \mathcal{D}_{,x\nu_1} &= 1 + x\left[2(\zeta+\nu_2) + x(3+4x\nu_2)\right] & \qquad \mathcal{D}_{,x\nu_2} &= 1 + x\left[2(\zeta+\nu_1) + x(3+4x\nu_1)\right] \end{aligned}$$

Since the main aim is to find critical curves in the parametric plane (ν_1,ν_2), particular values should be given for the fixed parameter ζ. For this example three values of the viscous damping ratio will be considered: $\zeta = \{0.90, 5.00, 8.00\}$. Additionally, initial conditions emerge from the solution of Eqs. (47) and (48) in terms of x and ν_2 for prescribed values of ζ (fixed parameter in the current case) and ν_1 (independent variable).

		INITIAL VALUES		
Fixed parameter	Critical curve	ν_{10}	ν_{20}	x_0
$\zeta = 0.90$	A-01	0.00000	0.17053	-1.14479
	A-02	0.00000	0.22155	-2.26820
$\zeta = 5.00$	A-03	0.00000	0.03443	-16.97040
	A-04	0.00000	1.32591	-2.83001
	A-05	0.05200	2.06935	-9.20408
	A-06	1.20000	0.01599	-3.10458
	A-07	1.20000	0.05344	-8.61426
$\zeta = 8.00$	A-08	0.00000	0.02148	-27.25100
	A-09	0.00000	0.76381	-4.62400
	A-10	0.03400	0.56548	-13.0963
	A-11	0.70000	0.00952	-5.03922
	A-12	0.70000	0.03329	-13.89100

Table 2: Initial conditions used to the computation of the critical damping curves shown in Fig. 3

In Table 2 a list of initial conditions obtained from the solution of the aforementioned algebraic equations is shown.

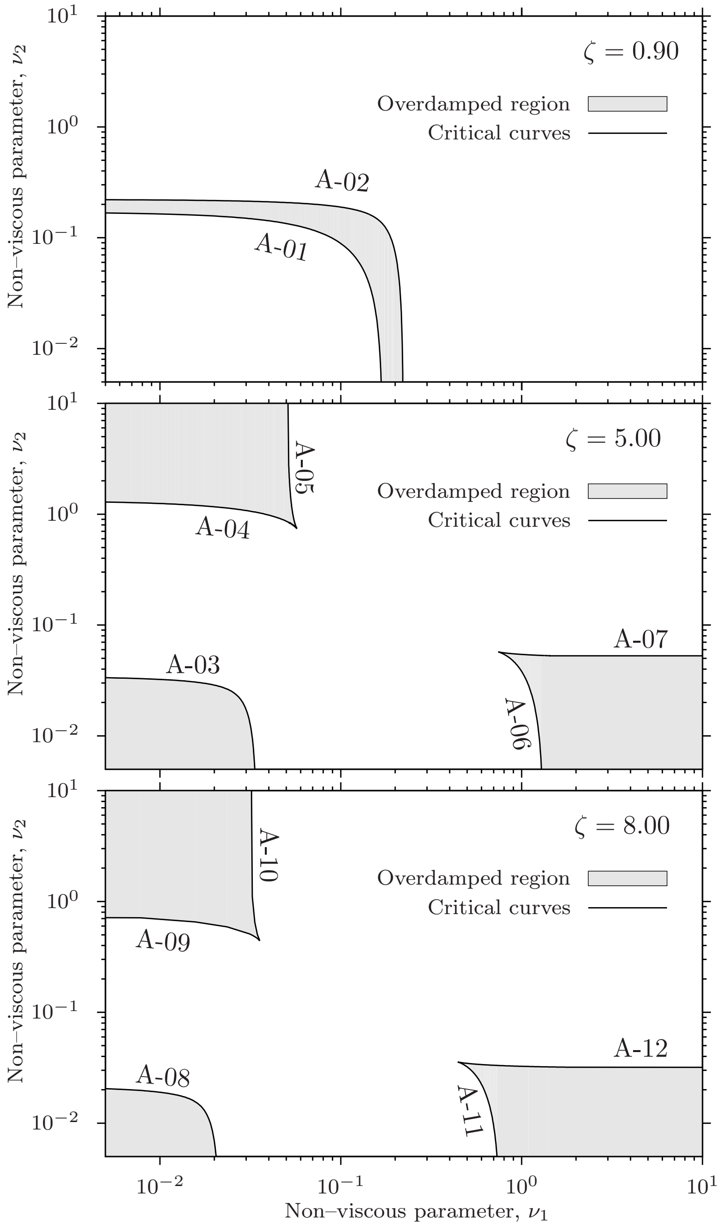

Figure 3: Example 2, Case (a): Single degree–of freedom nonviscous system with 2 hereditary kernels. Overdamped regions and critical curves in the domain of parameters ν_1 and ν_2 for three different damping rations. $\zeta = 0.90$ (top), $\zeta = 5.00$ (middle), $\zeta = 8.00$ (bottom)

Suitable pairs of the form (ν_{10}, x_0) and (ν_{10}, ν_{20}) allow us to find the critical curves using the proposed approach. The so found curves, named by A-01, A-02,...., have been represented in Fig. 3 within three different plots associated to each value of $\zeta = \{0.9, 5, 8\}$. Overdamped regions, represented by shaded areas within the critical curves, are indeed cross sections of 3D-manifolds defined in the space formed by the 3 parameters (ζ, ν_1, ν_2). It can be observed that the figures show symmetry respect to the line $\nu_1 = \nu_2$ due to the symmetry of the physical model respect to the parameters ν_1 and ν_2.

Case (b): Critical curves in plane (ζ, ν_2)

In this second case, critical curves in the plane (ζ, ν_2) will be determined, assuming now ν_1 as fixed parameter and consequently ζ will be considered as independent variable. Following the notation introduced in Sec. 2.2, it yields $\lambda \equiv x$, $q \equiv \nu_2$ and $p \equiv \zeta$. After assembling the proposed system of differential equations, the main matrix multiplying the derivatives $x'(\zeta)$ and $\nu_2'(\zeta)$ has the same form as that of the previous case. Indeed, it results

$$\begin{bmatrix} 0 & \mathcal{D}_{,\nu_2} \\ \mathcal{D}_{,xx} & \mathcal{D}_{,x\nu_2} \end{bmatrix} \begin{Bmatrix} x'(\zeta) \\ \nu_2'(\zeta) \end{Bmatrix} = - \begin{Bmatrix} \mathcal{D}_{,\zeta} \\ \mathcal{D}_{,x\zeta} \end{Bmatrix} \quad , \quad \begin{Bmatrix} x(\zeta_0) \\ \nu_2(\zeta_0) \end{Bmatrix} = \begin{Bmatrix} x_0 \\ \nu_{20} \end{Bmatrix} \tag{50}$$

where now

$$\mathcal{D}_{,\zeta} = x\,[2 + x(\nu_1 + \nu_2)] \quad , \quad \mathcal{D}_{,x\zeta} = 2\,[1 + x(\nu_1 + \nu_2)] \tag{51}$$

The method to construct the critical curves requires initial values which again are determined solving Eqs. (47) and (48) in terms of x and ν_2 fixing values of ν_1 and ζ. Three values are considered For the nonviscous parameter, say $\nu_1 = \{0.00, 0.05, 1.50\}$. Table 3 shows the chosen values for the viscous damping ratio ζ and their corresponding solutions ν_{20} and x_0. Each row in Table 3 leads to a critical curve, denoted in this case with the letter "B".

		INITIAL VALUES		
Fixed parameter	Critical curve	ζ_0	ν_{20}	x_0
$\nu_1 = 0.00$	B-01	1.00000	0.00000	-1.00000
	B-02	1.00000	0.19160	-2.76929
	B-03	4.00000	1.80565	-2.22076
$\nu_1 = 0.05$	B-04	0.95000	0.04736	-1.05573
	B-05	0.95000	0.19197	-2.72450
	B-06	5.20000	0.79903	-4.84245
	B-07	5.20000	1.93559	-9.57166
$\nu_1 = 1.50$	B-08	4.00000	0.03207	-2.61688
	B-09	4.00000	0.06738	-6.74685

Table 3: Initial conditions used to the computation of the critical damping curves shown in Fig. 4

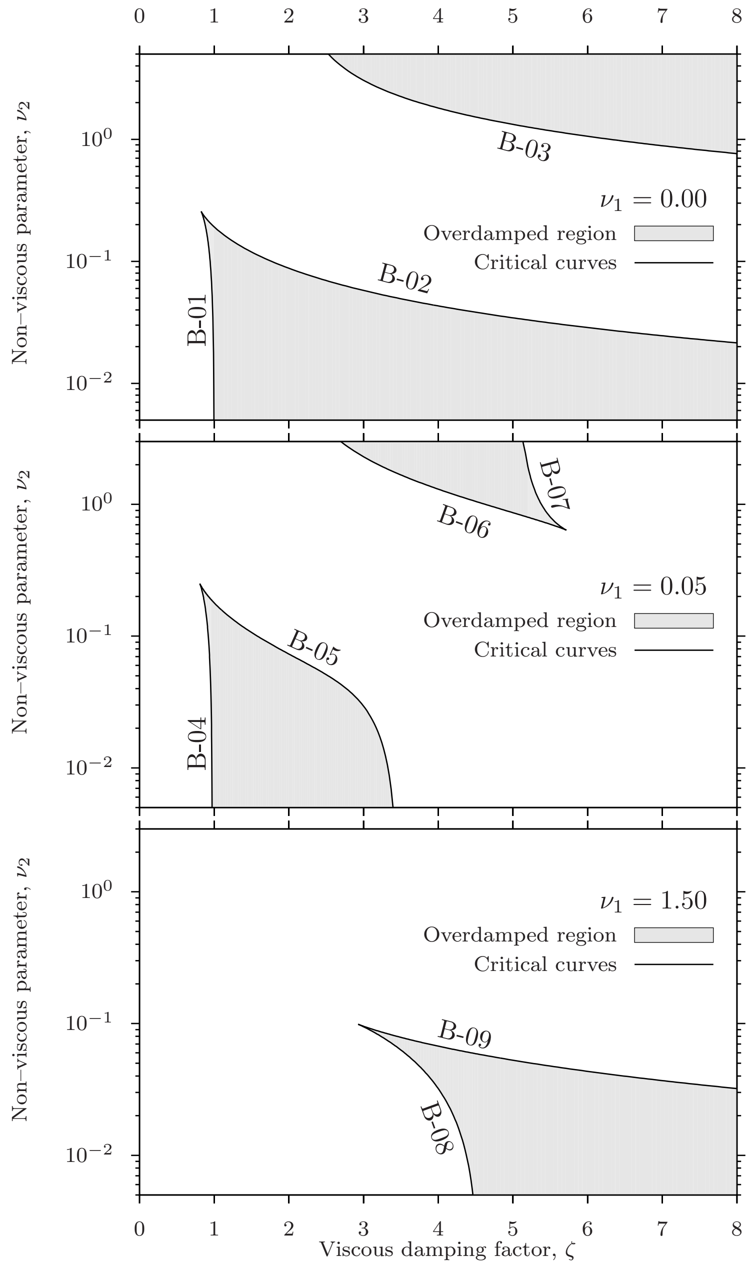

Figure 4: Example 2, Case (b): Single degree–of freedom nonviscous system with 2 hereditary kernels. Overdamped regions and critical curves in the domain of parameters ($\zeta,\ \nu_2$) for $\nu_1 = 0.00$ (top), $\nu_1 = 0.05$ (middle), $\nu = 1.50$ (bottom)

The critical curves determined from integration of Eqs. (50) with help of the initial values (Table 3) have been represented in Fig. 4 highlighting the enclosed overdamped regions by filled areas. As before, the three figures depict respectively three cross sections of the corresponding 3D critical manifolds, associated to the three given values of ν_1, say $\nu_1 = \{0.00, 0.05, 1.50\}$. The codes B-01, B-02,... assigned to each curve are in accordance with those shown in Table 3 with the corresponding initial values. It is interesting to observe that in Fig. 4 (top, for $\nu_1 = 0.00$), an overdamped subregion for values $\nu_2 \ll 1$ very similar to that of Fig. 2 arises. Additionally, a new overdamped subregion can be observed at the right–top corner of that plot (Fig. 4-top), which does not exist in Fig. 2. Presumably, according to this subregion, high values of ζ and ν_2 (simultaneously) lead the system to non-oscillatory motion. Let us see that the damper model in this case is formed by two dampers in parallel, one is viscous with coefficient c_1 and the other one is nonviscous with relaxation parameter μ_2. Indeed, at the limit $\mu_1 \to \infty$ ($\nu_1 = 0$) the damping function is transformed into

$$\lim_{\mu_1 \to \infty} \mathcal{G}(t) = c_1\,\delta(t) + c_2\,\mu_2\,e^{-\mu_2 t} \qquad \lim_{\mu_1 \to \infty} G(s) = c_1 + \frac{\mu_2\,c_2}{s + \mu_2} \tag{52}$$

Somehow, this overdamped subregion shown at right-top corner of Fig. 4($\nu_1 = 0$) can be interpreted as the effect produced by both the nonviscous parameter μ_2 (associated to the second kernel) and the viscous coefficcient c_1 (associated to the first kernel). This is the reason why such critical subregion did not appear in the Fig. 2 of Example 1, which was modeled exclusively with one kernel. Again, from the symmetry respect to the nonviscous parameters, this overdamped subregion also would appear in the plane (ζ, ν_1) for $\nu_2 = 0$. Furthermore, this effect is extended as a narrow volume in the approximate range $0 \leq \nu_1 \leq 0.07$ (respectively for symmetry in $0 \leq \nu_2 \leq 0.07$), see Fig. 3(middle and bottom). It seems clear that the hypothetical addition of new damping parameters would lead to a more difficult interpretation of the overdamped manifolds, specially because, as seen in this example, they do not follow regular geometrical structures. However, the proposed method could be sequentially applied to extract the most interesting curves for our analysis, for instance in artificial dampers design problems. Let us see now, in a final example, how to extract critical damping curves for a multiple dof system.

3.3. Multiple degree of freedom systems

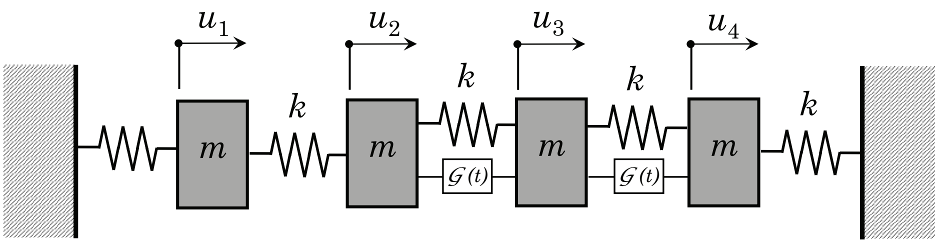

Figure 5: Example 3: The four degrees–of–freedom discrete system. $\mathcal{G}(t)$ represents the hereditary function of nonviscous dampers

In order to validate the proposed approach to find critical damping curves for multiple dof systems, a discrete lumped mass dynamical system with four dof is under consideration. The Fig. 5 represents the distribution of masses m, rigidities k and viscoelastic dampers with a hereditary function $\mathcal{G}(t)$. The mass matrix of the system is $\mathbf{M} = m\mathbf{I}_4$ while, according to the rigidities and dampers distribution, stiffness matrix yields

$$\mathbf{K} = k \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = k\,\boldsymbol{\mathcal{K}} \tag{53}$$

A damping function formed by one hereditary exponential kernel with viscous coefficient c and nonviscous relaxation parameter μ will be assumed. Hence, the damping matrix can be expressed as $\boldsymbol{\mathcal{G}}(t) = \mu\,\mathbf{C}\,e^{-\mu\,t}$ where

$$\mathbf{C} = c \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \equiv c\,\boldsymbol{\mathcal{C}} \tag{54}$$

With help of these dimensionless matrices, $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{C}}$, the nonlinear eigenvalue problem associated to this problem under dimensionless form can be expressed as

$$\left[x^2\mathbf{I}_4 + \frac{2\,x\,\zeta}{1+\nu\,x}\boldsymbol{\mathcal{C}} + \boldsymbol{\mathcal{K}}\right]\mathbf{u} = \mathbf{0} \tag{55}$$

where

$$x = \frac{s}{\omega_0} \ , \quad \nu = \frac{\omega_0}{\mu} \ , \quad \zeta = \frac{c}{2m\omega_0} \ , \quad \omega_0 = \sqrt{k/m} \tag{56}$$

Since $r = \operatorname{rank}(\boldsymbol{\mathcal{C}}) = 2$, the system has $2n + r = 10$ eigenvalues. Therefore, the determinant of the transcendental matrix can be transformed into a 10th order polynomial multiplying by the factor $(1+\nu x)^2$. Introducing then the function $\mathcal{D}(x,\nu,\zeta)$ as

$$\begin{aligned}
\mathcal{D}(x,\nu,\zeta) &= (1+\nu x)^2 \det\left[x^2\mathbf{I}_4 + \frac{2\,x\,\zeta}{1+\nu\,x}\boldsymbol{\mathcal{C}} + \boldsymbol{\mathcal{K}}\right] \\
&= 5 + 2(8\zeta + 5\nu)x + [(2\zeta+\nu)(6\zeta+5\nu)+20]\,x^2 + (54\zeta+40\nu)\,x^3 \\
&\quad + \left(32\zeta^2 + 54\zeta\nu + 20\nu^2 + 21\right)x^4 + (40\zeta+42\nu)\,x^5 + \left(12\zeta^2 + 40\zeta\nu + 21\nu^2 + 8\right)x^6 \\
&\quad + 8(\zeta+2\nu)x^7 + \left(1 + 8\nu\zeta + 8\nu^2\right)x^8 + 2\nu x^9 + \nu x^{10}
\end{aligned} \tag{57}$$

The objective is to determine overdamped regions enclosed by critical curves of the form $\nu = \nu(\zeta)$. For that, the proposed method needs the previous construction of the system given by Eqs. (16) and (17) in terms of $x(\zeta)$ and $\nu(\zeta)$ (the equivalences between the notation of Sec. 2 and that of the current example is $\lambda \equiv x$, $q \equiv \nu$ and $p \equiv \zeta$. Thus, the system in matrix form yields

$$\begin{bmatrix} 0 & \mathcal{D}_{,\nu} \\ \mathcal{D}_{,xx} & \mathcal{D}_{,x\nu} \end{bmatrix}\begin{Bmatrix} x'(\zeta) \\ \nu'(\zeta) \end{Bmatrix} = -\begin{Bmatrix} \mathcal{D}_{,\zeta} \\ \mathcal{D}_{,x\zeta} \end{Bmatrix} \ , \quad \begin{Bmatrix} x(\zeta_0) \\ \nu(\zeta_0) \end{Bmatrix} = \begin{Bmatrix} x_0 \\ \nu_0 \end{Bmatrix} \tag{58}$$

For the sake of clarity in the exposition, the expressions of the partial derivatives will not be written. Initial conditions can be found solving the system of algebraical equations for a particular value of ν_0 and ζ_0

$$\mathcal{D}(x_0,\nu_0,\zeta_0) = 0 \quad , \quad \mathcal{D}_{,x}(x_0,\nu_0,\zeta_0) = 0 \tag{59}$$

Testing for $\nu_0 = 0.06$ we obtain four different pairs (ζ_0, x_0), listed in Table 4. After numerically solving

	INITIAL VALUES		
Curve	ζ_0	ν_0	x_0
C-01	0.53258	0.06000	-2.05512
C-02	0.72949	0.06000	-7.88861
C-03	1.25218	0.06000	-1.45645
C-04	2.14493	0.06000	-8.07994

Table 4: Initial conditions used for the critical damping curves shown in Fig. 6

Eqs. (58), the curves are plotted in Fig. 6. It is observed that the so-obtained four curves enclose two overdamped regions which in turn intersect each other. The solid–filled region represents the set of values (ζ,ν) which lead the fourth mode to overdamping. On the other hand, lines–filled area corresponds to the overdamped region of the second mode. This can be checked following a root–locus plot varying parameters (ζ,ν) from underdamping to overdamping. Moreover, 2nd and 4th mode are precisely those modes in which the degrees of freedom attached to the viscoelastic dampers are most activated. Obviously, it follows then that the overlapping area (intersection between both types of filled-regions in Fig. 6) corresponds to the overdamping of both modes, simultaneously.

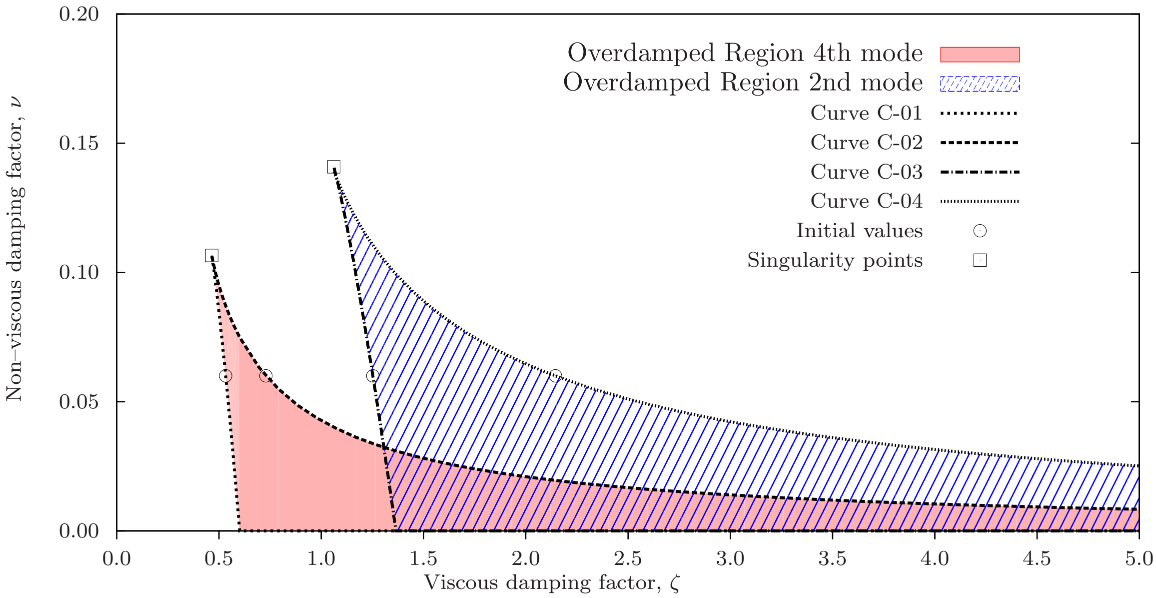

Figure 6: Example 3: Critical damping curves and the corresponding overdamped regions for the four degrees–of–freedom system. Overlapping areas represent two modes overcritically damped

A deeper inspection of Fig. 6 leads to the following question: Are there exist singularities in the domain (ζ, ν) which do not allow the application of the proposed method? To address the answer it is known that a system of differential equations like that one shown in Eq. (58) has solution (and it is unique) provided that the determinant of the matrix does not vanish in a neighborhood around the initial value. That determinant results to be $-\mathcal{D}_{,\nu}\,\mathcal{D}_{,xx}$, therefore possible singularities arise from the solution the systems of algebraic equations S1 or S2, in terms of the unknowns (x, ζ, ν), given by

$$\text{S1:} \begin{cases} \mathcal{D} = 0 \\ \mathcal{D}_{,x} = 0 \\ \mathcal{D}_{,\nu} = 0 \end{cases} \qquad \text{S2:} \begin{cases} \mathcal{D} = 0 \\ \mathcal{D}_{,x} = 0 \\ \mathcal{D}_{,xx} = 0 \end{cases} \tag{60}$$

These two problems can be solved numerically obtaining: (i) on one hand, complex solutions for some of the variables, x, ζ or ν for system S1. (ii) On the other hand, the solution of system S2 leads to suitable solutions for our interest, verifying $x < 0$ and $\zeta, \nu \geq 0$. For the latter case, the corresponding coordinates of the singularity points have been listed in Table 5. These two points, represented in Fig. 6, result to

	SINGULARITY POINTS		
	x	ζ	ν
POINT S1	-3.1059	0.46473	0.10658
POINT S2	-2.3221	1.06113	0.14088

Table 5: Example 3: Singularity points in the overdamped region of the 4-dof system. The points have been plotted in Fig. 6

be precisely the vertexes of the two overdamped regions, namely, intersection points of curves C01–C02 and C03–C04, respectively. They can not be used as initial values since, according to the implicit function theorem, equations $\mathcal{D} = 0$ and $\mathcal{D}_{,x} = 0$ do not define $x(\zeta)$ and $\nu(\zeta)$ unequivocally. Furthermore, the are triple roots, since for them the relations $\mathcal{D} = \mathcal{D}_{,x} = \mathcal{D}_{,xx} = 0$ hold.

We wonder now how does the solution behave in the intersection point between curves C-02 and C-03, located approximately at $\zeta = 1.30465$, $\nu = 0.03239$. This point satisfies $-\mathcal{D}_{,\nu}\, \mathcal{D}_{,xx} \neq 0$, therefore it should be valid as initial value of the proposed method. However, it belongs to both curves simultaneously, so that at a first sight the solution would not seem to be well defined. However, it turns out that two different solutions for the variable x ($x = -1.38042$ and $x = -15.2148$) are found, something that leads to two different initial values. In some way, this point does not correspond with a intersection point of the curves in the 3D domain (x, ν, ζ), while the *triple* roots of Table 5 effectively do.

As far as multiple degree-of-freedom systems concern, the success of the method lies on the availability of the transcendental matrix determinant and their derivatives. Therefore, from a numerical point of view, large systems will require high computational effort which limits the range of applicability for small or moderate order systems. Currently, our efforts are addressed to find out numerical procedures allowing to construct approximate critical curves but for larger systems, something that is under current research.

4. Conclusions

In this paper critical damping of nonviscously damped linear systems is studied. Nonviscous or viscoelastic vibrating structures are characterized by dissipative mechanisms depending on the history of response through hereditary functions. For certain values of the damping parameters, the response can become non-oscillatory. It is said then that some (or all) modes are overdamped. Particular values of the damping parameters which establish the limit between oscillatory and non-oscillatory motion are said to be on a critical surface (or critical manifold). In the present paper two methods to determine critical surfaces are presented:

Analytical method: it is proved that critical surfaces can analytically be obtained eliminating the eigenparameter λ from both the system determinant and its λ-derivative. That is, the two equations

$$\mathcal{D}(\lambda, \boldsymbol{\theta}) = 0 \quad , \quad \frac{\partial \mathcal{D}(\lambda, \boldsymbol{\theta})}{\partial \lambda} = 0 \tag{61}$$

where $\boldsymbol{\theta}$ denotes the array containing the damping parameters. This procedure is only affordable if the determinant can be reduced to a polynomial with order lower or equal than 5, since in such case the second equation is a fourth order polynomial.

Numerical method: In addition, a computational approach based on the transformation of the algebraical equations into a system of two ordinary differential equations is proposed. The system can be constructed from the available closed-form of the determinant $\mathcal{D}(\lambda, p, q)$, which is assumed to depend not only on the eigenparameter λ, but also on two damping parameters, p and q. Critical curves become relations of the form $q = q(p)$ and can be determined as solutions of the following system of differential equations

$$\begin{bmatrix} 0 & \dfrac{\partial \mathcal{D}}{\partial q} \\ \dfrac{\partial^2 \mathcal{D}}{\partial \lambda^2} & \dfrac{\partial^2 \mathcal{D}}{\partial \lambda \partial q} \end{bmatrix} \left\{ \begin{array}{c} \lambda'(p) \\ q'(p) \end{array} \right\} = - \left\{ \begin{array}{c} \dfrac{\partial \mathcal{D}}{\partial p} \\ \dfrac{\partial^2 \mathcal{D}}{\partial \lambda \partial p} \end{array} \right\} \quad , \quad \left\{ \begin{array}{c} \lambda(p_0) \\ q(p_0) \end{array} \right\} = \left\{ \begin{array}{c} \lambda_0 \\ q_0 \end{array} \right\}$$

The initial conditions are found solving the unknowns λ, q in Eqs. (61) for $p = p_0$.

To validate the theoretical results three numerical examples are analyzed. In the first example, the well-known overdamped region of a single degree-of-freedom system with one exponential kernel is resolved, showing perfect fitting between the proposed curves (obtained from the differential equations) and those of the analytical expressions. Moreover, simplified approximate expressions for the critical curves are also proposed. The second example is devoted to construct overdamped regions for single dof system with two exponential hereditary kernels. This problem involves three parameters, so that the solutions are 3D critical manifolds in the 3D-domain of the three parameters. This is undertaken plotting cross sections after leaving as fixed one of the three parameters. The third example shows how the method can be applied for multiple degrees of freedoms systems deriving overdamping regions for different modes and interpreting the obtained overlapping areas. Since this method is based on the evaluation and derivation of the determinant of the transcendental matrix, its range of validity is reduced to small or moderately sized systems. Encouraged by this limitation, the author is currently investigating how to extrapolate this method for larger systems.

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