DISTANCE FORMULAS ON WEIGHTED BANACH SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let v be a radial weight function on the unit disc or on the complex plane. It is shown that for each analytic function f_0 in the Banach space H_v^∞ of all analytic functions f such that $v|f|$ is bounded, the distance of f_0 to the subspace H_v^0 of H_v^∞ of all the functions g such that $v|g|$ vanishes at infinity is attained at a function $g_0 \in H_v^0$. Moreover a simple, direct proof of the formula of the distance of f to H_v^0 due to Perfekt is presented. As a consequence the corresponding results for weighted Bloch spaces are obtained.

1. INTRODUCTION AND NOTATION.

Let us introduce some notation and terminology. We set $R = 1$ (for the case of holomorphic functions on the unit disc) and $R = +\infty$ (for the case of entire functions). A *weight* v is a continuous function $v : [0, R[\to]0, \infty[$, which is non-increasing on $[0, R[$ and satisfies $\lim_{r\to R} r^n v(r) = 0$ for each $n \in \mathbb{N}$. We extend v to $\mathbb{D}$ if $R = 1$ and to $\mathbb{C}$ if $R = +\infty$ by $v(z) := v(|z|)$. For such a weight v, we define the Banach space H_v^∞ of analytic functions f on the disc $\mathbb{D}$ (if $R = 1$) or on the whole complex plane $\mathbb{C}$ (if $R = +\infty$) such that $\|f\|_v := \sup_{|z|<R} v(z)|f(z)| < \infty$. For an analytic function $f \in H(\{z \in \mathbb{C}; |z| < R\})$ and $r < R$, we denote $M(f, r) := \max\{|f(z)| \ ; \ |z| = r\}$. Using the notation O and o of Landau, $f \in H_v^\infty$ if and only if $M(f, r) = O(1/v(r)), r \to R$.

It is known that the closure of the polynomials in H_v^∞ coincides with the Banach space H_v^0 of all those analytic functions on $\{z \in \mathbb{C}; |z| < R\}$ such that $M(f, r) = o(1/v(r)), r \to R$. see e.g. [2].

Spaces of type H_v^∞ appear in the study of growth conditions of analytic functions and have been investigated in various articles since the work of Shields and Williams, see *e.g.* [2],[3], [5], [6], [9] and the references therein.

We recall some examples of weights:

For $R = 1$,
(*i*) $v(r) = (1 - r)^\alpha$ with $\alpha > 0$, which are the standard weights on the disc,
(*ii*) $v(r) = \exp(-(1 - r)^{-1})$, and
(*iii*) $v(z) = (\log \frac{e}{1-r})^{-\alpha}, \ \alpha > 0$

For $R = +\infty$,
(*i*) $v(r) = \exp(-r^p)$ with $p > 0$,
(*ii*) $v(r) = \exp(-\exp r)$, and
(*iii*) $v(r) = \exp\big(-(\log^+ r)^p\big)$, where $p \geq 2$ and $\log^+ r = \max(\log r, 0)$.

Given an analytic function f on $\mathbb{D}$ or $\mathbb{C}$, we denote by $\sigma_n f$ the n'th Cesaro mean of f; i.e. the arithmetic mean of the first n Taylor polynomials of f. In this case, one has $M(\sigma_n f, r) \leq M(f, r)$ for each $0 < r < R$.

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In this note we investigate the distance $d(f, H_v^0) = \inf_{g \in H_v^0} ||f - g||_v$ of a function $f \in H_v^\infty$ to the closed subspace H_v^0. Perfekt in Example 4.4 of [7] proved that $d(f, H_v^0) = \limsup_{r \to R} M(f, r)v(r)$ for each $f \in H_v^\infty$. This result follows from an abstract result [7, Theorem 2.3] with an argument using duality and measures. It implies Theorem 3.9 and Corollary 6.4 in Tjani [10] about the distance of a Bloch function to the little Bloch space. The result of Tjani only gives an estimate, not equality. There are some other recent papers dealing with distance formulas. See [11] and the references therein.

Our main result is Theorem 2.2. It shows that H_v^0 is a proximinal subspace of H_v^∞; that is, it proves that for each $f \in H_v^\infty$ the distance $d(f, H_v^0)$ is attained at a point $g \in H_v^0$. Moreover, it gives an elementary, direct, but not trivial, proof of the formula of the distance due to Perfekt [7]. The corresponding result for the case of Bloch type functions is obtained as a consequence in Corollary 2.5.

2. Results.

Given $f \in H_v^\infty$ we clearly have

$$\limsup_{|z| \to R} v(z)|f(z)| = \limsup_{r \to R} M(f, r)v(r) = \lim_{r \to R} \sup_{s \geq r} v(s)M(f, s).$$

Remark 2.1. It is easy to see that, for each $f \in H_v^\infty$,

$$\limsup_{r \to R} M(f, r)v(r) = \inf_{g \in H_v^0} \limsup_{r \to R} M(f - g, r)v(r)$$

Indeed, this follows from the fact that

$$\limsup_{r \to R} M(g, r)v(r) = 0 \quad \text{for every} \quad g \in H_v^0.$$

Theorem 2.2. *For every $f \in H_v^\infty$ there is $g \in H_v^0$ with*

$$d(f, H_v^0) = ||f - g||_v = \limsup_{r \to R} M(f, r)v(r).$$

To prove the theorem we begin with the following

Lemma 2.3. *Let $f \in H_v^\infty$ and assume that there is $\tau < 1$ with*

$$\tau ||f||_v \leq \limsup_{r \to R} M(f, r)v(r).$$

Then, for each $\varepsilon > 0$ and $m \in \mathbb{N}$ there is $n \in \mathbb{N}, n > m$, such that with $\rho = (1 - \tau)/(1 + \tau)$ we have

$$\left(\frac{1 + \tau}{2(1 + \varepsilon)} \right) ||f - \rho\sigma_n f||_v \leq \limsup_{r \to R} M(f, r)v(r) = \limsup_{r \to R} M(f - \rho\sigma_n f, r)v(r).$$

Proof. The last equality follows from the facts that $\sigma_n f \in H_v^0$ and that for each element $g \in H_v^0$ we have $\limsup_{r \to R} M(g, r)v(r) = 0$.

Fix $\varepsilon > 0$ and $m \in \mathbb{N}$. By the definition of lim sup there is $r_0 < R$ such that

$$\sup_{r_0 \leq r < R} M(f, r)v(r) \leq (1+\varepsilon) \inf_{0 < s < R} \sup_{s \leq r < R} M(f, r)v(r) \tag{1}$$

$$= (1 + \varepsilon) \limsup_{r \to R} M(f, r)v(r).$$

Since f is continuous on $r_0\overline{\mathbb{D}}$, the n'th Cesaro means of f satisfy $\sigma_n f \to f$ as $n \to \infty$ uniformly on $r_0\overline{\mathbb{D}}$. Put

$$\rho := \frac{1-\tau}{1+\tau}$$

and fix $0 < \delta$ such that

$$\left(\delta + \frac{2\tau}{1+\tau}\right) \leq (1+\varepsilon)\frac{2\tau}{1+\tau}. \tag{2}$$

For $0 \leq r \leq r_0$ we obtain $M(f - \sigma_n f, r)v(r) < \delta ||f||_v$ if $n > m$ is large enough. Hence

$$\begin{aligned} M(f - \rho\sigma_n f, r)v(r) &\leq (1-\rho)M(f,r)v(r) + \rho M(f - \sigma_n f, r)v(r) \\ &\leq ((1-\rho)+\delta)||f||_v. \end{aligned} \tag{3}$$

If $r_0 \leq s < R$ then we have, in view of (1),

$$M(f - \rho\sigma_n f, s)v(s) \leq (1+\rho)M(f,s)v(s) \leq (1+\varepsilon)(1+\rho)\limsup_{r\to R} M(f,r)v(r) \tag{4}$$

From the definition of ρ we get

$$(1+\varepsilon)(1+\rho) = \frac{2(1+\varepsilon)}{1+\tau}$$

and

$$1 - \rho = \frac{2\tau}{1+\tau}.$$

Hence (1), (2), (3), (4) and the assumption of the lemma yield

$$\begin{aligned} ||f - \rho\sigma_n f||_v &= \sup_{0\leq r<R} M(f - \rho\sigma_n f, r)v(r) \\ &\leq \max\left((\delta + (1-\rho))||f||_v, (1+\varepsilon)(1+\rho)\limsup_{r\to R} M(f,r)v(r)\right) \\ &\leq \max\left(\left(\delta + \frac{2\tau}{1+\tau}\right)||f||_v, \left(\frac{2(1+\varepsilon)}{1+\tau}\right)\limsup_{r\to R} M(f,r)v(r)\right) \\ &\leq \left(\frac{2(1+\varepsilon)}{1+\tau}\right)\limsup_{r\to R} M(f,r)v(r). \end{aligned}$$

The proof is complete. □

Proof. **(of Theorem 2.2)** Let $f \in H_v^\infty$. If $\limsup_{r\to R} M(f,r)v(r) = 0$ then $f \in H_v^0$ and $d(f, H_v^0) = 0$.

Now assume that $\limsup_{r\to R} M(f,r)v(r) > 0$ and find $\tau_0 < 1$ with

$$||f||_v \leq \frac{1}{\tau_0}\limsup_{r\to R} M(f,r)v(r).$$

Put $\rho_1 = (1-\tau_0)/(1+\tau_0)$ and $f_0 = f$.

We proceed by induction and suppose that we have already selected $\tau_0 < \tau_{m-1} < \tau_m < 1$, $\rho_m > 0$ and $f_m := f - \sum_{k=1}^m \rho_k \sigma_{n_k} f_{k-1}$ for some $n_m > n_{m-1}$ with $||f_m||_v < (1/\tau_m)\limsup_{r\to R} M(f_m, r)v(r)$.

A simple calculation shows

$$\frac{1-\tau_m}{3+\tau_m} < \left(\frac{2}{3}\right)\frac{1-\tau_m}{1+\tau_m}.$$

Find $\varepsilon_m > 0$ such that

$$\varepsilon_m < \frac{1}{m}, \qquad \frac{1+\tau_m}{2(1+\varepsilon_m)} > \tau_m \tag{5}$$

and

$$\frac{1 - \frac{1+\tau_m}{2(1+\varepsilon_m)}}{1 + \frac{1+\tau_m}{2(1+\varepsilon_m)}} = \frac{1+2\varepsilon_m - \tau_m}{3+2\varepsilon_m+\tau_m} < \left(\frac{2}{3}\right)\frac{1-\tau_m}{1+\tau_m}. \tag{6}$$

Put

$$\tau_{m+1} := \frac{1+\tau_m}{2(1+\varepsilon_m)} \quad \text{and} \quad \rho_{m+1} := \frac{1-\tau_m}{1+\tau_m} = \frac{1+2\varepsilon_m-\tau_{m-1}}{3+2\varepsilon_m+\tau_{m-1}}. \tag{7}$$

Observe that $\tau_m < \tau_{m+1} < 1$. Then Lemma 2.3 yields $n_{m+1} > n_m$ such that, with

$$f_{m+1} := f_m - \rho_{m+1}\sigma_{n_{m+1}} f_m = f - \sum_{k=1}^{m+1} \rho_k \sigma_{n_k} f_{k-1}, \tag{8}$$

we have

$$\begin{aligned} ||f_{m+1}||_v &\le \frac{1}{\tau_{m+1}} \limsup_{r\to R} M(f_{m+1}, r)v(r) \\ &= \frac{1}{\tau_{m+1}} \limsup_{r\to R} M(f, r)v(r). \end{aligned} \tag{9}$$

(5) and (7) yield $\lim_{m\to\infty} \tau_m = 1$ since (τ_m) is an increasing bounded sequence. On account of (6) we obtain

$$\rho_{m+1} \le \left(\frac{2}{3}\right)\rho_m \quad \text{for all } m,$$

hence,

$$\rho_m \le \left(\frac{2}{3}\right)^m \rho_0.$$

This implies that $\sum_{k=1}^\infty \rho_k\sigma_{n_k} f_{k-1}$ converges to an element $g \in H_v^0$, since $||\sigma_{n_k} f_{k-1}||_v \le ||f_{k-1}||_v \le \tau_{k-1}^{-1}||f||_v \le \tau_0^{-1}||f||_v$ for all k, as it follows from (9). Therefore, we can apply (8) and (9) to get

$$||f-g||_v \le ||f_{m+1}||_v + ||\sum_{k=m+2}^\infty \rho_k\sigma_{n_k} f_{k-1}||_v \le$$

$$\le \frac{1}{\tau_{m+1}} \limsup_{r\to R} M(f,r)v(r) + \rho_0\tau_0^{-1} \sum_{k=m+2}^\infty \left(\frac{2}{3}\right)^k.$$

Thus

$$||f-g||_v \le \limsup_{r\to R} M(f,r)v(r) = \inf_{h\in H_v^0} \limsup_{r\to R} M(f-h,r)v(r) \le d(f, H_v^0).$$

We conclude $d(f,H_v^0) = ||f-g||_v = \limsup_{r\to R} M(f,r)v(r)$. □

One of the referees pointed out that our construction reminded her/him of a construction in [1], where the authors prove a proximinality result for bounded operators.

Remark 2.4. The following simple examples show that the distance $d(f, H_v^0)$ can be attained at many points of H_v^0 for a given function $f \in H_v^\infty$.

(1) Consider the weight $v(r) = e^{-r}, r \in [0, \infty[$, on the complex plane and the analytic function $f(z) = e^z, z \in \mathbb{C}$. Clearly $f \in H_v^\infty$ and $||f||_v = 1$. Set $P_n(z) = \sum_{k=0}^n \frac{z^k}{k!}$ for each $n \in \mathbb{N}$. We have, for each n, $P_n \in H_v^0$ and

$$||f - P_n||_v = \sup_{r>0} e^{-r} \sum_{k=n+1}^{\infty} \frac{r^k}{k!} = 1 = d(f, H_v^0).$$

(2) Now define the weight $v(r) = 1 - r, r \in [0, 1[$, on the unit disc. The function $f(z) = \frac{1}{1-z} = \sum_{k=0}^\infty z^k$ belongs to H_v^∞ and $||f||_v = 1$. Set $P_n(z) = \sum_{k=0}^n z^k$ for each $n \in \mathbb{N}$. We have, for each n, $P_n \in H_v^0$ and

$$M(f - P_n, r) = \sum_{k=n+1}^{\infty} r^k = \frac{r^{n+1}}{1-r}.$$

Therefore

$$||f - P_n||_v = \sup_{r \in [0,1[} (1 - r) M(f - P_n, r) = 1 = d(f, H_v^0).$$

(3) The proximinality in Theorem 2.2, i.e. the existence of the minimizer g, also appears in Perfekt [8] as an abstract consequence of the fact that H_v^0 is an M-ideal of H_v^∞. Moreover, by further abstract M-ideal theory, the minimizer for a given $f \in H_v^\infty \setminus H_v^0$ is never unique; see [4]. This was pointed out to us by one of the referees, who also emphasized that we give a very explicit construction, which these references do not.

Let v be a weight on the unit disc $\mathbb{D}$; i.e. $R = 1$. The *weighted Bloch space* is defined by

$$\mathcal{B}_v = \{f \in H(\mathbb{D}) : f(0) = 0, \ \|f\|_{\mathcal{B}_v} = \sup_{z \in \mathbb{D}} v(z)|f'(z)| < \infty\},$$

and the *little Bloch space*

$$\mathcal{B}_{v,0} = \{f \in \mathcal{B} : \lim_{|z| \to 1} v(z)|f'(z)| = 0\}.$$

They are Banach spaces endowed with the norm $|| \cdot ||_{\mathcal{B}_v}$.

The classical Bloch space $\mathcal{B}$ and little Bloch space $\mathcal{B}_0$ correspond to the weight $v(z) := 1 - |z|^2$. Among the many references on these spaces, we mention Zhu [12], for example.

Define the bounded operators $S : \mathcal{B}_v \to H_v^\infty$, $S(h) = h'$ and $S^{-1} : H_v^\infty \to \mathcal{B}_v$, $(S^{-1}h)(z) = \int_0^z h(\xi) d\xi$. Then $SS^{-1} = id_{H_v^\infty}$, $S^{-1}S = id_{\mathcal{B}_v}$ and S, S^{-1} are isometric onto maps. These operators induce isometries between H_v^0 and $\mathcal{B}_{v,0}$.

The following result is a direct consequence of Theorem 2.2. It should be compared with Example 4.1 in [7]. It improves [10, Corollary 6.4].

Corollary 2.5. *For each $f \in \mathcal{B}_v$ there is $g \in \mathcal{B}_{v,0}$ such that*

$$d(f, \mathcal{B}_{v,0}) = ||f - g||_{\mathcal{B}_v} = \limsup_{r \to 1} M(f', r) v(r).$$

Finally we mention the weighted spaces of harmonic functions for a given weight v on $\{z \in \mathbb{C}; |z| < R\}$. Let h_v^∞ consist of all harmonic functions on $\{z \in \mathbb{C}; |z| < R\}$ with $||f||_v = \sup_{|z|<R} |f(z)|v(z) < \infty$ and let h_v^0 be the closure of all trigonometric polynomials in h_v^∞. Using the arguments of the proof of Theorem 2.2. word by word yields

Theorem 2.6. *For every $f \in h_v^\infty$ there is $g \in h_v^0$ with*

$$d(f, H_v^0) = ||f - g||_v = \limsup_{r \to R} M(f, r)v(r).$$

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