Contents

Prelude

In this paper, we study three different proofs of the Malgrange-Ehrenpreis Theorem. First, we give some historical and biographical details related to partial differential operators theory about the existence of fundamental solution of linear partial differential operators with constant coefficients. Later, we introduce several notations related to partial differential operators with constant coefficients and some properties of Banach spaces and distribution theory. Then, we show Rosay's proof using L^2 methods and the Mittag-Leffer procedure. Next, we present Rudin's proof, using Fourier transforms, complex analysis and the Hanh-Banach Theorem. Finally, we give a more recent proof by Wagner, using Fourier transforms and some properties of polynomials. We conclude the paper with some consequences on hypoelliptic operators.

Historic perspective

We begin this section with an anecdote that ends Tréves' article [**TPY03**]. In 1948 Schwartz visited Sweden to present his theory of distributions and had the opportunity to talk with Marcel Riesz. After Schwartz wrote the formula of integration by parts to explain the idea of the weak derivative, Riesz interrupted by saying "I hope you have found something else in your life. " Later, Schwartz told Riesz that he hoped to show that each linear partial differential equation with constant coefficients has a fundamental solution, a concept that could only be made precise with the theory of distributions. "Crazy!" Riesz said,"that's a project for the twenty-first century". However, the theorem was proved by Ehrenpreis and Malgrange in 1952. Schwartz knew that the positive solution meant that the equation $P(D)\,u = f$ had a solution $u \in \mathcal{D}'$ for each $f \in \mathcal{D}$ [**Sch50**].

Partial differential equations play a central role in pure and applied mathematics, and its study has provided results of great theoretical and practical interest. These equations express directly, for example, the Newton's fundamental laws of motion, which allowed the first quantitative description of planetary motion. They also led to the establishment of basic laws of many phenomena such as fluid motion, electric fields, heat transfer or mass, atmospheric motions, and many physical phenomena, chemical or technological. In fact, the consideration of partial differential equations was historically motivated by problems of physics and geometry. They appeared in hydrodynamic problems (D'Alembert, 1752), the vibrating membrane (Euler, 1766), and potential theory (Laplace, 1789). S. Kowalevsky, a student of Weierstrass, researched the existence of a solution to a system of partial differential equations with analytic coefficients around 1874. In the nineteenth century, the problems of elasticity and heat conduction, along with research pioneers like Fourier and

Heaviside, led to the introduction of new concepts, which later played a central role.

Malgrange's and Hörmander's theses, both from 1955, are the first comprehensive treatises on this topic. Malgrange studied distributions systematically and combined them with convolution operators in his thesis. On the other hand, Hörmander primarily used square-integrable functions, but distributions also appeared in his work.

Malgrange was born in Paris in 1928 and was a student at l'Ecole Normale Supérieure from 1947 to 1951. Along with Blanchard he spent a semester in 1948 at the Science Faculty of Nancy, where Delsarte, Dieudonné, Godement, Gauthier and Schwartz taught. During 1951 and 1952, after completing his studies, Jacques-Louis Lions and Malgrange spent a year in Nancy, where they met with Grothendieck and Malliavin. Schwartz wrote that, in his opinion, Nancy was one of the world's most important centers in mathematical analysis at the time. Under the direction of Schwartz, Malgrange completed his thesis in 1955. He then taught at Paris, Orsay and Grenoble. In 1977 he was elected a corresponding member of the French Academy of Sciences in Paris and was elected as full member in 1988.

On the other hand, Hörmander was born in Mjällby, Sweden in 1931. He studied analysis at the University of Lund under the direction of Marcel Riesz, who taught him function theory and harmonic analysis. He graduated in 1950 and began doing research with Riesz. Once Riesz retired, he began to work on partial differential equations. He completed his doctorate in 1955, visited several universities in the United States, and returned to accept a position as a professor at the University of Stockholm in 1957. In 1962, the International Congress of Mathematicians was in Stockholm; Hörmander was awarded the Fields Medal for his contributions to the theory of partial differential equations, and in particular his results on hypoeliptic partial differential operators. Between 1964 and 1968 he was at Princeton, but returned to the chair in mathematics from the University of Lund in 1968, where has been an emeritus professor since 1996. Between 1983 and 1985 he published his monumental work [**Hör09**] on the analysis of linear partial differential operators, which also includes the study of pseudo-differential operators.

One of the first great successes of the theory of distributions in connection with partial differential equations was the attractive and clear definition of the fundamental solution of a linear partial differential operator with constant coefficients $P(D)$.

As previously mentioned, the thesis of Malgrange [**Mal56**] and Hörmander [**Hör55**], both presented in 1955, were the first comprehensive treatment of the general theory of linear partial differential equations. The classical theory of partial differential equations selects a type of equation and studies the properties of its solutions. The general theory analyzes the relationship between the properties of a polynomial $P(z)$ and the properties of linear partial differential operator with constant coefficients $P(D)$ associated with it.

The first general theorem on the existence of fundamental solutions for linear partial operators with constant coefficients was obtained in 1953/54 by Malgrange and Ehrenpreis. Their proofs were based on the Hahn-Banach theorem. The method of Malgrange and Ehrenpreis is based on Malgrange's inequality involving the L^2 norm.

Although the idea of a fundamental solution appears only indirectly in the classical literature, the first use of fundamental solutions can be ascribed to d'Alembert in 1747 when he developed the solution of the problem of a vibrating string $\partial_t^2 u - c^2 \partial_x^2 u = f$. In 1789, Laplace used the fundamental solution $E = -\frac{1}{4\pi|x|}$ of the elliptic operator in three variables $\triangle_3 := \partial_x^2 + \partial_y^2 + \partial_z^2$ that has his name, and made the connection with Newton's gravitational potential. His work was completed by Poisson in 1813, showing $\triangle_3 (E * f) = f$. In 1809 Laplace considered the heat operator $\partial_t - \partial_x^2$ and calculated its fundamental solution. Poisson generalized the solution for arbitrary spatial dimension in 1818. In that year, Fourier calculated the fundamental solution of fourth order equation $\partial_t^2 - \partial_x^4$. The same year, Poisson generalized D'Alembert's formula for wave operators in three-spatial dimensions. The fundamental solution of the wave operator in two-spatial variables was not found until 1894 by Volterra. In 1849, Stokes obtained the fundamental matrix of the system of partial differential operators describing the elastic waves in an isotropic medium. Fredholm, in 1908, represented the fundamental solutions of elliptic operators in three variables using Abelian integrals, and proved his theory with the operator $\partial_x^4 + \partial_y^4 + \partial_z^4$. In 1911, his student Zeilon gave the first definition of a fundamental solution of a locally integrable function, extended the theory of Fredholm elliptic operators, and considered in particular the operator $\partial_x^3 + \partial_y^3 + \partial_z^3$, obtaining the singular support of the fundamental solution. An explicit representation of it was only obtained by P. Wagner in his paper [**Wag99**] published in 1999.

For more details and further developments we refer the reader to [**BS08**] and [**TPY03**].

Preliminaries

We denote the field of complex numbers as $\mathbb{C}$. We recall the multi-index notation. We denote by $z = (z_1, \dots, z_N) \in \mathbb{C}^N$ a complex vector, by $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{N}_0^N$ a multi-index and its order is defined by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$. We also define $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_N!$. Finally, given $z \in \mathbb{C}^N$, $p \in \mathbb{N}_0^N$ then $z^p := z_1^{p_1} \cdots z_N^{p_N}$.

The set of the N-variables polynomials $z_1, \dots, z_N$ over the field $\mathbb{C}$ is denoted by $\mathbb{C}[z_1, \dots, z_N]$. A polynomial of degree m is denoted by $P(z) := \sum_{|\alpha| \le m} a_\alpha z^\alpha$, $a_\alpha \in \mathbb{C}$; note that $P(z)$ has degree m if there exists α such that $|\alpha| = m$ with $a_\alpha \neq 0$, in this case $dg(P) = m$. If $P(z)$ is a polynomial of degree m, we say that the principal part is $P_m(z) := \sum_{|\alpha|=m} a_\alpha z^\alpha$.

We use this following notation for the partial derivatives $\partial_j := \frac{\partial}{\partial x_j}$ and $D_j := \frac{1}{i}\frac{\partial}{\partial x_j} = -i\frac{\partial}{\partial x_j}$ with $1 \le j \le N$. In multi-index, we set $\frac{\partial^{|\alpha|}}{\partial x^\alpha} := \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_N}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$ and $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$. For more details in the notation we refer to [Hör76].

We use the following notation for compactness. Let Ω an open set of $\mathbb{R}^N$, then $K \subset\subset \Omega$ is a compact set included in Ω. We denote the unit ball on $\mathbb{R}^N$ as $B_{\mathbb{R}^N}(0,1)$. We define on $\mathbb{R}^N$ the sphere S^{N-1} where $x \in S^{N-1}$ if $|x| = 1$.

Definition 0.1. Given $m \in \mathbb{N}, \Omega \subset \mathbb{R}^N$ we define the following spaces:

$$C^m(\Omega) := \{f : \Omega \to \mathbb{K} \text{ such that } \exists D^\alpha f \text{ continuous on } \Omega \text{ for all } |\alpha| \le m\}.$$

$$C^\infty(\Omega) = \mathcal{E}(\Omega) := \bigcap_{m \in \mathbb{N}} C^m(\Omega).$$

Note that $\mathcal{E}(\Omega)$ is the space of smooth functions.

Definition 0.2. Given a vector space E over a subfield of the complex numbers a *seminorm* on E is a function $p : E \to \mathbb{R}$ with the following properties for all $a \in \mathbb{R}$ and all $u, v \in E$,

(1) $p(av) = |a| \, p(v)$.
(2) $p(u + v) \leq p(u) + p(v)$.

A *system of seminorms* $\mathcal{P}$ on a vector space E is a set of seminorms such that

(1) $\forall \, p, q \in \mathcal{P}, \exists \, r \in \mathcal{P} : p \leq r, q \leq r$.
(2) $\forall \, 0 \neq x \in E, \exists \, p \in \mathcal{P} : p(x) > 0$.

Definition 0.3. Given a vector space E, we define $\mathcal{T}_{\mathcal{P}}$ by the locally convex topology associated to the system of seminorms $\mathcal{P}$ as follows: we claim that $G \in \mathcal{T}_{\mathcal{P}}$ if for every $x \in G$, exists $\varepsilon > 0$, $p \in \mathcal{P}$ with

$$B_p(x, \varepsilon) = \{y \in E : p(y - x) < \varepsilon\} \subset G \subset E.$$

Definition 0.4. We say that $(p_n)_n \subset \mathcal{P}$ is a fundamental sequence of seminorms for $\mathcal{T}_{\mathcal{P}}$ if:

(1) $p_1 \leq p_2 \leq \ldots$ (i.e., $p_i(x) \leq p_{i+1}(x)$ for every $x \in E$ and $i \in \mathbb{N}$).
(2) For each $p \in \mathcal{P}$ there exists $m \in \mathbb{N}$ and $\alpha \geq 0$ such that $p < \alpha p_m$.

Definition 0.5. Let Ω be an open set of $\mathbb{R}^N$

- If $\Omega = \mathbb{R}^N$,

$$\Omega_m = \left\{x \in \mathbb{R}^N : |x| < m\right\}.$$
$$K_m = \left\{x \in \mathbb{R}^N : |x| \leq m\right\}.$$

- If $\Omega \neq \mathbb{R}^N$,

$$\Omega_m = \left\{x \in \Omega : |x| < m, \, d\left(x, \complement\Omega\right) > \frac{1}{m}\right\}.$$
$$K_m = \left\{x \in \Omega : |x| \leq m, \, d\left(x, \complement\Omega\right) \geq \frac{1}{m}\right\}.$$

Therefore $(K_m)_m$ is a fundamental sequence of compact sets on Ω. It satisfies the following properties

(1) $\Omega_m \subset K_m \subset \Omega_{m+1} \subset K_{m+1} \subset \ldots \subset \Omega$.
(2) K_m is a compact set for any $m \in \mathbb{N}$.
(3) For each compact subset K of Ω there is $m \in \mathbb{N}$ such that $K \subset K_m$.

Remark 0.1. We give the space $\mathcal{E}(\Omega)$ the locally convex topology based on the following fundamental system of continuous seminorms

$$p_n(f) = \sum_{|\alpha|\leq n} \sup_{x\in K_n} |D^\alpha f(x)|, \qquad f \in \mathcal{E}(\Omega).$$

Definition 0.6. We define the support of a function as

$$supp(f) := \overline{\{x \in \mathbb{R}^N : f(x) \neq 0\}}^{\mathbb{R}^N}.$$

If $K \subset\subset \Omega$ is a compact subset, we denote by $\mathcal{D}^m(K) := \{f \in C^m(\Omega) : supp(f) \subset K\}$ with $m \in \mathbb{N}_0 \cup \{\infty\}$. Note that for $m = \infty$ we write $\mathcal{D}(K)$, where $\mathcal{D}(K)$ is a closed subspace of $\mathcal{E}(\Omega)$. If $\Omega \subset \mathbb{R}^N$ is an open subset, we set $\mathcal{D}^m(\Omega) := \bigcup_{K\subset\subset\Omega} \mathcal{D}^m(K)$ and $\mathcal{D}(\Omega) := \bigcup_{K\subset\subset\Omega} \mathcal{D}(K)$.

Definition 0.7. Given $P \in \mathbb{C}[z_1, \ldots, z_N]$, $P(z) = \sum_{|\alpha|\leq m} a_\alpha z^\alpha$, we define the following linear partial differential operator (P.D.O.) with constant coefficients associated with P by

$$P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha.$$

Example 0.1. Here are some examples of P.D.O.

(1) **The Cauchy-Riemman operator.**
Let $P(x, y) = \frac{i}{2}(x + iy)$, then the P.D.O. with constant coefficients associated with P is:

$$P(D) = \frac{i}{2}\left(\frac{1}{i}\frac{\partial}{\partial x} + i\frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial z}.$$

Given Ω be an open subset of $\mathbb{C}$, recall that for $u \in C^1(\Omega)$ and $z = x + iy$, we define:

$$\begin{cases} \dfrac{\partial u}{\partial z} & := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + \dfrac{1}{i}\dfrac{\partial}{\partial y}\right) \\ \dfrac{\partial u}{\partial \bar{z}} & := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - \dfrac{1}{i}\dfrac{\partial}{\partial y}\right) \end{cases}$$

Therefore, u is an analytic function if and only if $\frac{\partial u}{\partial \bar{z}} = 0$, by using Cauchy-Riemman equations.

(2) **The Laplacian.**
Let $N \geq 2$ and $P(x_1, \ldots, x_N) = -\sum_{j=1}^{N} x_j^2$.

The corresponding partial differential operator is

$$P(D) = -\sum_{j=1}^{N} i^2 \frac{\partial^2}{\partial x_j^2} = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} = \triangle$$

(3) **The Wave Operator.**
Let $N \geq 1$ and $P(x_1, \ldots, x_N, t) = \sum\limits_{j=1}^{N} x_j^2 - t^2$.
The corresponding partial differential operator is

$$P(D) = \frac{\partial^2}{\partial t^2} - \triangle_x.$$

(4) **The Heat Operator.**
Let $N \geq 1$ and $P(x_1, \ldots, x_N, t) = \sum\limits_{j=1}^{N} x_j^2 + it$.
The corresponding partial differential operator is

$$P(D) = \frac{\partial}{\partial t} - \triangle_x.$$

(5) **The Schrödinger Operator.**
Let $N \geq 1$ and $P(x_1, \ldots, x_N, t) = \sum\limits_{j=1}^{N} x_j^2 + t$.
The corresponding partial differential operator is

$$P(D) = \frac{1}{i}\frac{\partial}{\partial t} - \triangle_x.$$

Remark 0.2. There is a relationship between the polynomials and the partial differential operators. Let $x \in \mathbb{R}^N$ and $z \in \mathbb{C}^N$ with $\langle x, z\rangle = x_1 z_1 + \ldots + x_N z_N$. Then

$$P(D)e^{i\langle x,z\rangle} = P(D)e^{ix_1z_1+\ldots+ix_Nz_N} = \sum_{|\alpha|\leq m} a_\alpha \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} e^{i\langle x,z\rangle}$$

$$= \sum_{|\alpha|\leq m} a_\alpha \left(\frac{1}{i}\right)^{|\alpha|} i^{|\alpha|} z^\alpha e^{i\langle x,z\rangle} = P(z)e^{i\langle x,z\rangle}$$

Note that $e^{i\langle x,z\rangle}$ is an eigenvector of $P(D)$ with associated eigenvalue $P(z)$.

Definition 0.8. Given Ω an open subset of $\mathbb{R}^N$, we denote by $\mathcal{D}'(\Omega)$ the topological dual space of $\mathcal{D}(\Omega)$, when $\mathcal{D}(\Omega)$ is endowed with the inductive limit topology $\mathcal{D}(\Omega) = ind_{K\subset\subset\Omega}\mathcal{D}(K)$.
Hence, $u \in \mathcal{D}'(\Omega)$ if and only if

- $u : D(\Omega) \longrightarrow \mathbb{K}$ is linear and

- For each $K \subset\subset \Omega$ there exists $C > 0, m \in \mathbb{N} : |u(\varphi)| \leq C \sum\limits_{|\alpha| \leq m} \sup\limits_{x \in K} |D^\alpha \varphi(x)|$, for each $\varphi \in \mathcal{D}(K)$

The elements of $\mathcal{D}'(\Omega)$ are called *distributions on* Ω.

Proposition 0.9. $P(D) : \mathcal{E}(\Omega) \longrightarrow \mathcal{E}(\Omega)$ *is a linear and continuous operator.*

Proof: Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, $m \in \mathbb{N}$, be a P.D.O. It is easy to see that $P(D)$ is linear. Now, for any $n \in \mathbb{N}$, there exists $C := \sum_{|\alpha| \leq m} |a_\alpha|$ and $k = m + n$ such that:

$$\begin{aligned} p_n(P(D)f) &= \sum_{|\beta| \leq n} \sup_{x \in K_n} \left| D^\beta P(D) f(x) \right| \\ &= \sum_{|\beta| \leq n} \sup_{x \in K_n} \left| D^\beta \left(\sum_{|\alpha| \leq m} a_\alpha D^\alpha f(x) \right) \right| \\ &\leq C \sum_{|\gamma| \leq k} \sup_{x \in K_k} \left| D^\gamma f(x) \right|. \end{aligned}$$

□

Definition 0.10. We define the following space:

$L^p_{loc}(\mathbb{R}^N) := \{f : \mathbb{R}^N \to \mathbb{R}$ measurable with $f\chi_K \in L^p(\mathbb{R}^N)$, for each $K \subset\subset \mathbb{R}^N\}$

where

$$\chi_K = \begin{cases} \chi_K(x) = 1 & \text{if } x \in K \\ \chi_K(x) = 0 & \text{if } x \notin K. \end{cases}$$

Definition 0.11. Each $f \in L^1_{loc}(\Omega)$ defines a distribution

$$\begin{aligned} T_f : \ \mathcal{D}(\Omega) &\longrightarrow \ \mathbb{C} \\ \varphi &\longrightarrow \ T_f(\varphi) := \textstyle\int_\Omega f(x)\varphi(x)\,dx = \int_\Omega f\varphi \end{aligned}$$

Note that T_f is well defined. Let $\varphi \in \mathcal{D}(\Omega)$ and $K := supp(\varphi) \subset\subset \Omega$. Since $\int_\Omega f\varphi = \int_K f\varphi$, and $f \in L^1(K)$, $\varphi \in C(K), |\varphi| \leq M$, therefore there exists $\int_K f\varphi < +\infty$.

Clearly T_f is linear. Moreover, T_f is continuous since for each $\varphi \in \mathcal{D}(K)$

$$|T_f(\varphi)| = \left| \int_\Omega f\varphi \right| = \left| \int_K f\varphi \right| \leq \int_K |f|\,|\varphi| \leq \left(\int_K |f| \right) \sup_{x \in K} |\varphi(x)|.$$

Therefore, $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$.

Definition 0.12. Given $u \in \mathcal{D}'(\Omega)$ its partial derivatives are defined by

$$\langle \frac{\partial^{|\alpha|} u}{\partial x^\alpha}, \varphi \rangle := (-1)^{|\alpha|} \langle u, \frac{\partial^{|\alpha|}}{\partial x^\alpha} \varphi \rangle \quad \text{, for each } \varphi \in \mathcal{D}(\Omega).$$

In that case,

$$\langle D^\alpha u, \varphi \rangle := (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle \text{ , for all } \varphi \in \mathcal{D}(\Omega) \text{ and } \alpha \in \mathbb{N}_0^N.$$

Example 0.2. An important distribution is given by Dirac's delta function. This function is defined by

$$\begin{array}{rll} \delta: & D(\mathbb{R}^N) & \longrightarrow \mathbb{K} \\ & \varphi & \longrightarrow \langle \delta, \varphi \rangle := \varphi(0) \text{ , for all } \varphi \in \mathcal{D}(\mathbb{R}^N). \end{array}$$

Clearly δ is a distribution. We define the Heaviside function on $\mathbb{R}$ as

$$\begin{array}{rll} H: & \mathbb{R} \longrightarrow & \{0,1\} \\ & x \longrightarrow & H(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{such that } H \in L^1_{loc}(\mathbb{R}). \end{array}$$

We will prove that $H' = \delta$. Indeed, the associated distribution to H is defined by

$$\langle H, \varphi \rangle := \int_0^\infty \varphi(t)\, dt, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Using the estimate

$$|\langle H, \varphi \rangle| \leq A \sup_{x \in [-A,A]} |\varphi(x)| \text{ , for each } \varphi \in D([-A,A]),$$

we see that H defines a distribution. Moreover,

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^\infty \varphi'(t)\, dt = [-\varphi(t)]_0^A = \varphi(0) = \langle \delta, \varphi \rangle.$$

Remark 0.3. In $\mathbb{R}^N$, we define the Heaviside function as follows

$$\begin{array}{rll} H: & \mathbb{R}^N \longrightarrow & \{0,1\} \\ & x \longrightarrow & H_N(x_1, x_2, \ldots, x_N) := \begin{cases} 1 & \text{if } x_i \geq 0 \text{ with } 1 \leq i \leq N \\ 0 & \text{in other case} \end{cases}. \end{array}$$

In that case,

$$\frac{\partial^N H_N}{\partial x_1 \ldots \partial x_N} = \delta \text{ on } \mathcal{D}(\mathbb{R}^N).$$

We will need some previous results to prove Leibniz's general formula. Note that $P^{(\alpha)}$ is the α-derivative of P.

(1) We recall Leibniz's Formula with multi-index notation

$$\partial^\alpha (fg) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} f^{(\beta)} g^{(\alpha-\beta)}, \qquad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

(2) If we take $P(D) = D^\alpha$ (i.e., $P(z) = z^\alpha$) then $P^{(\beta)}(z) = \dfrac{\alpha!}{\beta!(\alpha-\beta)!} z^{\alpha-\beta}$.
Hence, $P^{(\beta)}(D) = \dfrac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}$.

(3) Recall that $P^{(\alpha)}(\eta) = \dfrac{\partial^{|\alpha|} P}{\partial \eta_1^{\alpha_1} \dots \partial \eta_N^{\alpha_N}}(\eta) = i^{|\alpha|} D^\alpha P(\eta)$.

Proposition 0.13. ***(Leibniz's general formula)*** *If $P \in \mathbb{C}[z_1, \dots, z_N]$ and $a, u \in \mathcal{E}(\Omega)$, then $P(D)(au) = \sum_\alpha \frac{1}{\alpha!} D^\alpha a P^{(\alpha)}(D)(u)$.*

Proof: Using Taylor's Formula,

$$P(z+\eta) = \sum_\alpha \frac{1}{\alpha!} \eta^\alpha P^{(\alpha)}(z). \tag{1}$$

By Leibniz's Formula (1)

$$P(D)(au) = \sum_\alpha \frac{1}{\alpha!} D^\alpha a R_\alpha(D)(u). \tag{2}$$

We will calculate the P.D.O. R_α. Let $\xi, \eta \in \mathbb{R}^N$, where:

$$\begin{cases} a(x) = e^{i\langle x,\xi\rangle} \\ u(x) = e^{i\langle x,\eta\rangle} \end{cases}.$$

Using this formula on (2) we obtain

$$\begin{aligned} P(D)(au) &= P(D)e^{i\langle x,\xi+\eta\rangle} = P(\xi+\eta)e^{i\langle x,\xi+\eta\rangle} = \\ &= \sum_\alpha \xi^\alpha e^{i\langle x,\xi\rangle} R_\alpha(\eta)\, e^{i\langle x,\eta\rangle} = e^{i\langle x,\xi+\eta\rangle} \sum_\alpha \xi^\alpha R_\alpha(\eta). \end{aligned}$$

Then, we conclude,

$$P(\xi+\eta) = \sum_\alpha \xi^\alpha R_\alpha(\eta).$$

By the uniqueness of polynomials in Taylor series, we obtain

$$R_\alpha(\eta) = \frac{1}{\alpha!} P^{(\alpha)}(\eta) \longrightarrow P(D)(au) = \sum_\alpha \frac{1}{\alpha!} D^\alpha a P^{(\alpha)}(D)(u).$$

Definition 0.14. We say that E *is a fundamental solution* of a linear partial differential operator with constant coefficients $P(D)$ if $P(D)E = \delta$.

Remark 0.4. The following results will be used in the calculation of some fundamental solution

(1) **Green's Formula**
Given $\varepsilon > 0$, define $\Omega_\varepsilon = \{x \in \mathbb{R}^N : |x| < \varepsilon\}$.
Let $f \in \mathcal{E}(\mathbb{R}^N \backslash \{0\})$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$.

$$\int_{\Omega_\varepsilon} \left(f(x)\triangle\varphi(x) - \triangle f(x)\varphi(x)\right) dx = \int_{|x|=\varepsilon} \left(f(x)\frac{\partial\varphi}{\partial r}(x) - \varphi(x)\frac{\partial f}{\partial r}(x) \right) d\sigma_\varepsilon$$

Where $\triangle = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian, $\frac{\partial}{\partial r}$ is the radial derivative, and $\partial\sigma_\varepsilon$ is the measure on the sphere $|x| = \varepsilon$.

(2) **Outher Normal**
Given $\Omega \subset \mathbb{R}^N$ an open subset with boundary $S = \delta\Omega$ of a differential manifold that is C^2 and connected, then $\nu(x) = (\nu_1(x), \ldots, \nu_N(x))$ is the Outher normal if exists $\varepsilon > 0$ such that $(x_1 + t\nu_1(x), \ldots, x_N + t\nu_N(x)) \notin \Omega$ for every $t \in [0, \varepsilon]$ (and it is normal).

(3) **Gauss's Formula**
Let $A(x) = (A_1(x), \ldots, A_N(x)) \in C^2\left(\overline{\Omega}\right)$ with $supp\,(A(x)) \subset\subset \mathbb{R}^N$

$$\int_{\Omega} div\,(A(x))\, dx = \int_{\delta\Omega} A(x)\nu(x)\, d\sigma$$

(4) **Second Green's Formula**
Let $f, g \in C^2(\overline{\Omega})$ such that $supp(f)$ or $supp(g)$ are compact.
If $grad\,(g) = \left(\frac{\partial g_1}{\partial x_1}, \ldots, \frac{\partial g_N}{\partial x_N}\right)$ and $\frac{\partial g}{\partial \nu} = grad(g) \cdot \nu(x)$, then

$$\int_{\delta\Omega} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu}\right) d\sigma = \int_{\Omega} (f \triangle g - g \triangle f)\, dx.$$

Example 0.3.
The Cauchy-Riemman operator
This operator is defined by

$$\overline{\partial} = P(D) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \text{ on } \mathbb{C} = \mathbb{R}^2.$$

We are going to study the fundamental solution. First, we define the following function

$$E : \begin{array}{ll} \mathbb{C} & \longrightarrow \mathbb{C} \\ z & \longrightarrow \frac{1}{\pi z} \end{array}.$$

This function satisfies $\pi E(x, y) = \frac{x - iy}{x^2 + y^2}$. Then, $|E(x, y)| = \frac{1}{\pi} \frac{1}{\sqrt{x^2 + y^2}}$ and this implies $E \in C(\mathbb{R}^2 \setminus \{(0, 0)\})$.

On the other hand, we know that $\frac{1}{|x|^p} \in L(B_{\mathbb{R}^N}(0, 1))$ if and only if $0 < p < N$. Then, $E \in L^1_{loc}(\mathbb{R}^2)$ (i.e., $E \in L^1(K)$ where $K \subset\subset \mathbb{R}^2$ is a compact subset).

Now recall, that E is a distribution defined by

$$\langle E, \varphi \rangle = \int_{\mathbb{R}^2} E(x, y)\varphi(x, y)\, dx\, dy.$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$. The polar coordinate change yields

$$\begin{cases} \dfrac{\partial}{\partial x} = \cos(\theta)\dfrac{\partial}{\partial r} - \dfrac{\sin(\theta)}{r}\dfrac{\partial}{\partial \theta} \\ \\ \dfrac{\partial}{\partial y} = \sin(\theta)\dfrac{\partial}{\partial r} - \dfrac{\cos(\theta)}{r}\dfrac{\partial}{\partial \theta} \end{cases}$$

and $\dfrac{1}{x+iy} = \dfrac{e^{-i\theta}}{r}$. If we define $\widetilde{\varphi}(r,\theta) = \varphi(r\cos(\theta), r\sin(\theta))$, therefore $\widetilde{\varphi}(0,\theta) = \varphi(0,0)$, for each $\theta \in [0,2\pi]$ and $\widetilde{\varphi}$ has period 2π.

After changing to the polar coordinates we have

$$\begin{aligned} \langle \overline{\partial} E, \varphi \rangle &= -\langle E, \overline{\partial}\varphi \rangle = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{x+iy}\left(\frac{\partial \varphi}{\partial x} + i\frac{\partial \varphi}{\partial y}\right) dx\, dy = \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{e^{-i\theta}}{r}\left(e^{i\theta}\frac{\partial \widetilde{\varphi}}{\partial r} + i\frac{e^{i\theta}}{r}\frac{\partial \widetilde{\varphi}}{\partial \theta}\right) r\, dr\, d\theta = \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \left(\frac{\partial \widetilde{\varphi}}{\partial r} + \frac{i}{r}\frac{\partial \widetilde{\varphi}}{\partial \theta}\right) dr\, d\theta = \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\left[\int_0^{\infty} \frac{\partial \widetilde{\varphi}}{\partial r}\, dr\right] d\theta - \frac{i}{2\pi}\int_0^{\infty}\frac{1}{r}\left[\int_0^{2\pi}\frac{\partial \widetilde{\varphi}}{\partial \theta}\, d\theta\right] dr = \\ &= -\frac{1}{2\pi}(2\pi)(-\varphi(0,0)) - \frac{i}{2\pi}\cdot 0 = \varphi(0,0) = \langle \delta, \varphi \rangle. \end{aligned}$$

Example 0.4.

The Laplacian

Let $N \geq 2$ and $P(x_1, \ldots, x_N) = -\sum_{j=1}^{N} x_j^2$. The Laplacian operator is defined by

$$P(D) = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} = \triangle.$$

We are going to study its fundamental solution. We consider $r := |x|$ with $x \in \mathbb{R}^N$, and we define:

$$E_N := \begin{cases} \ln(r) & \text{if } N = 2 \\ r^{2-N} & \text{if } N \geq 3 \end{cases}.$$

Note that, $E_N \in L^1_{loc}\left(\mathbb{R}^N\right)$ since

$$\int_{B_{\mathbb{R}^N}(0,1)} |E_N(x)|\, dx = \begin{cases} -2\pi\int_0^1 r\ln(r)\, dr = \dfrac{\pi}{2} & \text{if } N = 2 \\ 2\pi\int_0^1 r\, dr = \pi & \text{if } N \geq 3 \end{cases}.$$

Now recall, that E_N is a distribution defined by

$$\langle \triangle E_N, \varphi \rangle = \int_{\mathbb{R}^N} E_N(x)\triangle\varphi(x)\, dx \text{ , for each } \varphi \in \mathcal{D}\left(\mathbb{R}^N\right).$$

We cannot apply integration by parts, since the derivatives of E_N are not locally integrable. However we can apply Green's Theorem using Lebesgue's first Theorem

$$\langle \triangle E_N, \varphi\rangle = \lim_{\varepsilon\downarrow 0} I_\varepsilon \quad \text{where } I_\varepsilon := \int_{|x|\geq\varepsilon} E_N(x)\triangle\varphi(x)\,dx.$$

Since E_N is $\mathcal{E}\left(\mathbb{R}^N\right)$ if $|x|\geq\varepsilon$ and $\varphi\in\mathcal{D}(\mathbb{R}^N)$

$$I_\varepsilon := \int_{|x|\geq\varepsilon} \triangle E_N(x)\varphi(x)\,dx + \int_{|x|=\varepsilon}\left(E_N\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial E_N}{\partial r}\right)\,d\sigma_\varepsilon.$$

First, we calculate $\triangle E_N$ on $|x|\geq\varepsilon$.

- If $N=2$

$$\left.\begin{array}{l} \dfrac{\partial^2}{\partial x^2}\ln\left(x^2+y^2\right) = \dfrac{2y^2-2x^2}{\left(x^2+y^2\right)^2} \\ \dfrac{\partial^2}{\partial y^2}\ln\left(x^2+y^2\right) = \dfrac{2x^2-2y^2}{\left(x^2+y^2\right)^2} \end{array}\right\}$$

$$\triangle E_2 = \frac{2y^2-2x^2}{\left(x^2+y^2\right)^2} - \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2} = 0.$$

- If $N\geq 3$

$$\left.\begin{array}{l} \dfrac{\partial}{\partial x_i} r^{2-N} = (2-N)\,x_i r^{-N} \\ \dfrac{\partial^2}{\partial x_i^2} r^{2-N} = (2-N)\,r^{-N} + (2-N)\,x_i\dfrac{-N}{2}2x_i r^{-N-2} \end{array}\right\}$$

$$\triangle E_N = (2-N)\,Nr^{-N} - (2-N)\,N\left(\sum_{i=1}^N x_i^2\right)r^{-N-2} = 0.$$

Consequently,

$$I_\varepsilon = \int_{|x|=\varepsilon}\left(E_N\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial E_N}{\partial r}\right)\,d\sigma_\varepsilon.$$

We will use the following change of variables

$$\begin{array}{ll} x_i & = rf\left(\theta_1,\ldots,\theta_{N-1}\right)\ ,\ 1\leq i\leq N, \\ \dfrac{\partial x_i}{\partial r} & = f\left(\theta_1,\ldots,\theta_{N-1}\right) = -\dfrac{x_i}{r}\ ,\ 1\leq i\leq N, \\ \partial x & = F\left(\theta_1,\ldots,\theta_{N-1}\right)r^{N-1}\partial\theta_1\cdots\partial\theta_{N-1}, \\ \partial\sigma_\varepsilon & = \varepsilon^{N-1}F\left(\theta_1,\ldots,\theta_{N-1}\right)r^{N-1}\partial\theta_1\cdots\partial\theta_{N-1} = \varepsilon^{N-1}\partial\sigma_1 \text{ (measure of the unit sphere)}, \\ \dfrac{\partial}{\partial r} & = \sum\limits_{i=1}^N \dfrac{\partial}{\partial x_i}\dfrac{\partial x_i}{\partial r} = -\sum\limits_{i=1}^N \dfrac{x_i}{r}\dfrac{\partial}{\partial x_i} \text{ (other normal).} \end{array}$$

Therefore,

- If $N = 2$

$$I_\varepsilon = \int_{|x|=\varepsilon} \left(\ln(\varepsilon) \frac{\partial \varphi}{\partial r} + \varphi \frac{1}{\varepsilon} \right) \varepsilon \, d\sigma_1 =$$
$$= \int_{|x|=\varepsilon} \varepsilon \ln(\varepsilon) \frac{\partial \varphi}{\partial r} \, d\sigma_1 + \int_{|x|=\varepsilon} \varphi \, d\sigma_1 = (*) + (**) ,$$

using

$$-\frac{x_1}{r} \frac{\partial E_N}{\partial x_1} - \frac{x_2}{r} \frac{\partial E_N}{\partial x_2} = -\left(\frac{x_1^2}{r^3} + \frac{x_2^2}{r^3} \right) = -\frac{1}{r} = \frac{\partial E_N}{\partial r},$$

we define

$$\left| \frac{\partial \varphi}{\partial r} \right| \leq \sum_{i=1}^{N} \left| \frac{x_i}{r} \right| \left| \frac{\partial \varphi}{\partial x_1} \right| \leq \sum_{i=1}^{N} \sup_{\mathbb{R}^2} \left| \frac{\partial \varphi}{\partial x_i} \right| =: C.$$

Since

$$(*) = \int_{|x|=\varepsilon} \varepsilon \ln(\varepsilon) \frac{\partial \varphi}{\partial r} \, d\sigma_1 \leq \left| \int_{|x|=\varepsilon} \varepsilon \ln(\varepsilon) \frac{\partial \varphi}{\partial r} \, d\sigma_1 \right| \leq$$
$$= C \left| \varepsilon \ln(\varepsilon) \right| \left(\int d\sigma_1 \right) \overset{\varepsilon \to 0}{\longrightarrow} 0$$
$$(**) = -\int \widetilde{\varphi}(\varepsilon, \theta) \; d\sigma_1 \overset{\varepsilon \to 0}{\longrightarrow} -\widetilde{\varphi}(0, \theta) \int d\sigma_1 = -2\pi \varphi(0,0)$$

we get

$$\lim_{\varepsilon \to 0} I_\varepsilon = -2\pi \varphi(0,0) .$$

- If $N \geq 3$

$$I_\varepsilon = \int_{r=\varepsilon} \frac{1}{\varepsilon^{N-2}} \frac{\partial \varphi}{\partial r} \varepsilon^{N-1} \, d\sigma_1 - \int_{r=\varepsilon} \widetilde{\varphi}(\varepsilon, \theta_1, \ldots, \theta_{N-1}) (2-N) \frac{1}{\varepsilon^{N-1}} \varepsilon^{N-1} \, d\sigma_1$$
$$= \int_{r=\varepsilon} \varepsilon \frac{\partial \varphi}{\partial r} \, d\sigma_1 + (N-2) \int_{r=\varepsilon} \widetilde{\varphi}(\varepsilon, \theta_1, \ldots, \theta_{N-1}) \; d\sigma_1 = (***) + (****)$$

$(***)$ Since $\left| \frac{\partial \varphi}{\partial r} \right| \leq C$ and $\int_{r=\varepsilon} \varepsilon \frac{\partial \varphi}{\partial r} \varepsilon \, d\sigma_1 \overset{\varepsilon \to 0}{0}$

$(****)$ Using Lebesgue's Theorem $(N-2)\varphi(\widetilde{0})\omega_N$

we conclude

$$\triangle \left(\frac{1}{2\pi} \ln|x| \right) = \delta \qquad \text{if } N = 2$$
$$\triangle \left(\frac{1}{(N-2)\omega_N} \frac{1}{|x|^{N-2}} \right) = \delta \quad \text{if } N \geq 3$$

where $\omega_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$ is the volume of the unit sphere of $\mathbb{R}^N$.

Example 0.5.

The Wave Operator

Let $N \geq 2$ and $P(x_1, \ldots, x_N, t) = \sum_{j=1}^{N} x_j^2 - t^2$. The Wave Operator is defined by

$$P(D) = \frac{\partial^2}{\partial t^2} - \triangle_x.$$

We are going to study its fundamental solution. Taking $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we define

$$E(x,t) := \begin{cases} \frac{1}{2} & \text{if } t - |x| > 0 \\ 0 & \text{if } t - |x| < 0 \end{cases}$$

Therefore,

$$\begin{aligned}
\langle \frac{\partial^2}{\partial t^2} E - \frac{\partial^2}{\partial x^2}, \varphi \rangle &= \iint_{\mathbb{R}^2} E(x,t) \frac{\partial^2 \varphi}{\partial t^2}(x,t)\, dx\, dt - \iint_{\mathbb{R}^2} E(x,t) \frac{\partial^2 \varphi}{\partial x^2}(x,t)\, dx\, dt = \\
&= \frac{1}{2} \int\limits_{\mathbb{R}} \int_{|x|}^{\infty} \frac{\partial^2 \varphi(x,t)}{\partial t^2}\, dt\, dx - \frac{1}{2} \int_0^{\infty} \int_{-t}^{t} \frac{\partial^2 \varphi(x,t)}{\partial x^2}\, dx\, dt = \\
&= \frac{1}{2} \int_{\mathbb{R}} \left[\frac{\partial \varphi}{\partial t}(x,t) \right]_{t=|x|}^{\infty} dx - \frac{1}{2} \int_0^{\infty} \left[\frac{\partial \varphi}{\partial x}(x,t) \right]_{x=-t}^{x=t} dt = \\
&= -\frac{1}{2} \int_{\mathbb{R}} \frac{\partial \varphi}{\partial t}(x,|x|)\, dx - \frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial x}(t,t)\, dt + \frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial t}(-t,t)\, dt = \\
&= -\frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial t}(x,x)\, dx - \frac{1}{2} \int_{-\infty}^{0} \frac{\partial \varphi}{\partial t}(x,-x)\, dx \\
&\quad - \frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial x}(t,t)\, dt + \frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial t}(-t,t)\, dt = \\
&= -\frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial t}(x,x)\, dx - \frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial t}(-x,x)\, dx \\
&\quad - \frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial x}(t,t)\, dt + \frac{1}{2} \int_0^{\infty} \frac{\partial \varphi}{\partial t}(-t,t)\, dt = \\
&= -\frac{1}{2} \int_0^{\infty} \frac{\partial}{\partial y} \left[\varphi(y,y) \right] dy - \frac{1}{2} \int_0^{\infty} \frac{\partial}{\partial y} \left[\varphi(-y,y) \right] dy = \\
&= \frac{1}{2} \varphi(0,0) + \frac{1}{2} \varphi(0,0) = \varphi(0,0) = \langle \delta, \varphi \rangle.
\end{aligned}$$

Given $N = 1$ and $f \in C^2(\mathbb{R}) \backslash C^3(\mathbb{R})$, we define $g : \mathbb{R}^2 \to \mathbb{C}$ with $g(x,t) = f(x \pm t)$ then

$$\left.\begin{aligned} \frac{\partial^2 g}{\partial x^2} &= f''(x \pm t) \\ \frac{\partial^2 g}{\partial t^2} &= f''(x,t) \end{aligned}\right\} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) g = 0$$

The Wave operator has null-solutions which don't belong to $\mathcal{E}(\mathbb{R})$. This is related to the fact that $\{(x,t) \in \mathbb{R}^2 : P(x,t) = 0\}$ is not compact.

Example 0.6.
The Heat Operator
Let $N \geq 2$ and $P(x_1, \ldots, x_N) = \sum_{j=1}^{N} x_j^2 + it$. The Heat operator is defined by

$$P(D) = \frac{\partial}{\partial t} - \triangle_x.$$

We are going to study its fundamental solution. First, if $(x,t) \in (\mathbb{R} \times \mathbb{R}\backslash\{0\})$ we define the function:

$$E(x,t) := \frac{H(t)}{\sqrt{4\pi t}} \exp\left(\frac{-x^2}{4t}\right) \text{ , with } H(t) := \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

We are going to prove that $P(D)E = \delta$. By the following approximation we have that $E \in L^1_{loc}(\mathbb{R}^2) \subset \mathcal{D}'(\mathbb{R}^2)$

$$E(x,t) \leq \frac{H(t)}{\sqrt{4\pi t}}.$$

Now, taking $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we obtain

$$\begin{aligned}\langle\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) E, \varphi\rangle &= -\langle E, \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2}\rangle \\ &= -\iint_{[0,+\infty]\times\mathbb{R}} \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}} \left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2}\right) dx\, dt \\ &= \int_0^\infty \int_{\mathbb{R}} \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}} \frac{\partial \varphi}{\partial t} dx\, dt + \int_0^\infty \int_{\mathbb{R}} \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}} \frac{\partial^2 \varphi}{\partial x^2} dx\, dt.\end{aligned}$$

While we cannot apply integration by parts, we can apply Green's Theorem using Lebesgue's first Theorem. Consider

$$I_\varepsilon := \int_{\mathbb{R}} \int_\varepsilon^\infty \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}} \frac{\partial \varphi}{\partial t} dt\, dx,$$

with

$$\int_0^\infty \int_{\mathbb{R}} \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}} \frac{\partial \varphi}{\partial t} dx\, dt = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left[\int_\varepsilon^\infty \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}} \frac{\partial \varphi}{\partial t} dt\right] dx = \lim_{\varepsilon \to 0} I_\varepsilon,$$

we can apply Lebesgue's Theorem because

$$\left|\chi_{[\varepsilon,\infty]\times\mathbb{R}} \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}} \varphi(x,t)\right| \leq \frac{C\,|\varphi(x,t)|}{\sqrt{t}} \in L^1(\mathbb{R}^2).$$

Applying integration by parts we get

$$I_\varepsilon = -\int_{\mathbb{R}}\int_\varepsilon^\infty \frac{\partial}{\partial t}\left[\frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}}\right]\varphi(x,t)\,dx\,dt + \int_{\mathbb{R}}\left[\frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}}\varphi(x,t)\right]_{t=\varepsilon}^{t=\infty} dx$$

$$= -\frac{1}{4\sqrt{\pi}}\int_{\mathbb{R}}\int_\varepsilon^\infty \left(\frac{x^2}{2t^{\frac{5}{2}}} - \frac{1}{t^{\frac{3}{2}}}\right)\exp\left(\frac{-x^2}{4t}\right)\varphi(x,t)\,dx\,dt + \int_{\mathbb{R}}\frac{\exp\left(\frac{-x^2}{4\varepsilon}\right)}{\sqrt{4\pi\varepsilon}}\varphi(x,\varepsilon)\,dx.$$

Analogously, taking

$$J_\varepsilon := \int_\varepsilon^\infty\int_{\mathbb{R}} \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}}\frac{\partial^2\varphi}{\partial x^2}\,dx\,dt,$$

we get

$$\int_0^\infty\int_{\mathbb{R}} \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}}\frac{\partial^2\varphi}{\partial x^2}\,dx\,dt = \lim_{\varepsilon\to 0}\int_\varepsilon^\infty\int_{\mathbb{R}} \frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}}\,dx\,dt = \lim_{\varepsilon\to 0} J_\varepsilon.$$

We apply integration by parts twice, because

$$\frac{\partial}{\partial x}\left[\frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}}\right] = \frac{-x}{4\sqrt{\pi}t^{\frac{3}{2}}}\exp\left(\frac{-x^2}{4t}\right),\ \text{and}$$

$$\frac{\partial^2}{\partial x^2}\left[\frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}}\right] = \left(\frac{-1}{4\sqrt{\pi}t^{\frac{3}{2}}} + \frac{x^2}{8\sqrt{\pi}t^{\frac{5}{2}}}\right)\exp\left(\frac{-x^2}{4t}\right).$$

This gives us

$$J_\varepsilon = \frac{1}{4\sqrt{\pi}}\int_{\mathbb{R}}\int_\varepsilon^\infty \left(\frac{x^2}{2t^{\frac{5}{2}}} - \frac{1}{t^{\frac{3}{2}}}\right)\exp\left(\frac{-x^2}{4t}\right)\varphi(x,t)\,dx\,dt$$

$$+\int_\varepsilon^\infty \left[\frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi t}}\frac{\partial\varphi}{\partial x}(x,t)\right]_{x=-\infty}^{x=\infty} dt + \int_\varepsilon^\infty\left[\frac{x}{4\sqrt{\pi}t^{\frac{3}{2}}}\exp\left(\frac{-x^2}{4t}\right)\varphi(x,t)\right]_{x=-\infty}^{x=\infty} dt$$

$$= \frac{1}{4\sqrt{\pi}}\int_{\mathbb{R}}\int_\varepsilon^\infty \left(\frac{x^2}{2t^{\frac{5}{2}}} - \frac{1}{t^{\frac{3}{2}}}\right)\exp\left(\frac{-x^2}{4t}\right)\varphi(x,t)\,dx\,dt.$$

Then, we take K_ε as

$$I_\varepsilon + J_\varepsilon = -\int_{\mathbb{R}}\frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi\varepsilon}}\varphi(x,\varepsilon)\,dx =: K_\varepsilon.$$

Therefore,

$$\langle\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)E,\varphi\rangle = \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{\exp\left(\frac{-x^2}{4t}\right)}{\sqrt{4\pi\varepsilon}}\varphi(x,\varepsilon)\,dx = \lim_{\varepsilon\to 0}K_\varepsilon.$$

In K_ε we use the following change of variables $y = \frac{x}{2\sqrt{\varepsilon}}$, then

$$K_\varepsilon = \frac{2\sqrt{\varepsilon}}{2\sqrt{\pi}\sqrt{\varepsilon}} \int_{\mathbb{R}} e^{-y^2} \varphi(2\sqrt{\varepsilon}y, \varepsilon)\, dy = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \varphi(2\sqrt{\varepsilon}y, \varepsilon)\, dy.$$

When $\varepsilon \to 0$ we obtain $\varphi(2\sqrt{\varepsilon}y, \varepsilon)$ converges to $\varphi(0,0)$ and

$$\left| e^{-y^2} \varphi(2\sqrt{\varepsilon}y, \varepsilon) \right| \leq C e^{-y^2} \in \mathcal{L}^1(\mathbb{R}).$$

Applying Lebesgue's Theorem

$$\lim_{\varepsilon \to 0} K_\varepsilon = \frac{1}{\sqrt{\pi}} \left(\int_{\mathbb{R}} e^{-y^2}\, dy \right) \varphi(0,0) = \varphi(0,0) = \langle \delta, \varphi \rangle.$$

Therefore, we conclude

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) E = \delta.$$

Now we will introduce some definitions and properties of tempered distributions and Fourier Transform needed later.

Definition 0.15. We define the Schwartz space of rapidly decreasing functions. This space has the important property that the Fourier transform is an automorphism on this space.

$$\mathcal{S}\left(\mathbb{R}^N\right) := \left\{ f \in \mathcal{E}\left(\mathbb{R}^N\right) : q_k\left(f\right) := \sup_{|\alpha| \leq k} \left(1 + |x|^2\right)^{\frac{k}{2}} |f^{(\alpha)}| < \infty \text{ , for every } k \in \mathbb{N}_0 \right\}$$

Note that $\mathcal{S}\left(\mathbb{R}^N\right)$ has the following propierties:

a) $\mathcal{S}\left(\mathbb{R}^N\right)$ is a Fréchet space over the complex numbers.
b) The inclusions $\mathcal{D}\left(\mathbb{R}^N\right) \subset \mathcal{S}\left(\mathbb{R}^N\right) \subset \mathcal{E}\left(\mathbb{R}^N\right)$ are continuous and dense.
c) The inclusion $\mathcal{S}\left(\mathbb{R}^N\right) \subset L^p\left(\mathbb{R}^N\right)$ is continuous and dense.
d) The mapping $\dfrac{\partial}{\partial x_j} : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathcal{S}\left(\mathbb{R}^N\right)$ is continuous.

The space of tempered distributions is defined as the topological dual of the Schwartz space $\mathcal{S}\left(\mathbb{R}^N\right)$.

Definition 0.16. Given $f \in L^1(\mathbb{R}^N)$ we define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-ix\xi}\, dx \qquad \text{where } x\xi = \sum_{j=1}^{N} x_j \xi_j.$$

Proposition 0.17. *The mapping* $\wedge : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathcal{S}\left(\mathbb{R}^N\right), f \mapsto \widehat{f}$ *is a topological isomorphism and*

$$f(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \widehat{f}(\xi) e^{i\xi x}\, d\xi \quad \text{for every } f \in \mathcal{S}\left(\mathbb{R}^N\right) \text{ and } x \in \mathbb{R}^N.$$

Moreover,

a) $\widehat{D_j f}(\xi) = \xi_j \widehat{f}(\xi)$, $\widehat{x_j f} = D_j \widehat{f}$ *for every* $j = 1, \ldots, N$ *and* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$.
b) $\hat{\hat{f}} = (2\pi)^N \check{f}$, $\check{f}(x) := f(-x)$ *for each* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$.
c) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.

Definition 0.18. We define the Fourier Transform of a tempered distribution by

$$\begin{array}{rrcl} \mathcal{F}: & \mathcal{S}'\left(\mathbb{R}^N\right) & \longrightarrow & \mathcal{S}'\left(\mathbb{R}^N\right) \\ & U & \longrightarrow & \mathcal{F}(U)(\varphi) = \hat{U}(\varphi) := U(\hat{\varphi}) \text{ for each } \varphi \in \mathcal{S}\left(\mathbb{R}^N\right). \end{array}$$

Proposition 0.19. *The Fourier Transform* $\mathcal{F} : \mathcal{S}'\left(\mathbb{R}^N\right) \to \mathcal{S}'\left(\mathbb{R}^N\right)$ *has the following properties:*

a) $\mathcal{F}(u_f) = u_{\widehat{f}}$ *for each* $f \in L_1(\mathbb{R}^N)$.
b) $\mathcal{F}(D_j u) = x_j \mathcal{F}(u)$, $\mathcal{F}(x_j u) = -D_j \mathcal{F}(u)$ *for any* $u \in S'\left(\mathbb{R}^N\right)$ *and* $j = 1, \ldots, N$.
c) $\mathcal{F}^2(u) = (2\pi)^N \check{u}$.

Example 0.7. Now, we are going to see two examples of Fourier Transforms.

(1) $\mathcal{F}(\delta) = 1$ since

$$\mathcal{F}(\delta)(\varphi) = \delta(\widehat{\varphi}) = \widehat{\varphi}(0) = \int \varphi e^{-ix0}\, dx = \int \varphi\, dx = 1\,[\varphi]$$

such that $1 \in \mathcal{S}'\left(\mathbb{R}^N\right), \varphi \in \mathcal{S}\left(\mathbb{R}^N\right)$. As a consequence, $\mathcal{F}(D^\alpha \delta) = x^\alpha$ for each $\alpha \in \mathbb{N}_0^N$.

(2) Let $P \in \mathbb{C}[z_1, \ldots, z_N] \backslash \mathbb{C}$ and $u \in \mathcal{S}'\left(\mathbb{R}^N\right)$. Then $\mathcal{F}(P(D)u) = P(x)\mathcal{F}(u)$ since

$$\mathcal{F}(P(D)u) = \mathcal{F}\left(\sum a_\alpha (D^\alpha u)\right) = \sum a_\alpha x^\alpha \mathcal{F}(u) = P(x)\mathcal{F}(u).$$

Remark 0.5. Suppose that $P(D)$ has a fundamental solution $E \in \mathcal{S}'(\mathbb{R}^N)$ (i.e., $P(D)E = \delta$). Then

$$1 = \mathcal{F}(\delta) = \mathcal{F}(P(D)E) = P(x)\mathcal{F}(E).$$

No we will use the notation mentioned above to collect some elementary formulas that we will shall take for granted below. They follow from the classical analogues either by duality or by density arguments.

Proposition 0.20. *For all $\zeta \in \mathbb{C}^N, T \in \mathcal{D}'(\mathbb{R}^N), S \in \mathcal{S}'(\mathbb{R}^N)$ and $U \in \mathcal{E}'(\mathbb{R}^N)$, the following equations hold in $\mathcal{D}'(\mathbb{R}^N)$:*

(1) $P(\partial)(e^{\zeta x}T) = e^{\zeta x}(P(\partial + \zeta)T)$,
(2) $P(\partial)\mathcal{F}^{-1}S = \mathcal{F}_\xi^{-1}(P(i\xi)S)$,
(3) $(e^{\zeta x}U) * (e^{\zeta x}T) = e^{\zeta x}(U * T)$.

One of the classical theorems of Paley and Wiener characterizes the entire functions of exponential type (of one complex variable) whose restriction to the real axis is in L^2, as being exactly the Fourier Transforms of L^2-functions with compact support. We shall give two analogues of this (in several variables), one for $\mathcal{E}$-functions with compact support, and one for distributions with compact support.

Definition 0.21. If Ω is an open set in $\mathbb{C}^N$ and f is a continuous complex function on Ω, then f is said to be *holomorphic* in Ω if it is holomorphic in each variable separately. This means that if $(a_1, \ldots, a_n) \in \Omega$ and if

$$g_i(\lambda) = f(a_1, \ldots, a_{i-1}, a_i + \lambda, a_{i+1}, \ldots, a_n),$$

each of the functions $g_1, \ldots, g_n$ is holomorphic in some neighborhood of 0 in $\mathbb{C}$. A function that is holomorphic in all $\mathbb{C}^N$ is said to be *entire*.

Lemma 0.22. *If f is a entire function in $\mathbb{C}^N$ that vanishes on $\mathbb{R}^N$, then $f = 0$.*

Theorem 0.23. ***(Paley-Wiener)***

(a) *If $\phi \in \mathcal{D}(\mathbb{R}^N)$ has it is support in $rB_{\mathbb{R}^N}(0,1) = B_{\mathbb{R}^N}(0,r)$, and if*

$$f(z) = \int_{\mathbb{R}^N} \phi(t)\, e^{-iz\cdot t}\, dm_N(t) \quad z \in \mathbb{C}^N, \tag{3}$$

then f is entire, and there are constants $\gamma_N < \infty$ such that

$$|f(z)| \le \gamma_N (1 + |z|)^{-N} e^{r|Im(z)|} \quad z \in \mathbb{C}^N, N = 0, 1, 2, \ldots \tag{4}$$

(b) *Conversely, if an entire function f satisfies the condition (4), then there exists $\phi \in \mathcal{D}(\mathbb{R}^N)$, with support in $B_{\mathbb{R}^N}(0,r)$, such that (3) holds.*

Theorem 0.24. ***(Paley-Wiener-Schwartz)***

(a) *If* $u \in \mathcal{D}'(\mathbb{R}^N)$ *has it is support in* $B_{\mathbb{R}^N}(0,r)$*, if* u *has order* m*, and if*

$$f(z) = \int_{\mathbb{R}^N} \phi(t)\, e^{-iz\cdot t}\, dm_N(t) =: \widehat{u}(z) \quad z \in \mathbb{C}^N \tag{5}$$

then f *is entire the restriction of* f *to* $\mathbb{R}^N$ *is the Fourier transform of* u*, and there is a constant* $\gamma < \infty$ *such that*

$$|f(z)| \leq \gamma\,(1+|z|)^m\, e^{r|Im(z)|} \quad z \in \mathbb{C}^N \tag{6}$$

(b) *Conversely, if* f *is an entire function in* $\mathbb{C}^N$ *which satisfies* (6) *for some* m *and some* γ*, then there exists* $u \in \mathcal{D}'(\mathbb{R}^N)$*, with support in* $B_{\mathbb{R}^N}(0,r)$*, such that* (5) *holds.*

For more details about Topological Vector Spaces and Distribution Theory refer to [**MV97**] and [**Hor66**].

The Malgrange-Ehrenpreis Theorem

1. Rosay's proof

Rosay's proof of the Malgrange-Ehrenpreis theorem uses the Mittag-Leffer procedure, and he needed some previous results like the Hörmander Inequality 1.3 or the Hörmander Theorem 1.5 to prove the Approximation Theorem 1.9 that will be used in the Mittag-Leffer procedure as we can see in [**Ros91**].

Definition 1.1. For a P.D.O., $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ we define the adjoint operator $P^*(D)$ as

$$P^*(D) = \sum_{|\alpha|\leq m} \overline{a}_\alpha D^\alpha.$$

Remark 1.1. The adjoint operator has the following property: for a given $u \in L^2_{loc}(\mathbb{R}^N)$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$ (i.e., $P(D)\varphi \in L^2_{loc}(\mathbb{R}^N)$), then:

$$\begin{aligned}
(P\left(D\right)u|\varphi) &= \int_{\mathbb{R}^N} \sum_{|\alpha|\leq m} a_\alpha D^\alpha u\left(t\right)\overline{\varphi}\left(t\right)\,dt = \\
&= \sum_{|\alpha|\leq m} a_\alpha \int_{\mathbb{R}^N} D^\alpha u\left(t\right)\overline{\varphi}\left(t\right)\,dt = \\
&= \sum_{|\alpha|\leq m} a_\alpha \langle D^\alpha u\left(t\right), \overline{\varphi}\left(t\right)\rangle = \\
&= \sum_{|\alpha|\leq m} a_\alpha \left(-1\right)^{|\alpha|} \langle u\left(t\right), D^\alpha \overline{\varphi}\left(t\right)\rangle = \\
&= \langle u\left(t\right), \sum_{|\alpha|\leq m} a_\alpha \left(-1\right)^{|\alpha|} D^\alpha \overline{\varphi}\left(t\right)\rangle = \\
&= \int_{\mathbb{R}^N} u\left(t\right) \sum_{|\alpha|\leq m} a_\alpha \left(-1\right)^{|\alpha|} D^\alpha \overline{\varphi}\left(t\right)\,dt = \\
&\stackrel{(*)}{=} \int_{\mathbb{R}^N} u\left(t\right) \sum_{|\alpha|\leq m} a_\alpha \overline{D^\alpha \varphi}\left(t\right)\,dt =
\end{aligned}$$

$$= \int_{\mathbb{R}^N} u(t) \overline{\sum_{|\alpha| \le m} a_\alpha (-1)^{|\alpha|} D^\alpha \varphi (t)}\, dt =$$
$$= (u|P^*(D)\varphi).$$

To arrive at the equality $(*)$ we use the following:

$$\overline{D^\alpha \varphi(t)} = (-1)^{|\alpha|} D^\alpha \overline{\varphi}(t).$$

Definition 1.2. Let the P.D.O. $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ and $1 \le j \le N$ with $N \in \mathbb{N}$. We define $P_j(D)$ as the operator associated with $P^{(j)}(z)$, where

$$P^{(j)}(z) = \frac{\partial P}{\partial z_j}(z).$$

Observe that $P_j(D)$ has degree $< m$ and vanishes if in $P(D)$ does not appears x_j. Using Leibniz's general formula (0.13)

(7) $$P(D)(x_j \varphi) = x_j P(D)\varphi + P_j(D)\varphi \text{ , for each } \varphi \in \mathcal{D}(\mathbb{R}^N).$$

Theorem 1.3. ***(Hörmander inequality)*** *Let $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ be a linear P.D.O. with constant coefficients. For every open and bounded subset, $\Omega \subset \mathbb{R}^N$,*

$$\exists C > 0, \forall \varphi \in \mathcal{D}(\Omega) : \|P(D)(\varphi)\| \ge C\|\varphi\|,$$

where $C = |P|_m K_{m,\Omega} > 0$, with $|P|_m = \max(|a_\alpha| : |\alpha| = m)$ and $K_{m,\Omega}$ only depends on m and Ω.

Proof: First, we denote $A := \sup_{x \in \Omega} |x|$. We have to prove the next conditions:

(A) For every $m \in \mathbb{N}_0$ and $P(z)$ with $dg(P) \le m$ then

$$\|P_j(D)\varphi\| \le 2mA\|P(D)\varphi\| \text{ , for every } 1 \le j \le N \text{ and } \varphi \in \mathcal{D}(\Omega).$$

(B) For every $m \in \mathbb{N}_0$ and $P(z)$ with $dg(P) = m$ then

$$\|P(D)\varphi\| \ge |P|_m K_{m,\Omega}\|\varphi\| \text{ , for every } \varphi \in \mathcal{D}(\Omega).$$

Where $K_{m,\Omega}$ only depends on m and the diameter of Ω.

We begin by proving A. Let $P \in \mathbb{C}[z_1, \dots, z_N]$ and $\varphi \in \mathcal{D}(\Omega)$ We first show that $\|P(D)\varphi\| = \|P^*(D)\varphi\|$ as follows:

$$\|P(D)\varphi\|^2 = (P(D)\varphi|P(D)\varphi) = (\varphi|P^*(D)P(D)\varphi) =$$
$$= (\varphi|P(D)P^*(D)\varphi) = (P^*(D)\varphi|P^*(D)\varphi) =$$
$$= \|P^*(D)\varphi\|^2.$$

Assuming we have proved step (A) and using (7), we obtain

$$
\begin{aligned}
(8)\quad \|P(D)(x_j\varphi)\| &\leq \|x_jP(D)\varphi\| + \|P_j(D)\varphi\| \\
&\leq A\|P(D)\varphi\| + 2mA\|P(D)\varphi\| = (2m+1)A\|P(D)\varphi\|.
\end{aligned}
$$

Now, we are going to prove (A) by induction on m

- If $m = 0$, $P_j(D) = 0$, for each $1 \leq j \leq N$.
- The induction hypothesis is

$$\|P_j(D)\varphi\| \leq 2(m-1)A\|P(D)\varphi\|,$$

for every $\varphi \in \mathcal{D}(\Omega)$ with $P \in \mathbb{C}[z_1, \ldots, z_N]$ such that $dg(P) \leq m-1$.

We prove the inequality for $dg(P) = m$. First of all,

$$
\left.
\begin{aligned}
(P(D)(x_j\varphi)|P_j(D)\varphi) &= (x_jP(D)\varphi|P_j(D)\varphi) + \|P_j(D)\varphi\|^2 \\
(P(D)(x_j\varphi)|P_j(D)\varphi) &= \left(P_j^*(D)(x_j\varphi)|P^*(D)\varphi\right)
\end{aligned}
\right\}
$$

$$\|P_j(D)\varphi\|^2 = \left(P_j^*(D)(x_j\varphi)|P^*(D)\varphi\right) - (x_jP(D)\varphi|P_j(D)\varphi).$$

Using Cauchy-Schwartz's inequality

$$
\begin{aligned}
\|P_j(D)\varphi\|^2 &\leq \|P_j^*(D)(x_j\varphi)\|\|P^*(D)\varphi\| + \|x_jP(D)\varphi\|\|P_j(D)\varphi\| \\
&\overset{(*)}{\leq} (2m-1)A\|P_j(D)\varphi\|\|P(D)\varphi\| + A\|P(D)\varphi\|\|P_j(D)\varphi\|
\end{aligned}
$$

$(*)$ by (8) on $P_j^*(D)$.

Thus, we have

$$\|P_j(D)\varphi\|^2 \leq 2mA\|P_j(D)\varphi\|\|P(D)\varphi\|.$$

To conclude, we will also prove step (B) by induction on m

- If $m = 0$, then $P(D) = a_{(0,\ldots,0)}$ with $\max(|a_\alpha| : |\alpha| = m) = |a_0|$. Thus, $\|P(D)\varphi\| = \|a_0\varphi\| = |a_0|\,\|\varphi\|$ with $|P_m| = |a_0|$ and $K_{0,\Omega} = 1$, for every $\varphi \in \mathcal{D}(\Omega)$.
- Assuming that (B) holds for every $P \in \mathbb{C}[z_1, \ldots, z_N]$ with $dg(P) \leq m-1$ by induction hypothesis, we show (B) for all $P \in \mathbb{C}[z_1, \ldots, z_N]$ with $dg(P) = m$.

  Denote by $|P^j|$ the maximum of the modules of the coefficients of higher degree of $P^{(j)}$. Then

$$|P^j|_{m-1} = |P^{(j)}|_{m-1} \geq |P|_m \text{ and } dg\left(P^{(j)}\right) = m-1.$$

Using step (A) and the induction hypothesis to $P^{(j)}: 1 \leq j \leq N$, we get

$$2mA\|P(D)\varphi\| \geq \|P_j(D)\varphi\| \geq |P^j|_{m-1}K_{m-1,\Omega}\|\varphi\| \geq |P|_m K_{m-1,\Omega}\|\varphi\|.$$

Then, we conclude

$$|P|_m \frac{K_{m-1,\Omega}}{2mA}\|\varphi\| = |P|_m K_{m,\Omega}\|\varphi\| \leq \|P(D)\varphi\| \text{ , for each } \varphi \in \mathcal{D}(\Omega).$$

□

In the proof of the next theorem we will use the Hilbert space Riesz representation theorem.
Let H be a Hilbert space with a closed subset $H_0 \subset H$ and let $f: H_0 \to \mathbb{K}$ be antilinear and continuous. Then, there exists $u \in H$ such that $f(x) = (u|x)$ for all $x \in H_0$.

Proposition 1.4. *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded set then:*

$$\exists C_1 > 0\, \forall g \in L^2(\Omega)\, \exists u \in L^2(\Omega) : P(D)u = g \text{ and } \|u\| \leq C_1\|g\|.$$

Proof: Denote by $E = P^*(D)(\mathcal{D}(\Omega)) \subset L^2(\Omega)$. Fix $g \in L^2(\Omega)$ and define the following operator

$$\begin{array}{rrcl} T_g : & E & \longrightarrow & \mathbb{K} \\ & P^*(D)\varphi & \longrightarrow & T_g(P^*(D)\varphi) := (g|\varphi) \quad \text{, for each } \varphi \in \mathcal{D}(\Omega) \end{array}$$

Observe that T_g is well defined. Too see this, we must show that $P^*(D)\varphi_1 = P^*(D)\varphi_2 \in E$ implies $(g|\varphi_1) = (g|\varphi_2)$. Assuming $P^*(D)\varphi_1 = P^*(D)\varphi_2$, then $P^*(D)(\varphi_1 - \varphi_2) = 0 : \varphi_1 - \varphi_2 \in D(\Omega)$. By theorem 1.3, then $C\|\varphi_1 - \varphi_2\| \leq \|P^*(\varphi_1 - \varphi_2)\| = 0$ with $C > 0$. We conclude that $\varphi_1 = \varphi_2$ on $L^2(\Omega)$, and therefore $\varphi_1 = \varphi_2$ on $\mathcal{D}(\Omega)$.

Note that T_g is antilinear, since T_g satisfy the following properties

$$\begin{cases} T_g(\varphi + \psi) & = T_g(\varphi) + T_g(\psi) \\ T_g(\lambda\varphi) & = \overline{\lambda} T_g(\varphi) \text{ where } \lambda \in \mathbb{C} \end{cases}.$$

Moreover, T_g is L^2-continuous. Indeed,

$$\|T_g(P^*(D)\varphi)\| = |(g|\varphi)| \leq \|g\|\|\varphi\| \leq C^{-1}\|g\|\|P^*(D)\varphi\|.$$

There exists a unique antilinear and continuous extension

$$\overline{T}_g : \overline{E} \longrightarrow \mathbb{K} \text{ such that } \overline{T}_g\big|_E = T_g.$$

Using Hilbert space Riesz representation theorem we find $u \in L^2(\Omega)$ such that

$$(u|h) = \overline{T}_g(h),\ h \in \overline{E} \text{ and } \|u\| = \|T_g\|.$$

Now we only have to prove that $P(D)u = g$ (on $\mathcal{D}'(\Omega)$). Since $(u|h) = T_g(h)$, for each $h \in \overline{E}$,

$$(g|\varphi) = T_g(P^*(D)\varphi) = \overline{T}_g(P^*(D)\varphi) = (u|P^*(D)\varphi)\ , \text{ for each } \varphi \in \mathcal{D}(\Omega).$$

On the other hand,

$$\langle P(D)u, \overline{\varphi}\rangle = \int_{\mathbb{R}^N} u\overline{P^*(D)}\varphi = (u|P^*(D)\varphi) = (g|\varphi) = \langle g|\overline{\varphi}\rangle.$$

Which implies $P(D)u = g$ on $\mathcal{D}'(\Omega)$.

□

Theorem 1.5. ***(Hörmander Theorem)*** *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. For each $P \in \mathbb{C}[z_1, \ldots, z_N]$, there exists $C' > 0$, such that, for every $\eta \in \mathbb{R}$*

$$\int_\Omega e^{\eta x_1}|P(D)\varphi|^2 \geq C' \int_\Omega e^{\eta x_1}|\varphi|^2\ , \textit{ for each } \varphi \in \mathcal{D}(\Omega)$$

Proof: Let $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ be a P.D.O. for $\eta \in \mathbb{R}$. We define

$$Q_\eta(z) = P(z) + \sum_{|\alpha|\leq m, \alpha_1\neq 0} a_\alpha D^\alpha \frac{1}{\alpha!}\left(\frac{-\eta}{2}\right)^{x_1} P^{(\alpha)}(z).$$

The principal part of $Q_\eta(z)$ and $P(z)$ are equals. We use Leibniz's general formula in Remark 0.13 (for $\psi \in \mathcal{D}(\mathbb{R}^N)$) to get

$$Q_\eta(D)\psi = P(D)\psi + \sum_{|\alpha|\leq m, \alpha_1\neq 0} \frac{1}{\alpha!}\left(\frac{-\eta}{2}\right)^{x_1} P^{(\alpha)}(D)\psi =$$

$$= e^{\left(\frac{-\eta}{2}\right)x_1} \sum_{|\alpha|\leq m, \alpha_1\neq 0} \frac{1}{\alpha!} D^\alpha \left(e^{\left(\frac{-\eta}{2}\right)x_1}\right) P^{(\alpha)}(D)\varphi =$$

$$= e^{\left(\frac{-\eta}{2}\right)x_1} P(D)\left(e^{\left(\frac{-\eta}{2}\right)x_1}\psi\right),$$

using Hörmander inequality 1.3 for $Q_\eta(D)$ with $\psi := e^{\left(\frac{-\eta}{2}\right)x_1}\varphi$, $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega Q_\eta(D)\left(e^{\left(\frac{\eta}{2}\right)x_1}\varphi\right)^2 \geq C^2 \int_\Omega \left(e^{\left(\frac{\eta}{2}\right)x_1}|\varphi|\right)^2,$$

thus

$$\int_\Omega e^{\eta x_1}|P(D)\varphi|^2 \geq C^2 \int_\Omega e^{\eta x_1}|\varphi|^2.$$

Observe that C does not depend on η and φ.

$\square$

Corollary 1.6. *Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ satisfying $P(D)\varphi = 0$ over the set $H_{1,+} := \{x \in \mathbb{R}^N : x_1 > 0\}$. Then $\varphi = 0$ over $H_{1,+}$.*

Proof: Let Ω be an open and bounded subset with $\Omega \supset supp\,(\Omega)$. By Hörmander Theorem 1.5, we have

$$\exists C' > 0, \forall \eta > 0 : \int_\Omega e^{\eta x_1} |P(D)\varphi|^2 \geq C' \int_\Omega e^{\eta x_1} |\varphi|^2.$$

We observe the values of the function:

$$e^{\eta x_1}\,|P(D)\varphi| \quad \begin{cases} = 0 & \text{if } x_1 > 0. \\ \text{converges to } 0 & \text{if } x_1 \leq 0 \text{ (pointwise convergence).} \end{cases}$$

$$|e^{\eta x_1}\,|P(D)\varphi|| \quad \begin{cases} = 0 & \text{if } x_1 > 0. \\ \leq \sup_\Omega |P(D)\varphi| =: M & \text{if } x_1 \leq 0. \end{cases}$$

We can apply the dominated convergence theorem to get

$$\lim_{\eta \to \infty} e^{\eta x_1}\,|P(D)\varphi|^2 = 0.$$

Assuming that there exists $x_0 \in H_{1,+}$ such that $\varphi(x_0) \neq 0$, we can find a closed ball B centered at x_0 included in $\Omega \cap H_{1,+}$, such that if $x \in B$ then $|\varphi(x)| > \frac{|\varphi(x_1^0)|}{2}$ with $x_1 > x_1^0$. Therefore

$$\int_\Omega e^{\eta x_1}\,|\varphi(x)|^2 \geq \int_B e^{\eta x_1}\,|\varphi(x)|^2 \geq e^{\eta \frac{x_1^0}{2}} \frac{|\varphi(x_1^0)|^2}{4} \mu(B) \overset{\mu \to \infty}{\longrightarrow} \infty.$$

This fact contradict the last identity.

$\square$

Now we introduce the following notation:

- We define $B_r := \{x \in \mathbb{R}^N \text{ such that } |x| \leq r\}$.
- We are going to introduce *regular sequences* as follows. Let $\rho \in \mathcal{E}(\mathbb{R}^N)$ with $supp(\rho) \subset B_1$, $\int_{\mathbb{R}^N} \rho = 1$, $\rho \geq 0$. For $\varepsilon > 0$, we set $\rho_\varepsilon(x) = \varepsilon^{-N}\rho\left(\dfrac{x}{\varepsilon}\right)$ then $supp(\rho_\varepsilon) \subset B_\varepsilon$, $\int_{\mathbb{R}^N} \rho_\varepsilon = 1$. For $u \in L^1_{loc}(\mathbb{R}^N)$, we set
$$u_\varepsilon := \int_{\mathbb{R}^N} u(x-y)\rho_\varepsilon(y)\,dy = \int_{\mathbb{R}^N} u(y)\rho_\varepsilon(x-y)\,dy.$$

Remark 1.2. Let $u \in L^1_{loc}(\mathbb{R}^N)$ then:

- $u_\varepsilon \in \mathcal{E}(\mathbb{R}^N)$ for each $\varepsilon > 0$.
- $supp(u_\varepsilon) \subset supp(u) + B_\varepsilon$ where $supp(u) \subset\subset \mathbb{R}^N$.

- If u is continuous, then $u_\varepsilon \to u$ $(\varepsilon \downarrow 0)$ converges uniformly on the compact subsets of $\mathbb{R}^N$.
- If $u \in L^p(\mathbb{R}^N)$, then $u_\varepsilon \to u$ on $L^p(\mathbb{R}^N)$ with $1 \le p < +\infty$.

Corollary 1.7. *Let $\varphi \in L^2(\mathbb{R}^N)$ with supp$(\varphi) \subset\subset \mathbb{R}^N$. If supp$(P(D)\varphi) \subset B_r$, then supp$(\varphi) \subset B_r$.*

Proof: First, we study the case with $\varphi \in \mathcal{E}(\mathbb{R}^N)$, *supp*$(\varphi) \subset\subset \mathbb{R}^N$ and *supp*$(P(D)) \subset B_r$. Using Corollary 1.6 and using that B_r is a intersection subspace, we obtain the first case.
Now, we study the general case. Take $\varphi \in L^2(\mathbb{R}^N)$ with *supp*$(\varphi) \subset\subset \mathbb{R}^N$ and *supp*$(P(D)\varphi) \subset B_r$. Given $\varepsilon > 0$, and define $\varphi_\varepsilon := \varphi * \rho_\varepsilon$, where ρ_ε is a regular sequence. Then, $\varphi_\varepsilon \in L^2(\mathbb{R}^N) \cap \mathcal{E}(\mathbb{R}^N)$ with *supp*$(\varphi_\varepsilon) \subset\subset \mathbb{R}^N$ and

$$P(D)\varphi_\varepsilon = P(D)(\varphi * \rho_\varepsilon) = (P(D)\varphi) * \rho_\varepsilon.$$

Therefore, we conclude that *supp*$(P(D)\varphi_\varepsilon) \subset B_{r+\varepsilon}$. Using the first case, as *supp*$(\varphi_\varepsilon) \subset B_{r+\varepsilon}$ we get *supp*$(\varphi) \subset B_r$ since $\varphi_\varepsilon \overset{\varepsilon\downarrow 0}{\to} \varphi$ on L^2.

□

Lemma 1.8. *Let $0 < r < R$ and $H := \{u \in L^2(B_R) : P(D)u = 0 \text{ on } B_R\}$. Then, for every $g \in L^2(B_r), g \in H^\perp$ on $L^2(B_r)$, there exists $\omega \in L^2(B_R)$ such that $(\varphi|g)_{L^2(B_r)} = (P(D)\varphi|\omega)_{L^2(B_r)}$ for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$.*

Proof: Let $E := \left\{P(D)\varphi : \varphi \in \mathcal{D}(\mathbb{R}^N)\right\} \subset L^2(B_R)$. We define the following operator:

$$\begin{aligned} T: \quad E \subset L^2(B_R) &\longrightarrow \mathbb{K} \\ P(D)\varphi &\longrightarrow T(P(D)\varphi) := (\varphi|g)_{L^2(B_r)} = \textstyle\int_{B_r} \varphi\bar{g}. \end{aligned}$$

Note that T is well defined. Indeed, if $P(D)\varphi_1 = P(D)\varphi_2$ with $\varphi_i \subset \mathcal{D}(\mathbb{R}^N), i = \{1,2\}$ on B_R, then $P(D)(\varphi_1 - \varphi_2) = 0$ on B_R. Since $g \in H^\perp$, then $(\varphi_1 - \varphi_2|g)_{L^2(B_r)} = 0$, and we can conclude that $(\varphi_1|g)_{L^2(B_r)} = (\varphi_2|g)_{L^2(B_r)}$.

Moreover, T is linear and continuous. We apply theorem 1.4 to get

$$\exists C_1 > 0 : \forall h \in L^2(B_R) \exists k \in L^2(B_r) : P(D)h = k, \|k\|_{L^2(B_r)} \le C_1\|h\|_{L^2(B_R)}.$$

Writing $C := C_1\|g\|_{L^2(B_r)}$, we should prove

$$\left|(\varphi|g)_{L^2(B_r)}\right| \le C\|P(D)\varphi\|_{L^2(B_R)} \text{ , for each } \varphi \in \mathcal{D}(\mathbb{R}^N).$$

We study the next two cases:

- If $P(D)\varphi = 0$, then $T(P(D)\varphi) = 0$ because $g \in H^{\perp}$ and T is linear.
- If $P(D)\varphi \neq 0$, we apply Theorem 1.4 to get $\psi \in L^2(B_R)$ such that $P(D)\psi = P(D)\varphi$ and $\|\psi\|_{L^2(B_R)} \leq C_1\|P(D)\varphi\|_{L^2(B_R)}$.

$$\begin{aligned}\left|(\varphi|g)_{L^2(B_r)}\right| &= \left|(\varphi-\psi|g)_{L^2(B_r)}\right| + \left|(\psi|g)_{L^2(B_r)}\right| \overset{(*)}{=} \left|(\psi|g)_{L^2(B_r)}\right| \\ &\leq \|\varphi\|_{L^2(B_r)}\|g\|_{L^2(B_r)} \leq C_1\|P(D)\varphi\|_{L^2(B_r)}\|g\|_{L^2(B_r)}.\end{aligned}$$

$(*)$ Follows since $P(D)(\varphi - \psi) = 0$

Furthermore, the extension is unique by continuity $T : \overline{E} \subset L^2(B_R) \to \mathbb{K}$ is linear and continuous as well. By Riesz's Theorem, there exists $\omega \in \overline{E} \subset L^2(B_R) : T(h) = (h|\omega)_{L^2(B_R)}$ for every $h \in \overline{E}$. Therefore, $(P(D)\varphi|\omega)_{L^2(B_R)} = (\varphi|g)_{L^2(B_r)}$, for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$.

□

Theorem 1.9. ***(Aproximation Theorem)*** *Let $0 < r < r' < R$, for any $v \in L^2(B_{r'})$ with $P(D)v = 0$ on $B_{r'}$. Then, there exists $(v_j)_j \subset L^2(B_R)$ with $P(D)v_j = 0$ on B_R and v_j converges to 0 on $L^2(B_r)$ when j tends to infinity.*

Proof: We may assume that $v \in \mathcal{D}(\mathbb{R}^N)$ with $P(D)v = 0$ on $B_{r''}$, where $0 < r < r'' < r'$. Since we can multiply by functions that vanish outside $B_{r'}$ and are 1 in B_r. It is enough to prove that v belongs to the closure of $L^2(B_r)$ of the subspace

$$H := \left\{u \in L^2(B_R) : P(D)u = 0 \text{ on } B_R\right\} \subset L^2(B_r) \text{ with } \overline{H} = H^{\perp\perp}.$$

In order to see this, we will show that, for each $g \in L^2(B_r)$ with $(\alpha|g)_{L^2(B_r)} = 0$ and for every $\alpha \in L^2(B_R)$ such that $P(D)\alpha = 0$ on B_R (i.e., $g \in H^{\perp}$), then $(v|g)_{L^2(B_r)} = 0$.

First, we define the following functions:

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{if } x \in B_r \\ 0 & \text{if } x \notin B_r \end{cases} \qquad \widetilde{\omega}(x) = \begin{cases} \omega(x) & \text{if } x \in B_R \\ 0 & \text{if } x \notin B_R. \end{cases}$$

By Lemma 1.8, $(\varphi|g)_{L^2(B_r)} = (P(D)\varphi|\omega)_{L^2(B_R)}$, for each $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Then:

$$\int_{\mathbb{R}^N} \varphi\overline{\widetilde{g}} = \int_{\mathbb{R}^N} P(D)\varphi\overline{\widetilde{\omega}} \text{ , for each } \varphi \in \mathcal{D}(\mathbb{R}^N).$$

Therefore, $P^*(D)\widetilde{\omega} = \widetilde{g}$ with $supp(\widetilde{\omega}) \subset\subset \mathbb{R}^N$ and $supp(P^*(D)\widetilde{\omega}) = supp(\widetilde{g}) \subset B_r$. By corollary 1.7, $supp(\widetilde{\omega}) \subset B_r$, hences $\omega = 0$ on $B_R \backslash B_r$. Let $v \in \mathcal{D}(\mathbb{R}^N)$

with $P(D)v = 0$ on $B_{r'}$. We apply Lemma 1.8 to conclude

$$(\varphi|g)_{\mathcal{L}^2(B_r)} = (P(D)\varphi|\omega)_{\mathcal{L}^2(B_R)} = \int_{B_R} P(D)v\overline{\omega} =$$

$$\stackrel{(*)}{=} \int_{B_r} P(D)v\overline{\omega} = (P(D)\varphi|\omega)_{\mathcal{L}^2(B_r)} = 0.$$

$(*)$ Follows using $\omega = 0$ on $B_R \backslash B_r$. This vanishes because $P(D)v = 0$ on $B_r \subseteq B_{r''}$.

□

Theorem 1.10. *For each $g \in L^2_{loc}(\mathbb{R}^N)$, there exists $u \in L^2_{loc}(\mathbb{R}^N)$ such that $P(D)u = g$*

Proof: In the proof, we are going to use Mittag-Leffer procedure. First, we recall that $L^p_{loc}(\mathbb{R}^N)$ is a Fréchet-space with a topology defined by the increasing sequence of seminorms.

$$\|f\|_k := \left(\int_{B_k} |f|^p\right)^{\frac{1}{p}}, \quad k = 1, 2, 3, \dots$$

Fix $g \in L^2_{loc}(\mathbb{R}^N)$. We are going to find a sequence $\{u_p\}$ such that $u_p \in L^2_{loc}(B_p)$ by induction. By proposition 1.4, for $g|_{B_2} \in L^2(B_2)$, we find $u_1 \in L^2(B_2)$ with $P(D)u_1 = g$ on B_2. We can extend u_1 to $\mathbb{R}^N \backslash B_2$ setting $u_1 = 0$. Therefore, $u_1 \in L^2_{loc}(\mathbb{R}^N)$.

Assume the following induction hypothesis $u_1, \dots, u_p$ with $u_j \in L^2(B_{j+1})$ chosen such that it satisfies $P(D)u_j = g$ on B_{j+1} and $\|u_{j+1} - u_j\|_{L^2(B_j)} \leq 2^{-j}$. We have to find $\omega \in L^2(B_{p+1})$ such that $P(D)\omega = g$ on B_{p+2}. There exists $\omega \in L^2(B_{p+1})$, by Proposition 1.4. Therefore, on B_{p+1} we obtain $P(D)(u_p - \omega) = P(D)u_p - P(D)\omega = g - g = 0$. Applying the Approximation Theorem 1.9, there exists

$$v \in L^2(B_{p+2}) \text{ with } P(D)v = 0 \text{ on } B_{p+2} \text{ and } \|(u_p - \omega) - v\|_{L^2(B_p)} \leq 2^{-p}.$$

Writing $u_{p+1} = w + v$, we have $u_{p+1} \in L^2(B_{p+2})$,

$$P(D)u_{p+1} = P(D)w + P(D)v = g \text{ on } B_{p+2} \text{ and } \|u_{p+1} - u_p\|_{L^2(B_p)} \leq 2^{-p}.$$

We can extend it considering $u_p = 0$ outside B_{p+1}, for each $p \in \mathbb{N}$. Then $u_p \in L^2_{loc}(\mathbb{R}^N)$.

Note that $(u_p)_{p\in\mathbb{N}}$ is a Cauchy sequence on $L^2_{loc}(\mathbb{R}^N)$. Fix $k \in \mathbb{N}$ with $p > k, q \in \mathbb{N}$, then:

$$\|u_{p+q} - u_p\|_k = \|u_{p+q} - u_p\|_{L^2(B_k)} \leq \sum_{j=p}^{p+q-1} \|u_{j+1} - u_j\|_{L^2(B_j)} \leq \sum_{j=p}^{p+q-1} 2^{-j}.$$

Therefore, $u \in L^2_{loc}(\mathbb{R}^N)$ and $\lim\limits_{p\to\infty} u_p = u$ on $L^2_{loc}(\mathbb{R}^N)$.

This implies

$$\lim_{p\to\infty} \int_{\mathbb{R}^N} u_p P(-D)\varphi = \int_{\mathbb{R}^N} uP(-D)\varphi.$$

Therefore

$$\lim_{p\to\infty} \langle P(D)u_p, \varphi\rangle = \langle P(D)u, \varphi\rangle \text{ , for each } \varphi \in \mathcal{D}(\mathbb{R}^N) \text{ with } P(-D)\varphi \in \mathcal{D}(\mathbb{R}^N).$$

Then, $\lim_{p\to\infty} P(D)u_p = P(D)u$ on $\mathcal{D}'(\mathbb{R}^N)$. Using $P(D)u_p = g$ on B_p, we get $supp(\varphi) \subset B_p$ with $p \in \mathbb{N}$. Then $\langle P(D)u_p, \varphi\rangle = \langle g, \varphi\rangle$, for each $\varphi \in \mathcal{D}(\mathbb{R}^N)$.

Yields $\lim_{p\to\infty}\langle P(D)u_p, \varphi\rangle = \langle g, \varphi\rangle$ for each $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Thus, $P(D)u = g$ on $\mathcal{D}'(\mathbb{R}^N)$.

Theorem 1.11. ***(Malgrange-Ehrenpreis)*** [Ros91]*Each linear partial differential operator with constant coefficients $P(D)$ has a fundamental solution.*

Proof. First, we define the Heaviside function on $\mathbb{R}^N$ with $x = (x_1, \ldots, x_N)$

$$\begin{aligned} H: \ \mathbb{R}^N &\longrightarrow \mathbb{R} \\ x &\longrightarrow H(x) := \begin{cases} 1 & \text{if } x_i \geq 0,\ 1 \leq i \leq N \\ 0 & \text{in other case} \end{cases} \end{aligned}$$

We have

$$\frac{\partial^N H}{\partial x_1 \cdots \partial x_N} = \delta \text{ on } \mathcal{D}'(\mathbb{R}^N) \text{ and } H \in L^2_{loc}(\mathbb{R}^N).$$

By Theorem 1.10, there is $u \in L^2_{loc}(\mathbb{R}^N)$ with $P(D)u = H$.
Writing $E := \frac{\partial^N u}{\partial x_1 \cdots \partial x_N} \in \mathcal{D}'(\mathbb{R}^N)$, we obtain:

$$P(D)E = P(D)\left(\frac{\partial^N u}{\partial x_1 \cdots \partial x_N}\right) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N}(P(D)u) = \delta,$$

and E is a fundamental solution.

2. Rudin's proof

Rudin's proof of Malgrange-Ehrenpreis Theorem is very similar to the original proof by Malgrange as we can see in [Rud91], and he needed some previous lemmas about polynomials with complex coefficients and complex analysis. He also used Fourier Transforms.

Remark 2.1. We denote by T^N is the torus in $\mathbb{C}^N$:

$$T^N := \left\{w = \left(e^{i\theta_1}, \dots, e^{i\theta_N}\right) \in \mathbb{C}^N,\, \theta_i \in \mathbb{R},\, 1 \leq i \leq N\right\}$$

and by σ_N the Haar measure of T^N [Rud66] , that is, Lebesgue measure divided by $(2\pi)^N$.

Lemma 2.1. *If P is a polynomial in $\mathbb{C}^N$ of degree m, then there is a constant $A > 0$, depending only on P, such that*

$$|f(z)| \leq Ar^{-m} \int_{T^N} |(fP)(z + rw)| \, d\sigma_N(w) \tag{9}$$

for every entire function f, for every $z \in \mathbb{C}^N$, and for every $r > 0$.

Proof: Assume first that F is an entire function of one complex variable. We define the polynomial

$$Q(\lambda) := c \prod_{i=1}^{m} (\lambda + a_i)\,, \quad \lambda, c, a_i \in \mathbb{C} \text{ for every } 1 \leq i \leq m,\, m \in \mathbb{N}.$$

Put $Q_0(\lambda) = c \prod (1 + \overline{a_i}\lambda)$. Then, $cF(0) = (FQ_0)(0)$. Since $|Q_0| = |Q|$ on the unit circle, by Cauchy's integral formula we have

$$|cF(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|(FQ)(e^{i\theta})\right| \, d\theta. \tag{10}$$

The polynomial P can be written in the form $P = P_0 + P_1 + \ldots + P_m$, where each P_j is a homogeneous polynomial of degree j. Define A by

$$\frac{1}{A} := \int_{T^N} |P_m| \, d\theta_N.$$

Since P has degree m, this integral is positive. If $z \in \mathbb{C}^N$ and $w \in T^N$, define:

$$F(\lambda) := f\,(z + r\lambda w) \text{ and } \quad Q(\lambda) := P(z + r\lambda w), \quad \lambda \in \mathbb{C} \text{ and } r > 0.$$

The leading coefficient of Q is $r^m P_m(w)$. Hence (10) implies

$$r^m \left|P_m\,(w)\right| \left|f\,(z)\right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|(fP)(z + re^{i\theta} w)\right| \, d\theta. \tag{11}$$

If we integrate (11) with respect to σ_N, we get

$$|f(z)| \leq Ar^{-m}\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|(fP)(z+re^{i\theta}w)\right| \, d\theta_N(w). \tag{12}$$

The measure σ_N is invariant under the change of variables $w \to e^{i\theta}w$. Therefore, the inner integral in (12) is independent of θ. This gives (9).

□

Theorem 2.2. *Suppose P is a polynomial in N variables, and $v \in \mathcal{D}'(\mathbb{R}^N)$ with compact support. Then, the equation*

$$P(D)u = v \tag{13}$$

has solution with compact support if and only if there is an entire function g in $\mathbb{C}^N$ such that

$$Pg = \widehat{v}. \tag{14}$$

When this condition is satisfied, (13) has a unique solution u with compact support; the support of u lies in the convex hull of the support of v.

Proof: If (13) has a solution u with compact support, (a) in the Paley-Wiener-Schwartz Theorem 0.24 shows that (14) holds with $g = \widehat{u}$.

Conversely, suppose that (14) holds for some entire function g. Choose $r > 0$ so that v has its support in B_r. By Lemma 2.1, (14) implies that

$$|g(z)| \leq A\int_{T^N} |\widehat{v}(z+w)| \, d\sigma_N(w) \quad z \in \mathbb{C}^N.$$

By part (a) of the theorem of Paley-Wiener-Schwartz 0.24, there exist m and γ such that

$$|\widehat{v}(z+w)| \leq \gamma \, (l + |z+w|)^N \exp\{r\,|Im(z+w)|\}\,.$$

Moreover, there are constants $c_1 > 0$ and $c_2 > 0$ satisfying, for all $z \in \mathbb{C}^N$ and all $w \in T^N$,

$$l + |z+w| \leq c_1\,(l+|z|) \text{ and}$$
$$|Im(z+w)| \leq c_2 + |Im\,(z)|\,.$$

From these inequalities it follows that

$$|g(z)| \leq B\,(1+|z|)^N \exp r\,|Im\,(z)|, \quad z \in \mathbb{C}^N, \tag{15}$$

where B is another positive constant (depending on γ, A, N, c_1, c_2, and r). By (15) and by part (b) of Paley-Wiener-Schwartz Theorem 0.24, $g = \widehat{u}$ for some distribution u with support in B_r. Hence, (14) implies $P\widehat{u} = \widehat{v}$, which is equivalent to (13). The uniqueness of u is obvious, since there is at most one entire function $\widehat{u}$ that satisfies $P\widehat{u} = \widehat{v}$. The preceding argument shows that the support S_u of u lies in every closed ball centered at the origin that contains the support S_v of v. Since (13) implies

$$P(D)(\tau_x u) = \tau_x u, \qquad x \in \mathbb{R}^N,$$

the same statement is true for $x + S_u$ and $x + S_v$. Consequently, S_u lies in the intersection of all the closed balls (centered anywhere in $\mathbb{R}^N$) that contain S_v. Since this intersection is the convex hull of S_v, the proof is complete.

□

Theorem 2.3. ***(Malgrange-Ehrenpreis)*** *If P is a polynomial in N variables of degree m, then the differential operator $P(D)$ has a fundamental solution E that satisfies*

$$|E(\psi)| \leq Ar^{-m} \int\limits_{T^N} d\sigma_N(w) \int\limits_{\mathbb{R}^N} \left|\widehat{\psi}(1+rw)\right| dm_N(t) \tag{16}$$

for every $\psi \in \mathcal{D}\left(\mathbb{R}^N\right)$ and for every $A, r > 0$.

Proof: A is the constant that appears in Lemma 2.1. The main point of the theorem is the existence of a fundamental solution, rather than the estimate (16) which arises from the proof. Fix $r > 0$, and define

$$\|\psi\| := \int_{T^N} d\sigma_N(w) \int\limits_{\mathbb{R}^N} \left|\widehat{\psi}(1+rw)\right| dm_N(t). \tag{17}$$

To prepare for the main part of the proof, let us first show that

$$\lim_{j\to\infty} \|\psi_j\| = 0 \quad \text{ if } \psi_j \to 0 \text{ in } \mathcal{D}(\mathbb{R}^N). \tag{18}$$

Observe that $\widehat{\psi}(t+w) = \widehat{\psi}(e^{-wt})$ if $t \in \mathbb{R}^N$ and $w \in \mathbb{C}^N$. Hence

$$\|\psi\| = \int_{T^N} d\sigma_N(w) \int\limits_{\mathbb{R}^N} \left|\psi(e^{-wt})\right|. \tag{19}$$

If $\psi_j \to 0$ in $\mathcal{D}(\mathbb{R}^N)$, every ψ_j has their support in some compact set K. The functions e^{-rwt} such that $w \in T^N$ are uniformly bounded on K. It follows from the Leibniz formula that

$$\left\| D^{\alpha}(\psi_j)(e^{-rwt}) \right\|_{\infty} \leq C(K,\alpha) \max_{\beta \leq \alpha} \left\| D^{\beta}\psi_j \right\|_{\infty}. \tag{20}$$

The right side of (20) tends to 0, for every α. Hence, given $\varepsilon > 0$, there exists j_0 such that

$$\left\| (I - \triangle)^N ((e^{-rwt})\psi_j) \right\|_2 \leq \varepsilon, \qquad \text{where } j > j_0,\ w \in T^N. \tag{21}$$

and $\triangle = D_1^2 + \ldots + D_N^2$ is the Laplacian. By the Plancherel theorem, (21) is the same as

$$\int_{\mathbb{R}^N} \left| \left(1 + |t|^2\right)^N \widehat{\psi}_j(t + rw) \right|^2 dm_N(t) < \varepsilon^2, \tag{22}$$

from which it follows, by the Schwarz inequality and (17), that $\|\psi_j\| < C\varepsilon$ for all $j > j_0$, where

$$C^2 = \int_{\mathbb{R}^N} \left(1 + |t|^2\right)^{-2N} dm_N(t) < \infty. \tag{23}$$

This proves (18). Suppose now that $\phi \in \mathcal{D}(\mathbb{R}^N)$ and that

$$\psi = P(D)\phi. \tag{24}$$

Then $\widehat{\psi} = P(D)\widehat{\phi}$,and $\widehat{\psi}$ and $\widehat{\phi}$ are entire. Thus, ψ determines ϕ. In particular, $\phi(0)$ is a linear functional of ψ, defined on the range of $P(D)$. The main point of the proof consists in showing that this functional is continuous, i.e., that there is a distribution $u \in \mathcal{D}'(\mathbb{R}^N)$ that satisfies

$$u(P(D)\phi) = \phi(0), \qquad \phi \in \mathcal{D}(\mathbb{R}^N), \tag{25}$$

because then, the distribution $E = \check{u}$ satisfies

$$\begin{aligned} (P(D)E)(\phi) &= E(P(-D)\phi) = u\left((P(-D)\phi)^{\vee}\right) \\ &= u\left(P(D)\check{\phi}\right) = \check{\phi}(0) = \phi(0) = \delta(\phi). \end{aligned}$$

So, $P(D)E = \delta$, as desired.

Using lemma 2.1 with $\widehat{\psi} = P(D)\widehat{\phi}$ yields

$$\left| \widehat{\phi}(t) \right| \leq Ar^{-m} \int_{T^N} \left| \widehat{\psi}(1 + rw) \right| dm_N(t)\, d\sigma_N(w). \tag{26}$$

By the inversion theorem, $\phi(0) = \int_{\mathbb{R}^N} \widehat{\phi}\, dm_N$. Thus, (26), (17), and (24) yield:

$$|\phi(0)| \leq Ar^{-m} \|P(D)\phi\|. \tag{27}$$

Let Y be the subspace of $\mathcal{D}(\mathbb{R}^N)$ consisting in the functions $P(D)\phi$, $\phi \in \mathcal{D}(\mathbb{R}^N)$. By (27), the Hahn-Banach theorem shows that the linear functional defined on Y by $P(D)\phi \to \phi(0)$ extends to a linear functional u on $\mathcal{D}(\mathbb{R}^N)$ that satisfies (25), as well as

$$|u(\psi)| \leq Ar^{-m} \|\psi\| \qquad \text{for each } \psi \in \mathcal{D}(\mathbb{R}^N). \tag{28}$$

By (18), $u \in \mathcal{D}'(\mathbb{R}^N)$. This completes the proof.

□

3. Wagner's proof

Wagner's proof is the most recent proof of Malgrange-Ehrenpreis Theorem [Wag09]. For the proof he needed some previous lemmas about polynomials with complex coefficients and Fourier Transforms.

Lemma 3.1. *If P is a polynomial such that $P = \prod_{j=1}^{l} Q_j^{k_j}$ where $Q_1, \ldots, Q_l$ irreducible for each $k_j \in \mathbb{N}$. Then, for every $a \in \mathbb{R}$, the polynomial given by $x' \to P(x', a)$ is not the zero polynomial*

Proof: To arrive at a contradiction, assume $a \in \mathbb{R}$ such that $P(x', a) = 0$ for every $x' \in \mathbb{R}^N$. Using the Taylor's Formula, we have

$$P(x', x_{N+1}) = P(x', a) + \sum_{j=1}^{m} \frac{1}{j!} \frac{\partial^j P}{\partial x_{N+1}^j}(x', a)(x_{N+1} - a)^j = (x_{N+1} - a)S(x).$$

This is not possible, since P is irreducible.

□

Lemma 3.2. *If P is a polynomial in $\mathbb{C}[z_1, \ldots, z_N]$ of degree $m \geq 1$, and $\eta \in S^{N-1}$ (with $|\eta| = 1$), then $V_{\mathbb{R}}^N(P) := \left\{x \in \mathbb{R}^N : P(x) = 0\right\}$ is a Lebesgue null-set in $\mathbb{R}^N$.*

Proof: We will prove this statement by induction on N. Note that the step $N = 1$ is trivial, since $Card(V_{\mathbb{R}}^N(P)) < \infty$. Now we assume that the result holds for N and we write $P := \prod_{j=1}^{l} Q_j^{k_j}$ with $Q_1, \ldots, Q_l$ irreducible. Note that, $V_{\mathbb{R}}(P) = \bigcup_{j=1}^{l} V_{\mathbb{R}}(Q_j)$. We may assume without loss of generality that P is irreducible. Using the former lemma 3.1, we obtain that the Lebesgue Measure in $\mathbb{R}^N$ is $m_N\left(\left\{x' \in \mathbb{R}^N : P(x', a) = 0\right\}\right) = 0$ for any $a \in \mathbb{R}$ due to $P(x', a)$ is not the zero polynomial. By Fubini's Theorem

$$m_{N+1}(V_{\mathbb{R}}^N(P)) = \int_{-\infty}^{+\infty} m_N\left(\left\{x' \in \mathbb{R}^N : P(x', a) = 0\right\}\right) da = 0.$$

□

Lemma 3.3. *If P is a polynomial in $\mathbb{C}[z_1, \ldots, z_N]$ of degree $m \geq 1$ such that for each $\eta \in S^{N-1}$ (satisfying $|\eta| = 1$) then $P_m(\eta) \neq 0$ and there exists $Q_1, \ldots, Q_{m-1} \in \mathbb{C}[z_1, \ldots, z_N]$ such that*

$$P(z + \lambda\eta) = \lambda^m P_m(\eta) + \sum_{k=0}^{m-1} \lambda^k Q_k(z).$$

Proof: Write the polynomial P as

$$P(\xi) = \sum_{|\alpha \leq m} a_\alpha \xi^\alpha = \sum_{k=0}^{m} \sum_{|\alpha|=k} a_\alpha \xi^\alpha = \sum_{k=0}^{m} P_k(\xi).$$

Note that P_k is homogeneous of degree k, thus, $P_k(t\xi) = t^k P_k(\xi)$ for any $t \in \mathbb{C}$. Therefore,

$$P_k(z + \lambda\eta) = \lambda^k P_k\left(\frac{z}{\lambda} + \eta\right) = \lambda^k \sum_{|\alpha|\leq k} \left(\frac{1}{\alpha!}\right) P_k^{(\alpha)}(\eta) \left(\frac{z}{\lambda}\right)^\alpha =$$
$$= \sum_{|\alpha|\leq k} \frac{1}{\alpha!} P_k^{(\alpha)}(\eta) z^\alpha \lambda^{k-\alpha}.$$

We can see that the term constant λ^m only appears $k = m$ and $\alpha = 0$. The coefficient in this case is $P_m(\eta)$.

□

Lemma 3.4. *If $\lambda_0, \ldots, \lambda_m \in \mathbb{C}$ are pairwise different, then the unique solution of the linear system of equations*

$$\sum_{j=0}^{m} a_j \lambda_j^k = \begin{cases} 0 & \textit{if } k = 0, \ldots, m-1 \\ 1 & \textit{if } k = m \end{cases}$$

is given by $a_j = \prod_{k=0,k\neq j}^{m} (\lambda_j - \lambda_k)^{-1}$.

Proof: Since Vandermonde's determinant does not vanish for pairwise distinct λ_j, the vector $(a_0, \ldots, a_m) \in \mathbb{C}^{m+1}$ is uniquely determined. Furthermore, if $p(z) = \prod_{j=0}^{m} (z - \lambda_j)$, then the Residue Theorem implies that

$$\sum_{j=0}^{m} p'(\lambda_j)^{-1} \lambda_j^k = \frac{1}{2\pi i} \lim_{N\to\infty} \int_{|z|=N} \frac{z^k}{p(z)}\, dz = \begin{cases} 0 & \text{if } k = 0, \ldots, m-1 \\ 1 & \text{if } k = m \end{cases}.$$

On the other hand, $p'(\lambda_j) = \prod_{k=0,k\neq j}^{m} (\lambda_j - \lambda_k)$. Observe that if $P(z) = (z - \lambda_0) \cdot (z - \lambda_1) \cdot \ldots \cdot (z - \lambda_m)$ with $N > \max(|\lambda_0|, \ldots, |\lambda_m|)$ then

- If $0 \leq k < m$, $\frac{1}{2\pi i} \lim_{N\to\infty} \int_{|z|=N} \frac{z^k}{p(z)}\, dz = 0$ since

$$\left|\frac{1}{2\pi i} \int_{|z|=N} \frac{z^k}{p(z)}\, dz\right| = \left|\frac{1}{2\pi i} \int_0^{2\pi} \frac{N^k e^{ik\theta}}{P(Ne^{i\theta})} N e^{i\theta}\, d\theta\right| \leq \frac{N^{k+1}}{|P(Ne^{i\theta})|} \overset{N\to+\infty}{\longrightarrow} 0.$$

Using $P(Ne^{i\theta}) = \prod_{j=0}^{m} (Ne^{i\theta} - \lambda_j)$, ($grad(P) = m + 1$) we have $|P(Ne^{i\theta})| \sim N^{m+1}$.

- If $k=m$, $\frac{1}{2\pi i}\lim_{N\to\infty}\int_{|z|=N}\frac{z^m}{p(z)}\,dz=1$, since

$$\frac{1}{2\pi i}\int_{|z|=N}\frac{z^m}{p(z)}\,dz=\frac{1}{2\pi i}\int_0^{2\pi}\frac{N^m e^{im\theta}}{P(Ne^{i\theta})}iNe^{i\theta}\,d\theta=$$

Making a change of variable $\theta=-t$

$$=\frac{1}{2\pi i}\int_0^{2\pi}-\frac{N^m e^{-imt}}{P(Ne^{-it})}iNe^{-it}\,dt=\frac{1}{2\pi i}\int_{-2\pi}^{0}\frac{N^m e^{-imt}}{P(Ne^{-it})}iNe^{-it}\,dt=$$

Doing $z=Re^{it}$

$$=\frac{1}{2\pi i}\int_{|z|=N}\frac{R^{2m}}{z^m}\frac{R^2}{z^2}\frac{dz}{P(\frac{R^2}{z})}$$

Now, we denote by $\varphi(z):=z^{m+2}P(\frac{R^2}{z})=z(R^2-\lambda_0 z)\cdot\ldots\cdot(R^2-\lambda_m z)$. Then, φ has the only root, $z=0$, using $R^2-\lambda_j z=0$ if and only if $|\lambda_j||z|=R^2$ and $|\lambda_j|<R$. Using the Residue Theorem

$$\frac{1}{2\pi i}\int_{|z|=N}\frac{R^{2m}}{z^m}\frac{R^2}{z^2}\frac{dz}{P(\frac{R^2}{z})}=N^{2m+2}Res\left(\frac{1}{\varphi(z)},0\right)=$$
$$=N^{2m+2}\lim_{z\to 0}\frac{z}{\varphi(z)}=$$
$$=\lim_{N\to\infty}\frac{N^{2m+2}}{N^{2m+2}}=1.$$

□

Theorem 3.5. ***(Malgrange-Ehrenpreis)*** *Each non-constant linear partial differential operator with constant coefficients $P(D)$ has a fundamental solution. Moreover, let $P(\xi)=\sum_{|\alpha|\le m}c_\alpha\xi^\alpha\in\mathbb{C}[\xi]\backslash\{0\}$ be a not identically vanishing polynomial on $\mathbb{R}^N$ of degree m; if $\eta\in S^{N-1}$ where $P_m(\eta)\neq 0$, the real numbers $\lambda_0,\ldots,\lambda_m$ are pairwise different, and $a_j=\prod_{k=0,k\neq j}^m(\lambda_j-\lambda_k)^{-1}$, then*

$$E=\frac{1}{\overline{P_m(2\eta)}}\sum_{j=0}^m a_j e^{\lambda_j\eta x}\mathcal{F}_\xi^{-1}\left(\frac{\overline{P(i\xi+\lambda_j\eta)}}{P(i\xi+\lambda_j\eta)}\right)$$

is a fundamental solution of $P(D)$ (i.e. $P(D)E=\delta$).

Proof: By Lemma 3.2, $m_N\left(\{\xi\in\mathbb{R}^N : P(i\xi+\lambda\eta)=0\}\right)=0$. Now, we can define

$$S(\xi)=\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}\in L^\infty(\mathbb{R}^N)\subset\mathcal{S}'(\mathbb{R}^N).$$

On the other hand, for $S \in \mathcal{S}'(\mathbb{R}^N)$ and $\eta \in \mathbb{C}^N$, we have by part (1) and (2) of the Proposition 0.20

$$P\left(D\right)\left(e^{\eta x}\mathcal{F}^{-1}S\right) \overset{1)}{=} e^{\eta x}P(D+\eta)\mathcal{F}^{-1}S \overset{2)}{=} e^{\eta x}\mathcal{F}_{\xi}^{-1}(P(i\xi+\eta)S)$$

Then, for $S = \overline{P(i\xi+\lambda\eta)}/P(i\xi+\lambda\eta)$ with $\lambda \in \mathbb{R}$

$$P\left(D\right)\left(e^{\lambda\eta x}\mathcal{F}^{-1}\left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}\right)\right) = e^{\lambda\eta x}\mathcal{F}_{\xi}^{-1}\left(\overline{P(i\xi+\lambda\eta)}\right).$$

Furthermore,

$$\mathcal{F}_{\xi}^{-1}(\overline{P(i\xi+\lambda\eta)}) = \mathcal{F}_{\xi}^{-1}(\overline{P}(-i\xi+\lambda\eta)) = \overline{P}(-D+\lambda\eta)\delta,$$

and hence

$$\begin{aligned} P\left(D\right)\left(e^{\lambda\eta x}\mathcal{F}^{-1}\left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}\right)\right) &= e^{\lambda\eta x}\overline{P}(-D+\lambda\eta)\delta = \overline{P}(-D+2\lambda\eta)(e^{\lambda\eta x}\delta) = \\ &= \overline{P}(-D+2\lambda\eta)\delta = \lambda^m\overline{P_m(2\eta)}\delta + \sum_{k=0}^{m-1}\lambda^k\overline{Q_k}(-\partial)\delta = \\ &= \lambda^m\overline{P_m(2\eta)}\delta + \sum_{k=0}^{m-1}\lambda^k T_k\delta \end{aligned}$$

for certain distributions $T_k \in \mathcal{E}'(\mathbb{R}^N)$. By our choice of a_j, we conclude from Lemma 3.4 that $P(D)E = \delta$.

□

Consequences of the existence of fundamental solutions

Definition 3.6. A linear partial differential operator P on $\Omega \subset \mathbb{R}^N$ is called *hypoelliptic* if for every $U \subset \Omega$ and for every $u \in \mathcal{D}'(U)$, $u \in \mathcal{E}(U)$ if $P(D)u \in \mathcal{E}(U)$.

Theorem 3.7. ***(Schwarz)*** *Let $P(D)$ be a partial differential operator with constant coefficients. $P(D)$ is hypoelliptic on $\mathbb{R}^N$ if and only if it has a fundamental solution E, with $E|_{\mathbb{R}^N\setminus\{0\}} \in \mathcal{E}(\mathbb{R}^N\setminus\{0\})$.*

Proof: By the Malgrange-Ehrenpreis Theorem 1.11, there is an $E \in \mathcal{D}'(\mathbb{R}^N)$ such that $P(D)E = \delta$. Since $P(D)$ is hypoelliptic on $\mathbb{R}^N$ and $P(D)E|_{\mathbb{R}^N\setminus\{0\}} = \delta|_{\mathbb{R}^N\setminus\{0\}} \in \mathcal{E}(\mathbb{R}^N\setminus\{0\})$, we have that $E|_{\mathbb{R}^N\setminus\{0\}} \in \mathcal{E}(\mathbb{R}^N\setminus\{0\})$.

In order to show the converse, let $U \subset \mathbb{R}^N$ be an open subset and $u \in \mathcal{D}'(U)$ with $f := P(D)u \in \mathcal{E}(U)$. Fix $x_0 \in U$ and $g \in \mathcal{D}(U)$ such that $g \equiv 1$ on a neighborhood U_0 of x_0 included on U. Using $P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ we get

$$\begin{aligned} P(D)(gu) &= \sum_{|\alpha|\le m} a_\alpha D^\alpha(gu) = gP(D)u + \sum_{0<|\alpha|\le m} a_\alpha \sum_{\gamma<\alpha} \tbinom{\alpha}{\gamma} D^{\alpha-\gamma} g D^\gamma u = \\ &= gP(D)u + v, \end{aligned}$$

where $v \in \mathcal{E}'(U)$ and $supp(v) \subset \complement U_0$, since $D^{\gamma-\alpha}g$ vanishes on U_0 for $\gamma < \alpha$. Since g has compact support, $gu \in \mathcal{E}'(U)$. Moreover, we have

$$\begin{aligned} E * (P(D)(gu)) = P(D)E * (gu) = \delta * (gu) = gu = \\ = E * (gP(D)u + v) = E * (gf) + E * v. \end{aligned}$$

We know that $gf \in \mathcal{D}(U)$, and this implies that $E * (gf) \in \mathcal{E}(\mathbb{R}^N)$. Then, is sufficient to show that $E * v$ is a $\mathcal{E}$-function on a neighborhood of x_0. For $\varepsilon > 0$ take $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^N)$ with $\varphi_\varepsilon \equiv 1$ on $B_{\frac{\varepsilon}{2}}$ (i.e. $B_{\mathbb{R}^N}\left(0, \frac{\varepsilon}{2}\right)$) and $supp(\varphi_\varepsilon) \subset B_\varepsilon$. Thus we have

$$E * v = (\varphi_\varepsilon E) * v + ((1-\varphi_\varepsilon)E) * v, \text{ with } supp\,(\varphi_\varepsilon E) * v \subset B_\varepsilon + supp(v).$$

Hence, for $\varepsilon > 0$ small enough we have $supp\,(\varphi_\varepsilon E) * v \subset \complement V$, where V is a neighborhood of x_0.

On the other hand (by assumption) , $(1-\varphi_\varepsilon)E$ is a $\mathcal{E}$-function , and therefore $(1-\varphi_\varepsilon)E * v$ is also a $\mathcal{E}$-function. Consequently, $gu \in \mathcal{E}(V)$. As $u|_V = gu|_V$, u is a $\mathcal{E}$-function on a neighborhood of x_0.

□

Corollary 3.8. *If $P(D)$ is a hypoelliptic partial differential operator with constant coefficients on $\mathbb{R}^N$, then for each open subset U of $\mathbb{R}^N$, the topologies of $C^\infty(U), C(U), \mathcal{D}'_b(U)$ induced on*

$$\mathcal{N}_{P(D)}(U) := \{f \in \mathcal{E}(U) : P(D)f = 0\}$$

coincide.

Proof: Certainly $\mathcal{T}_{\mathcal{E}(U)} \succeq \mathcal{T}_{C(U)} \succeq \mathcal{T}_{\mathcal{D}'_b(U)}$ (i.e., the topology $\mathcal{T}_{\mathcal{D}'_b(U)}$ is finer that $\mathcal{T}_{C(U)}$, and the topology $\mathcal{T}_{C(U)}$ is finer that $\mathcal{T}_{\mathcal{E}(U)}$) . Take a net $(u_i)_{i\in I} \subset \mathcal{N}_{P(D)}(U)$ such that $u_i \to 0$ in $\mathcal{D}'_b(U)$. It is sufficient to show that for every $x_0 \in U$, there exists a neighborhood of x_0, V, such that $u_i \to 0$ on $\mathcal{E}(V)$. To do this, we fix $x_0 \in U$, U_0 a neighborhood of x_0 with $U_0 \subset U$, and $g \in \mathcal{D}(U)$ with $g \equiv 1$ on U_0. As saw in the proof of Theorem 3.7

$$\begin{aligned} P(D)(gu) &= \sum_{|\alpha|\leq m} a_\alpha D^\alpha(gu) = gP(D)u + \sum_{0<\alpha|\leq m} a_\alpha \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} D^{\alpha-\gamma} g D^\gamma u = \\ &= gP(D)u + v. \end{aligned}$$

Hence $supp(u_i) \subset \complement U_0$ and $supp(v_i) \subset supp(g)$ for every $i \in I$. Additionally, as we have seen before,

$$\begin{aligned} E * (P\,(D)\,(gu_i)) = E * v_i = P(D)E * (gu_i) = \delta * (gu_i) = gu_i = \\ = (\varphi_\varepsilon E) * v_i + ((1-\varphi_\varepsilon)\,E) * v_i. \end{aligned}$$

Chosing $\varepsilon > 0$ small enough, there exists a neighborhood of x_0, V, with $V \subset U_0$ and $supp((\varphi_\varepsilon E) * v_i) \subset \complement V$. By the definition of v_i, $v_i \to 0$ in $\mathcal{D}'_b(V)$. Because $supp(v_i) \subset supp(g)$ for each $i \in I$, $v_i \to 0$ in $\mathcal{E}'_b(\mathbb{R}^N)$. Now,

$$u_i|_V = gu_i|_V = ((1-\varphi_\varepsilon)\,E) * v_i|_V + \underbrace{(\varphi_\varepsilon E) * v_i|_V}_{=0} = ((1-\varphi_\varepsilon)\,E) * v_i|_V\,.$$

Since $((1-\varphi_\varepsilon)\,E) \in \mathcal{E}(\mathbb{R}^N)$ and $v_i \to 0$ in $\mathcal{E}'_b(\mathbb{R}^N)$, we have $((1-\varphi_\varepsilon)\,E)\,v_i \to 0$ in $\mathcal{E}(\mathbb{R}^N)$. Therefore, we conclude $u_i \to 0$ in $\mathcal{E}(V)$, and this proof is complete.

□

Corollary 3.9. ***(Weyl's Lemma)***

(1) *Let $\Omega \subset \mathbb{C}$ and $T \in D'(\Omega)$ such that $\frac{\partial}{\partial \bar{z}} T = 0$. Then $T \in H(\Omega)$.*
(2) *Let $\Omega \subset \mathbb{R}^N$ and $T \in D'(\Omega)$ such that $\triangle T = 0$. Then T is harmonic in Ω.*

Proof: By the Malgrange-Ehrenpreis Theorem, both operators them have a fundamental solution E, and E satisfies

(1) By Corollary 3.8, if $T \in \mathcal{D}'(\Omega)$ with $\frac{\partial T}{\partial \bar{z}} = 0$, then $T \in \mathcal{E}$ and $\frac{\partial T}{\partial \bar{z}} = 0$. Thus satisfies Cauchy-Riemann condition, and $T \in H(\Omega)$
(2) By Corollary 3.8, if $T \in D'(\Omega)$ with $\triangle T = 0$. Then $T \in \mathcal{E}(\Omega)$ and $\triangle T = 0$. Thus T is harmonic in Ω.

□

Corollary 3.10. *Let P be a nonzero polynomial in $\mathbb{C}[z_1, \ldots, z_N]$, then*

a) *For every $T \in \mathcal{E}(\mathbb{R}^N)$ there exists $S \in \mathcal{D}'(\mathbb{R}^N)$ such that $P(D)S = T$.*
b) *For every $g \in \mathcal{D}(\mathbb{R}^N)$ there exists $f \in \mathcal{E}(\mathbb{R}^N)$ such that $P(D)f = g$.*
c) *Let ω, Ω be open subsets in $\mathbb{R}^N$ with $\overline{\omega} \subset\subset \Omega$. Then, for every $g \in \mathcal{E}(\Omega)$ there exists $f \in \mathcal{E}(\Omega)$ such that*

$$P(D)f|_\omega = g|_\omega \quad \textit{(local solvability)}.$$

Proof: Let E be a fundamental solution of $P(D)$.

a) Given $T \in \mathcal{E}(\mathbb{R}^N)$, we take $S := E * T \in \mathcal{D}'(\mathbb{R}^N)$. Then

$$P(D)S = (P(D)E) * T = \delta * T = T.$$

b) Given $g \in \mathcal{D}(\mathbb{R}^N)$, we take $f := E * g \in \mathcal{E}(\mathbb{R}^N)$. Then

$$P(D)f = (P(D)E) * g = \delta * g = g.$$

c) Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ with $\varphi \equiv 1$ on a neighborhood of $\overline{\omega}$. Given $g \in \mathcal{E}(\Omega)$, $\varphi g \in \mathcal{D}(\mathbb{R}^N)$. By *b*) there exists $\widetilde{f} \in \mathcal{E}(\mathbb{R}^N)$ such that $P(D)\widetilde{f} = \varphi g$. Then $f := \widetilde{f}\Big|_\Omega \in \mathcal{E}(\Omega)$ and

$$P(D)f|_\omega = \varphi g|_\omega = g|_\omega \quad \text{(local solutions)}.$$

□

Corollary 3.11. *Let P be a nonzero polynomial in $\mathbb{C}[z_1, \ldots, z_N]$. Then,*

a) *There exists a fundamental solution of $P(D)$ in $\mathcal{E}'(\mathbb{R}^N)$ if and only if $P(D)$ is constant.*

b) *If* $T \in \mathcal{E}'\left(\mathbb{R}^N\right)$ *with* $P(D)T = 0$, *then* $T = 0$.

Proof: Let E be a fundamental solution of $P(D)$.

a) Suppose that P is not constant and there exists $E \in \mathcal{E}'\left(\mathbb{R}^N\right)$ with $P(D)E = \delta$. Then

$$P(D) : \mathcal{E}'\left(\mathbb{R}^N\right) \longrightarrow \mathcal{E}'\left(\mathbb{R}^N\right)$$

is surjective. Indeed, for every $T \in \mathcal{E}'\left(\mathbb{R}^N\right)$ we have $P(D)(E * T) = T$ with $E * T \in \mathcal{E}'\left(\mathbb{R}^N\right)$. Note that

$$P(-D) = P(D)^t : \mathcal{E}\left(\mathbb{R}^N\right) \longrightarrow \mathcal{E}\left(\mathbb{R}^N\right)$$

is injective, because it is the transpose of $P(D)$. Indeed, if $P(D)^t f = 0$, this implies, for every $T \in \mathcal{E}'\left(\mathbb{R}^N\right)$, $\langle P(D)T, f\rangle = 0$, since $P(D)\mathcal{E}'\left(\mathbb{R}^N\right) = \mathcal{E}'\left(\mathbb{R}^N\right)$. Then, $\langle u, f\rangle = 0$ for every $u \in \mathcal{E}'\left(\mathbb{R}^N\right)$, and we conclude that $f = 0$.
Since P is not constant then $\check{P}(z) := P(-z)$ is not neither, if we take a root of $\check{P}$, $z_0 \in \mathbb{C}^N$, then $f := e^{i\langle x, z_0\rangle}$ is a nonzero $\mathcal{E}$ function with $P(-D)f = \check{P}(z_0)f = 0$. Consequently, $P(-D)$ is not injective. Thus, we conclude that P must be constant.

The converse holds taking $E = \delta/P(0)$.

b) It comes easily from

$$0 = E * 0 = E * P(D)T = P(D)E * T = \delta * T = T.$$

□

Definition 3.12. Let $P(z) = \sum_{|\alpha|\leq m} a_\alpha z^\alpha$ be a polynomial in $\mathbb{C}[z_1, \ldots, z_N]$. Then,

$$V(P) := \left\{z \in \mathbb{C}^N : P(z) = 0\right\}$$

is a 0-variety of P and

$$W_P(\Omega) := \{f \in C^\infty(\Omega) : P(D)f = 0\}$$

is the set of solutions of the homogeneous equation.
We define the *exponential solutions*:

$$ES_P := \left\{u(x) := Q(x)e^{i<x,z>} : Q \in \mathbb{C}[z_1, \ldots, z_N], z \in \mathbb{C}^N \text{ and } P(D)u = 0\right\}$$

satisfying the next properties:

a) Given $z \in V(P)$, then $P(D)e^{i\langle x,z\rangle} = P(z)e^{i\langle x,z\rangle} = 0$.
Therefore, $\{e^{i\langle x,z\rangle} : z \in V(P)\} \subset ES_P$.

b) Given $v(x) := Q(x)e^{i\langle x,\xi\rangle} \in ES_P$, then $\xi \in V(P)$. Therefore, $P(\xi) = 0$.

Theorem 3.13. ***(Malgrange Theorem, 1956)*** *Let Ω a convex subset then* $\overline{span\,(ES_P)}^{\mathcal{E}(\Omega)} = \mathcal{N}_P(\Omega)$.

The former theorem is not a trivial theorem and its proof is very difficult. This theorem implies that every 0-solution of $P(D)u = 0$ can be approximated by global 0-solutions (on $\mathbb{R}^N$).

Theorem 3.14. *If Ω is convex, then for every $g \in \mathcal{E}(\Omega)$ there exists $f \in \mathcal{E}(\Omega)$ such that $P(D)f = g$ (i.e., $P(D) : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ is surjective).*

Proof: Let $K_1 \subset \overset{\circ}{K}_2 \subset K_2 \subset \overset{\circ}{K}_3 \subset K_3 \subset \dots$ be a fundamental sequence of compact sets in Ω. We select $\psi \in \mathcal{D}(\Omega)$ such that $\psi_j \equiv 1$ on a neighborhood of K_j. We set $\varphi_1 = \psi_1$, $\varphi_j := \psi_j - \psi_{j-1}$ with $j > 1$. Note that, $\varphi \equiv 0$ on a neighborhood of K_{j-1} with $j > 1$ and $\sum_{j=1}^{\infty} \varphi_j = 1$ on Ω (and locally finite sum).
Using the fundamental solution, as $g\varphi_j \in \mathcal{D}\left(\mathbb{R}^N\right)$, for every j there exists $f_j \in C^\infty(\mathbb{R}^N)$ with $P(D)f_j = g\varphi_j$. Given f_j, $P(D)f_j = 0$ on a neighborhood of K_{j-1}, then, by 3.13, there exists $h_j \in C^\infty(\mathbb{R}^N)$ with $P(D)h_j = 0$ and

$$\sup_{x \in K_{j-1}} \sup_{|\alpha| \leq j-1} \left| (f_j - h_j)^{(\alpha)}(x) \right| \leq \frac{1}{2^j}.$$

Set $h_1 = 0$ and consider the series

$$f := \sum_{j=1}^{\infty} (f_j - h_j).$$

We have, for $n > m$ and $p \in \mathbb{N}$

$$\sup_{x \in K_m} \sup_{|\alpha| \leq m} \left(\sum_{\nu=n+1}^{n+p} (f_\nu - h_\nu)^{(\alpha)}(x) \right) \leq \sum_{\nu=n+1}^{n+p} \frac{1}{2^\nu}.$$

Then, we conclude that $f \in C^\infty(\Omega)$ and

$$P(D)f = \sum_{j=1}^{\infty} (P(D)f_j - P(D)h_j) = \sum_{j=1}^{\infty} P(D)f_j = \sum_{j=1}^{\infty} g\varphi_j = g.$$

$\square$

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Thesis

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