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COMPORTAMIENTO CAÓTICO EN COMPACTOS INVARIANTES PARA OPERADORES LINEALES

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Contents

Sumario

En este trabajo vamos a estudiar el comportamiento caótico en compactos invariantes para operadores lineales. Veremos algunos teoremas y resultados que nos permiten caracterizar a estos operadores y nos proporcionan algunos ejemplos concretos. Previamente mostraremos resultados similares que ya han sido estudiados para subespacios en general, y que nosotros adaptaremos para el estudio de la hiperciclicidad en operadores sobre compactos absolutamente convexos.

Primeramente, en el Capítulo 1 daremos unas nociones básicas sobre análisis funcional que nos serán de gran utilidad a lo largo del presente trabajo, en particular sobre espacios de Banach, así como teoría de operadores. Describiremos el concepto de sistema dinámico discreto y sus propiedades fundamentales. Presentaremos distintos teoremas y resultados que caracterizan dichas propiedades y que serán empleados en los siguientes capítulos.

En el Capítulo 2 introduciremos el concepto de operador sub-hipercíclico respecto a un subespacio que Blair F.Madore y Rubén A.Martínez-Avendaño estudiaron en [**Blair010**] y Can Le en [**Le**]. Para ello haremos una recopilación de los resultados más importantes de estos artículos.

En el Capítulo 3 , mostraremos el resultado obtenido por Nathan S.Feldman en [**NathF**] según el cual existe un operador lineal T con la propiedad de que cualquier función continua f en un espacio métrico compacto X, es topológicamente conjugada a la restricción de T sobre un compacto invariante, lo que motiva nuestro trabajo y refleja la importancia del estudio de operadores sobre compactos invariantes.

Finalmente en el Capítulo 4, mostraremos algunos resultados nuevos sobre operadores definidos en compactos invariantes . Para ello nos centraremos en compactos absolutamente convexos y veremos teoremas que nos permiten generalizar propiedades que dadas sobre un operador en un compacto invariante también se verifican si extendemos el operador a su envoltura convexa. Además presentaremos ejemplos dónde se ponen de manifiesto dichos resultados, y mostraremos teoremas que nos caractericen cuando un operador sobre un compacto es transitivo, mezclante o caótico.

Summary

In this work we will study the chaotic behavior in compact invariant sets for linear operators. We will see several theorems and results that will allow us to characterize these operators and will provide us some concrete examples. To lay a foundation for these results, we will show similar results for subspaces in general that have already been studied and that we will adapt to the study of hypercyclicity for operators defined on invariant, compact and absolutely convex sets.

To begin, in Chapter 1 we will give some basic notions of funtional analysis that will be useful throughout this work, in particular while studying Banach spaces and operator theory. We will also describe the concept of discrete dynamical systems and their fundamental properties. We will establish different theorems and results that characterize these properties, which will be used in the following chapters.

Later, in Chapter 2 we will introduce the concept of subspace-hypercyclicity that Blair F.Madore and Rubén A.Martínez-Avendaño studied in [**Blair010**] and that Can Le examined in [**Le**]. To this end, we will display the most important results obtained in these articles.

In Chapter 3, we will demonstrate Nathan S.Feldman's result obtained on the Hilbert space, in [**NathF**], which gives us that there exists an operator T such that any function f in a compact metric space X is topologically conjugate to the restriction of T to a invariant compact set. This achievement reflects the importance of studying operators defined on invariant compact sets.

Finally, in Chapter 4, we will prove some new results about operators defined on absolutely convex invariant compact sets. We will establish some theorems that will let us conclude when a property that is satisfied by an operator defined on a invariant compact set is also verified if we extend the operator to its convex envelope. Additionally, we will illustrate this situation with several examples, and we will show some theorems that characterize when an operator defined on a compact invariant set is chaotic, transitive, or mixing.

CHAPTER 1

Preliminaries

In this section we set up basic definitions, theorems and some tools that will be helpful in this work. The main references can be found in the books [**Rud74, MV97, HP57, Rob**].

1. Metric, Banach, Frèchet and Hilbert spaces

We can start with the notion of metric, Banach and Hilbert spaces and their properties:

Definition 1.1 (Metric space)**.** A real-valued function $d : X \times X \to \mathbb{R}$, defined for each pair of elements $x, y \in X$ is called a *metric* if it satisfies:

(i) $d(x, y) \geq 0$, $d(x, x) = 0$ and $d(x, y) > 0$ if $x \neq y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$, the triangle inequality.

A set X provided with a metric is called *metric space* and $d(x, y)$ is called the distance between x and y.

We will understand by *neighborhood* of a point $p \in X$ a set $\mathcal{U} \subset X$, which contains an open set $\mathcal{V}$ containing p.

A point x in a metric space X is called *isolated* if some neighbourhood of x contains no other point in X.

A metric space is said to be *locally compact* if each point has a compact neighbourhood. Finally, we say that a metric space is *complete* if every Cauchy sequence in X converges to an element of X.

Theorem 1.2 (Baire category theorem)**.** *Let (X, d) be a complete metric space and $\{G_n\}_n$ a sequence of nonempty dense open sets. Then $G := \bigcap_{n=1}^{\infty} G_n$, is a dense G_δ-set in X.*

Definition 1.3. A functional $p : X \to \mathbb{R}_+$ on a vector space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is called a *seminorm* if for all $x, y \in X$ and $\lambda \in \mathbb{K}$:

(i) $p(x + y) \leq p(x) + p(y)$
(ii) $p(\lambda x) = |\lambda| p(x)$.
If, in addition,
(iii) $p(x) = 0$ implies that $x = 0$ then p is called a *norm*.

Definition 1.4 (Fréchet space)**.** A Fréchet space is a vector space X endowed with a separating increasing sequence $(p_n)_n$ of seminorms which is complete in the metric given by:

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} min(1, p_n(x-y)), x, y \in X.$$

Definition 1.5 (Normed space)**.** The pair $(X, ||\bullet||)$ is called a *normed space* where X is a vector space endowed with a norm $|| \bullet ||$.

Every normed linear space may be regarded as a metric space, being $\|x-y\|$ the distance between x and y. A *Banach space* is a normed linear space which is complete with the metric defined by its norm.

Definition 1.6 (Hilbert space)**.** A *Hilbert space* H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product. So H is a complex vector space on which there is an inner product $\langle x, y \rangle$ associating a complex number to each pair of elements x, y of H that satisfies the following properties:

(i) $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

(ii) It is linear in its first argument. For all complex numbers:

$$\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle.$$

(iii) It is positive definite: $\langle x, y \rangle \geq 0$ and it's equal to 0 if and only if $x = 0$.

The norm defined by the inner product $\langle \bullet, \bullet \rangle$ is the real-valued function:

$$\|x\| = \sqrt{\langle x, x \rangle}$$

and the distance between two points x,y in H is defined in terms of the norm by:

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle}.$$

Proposition 1.7. *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator. The following four statements are equivalent:*

(i) *T is continuous at 0.*

(ii) *T is continuous.*

(iii) *T is uniformly continuous.*

(iv) *T is bounded, i.e., there exists a constant $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$.*

Definition 1.8. Let X and Y be Banach spaces and $T : X \to Y$ be a continuous linear operator. We define

$$\|T\| := \inf\{C > 0 : \|Tx\|_Y \leq C\|x\|_X \text{ for all } x \in X\},$$

and we refer to $\|T\|$ as the operator norm of T.

Some equivalent formulations are the following:

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|_Y = \sup_{\|x\|=1} \|Tx\|_Y.$$

In this work there appears some results of characterization of hypercyclicity based on spectral theory. Some basic results of functional analysis that will be useful in understanding some of these concepts are the following:

Definition 1.9. Let X be a complex Banach space X, and let T be an operator on X. The *spectrum* $\sigma(T)$ of T is defined as:

$$\sigma(T) = \{\lambda \in \mathbb{C} \; ; \; \lambda I - T \text{ is not invertible}\}.$$

Moreover, each $0 \neq x \in X$ satisfying $Tx = \lambda x$ is an eigenvector for T corresponding to λ.

The *point spectrum* $\sigma_p(T)$ is the set of eigenvalues of T.
The number

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

is called the *spectral radius* of T. We also have that

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

Theorem 1.10 (Riesz decomposition theorem)**.** *If $\sigma(T) = \sigma_1(T) \cup \sigma_2(T)$, where σ_1 and σ_2 are two disjoint non-empty closed sets, there exist non-trivial T-invariant closed subspaces M_1 and M_2 of X such that $X = M_1 \oplus M_2$, and*

$$\sigma(T \mid M_1) = \sigma_1 \quad \textit{and} \quad \sigma(T \mid M_2) = \sigma_2.$$

Definition 1.11. An operator T on a complex Banach space X is called *compact* if for every $(x_n)_n$ in X with $\|x_n\| \leq 1$, $n \geq 1$, the sequence $(Tx_n)_n$ has a convergent subsequence. This is equivalent to saying that the image of the closed unit ball under T is *relatively compact*; that is, its closure is compact.

Theorem 1.12. *Let T be compact and $\lambda \neq 0$. Then:*

(1) *Every nonzero $\lambda \in \sigma(T)$ is an eigenvalue of T.*
(2) *For every $n \geq 1$, $Ker(\lambda I - T)^n$ is finite-dimensional.*
(3) *For every $n \geq 1$, the image of $(\lambda I - T)^n$ is closed.*
(4) *There is an $n \geq 1$ such that, for all $k \geq 0$,*

$$Im(\lambda I - T)^n = Im(\lambda I - T)^{t+k}.$$

(5) *$\sigma(T)$ is countable.*
(6) *The eigenvalues can only accumulate at 0. If the dimension of X is not finite, then $\sigma(T)$ must contain 0.*

Definition 1.13. Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators $T : X \to Y$ endowed with the operator norm. This space turns into a Banach space whenever Y is a Banach space. If $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$, the *dual* $X^* = \mathcal{L}(X, \mathbb{K})$ of a Banach space X is the space of all continuous linear functionals on X. If $x^* \in X^*$ then we write,

$$x^*(x) = \langle x, x^* \rangle, \quad x \in X.$$

The *adjoint* $T^* : X^* \to X^*$ of an operator T on X is defined by $T^* x^* = x^* \circ T$; that is,

$$\langle x, T^* x^* \rangle = \langle Tx, x^* \rangle, \quad x \in X, \; x^* \in X^*.$$

2. Classical Banach spaces

Now we introduce the classical sequence and function spaces that we will use in this work. Here, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$. The symbol X will always stand for a Banach space.

If $p \in [1, +\infty)$, we will use the following notation:

$$\ell^\infty := \ell^\infty(X) := \{(x_n)_{n\in\mathbb{N}} \subset X : \sup_{n\in\mathbb{N}} \|x_n\|_X < \infty\}$$

with the norm $\|(x_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\|_X$,

$$\ell^p := \ell^p(X) := \{(x_n)_{n\in\mathbb{N}} \subset X : \sum_{n\in\mathbb{N}} \|x_n\|_X^p < \infty\}$$

with the norm $\|(x_n)_{n\in\mathbb{N}}\|_p := \left(\sum_{n\in\mathbb{N}} \|x_n\|_X^p\right)^{1/p}$ and

3. Linear Discrete Dynamical systems

In this section we will introduce some basic definitions and results of linear dynamical systems. The results that are shown in this chapter can be found in [**BM09**] and [**GP11**].

Dynamical systems are used to study the behavior of evolving systems. Let X be a set of elements that describes the different acceptable states of a system. If $x_n \in X$ is the state of the system at time $n \geq 0$, then its evolution will be given by a linear map $T : X \to X$ such that $x_{n+1} = T(x_n)$. We will work in a Banach space X and a continuous map T.

Definition 1.14 (Discrete dynamical system)**.** Let X be an metric space and let T be a continuous map $T : X \to X$. A *discrete dynamical system* is a pair (X, T). We define the *orbit* of a point $x \in X$ as the set $Orb(x, T) = \{T^n(x) : n \in \mathbb{N}\}$, where T^n denotes the n-th iterate of a map T. We will often simply say that T or $T : X \to X$ is a dynamical system.

Definition 1.15. Let $S : Y \to Y$ and $T : X \to X$ be dynamical systems.

(1) Then T is called quasi-conjugate to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$; that is, the following diagram commutes.

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \downarrow\Phi & & \downarrow\Phi \\ X & \xrightarrow{T} & X \end{array}$$

(2) If ϕ can be chosen to be a homeomorphism, then S and T are called conjugates.

Definition 1.16. We say that a property $\wp$ for dynamical systems is preserved under(quasi-) conjugacy if the following holds: if a dynamical system $S : Y \to Y$ has property $\wp$ then every dynamical system $T : X \to X$ that is (quasi-) conjugate to S also has property $\wp$.

Definition 1.17. Let $T : X \to X$ be a dynamical system. Then $Y \subset X$ is called T-invariant or invariant under T if $T(Y) \subset Y$.

Definition 1.18. We say that $x \in X$ is a *fixed point for the dynamical system* $T : X \to X$ if $Tx = x$, and we say that $x \in X$ is a *periodic point for the dynamical system* T if $T^n x = x$ for some $n \in \mathbb{N}$. The set of all periodic points is denoted by $Per(T)$. If $x \in Per(T)$ then the smallest positive integer n such that $T^n x = x$ is called a *primary period* of x.

Definition 1.19. A dynamical system $T : X \to X$ is:

(i) *topologically transitive* if for any pair of nonempty open sets $U, V \subset X$ there exists an $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$;
(ii) *weakly mixing* if the map $T \times T$ is topologically transitive;
(iii) *mixing* if for any pair of nonempty open sets $U, V \subset X$ there exists some $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for every integer $n \geq n_0$;

Observation 1. *For any linear dynamical system,*

$$\textit{mixing} \Longrightarrow \textit{weak mixing} \Longrightarrow \textit{topological transitivity}$$

In 1989 Robert L. Devaney proposed the first good definition of chaos; see [**Dev89**]. This concept reflects the unpredictability of chaotic systems because the definition contains a *sensitive dependence on initial conditions*, i.e.:

Definition 1.20. Let (X, d) be a metric space without isolated points. Then the dynamical system $T : X \to X$ is said to have *sensitive dependence on initial conditions* if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon$ such that, for some $n \geq 0$, $d(T^n x, T^n y) > \delta$. The number δ is called a *sensitivity constant* for T.

Definition 1.21 (Devaney chaos)**.** A dynamical system $T : X \to X$ is called chaotic in the sense of *Devaney* if it satisfies the following properties:

(i) T is topologically transitive,
(ii) $Per(T)$ is dense in X,
(iii) T has sensitive dependence on initial conditions.

However, Banks, Brooks, Cairns, Davis and Stacey proved in 1992, through their work in [**BBCDS92**], that if X is an infinite set, the sensitivity is a consequence of transitivity and dense periodicity.

Theorem 1.22 ([**BBCDS92**])**.** *Let X be a non-finite metric space. If a dynamical system $T : X \to X$ is topologically transitive and has a dense set of periodic points then T has sensitive dependence on initial conditions with respect to any metric defining the topology of X.*

A link between chaos theory and linear operator theory was established by by Birkhoff's Transitivity Theorem in 1920. In this theorem, he showed that the topological transitivity was equivalent to the notion of hypercyclicity that Beauzamy established in 1986:

Definition 1.23 ([**Bea86**])**.** Let X be a topological vector space.
An operator $T : X \to X$ is said to be *hypercyclic* if there is an $x \in X$ whose orbit $Orb(x, T)$ is dense in X. In that case, x is called a *hypercyclic vector* for T. The set of hypercyclic vectors is denoted by $HC(T)$.

Theorem 1.24 (Transitivity theorem, [**Bir20**]). *Let X be a separable complete metric space without isolated points, and let $T : X \to X$ be a continuous map. Then the following assertions are equivalent:*

(i) *T is topologically transitive;*
(ii) *T is hypercyclic operator.*

If one of these conditions holds then, by Theorem 1.2, the set $HC(T)$ of hypercyclic vectors is a dense G_δ-set; i.e., $HC(T)$ is a countable intersection of open dense sets.

In 1991 Godefroy and Shapiro also adopted Devaney's definition for linear chaos.

Definition 1.25 ([**GS91**]). Let X be a complete metric vector space. An operator $T : X \to X$ is called chaotic in the sense of Devaney, if:

(i) T is hypercyclic.
(ii) $Per(T)$ is dense in X.

Proposition 1.26. *The following properties are preserved by quasi-conjugacy:*

(i) *Topological transitivity.*
(ii) *The property of having a dense orbit.*
(iii) *The property of having a dense set of periodic points.*
(iv) *Devaney Chaos.*
(v) *The mixing property.*
(vi) *The weak-mixing property.*

The reader can find the proofs of the following results in [**BM09**]. Additionally, the original proofs of some of these results can be found in [**Kit82**]:

Proposition 1.27. *Let T be a hypercyclic operator on a (real or complex) Banach space X. Then we have:*

(i) *T^* has no eigenvalues, that is, $\sigma_p(T^*) = \emptyset$;*
(ii) *the orbit of every $x^* \neq 0$ in X^* under T^* is unbounded.*

Theorem 1.28. *Let T be a hypercyclic operator on a complex Banach space X. Then every connected component of $\sigma(T)$ meets the unit circle S^1, i.e., $\sigma(T) \cap S^1 \neq \emptyset$.*

Proposition 1.29. *Let T be a linear map on a complex vector space X. Then the set $Per(T)$ of periodic points of T is given by*

$$span\{x \in X \ ; \ Tx = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ with } \lambda^n = 1 \text{ for some } n \in \mathbb{N}\}.$$

Proposition 1.30. *Let T be a chaotic operator on a complex Banach space X. Then its spectrum has no isolated points and it contains infinitely many roots of unity; in particular, $\sigma(T) \cap S^1$ is infinite.*

Theorem 1.31. *No compact operator is hypercyclic.*

4. Hypercyclicity criterion

There are some criteria under which an operator is chaotic, mixing or weakly mixing. These criteria are the following:

Theorem 1.32 (Godefroy-Shapiro criterion). *Let T be an operator. Suppose that the subspaces*

$$X_0 := span\{x \in X; \quad Tx = \lambda x \quad \text{for some} \quad \lambda \in \mathbb{K} \quad \text{with} \quad |\lambda| < 1\}$$

$$Y_0 := span\{x \in X; \quad Tx = \lambda x \quad \text{for some} \quad \lambda \in \mathbb{K} \quad \text{with} \quad |\lambda| > 1\}$$

are dense in X.
Then T is mixing, and in particular hypercyclic.

If, moreover, X is a complex space and the subspace

$$Z_0 := span\{x \in X; \quad Tx = \lambda x \quad \text{for some} \quad \lambda \in \mathbb{C} \quad \text{with} \quad |\lambda|^n = 1 \quad \text{for some} \quad n \in \mathbb{N}\}$$

is dense in X, then T is chaotic.

Theorem 1.33 (Kitai's criterion). *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$ and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^n x \to 0$,
(ii) $S^n y \to 0$,
(iii) $TSy \to y$,

then T is mixing.

Theorem 1.34 (Gethner-Shapiro criterion). *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^{n_k} x \to 0$,
(ii) $S^{n_k} y \to 0$,
(iii) $TSy \to y$,

then T is weakly mixing.

Theorem 1.35 (Hypercyclicity criterion). *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and maps $S_{n_k} : Y_0 \to X$, $k \geq 1$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^{n_k} x \to 0$,
(ii) $S^{n_k} y \to 0$,
(iii) $T^{n_k} S^{n_k} y \to y$,

then T is weakly mixing, and in particular hypercyclic.

5. Weighted shifts

In this section we include some basic results of weighted shifts, which make up an important class of hypercyclic and chaotic operators.

Definition 1.36. The basic model of all shifts is the *backward shift*

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Another shift is the *weighted backward shift* which is defined as:

$$B_w(x_1, x_2, x_3, \ldots) = (w_2 x_2, w_3 x_3, w_4 x_4, \ldots),$$

where $w = (w_n)_n$ is called a *weight sequence*. The weights w_n will always be assumed to be non-zero.

These operators can be defined on an arbitrary sequence space X, that is , a linear space of sequences or, in other words, a subspace of $w = \mathbb{K}^{\mathbb{N}}$. Moreover, X should carry a topology that is compatible with the sequences space structure of X. We interpret this by demanding that convergence in X should imply coordinatewise convergence. A Banach (Fréchet) space of this kind is called a Banach (Frèchet) sequence space.

Proposition 1.37. *Let X be a Fréchet sequence space. Then every weighted shift $B_w : X \to X$ is continuous.*

Theorem 1.38. *Let X be a Fréchet sequence space in which $(e_n)_n$ (where $e_n = (0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots)$) is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:*

(i) *B is hypercyclic;*
(ii) *B is weakly mixing;*
(iii) *there is an increasing sequence $(n_k)_k$ of positive integers such that $e_{n_k} \to 0$ in X as $k \to \infty$.*

Theorem 1.39. *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:*

(i) *B is chaotic;*
(ii) *$\sum_{n=1}^{\infty} e_n$ converges in X;*
(iii) *The constant sequences belong to X;*
(iv) *B has a non-trivial periodic point.*

Theorem 1.40. *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the backward shift B is an operator on X.*

(1) *The following assertions are equivalent:*
 (i) *B_w is hypercyclic;*
 (ii) *B_w is weakly mixing;*
 (iii) *There is an increasing sequence $(n_k)_k$ of positive integers such that*

$$\left(\prod_{\nu=1}^{n_k} w_\nu\right)^{-1} e_{n_k} \to 0$$

 in X as $k \to \infty$.
(2) *The following assertions are equivalent:*
 (i) *B_w is mixing;*
 (ii) *We have*

$$\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n \to 0$$

 in X as $n \to \infty$;
(3) *Suppose that the basis $(e_n)_n$ is unconditional (that is, for every $x \in X$, the representation $x = \sum_{n=1}^{\infty} a_n e_n$ converges unconditionally). Then the following assertions are equivalent:*
 (i) *B_w is chaotic;*

(ii) *The series*

$$\sum_{n=1}^{\infty}\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$$

converges in X*;*

(iii) *The sequence*

$$\left(\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1}\right)_n$$

belongs to X*;*

(iv) B_w *has a non-trivial periodic point.*

Remark 1.41. A weighted shift B_w is an operator on a sequence space ℓ^p, $1 \leq p \leq \infty$, if and only if the weights $w_n, n \geq 1$, are bounded. The respective characterizing conditions for B_w to be hypercyclic, mixing, or chaotic on ℓ^p, $1 \leq p \leq \infty$, are the following:

$$sup_{n\geq 1}\prod_{\nu=1}^{n}|w_\nu| = \infty; \quad lim_{n\to\infty}\prod_{\nu=1}^{n}|w_\nu| = \infty; \quad \sum_{n=1}^{\infty}\frac{1}{\prod_{\nu=1}^{n}|w_\nu|^p} < \infty$$

CHAPTER 2

Subspace hypercyclicity

The main purpose of this chapter is to formally introduce the concept of subspace-hypercyclicity that Blair F.Madore and Rubén A.Martínez-Avendaño studied in [**Blair010**]. We will first introduce the concept of subspace-transitivity and present some trivial examples.

Next, we will show the existence of nontrivial examples. For this, we will introduce the concept of subspace-transitivity and we will show that a subspace -hypercyclicity criterion holds. Finally, in the last section of this chapter we will prove that subspace-hypercyclicity like hypercyclicity, is a purely infinite-dimensional concept.

1. Definition and some trivial examples

In this chapter X always denotes a separable Banach space over the field of complex numbers $\mathbb{C}$. We will usually work in an infinite-dimensional space X, so we will explicitly indicate when a result or a definition only holds for finite or infinite dimensions.

Whenever we talk about a subspace M of X we will assume that M is topologically closed. We will denote by $\mathbf{B}(X)$ the set of all bounded linear operators on X. We will usually refer to elements of $\mathbf{B}(X)$ as just operators.

Definition 2.1. Let $T \in \mathbf{B}(X)$ and let M be a nonzero subspace of X. We say that T is *subspace-hypercyclic* for M if there exists an $x \in X$ such that $Orb(T, x) \cap M$ is dense in M. We call x a *subspace-hypercyclic vector.*

The definition above reduces to the classical definition of hypercyclicity if $M = X$. We start by showing the simplest example of a subspace-hypercyclic operator that is not hypercyclic.

Example 2.2. Let T be a hypercyclic operator on X with hypercyclic vector x and let I be the identity operator on X. Then the operator $T \oplus I : X \oplus X \to X \oplus X$ is subspace-hypercyclic for the subspace $M := X \oplus 0$ with subspace-hypercyclic vector $x \oplus 0$. Clearly, $T \oplus I$ is not hypercyclic.

The above example is trivial in the sense that M is an invariant subspace for $T \oplus I$. It is obvious that $T \oplus I|_M$ is, in fact, a hypercyclic operator. The following example is trivial in the same sense.

Example 2.3. Let T be a hypercyclic operator on H with hypercyclic vector x, and assume that $C \in \mathbf{B}(X)$ is nonzero and has closed range M. If $A \in \mathbf{B}(X)$ satisfies the equation $AC = CT$, then it can easily be checked that A is subspace-hypercyclic for M with subspace-hypercyclic vector Cx.

2. Some nontrivial examples and a subspace-hypercyclicity criterion

We would like to find examples of subspace-hypercyclic operators for a subspace M such that M is not invariant under the operator. To look for these examples, we will introduce a subspace-hypercyclicity criterion.

Let us denote the set of subspace-hypercyclic vectors for M by:

$$HC(T, M) := \{x \in H : Orb(T, x) \cap \mathrm{M} \text{ is dense in M}\}.$$

Lemma 2.4. *Let $T \in \boldsymbol{B}(X)$ and let M be a nonzero subspace of X. Then*

$$HC(T, M) = \bigcap_{j=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(B_j),$$

where $\{B_j\}$ is a countable open basis for the relative topology of M as a subspace of X.

Proof. We have that $x \in \bigcap_{j=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(B_j)$ if and only if, for any $j \in \mathbb{N}_0$, there exists a number $n \in \mathbb{N}_0$ such that $T^n x \in B_j$. But since $\{B_j\}$ is a basis for the relative topology of M, this occurs if and only if $Orb(T, x) \cap M$ is dense in M or, equivalently, if $x \in HC(T, M)$. □

Hence, if the set in the display above is nonempty, T is subspace-hypercyclic for M. Our following lemma will obtain much more than what is needed to imply the nonemptiness of said set. The following definition will be convenient in this process.

Definition 2.5. Let $T \in \mathbf{B}(X)$ and let M be a nonzero subspace of X. We say that T is *subspace-transitive with respect to M* if for all nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exists an $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ contains a relatively open nonempty subset of M.

Theorem 2.6. *Let $T \in \boldsymbol{B}(X)$ and let M be a nonzero subspace of X. Then the following conditions are equivalent:*

(i) *The operator T is subspace-transitive with respect to M.*
(ii) *For any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exists an $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M.*
(iii) *For any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exists an $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ is nonempty and $T^n(M) \subseteq M$.*

Proof. The implication $(ii) \Longrightarrow (i)$ is obvious. The implication $(iii) \Longrightarrow (ii)$ is obvious once we notice that the operator $T^n\mid_M \colon M \longrightarrow M$ is continuous and hence $T^{-n}(U)$ is relatively open in M if U is relatively open in M.

We will show that $(i) \Longrightarrow (iii)$. Let T be subspace-transitive with respect to M and let U and V be nonempty relatively open subsets of M. By Definition 3.2, it follows that there exists an $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ contains a relatively open nonempty set, say W. Thus, in particular, $T^{-n}(U) \cap V$ is nonempty.

Now, let $x \in M$. Since $W \subseteq T^{-n}(U)$, it follows that $T^n(W) \subseteq M$. Take $x_0 \in W$. Since W is relatively open and $x \in M$, for $r > 0$ small enough, we have $x_0 + rx \in W$, and hence $T^n(x_0 + rx) = T^n(x_0) + rT^n(x) \in M$. Since $T^n(x_0) \in M$, subtracting it and dividing by r leads to $T^n(x) \in M$, showing that $T^n(M) \subseteq M$ and finishing the proof. □

The following lemma will achieve 'half' of the classical equivalence of topological transitivity and hypercyclicity.

Lemma 2.7. *Let $T \in \boldsymbol{B}(X)$ and let M be a nonzero subspace of X. Assume that T is subspace-transitive with respect to M. Then*

$$\bigcap_{j=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(B_j) \cap M$$

is a dense subset of M. Here $\{B_j\}$ is a countable open basis for the (relative) topology of M.

Proof. By Theorem 2.6, for each j and k, there exists an $n_{j,k} \in \mathbb{N}_0$ such that the set $T^{n_{j,k}}(B_j) \cap B_k$ is nonempty and relatively open. Hence, the set

$$A_j := \bigcup_{k=1}^{\infty} T^{-n_{j,k}}(B_j) \cap B_k$$

is relatively open. Furthermore, each A_j is dense, since it intersects each B_k. By the Baire category theorem, this implies that

$$\bigcap_{j=1}^{\infty} A_j = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} T^{-n_{j,k}}(B_j) \cap B_k$$

is a dense set. But clearly,

$$\bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} T^{-n_{j,k}}(B_j) \cap B_k \subseteq \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} T^{-n}(B_j) \cap M,$$

and the result follows. □

The previous lemmas imply the following theorem:

Theorem 2.8. *Let $T \in \boldsymbol{B}(X)$ and let M be a nonzero subspace of X. If T is subspace-transitive for M then T is subspace-hypercyclic for M.*

The following theorem is a subspace-hypercyclicity criterion, stated in the style of Theorem 1.35.

Theorem 2.9. *Let $T \in \boldsymbol{B}(X)$ and let M be a nonzero subspace of X. Assume that there exist X_0 and Y_0, dense subsets of M, and an increasing sequence of positive integers $\{n_k\}$ such that*

(i) *$T^{n_k} \longrightarrow 0$ for all $x \in X_0$,*
(ii) *for each $y \in Y_0$, there exists a sequence $\{x_k\}$ in M such that $x_k \to 0$ and $T^{n_k} x_k \to y$,*
(iii) *M is an invariant subspace for T^{n_k} for all $k \in \mathbb{N}$.*

Then T is subspace-transitive with respect to M, and hence T is subspace-hypercyclic for M.

Proof. Let U and V be nonempty relatively open subsets of M. We will show that there exists a $k \in \mathbb{N}_0$ such that $T^{-n_k}(U) \cap V$ is nonempty. By Theorem 2.6, since $T^{n_k}(M) \subseteq M$, it will follow that T is subspace-transitive with respect to M.

Since X_0 and Y_0 are dense in M, there exists a $v \in X_0 \cap V$ and $u \in Y_0 \cap U$. Furthermore, since U and V are relatively open, there exists an $\epsilon > 0$ such that the M-ball centered at v of radius ϵ is contained in V and the M-ball centered at u of radius ϵ is contained in U.

By hypothesis, given these $v \in X_0$ and $u \in Y_0$, one can choose k large enough such that there exists $x_k \in M$ with $\|T^{n_k}v\| < \frac{\epsilon}{2}$, $\|x_k\| < \epsilon$ and $\|T^{n_k}x_k - u\| < \frac{\epsilon}{2}$. We have:

- $v + x_k \in V$. Indeed, since $v \in M$ and $x_k \in M$, it follows that $v + x_k \in M$. Also, since
$$\|(v + x_k) - v\| = \|x_k\| < \epsilon,$$
it follows that $v + x_k$ is in the M-ball centered at v of radius ϵ and hence $v + x_k \in V$.
- $T^{n_k}(v + x_k) \in U$. Indeed, since v and x_k are in M and M is invariant under T^{n_k}, it follows that $T^{n_k}(v + x_k) \in M$. Also,
$$\|T^{n_k}(v + x_k) - u\| \leq \|T^{n_k}v\| + \|T^{n_k}x_k - u\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$
and hence $T^{n_k}(v + x_k)$ is in the M-ball centered at u of radius ϵ. Thus $T^{n_k}(v + x_k) \in U$.

The two facts above imply that $v + x_k \in T^{-n_k}(U) \cap V$ and hence this set is nonempty. □

A natural question to ask is whether condition (iii) in the theorem above is really necessary. Le [**Le**] proved an alternative Subspace-Hypercyclicity Criterion with a condition weaker than condition (iii) above.

Theorem 2.10. *Let $T \in \boldsymbol{B}(X)$ and let M be a nonzero subspace of X. Assume there exist X_0 and Y_0, dense subsets of M, and an increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ such that*

(i) *$T^{n_k} \longrightarrow 0$ for all $x \in X_0$,*
(ii) *for each $y \in Y_0$, there exists a sequence $\{x_k\}$ in M such that $x_k \to 0$ and $T^{n_k}x_k \to y$,*
(iii) *$X_0 \subset \bigcap_{k=1}^{\infty} T^{-n_k}(M)$.*

Then T is subspace-hypercyclic for M.

Proof. Let $(\varepsilon_j)_{j=1}^{\infty}$ be a sequence of positive numbers such that
$$lim_{j\to\infty}(j\varepsilon_j + \sum_{i=j+1}^{\infty} \varepsilon_i) = 0.$$
Since X is separable, we can assume that $Y_0 = (y_j)_{j=1}^{\infty}$ for some sequence $(y_j)_{j=1}^{\infty}$. We construct a sequence $(x_j)_{j=1}^{\infty} \subset X_0$ and a subsequence $(n_{k_j})_{j=1}^{\infty}$ of $(n_k)_{k=1}^{\infty}$ by induction. Let $x_1 \in X_0$ and n_{k_1} be such that $\|x_1\| + \|T^{n_{k_1}}x_1 - y_1\| < \epsilon_1$. For each j

choose n_{k_j} and $x_j \in X_0$ such that $\|x_j\| + \|T^{n_{k_j}} x_i\| + \|T^{n_{k_i}} x_j\| + \|T^{n_{k_j}} x_j - y_j\| < \epsilon_j$ for all $i < j$.

Let $x = \sum_{i=1}^{\infty} x_i$. Since $\sum_{i=1}^{\infty} \|x_i\| \leq \sum_{i=1}^{\infty} \varepsilon_i$, x is well defined. For every j, we have $T^{n_{k_j}} x \in M$ and

$$\|T^{n_{k_j}} x - y_j\| = \left\| T^{n_{k_j}} x_j - y_j + \sum_{i=1}^{j-1} T^{n_{k_j}} x_i + \sum_{i=j+1}^{\infty} T^{n_{k_j}} x_i \right\|$$

$$\leq \|T^{n_{k_j}} x_j - y_j\| + \sum_{i=1}^{j-1} \|T^{n_{k_j}} x_i\| + \sum_{i=j+1}^{\infty} \|T^{n_{k_j}} x_i\| \leq j\varepsilon_j + \sum_{i=j+1}^{\infty} \varepsilon_i.$$

Thus $lim_{j\to\infty} \|T^{n_{k_j}} x - y_j\| = 0$ and T is M-hypercyclic. □

Note that Theorem 2.9 is stronger than Theorem 2.10 because it shows that T is, in fact, subspace-transitive for M. In general, let $T : X \to X$ be an operator for which there exists an increasing sequence $\{n_k\}$ of natural numbers such that $T^{n_k}(M) \subseteq M$. If the sequence $T^{n_k}|_M : M \to M$ is universal, it follows that T is subspace-hypercyclic for M. Contrast this with example 2.12, which will show that an operator T can be subspace-hypercyclic for a subspace M not invariant for any power of the operator.

The following is our first example of a subspace-hypercyclic operator for a subspace M such that M is not invariant for the operator. Recall that: the *forward shift* S on ℓ^2 is the operator defined by

$$S(x_0, x_1, x_2, x_3, \ldots) := (0, x_0, x_1, x_2, \ldots),$$

and the *backward shift* B on ℓ^2 is the operator defined by

$$B(x_0, x_1, x_2, x_3, \ldots) := (x_1, x_2, x_3, \ldots).$$

Clearly BS is the identity on ℓ^2.

Example 2.11. Let $\lambda \in \mathbb{C}$ be of modulus greater than 1 and consider $T := \lambda B$ where B is the backward shift on ℓ^2. Let M be the subspace of ℓ^2 consisting of all sequences with zeroes on the even entries; that is,

$$M := \{(a_n)_{n=0}^{\infty} \in \ell^2 : a_{2k} = 0 \quad \text{for} \quad \text{all} \quad k\}.$$

Then T is subspace-hypercyclic for M.

Proof. We will apply 2.6 to give a proof. Let $X_0 = Y_0$ be the subset of M consisting of all finite sequences ; i.e, those sequences that only have a finite number of nonzero entries: this clearly is a dense subset of M. Let $n_k := 2k$. Let us check that conditions (i),(ii) and (iii) hold.

Let $x \in X_0$. Since x only has finitely many nonzero entries, $T^{2k}x$ will be zero eventually for k large enough. Thus (i) holds. Let $y \in Y_0$ and define $x_k := \frac{1}{\lambda^{2k}} S^{2k} y$, where S is the forward shift on ℓ^2. Each x_k is in M since the even entries of y are shifted by S^{2k} into the even entries of x_k. We have

$$\|x_k\| = \frac{1}{|\lambda^{2k}|} \|y\|,$$

and thus it follows that $x_k \to 0$, since $|\lambda| > 1$. Also, because

$$T^{2k}x_k = (\lambda B)^{2k}x_k = (\lambda B)^{2k}\frac{1}{\lambda^{2k}}S^{2k}y = y,$$

we have that condition *(ii)* holds. That condition *(iii)* holds, follows from the fact that if a vector has a zero entry on all even positions then it will also have a zero entry on all even positions after the application of the backward shift any even number of times. The subspace-hypercyclicity of T now follows.

□

As we commented before, we have just given our first example of a subspace-hypercyclic operator T for which M is not invariant under T. Observe that, nevertheless, the subspace M in the above example is invariant for T^2.

The example above can be generalized. Let $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$ be some fixed numbers with $a < b$ and consider the subspace M of ℓ^2 consisting of all sequences with zeroes on the entries indexed by the set $\{a + kb : k \in \mathbb{N}_0\}$. Then, the argument above holds for X and Y both equal to the set of all sequences in M with finitely many nonzero entries and $n_k := bk$. Hence, for $|\lambda| > 1$, the operator λB is subspace-hypercyclic for M. In this case, the space M is T^b invariant but not T invariant.

All examples given so far are in some sense trivial, because they contain some more or less hidden element of hypercyclicity.

In Examples 2.2 and 2.3, the subspace is invariant under the operator, and hence the operator restricted to the subspace is hypercyclic. In Example 2.11 the subspace is invariant under a power of the operator. In Theorem 2.9, the subspace is invariant for the operators T^{n_k}.

The next example, although based on a hypercyclic operator, does not fit into any of the categories above, since the subspace is not invariant for any power of the operator.

Example 2.12. Let $\lambda \in \mathbb{C}$ be of modulus greater than 1 and let B be the backward shift on ℓ^2. Let $m \in \mathbb{N}$ and M be the subspace of ℓ^2 consisting of all sequences with zeroes in the first m entries; that is,

$$M := \{(a_n)_{n=0}^{\infty} \in \ell^2 : a_n = 0 \quad \text{for} \quad n < m\}.$$

Then λB is subspace-hypercylic for M.

Proof. The argument used here is really the same as the one Rolewicz [**Rol**] originally used to show the hypercyclicity of λB. We will base our proof on the expositions of Halmos [**HALM82**] and of Jiménez-Munguía [**MUNG07**].

First of all, for a complex sequence $h = \{h_j\}_{j=0}^{\infty}$ with finitely-many nonzero entries, we define its *length* as

$$|h| := \min\{s \in \mathbb{N}_0 : h_k = 0 \quad \text{for all} \quad k \geq s\}.$$

We can choose a countable dense subset of M, called $\{f_j\}$, consisting of sequences which have at most a finite number of nonzero entries. Define a (necessarily increasing) sequence of integers k_j inductively as follows. Let $k_0 = 0$, and for each $j \in \mathbb{N}$ choose k_j in such a way that

$$\frac{\|f_j\|}{|\lambda|^{k_j - k_{j-1}}} \leq \frac{1}{|\lambda|^j}, \tag{2.1}$$

and such that $k_j > k_{j-1} + |f_{j-1}|$.

Define the vector f by :

$$f := \sum_{j=0}^{\infty} \frac{S^{k_j} f_j}{\lambda^{k_j}},$$

where S is the forward shift.
We must first show that $f \in \ell^2$. It follows from inequality 2.1 that

$$\left\| \frac{S^{k_j} f_j}{\lambda^{k_j}} \right\| = \frac{\|f_j\|}{|\lambda|^{k_j}} \leq \frac{1}{|\lambda|^{j+k_{j-1}}} \leq \frac{1}{\lambda^j},$$

and hence the infinite sum converges in norm to an ℓ^2 vector. Let $n \in \mathbb{N}$. Since for all j the condition $k_j > k_{j-1} + |f_{j-1}|$ holds, it follows that $k_n > k_j + |f_j|$ for all $j < n$ and hence

$$(\lambda B)^{k_n} \frac{S^{k_j} f_j}{\lambda^{k_j}} = 0.$$

This implies that $(\lambda B)^{k_n} f$ starts with the vector f_n and hence that $(\lambda B)^{k_n} f$ is in M. The condition $k_j > k_{j-1} + |f_{j-1}|$ also implies that the norm of the difference $(\lambda B)^{k_n} f - f_n$ is given by

$$\|(\lambda B)^{k_n} f - f_n\|^2 = \sum_{j=n+1}^{\infty} \left\| \frac{S^{k_j - k_n} f_j}{|\lambda|^{k_j - k_n}} \right\|^2 = \sum_{j=n+1}^{\infty} \left(\frac{\| f_j \|}{|\lambda|^{k_j - k_n}} \right)^2$$

$$\leq \sum_{j=n+1}^{\infty} \left(\frac{\| f_j \|}{|\lambda|^{k_j - k_{j-1}}} \right)^2 \leq \sum_{j=n+1}^{\infty} \frac{1}{|\lambda|^{2j}}.$$

Let $h \in M$. Given $\epsilon > 0$, choose N such that $\| h - f_N \| < \frac{\epsilon}{2}$ and such that

$$\left(\sum_{j=N+1}^{\infty} \frac{1}{|\lambda|^{2j}} \right)^{1/2} < \frac{\epsilon}{2}.$$

It then follows that

$$\| (\lambda B)^{k_N} f - h \| < \epsilon,$$

and hence that $Orb(\lambda B, f) \cap M$ is dense in M. □

In the example above, it is clear that it is impossible to find an increasing sequence of integers n_k such that M is invariant for T^{n_k} (since clearly, M is not invariant for T^n for any n). Thus, condition (iii) in Theorem 2.6 is not necessary.

Observe that the operator above does not satisfy Theorem 2.6. Thus subspace-hypercyclicity for a subspace M does not imply subspace-transitivity for M. Another example of a subspace-hypercyclic operator for a subspace M which is not subspace-transitive for M was given in [**Le**]. It should be noted that the procedure used in Example 2.12 above, could have been used to find a subspace-hypercyclic vector for the operator in Example 2.11.

Let us contrast the behaviour of the subspace-hypercyclic vector in the previous two examples. In Example 2.11, the orbit of any subspace-hypercyclic vector x under T goes in and out of the space M at regular intervals. In Example 2.11, by choosing the sequence of finite vectors $\{f_j\}$ and the sequence of natural numbers $\{kj\}$, the orbit of f under T goes in and out of the space M at irregular intervals; namely, one can find arbitrarily long consecutive elements of the orbit that stay inside the space and arbitrarily long consecutive elements of the orbit that stay outside the space.

We do not know if the vector f in Example 2.12 is hypercyclic. We can easily obtain a subspace-hypercyclic operator which is not hypercyclic and has the properties of Example 2.12 as follows:

Example 2.13. Let $|\lambda| > 1$, and consider the operator $T := (\lambda B) \oplus I$ on $X := l^2 \oplus l^2$. Let M be as in Example 2.12, and let f be a subspace-hypercyclic vector for M. Define $N := M \oplus 0$. Then $f \oplus 0$ is a subspace-hypercyclic vector for N, but $f \oplus 0$ is not hypercyclic for X. Also, N is not an invariant subspace for T^k for any k.

3. Finite dimensions

The following easy observation will be useful.

Proposition 2.14. *Let $T \in \boldsymbol{B}(X)$ be subspace-hypercyclic for M. If N is an invariant subspace for T and $M \subseteq N$, then $T\mid_N \colon N \to N$ is subspace-hypercyclic for M.*

We recall the following well-known definitions.

Definition 2.15. Let M and N be subspaces of X. If $M \cap N = \{0\}$ and $M + N = X$ we say that M and N are *complementary.*

Definition 2.16. Let M and N be complementary subspaces of X. The *projection onto M along N* is the function $P : X \to X$ defined as

$$P(x + y) = x,$$

where $x \in M$ and $y \in N$.

Theorem 2.17. *Let M and N be complementary subspaces of X and let P be the projection onto M along N. Let $T \in \boldsymbol{B}(X)$ and suppose that N is invariant under T. If T is subspace-hypercyclic for some $L \subseteq M$, then $PT|_M$ is subspace-hypercyclic for L.*

Proof. Assume T is subspace-hypercyclic for L with subspace-hypercyclic vector $x \in L$. Since $Orb(T, x) \cap L$ is dense in L, and $L \subseteq M$ it follows that

$P(Orb(T,x)) \cap L$ is dense in L. Also, since N is an invariant subspace for T, we have that $PTP = PT$ and hence that $(PT)^k = PT^k$ for all $k \in \mathbb{N}$. Thus

$$P(Orb(T,x)) = Orb(PT\mid_M, x).$$

It follows that $Orb(PT\mid_M, x) \cap L$ is dense in L, as desired. □

The following proposition is due to A.Peris.

Theorem 2.18. *Let $T \in \boldsymbol{B}(X)$. If T is subspace-hypercyclic for some subspace then $\sigma(T) \cap S^1 \neq \emptyset$.*

Proof. Assume the intersection is empty. Then, there exist (possibly empty) sets K_1 and K_2 such that $\sigma(T) = K_1 \cup K_2$ with $K_1 \subseteq \mathbb{D}$ and $K_2 \subseteq \overline{\mathbb{D}}^c$. By the Riesz Decomposition Theorem 1.10, there exist complementary invariant subspaces M_1 and M_2 such that

$$\sigma(T\mid_{M_1}) \subseteq K_1$$

and

$$\sigma(T\mid_{M_2}) \subseteq K_2.$$

Let $x \in X$. Then, there exist $x_1 \in M_1$ and $x_2 \in M_2$ such that $x = x_1 + x_2$. If x_2 were equal to 0, then

$$T^n x = T^n x_1 = (T\mid_{M_1})^n x_1$$

which converges to zero because $\sigma(T\mid_{M_1}) \subseteq \mathbb{D}$. Thus, $Orb(T,x)$ is bounded and hence its intersection with any subspace cannot be dense in that subspace. Assume x_2 is not equal to zero. We have

$$\|T^n x\| = \|T^n x_1 + T^n x_2\| \geq \|T^n x_2\| - \|T^n x_1\|,$$

and as before, $\|T^n x_1\|$ converges to zero. Since $\sigma(T\mid_{M_2}) \subseteq \overline{\mathbb{D}}^c$, it follows that $\|T^n x_2\|$ goes to infinity, and hence $\|T^n x\|$ also converges to infinity. Thus only finitely many elements of $Orb(T,x)$ intersect any bounded set, and hence $Orb(T,x)$ intersected with a subspace cannot be dense in that subspace. □

In the following propositions M° will denote the annihilator of M, that is; $M^\circ = \{f \in X^* : f(x) = 0 \text{ if } x \in M\}$

Proposition 2.19. *Let $T \in \boldsymbol{B}(X)$ be subspace-hypercyclic for M. Then $Ker(T^* - \lambda I) \subseteq M^\circ$ for all $\lambda \in \mathbb{C}$.*

Proof. Assume that $Orb(T,x) \cap M$ is dense in M. Fix $\lambda \in \mathbb{C}$ and let y be in $Ker(T^* - \lambda I)$. Suppose $y \notin M^\circ$ and let $\phi : M \to \mathbb{C}$ be the functional defined by $\phi(x) = \langle x, y\rangle$. Clearly ϕ is surjective because $y \notin M^\circ$. Observe that

$$\langle T^n x, y\rangle = \langle x, T^{*n} y\rangle = \langle x, \lambda^n y\rangle = \lambda^n \langle x, y\rangle \tag{2.2}$$

and hence

$$\phi(Orb(T,x) \cap M) = \{\lambda^n \langle x, y\rangle : T^n x \in M\}.$$

But if $Orb(T,x) \cap M$ is dense in M, then $\phi(Orb(T,x) \cap M)$ must be dense in $\mathbb{C}$. Since the set in 2.2 is clearly not dense in $\mathbb{C}$, it follows that ϕ is not surjective, and hence $y \in M^\circ$.

□

The result above can be generalized.

Proposition 2.20. *Let $T \in \boldsymbol{B}(X)$ be subspace-hypercyclic for M. Then $Ker(T^* - \lambda I)^p \subseteq M^\circ$ for all $\lambda \in \mathbb{C}$.*

Theorem 2.21. *Let X be finite-dimensional. If $T \in \boldsymbol{B}(X)$ then T is not subspace-hypercyclic for any M.*

PROOF. Since X is finite-dimensional and $T^* \in \mathbf{B}(X^*)$, there exist complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_s$ and natural numbers $p_1, p_2, \ldots, p_s$ such that X^* is the direct sum of the subspaces

$$ker(T^* - \lambda_1 I)^{p_1}, ker(T^* - \lambda_2 I)^{p_2}, \ldots, ker(T^* - \lambda_s I)^{p_s}.$$

If T were subspace-hypercyclic for some $M \neq \{0\}$, then by Proposition 2.20 for each j, we have

$$ker(T^* - \lambda_j I)^{p_j} \subseteq M^\circ.$$

Hence $X^* \subseteq M^\circ$ and thus $M = \{0\}$, a contradiction. □

Theorem 2.22. *Let $T \in \boldsymbol{B}(X)$. If T is subspace-hypercyclic for M, then M is not finite-dimensional.*

PROOF. Assume T is subspace-hypercyclic for a finite-dimensional subspace M and let $x \in M$ such that $Orb(T, x) \cap M$ is dense in M. It then follows that the infinite set $Orb(T, x) \cap M$ in the finite-dimensional space M has a finite subset of nonzero linearly dependent vectors, say $\{T^{n_1}x, T^{n_2}x, \ldots, T^{n_k}x\}$. Let $m = \max\{n_1, n_2, \ldots, n_k\}$.

An easy induction argument shows that $Orb(T, x) \subseteq span\{x, Tx, T^2x, \ldots, T^{m-1}x\}$. Hence the closed linear span N of $Orb(T, x)$ is finite-dimensional. Observe that the density in M of $Orb(T, x) \cap M$ implies that $M \subseteq N$.

Since clearly N is an invariant subspace for T, Proposition 2.14 shows that $T\mid_N$ is subspace-hypercyclic for M. But Theorem 2.21 contradicts this, and so the proof is finished. □

Lemma 2.23. *Let $T \in \boldsymbol{B}(X)$ be an operator that is subspace-hypercyclic for a subspace M, such that $X = X_1 \oplus X_2$, where X_1, X_2 are T-invariant sets and $T = T_1 \oplus T_2$ where $T_1 = T\mid_{X_1}$ and $T_2 = T\mid_{X_2}$. Then T_1 is subspace-hypercyclic for the projection of M onto X_1 and T_2 is subspace-hypercyclic for the projection of M onto X_2*

PROOF. Let M_1 and M_2 be the projections of M onto X_1 and X_2 respectively. Since T is subspace-hypercyclic for a subspace M, there exists $x \in X$ such that $Orb(T, x) \cap M$ is dense in M. Let $N(x, M) = \{n \in \mathbb{N} \mid T^n x \in M\} = \{n_j\}_j$. As $M = M_1 \oplus M_2$, we have that $T^{n_j}x \in M = M_1 \oplus M_2$ for each $n_j \in N(x, M)$, so $T^{n_1}x = x_{1,n_1} + x_{2,n_1}$ and $T^{n_j}x = T^{n_j - n_1}(T^{n_1}x) = T^{n_j - n_1}x_{1,n_1} + T^{n_j - n_1}x_{2,n_1} \in M$.

Then necessarily, as $T^{n_j - n_1}x_{1,n_1} \in M_1$, $T^{n_j - n_1}x_{2,n_1} \in M_2$, and the projection of a dense subset is dense, we have that T_1 is subspace-hypercyclic for the projection of M onto X_1 and T_2 is subspace-hypercyclic for the projection of M onto X_2. □

Theorem 2.24. *Let $T \in \boldsymbol{B}(X)$. If T is compact, then T is not subspace-hypercyclic for any subspace.*

PROOF. Assume that T is compact and subspace-hypercyclic for some M. Since T is compact, Theorem 2.18 implies that $A = \sigma(T) \cap S^1 \neq \emptyset$ and A is an isolated set of $\sigma(T)$. By Theorem 1.10 there are non-trivial T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$,

$$\sigma(T\mid_{X_1}) = A \quad \text{and} \quad \sigma(T\mid_{X_2}) = \sigma(T)\backslash A,$$

where $T = T_1 \oplus T_2$ with $T_1 = T\mid_{X_1}$ and $T_2 = T\mid_{X_2}$. As T_1 is compact and $0 \notin \sigma(T)$, T_1 has finite rank. On the other hand, $\sigma(T_2) \cap S^1 = \emptyset$. Since Theorem 2.22 gives us that T_1 is not subspace-hypercylic, we have that T_2 is subspace hypercyclic for the projection of M onto X_2 by Lemma 2.23. But this result contradicts Theorem 2.18.

□

CHAPTER 3

Universal Linear Dynamics

The main purpose of this chapter is to show that linear dynamics can be as complex as non-linear dynamics. It has been known for sometime that continuous linear operators on a Hilbert space can actually be chaotic.

In fact, we shall show that the orbits of linear operators can be as complicated as the orbits of any continuous function.

More precisely, there is a bounded linear operator T with the property that any continuous function f on a compact metric space X, is topologically conjugate to the restriction of T to an invariant compact set. This result was studied by Nathan S.Feldman in [**NathF**]

In this chapter we will work in an n-dimensional complex Hilbert space, that will be denoted by H_n.

The Backward Shift The set of all sequences $x = (x_0, x_1, x_2, \ldots)$ of complex numbers such that $\| x \|^2 = \sum_{n=0}^{\infty} |x_n|^2 < \infty$ is denoted by ℓ^2. There is a natural inner product on ℓ^2: if $x, y \in \ell^2$, then $< x, y >= \sum_{n=0}^{\infty} x_n \overline{y_n}$, where x_n and y_n denote the n^{th} coordinates of x and y, respectively. With the above inner product, ℓ^2 becomes a separable infinite dimensional complex Hilbert space. An important linear operator on ℓ^2 is the Backward Shift B. The Backward shift acts as follows:

$$B(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

It is clear that $\| Bx \| \leq \| x \|$ for all $x \in \ell^2$. It follows that B is a continuous linear operator from ℓ^2 into ℓ^2. It is also easy to check that the orbit of any vector under B converges to zero. For if $x \in \ell^2$, then $\| B^n x \|^2 = \sum_{k=n}^{\infty} |x_k|^2 \to 0$ as $n \to \infty$. Hence the dynamics of the Backward Shift are fairly simple (all orbits converge to the fixed point 0). However, in 1969 Rolewicz [**Rol**] proved the following surprising result.

Theorem 3.1. *If B is the Backward shift on ℓ^2, then $2B$ is chaotic on ℓ^2.*

Proof. Let $T = 2B$. Notice that $T^n(x_0, x_1, x_2, \ldots) = 2^n(x_n, x_{n+1}, \ldots)$. In order to show that the periodic points for T are dense, let $y \in \ell^2$. If $y = (y_0, y_1, \ldots)$, then we define vectors x_n as follows:

$$x_n = (y_0, \ldots, y_{n-1}, \frac{y_0}{2^n}, \ldots, \frac{y_{n-1}}{2^n}, \frac{y_0}{2^{2n}}, \ldots, \frac{y_{n-1}}{2^{2n}}, \frac{y_0}{2^{3n}}, \ldots, \frac{y_{n-1}}{2^{3n}}, \ldots).$$

It is easy to check that $x_n \in \ell^2$ and that x_n is a periodic point for T with period n. Furthermore, $x_n \to y$, and hence the periodic points for T are dense. To show that T is transitive, let U and V be two open sets in ℓ^2 and choose vectors $x \in U$ and $y \in V$. Now let

$$z_n = (x_0, x_1, \ldots, x_{n-1}, \frac{y_0}{2^n}, \ldots, \frac{y_{n-1}}{2^n}, 0, 0, \ldots).$$

Then $z_n \to x$ and $T^n z_n = (y_0, \ldots, y_{n-1}, 0, 0, \ldots) \to y$. Hence for large n, $z_n \in U$ and $T^n z_n \in V$. Thus, T is transitive and therefore chaotic on ℓ^2. $\square$

Naturally, Rolewicz [**Rol**] did not use this terminology, and actually he only proved that twice the Backward Shift is transitive, however this is the crucial part in the definition of chaos. This result surprisingly shows us that linear operators can be chaotic.

The Backward Shift of Higher Multiplicity: Let us consider a natural generalization of the Backward Shift. Suppose H_n is a separable complex Hilbert space with dimension n ($1 \leq n \leq \infty$), then $\ell^2(H_n)$ will denote the set of all sequences of vectors $\{x_k\}_{k=0}^{\infty}$ in H_n satisfying $\sum_{k=0}^{\infty} \parallel x_k \parallel^2 < \infty$. If we define a norm on $\ell^2(H_n)$ by $\parallel \{x_k\}_{k=0}^{\infty} \parallel^2 = \sum_{k=0}^{\infty} \parallel x_k \parallel^2$. Then $\ell^2(H_n)$ becomes a separable infinite dimensional Hilbert space. The Backward Shift of multiplicity n is the operator B_n on $\ell^2(H_n)$ defined as:

$$B_n(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

Thus B_n takes a sequence of vectors and produces another sequence of vectors. If $n < \infty$, then $H_n = \mathbb{C}^n$; so each of the coordinates x_k is a vector in $\mathbb{C}^n$. In particular, $\ell^2(H_1) = \ell^2(\mathbb{C}) = \ell^2$, and so $B_1 = B$. If $n = \infty$, then since all separable infinite dimensional Hilbert spaces are isomorphic, one may think of H_∞ as being ℓ^2. It is easy to show that B_n is unitarily equivalent to $B \oplus B \oplus \ldots \oplus B$ (n times). The same proof as given in Theorem 3.1 also gives the following result:

Theorem 3.2. *If $1 \leq n \leq \infty$ and B_n is the Backward shift with multiplicity n, then $2B_n$ is chaotic on $\ell^2(H_n)$.*

Now that we know that there are chaotic linear operators, one may naturally ask how many linear operators are chaotic. Surprisingly, there are more than one might think at first. In fact, it follows by the works of Herrero [**DH92**] and Chan [**KCH**] that the chaotic linear operators on a Hilbert space H are dense in $B(H)$ in the strong operator topology, that is the topology of pointwise convergence. Indeed, within various natural classes of linear operators there are many examples of linear operators that are chaotic; among these operators are shift operators, such as twice the Backward shift, composition operators, and adjoints of multiplication operators. More information on these operators can be found in [**HNS**],[**PSBJHS**] and [**RMGJHS**].

Now that we have seen that linear operators may have vectors with dense orbits, we are naturally lead to ask the following question:

Question: For a linear operator, how complicated can an orbit be that is not dense?

For example, can an orbit for a linear operator be dense in the unit ball of the Hilbert space, or dense in a Cantor set, or some other fractal-like subset of the

Hilbert space? The following result by Bourdon and Feldman sheds some light on these types of questions.

Theorem 3.3. *If $T \in B(H)$, $x \in H$, and the orbit of x is somewhere dense in H, then the orbit of x must be dense in H.*

Recall that a set is somewhere dense if its closure has non-empty interior, otherwise it is called nowhere dense. Thus, for a linear operator an orbit is either nowhere dense or everywhere dense. In particular, a linear operator cannot have an orbit whose closure is the closed unit ball. On the other hand, the following result will show that orbits of linear operators can be as complicated as the orbit of any continuous function on a compact metric space.

Theorem 3.4. *Suppose that $1 \leq n \leq \infty$ and B_n is the Backward Shift with multiplicity n on $\ell^2(H_n)$; also let $T = 2B_n$. If $f : X \to X$ is any continuous function on a closed and bounded set $X \subseteq H_n$, then there is an invariant closed set $K \subseteq \ell^2(H_n)$ such that $T\mid_K$ is topologically conjugate to f.*

Proof. Suppose that $T = 2B_n$ and $f : X \to X$ is a continuous function on a closed bounded set $X \subseteq H_n$. Define $h : X \to \ell^2(H_n)$ by:

$$h(x) = (x, \frac{f(x)}{2}, \frac{f^2(x)}{2^2}, \frac{f^3(x)}{2^3}, \ldots).$$

Notice that $x, f(x), f^2(x), \ldots$ are all vectors in $X \subseteq H_n$ and since $Orb(f,x) \subseteq X$ and X is a bounded set in H_n it follows easily that $\|h(x)\| < \infty$. Thus h does map X into $\ell^2(H_n)$. It is also clear that h is one-to-one and that $h(f(x)) = T(h(x))$ for all $x \in X$. Thus if we set $K = h(X)$, then K is invariant for T and $h \circ f \circ h^{-1} = T\mid_K$. So, it suffices to show that $h : X \to K$ is a homeomorphism.

First let us show that h is continuous. So, suppose that $x_0 \in X$ and $\epsilon > 0$. Let $d > 0$ be the diameter of X, that is $\|x - y\| \leq d$ for all $x, y \in X$. Then choose an $m \geq 1$ such that $\sum_{k=m}^{\infty} \frac{d^2}{4^k} \leq \frac{\epsilon^2}{4}$. Since $f^0, f^1, f^2, \ldots, f^m$ are all continuous at x_0, there exists a $\delta > 0$ such that if $\|x - x_0\| < \delta$, then $\|f^k(x) - f^k(x_0)\| < \frac{\epsilon}{2\sqrt{(m+1)}}$ for all $k \in \{0, \ldots, m\}$. Thus if $\|x - x_0\| < \delta$, then

$$\begin{aligned}\|h(x) - h(x_0)\|^2 &= \sum_{k=0}^{m} \frac{\|f^k(x) - f^k(x_0)\|^2}{4^k} + \sum_{k=m+1}^{\infty} \frac{\|f^k(x) - f^k(x_0)\|^2}{4^k} \\ &\leq \sum_{k=0}^{m} \frac{\epsilon^2}{4(m+1)} + \sum_{k=m+1}^{\infty} \frac{d^2}{4^k} \leq \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2} < \epsilon^2.\end{aligned}$$

Thus $\|h(x) - h(x_0)\| < \epsilon$. Hence h is continuous.

Also, since it is clear that $\|h(x) - h(y)\| \geq \|x - y\|$ we see that $h^{-1} : K \to X$ is also continuous and, in fact, Lipschitz. Thus h is a homeomorphism, and the result follows.

□

We next notice that if the continuous function f is Lipschitz on $X \subseteq H_n$, that is, $\|f(x)-f(y)\| \leq C\|x-y\|$, then the conjugating homeomorphism h is bi-Lipschitz (that is, h and h^{-1} are both Lipschitz).

Theorem 3.5. *Suppose $1 \leq n \leq \infty$ and $f : X \to X$ is a Lipschitz function on a closed bounded subset $X \in H_n$ with Lipschitz constant M. If $\lambda > \max\{M, 1\}$, then there is a closed set $K \in \ell^2(H_n)$ that is invariant for $T = \lambda B_n$ such that f is topologically conjugate to $T\mid_K$ via a bi-Lipschitz homeomorphism.*

If $1 \leq n < \infty$, then Theorem 3.4 implies that any continuous function $f : X \to X$ on a compact set $X \subseteq \mathbb{C}^n$ is topologically conjugate to the restriction of a linear operator to an invariant closed set. In particular, consider the logistic function $f(x) = \lambda x(1-x)$. It is known that for $\lambda > 4$ there exists a Cantor set $\wedge \subseteq [0, 1]$ such that $f(\wedge) = \wedge$ and $f : \wedge \to \wedge$ is chaotic.

Since f is chaotic on a Cantor set $\wedge \subseteq \mathbb{R} \subseteq \mathbb{C}$, it follows that f is topologically conjugate to a restriction of $T = 2B_1 =$ twice the Backward shift (with multiplicity one). Thus, there is a Cantor set $K \subset \ell^2$ such that $T(K) \subseteq K$ and $T\mid_K$ is conjugate to f. In particular, since f is transitive, it has a dense orbit, thus $T\mid_K$ also has a dense orbit. It follows that T has an orbit that is dense in a Cantor set.

Similarly, $T = 2B_1$ has orbits that are dense in compact sets that are homeomorphic to products of intervals and Cantor sets, Julia sets, and other fractal-like sets. A similar situation occurs in higher dimensions.

Lemma 3.6. *If X is a compact metric space, then X is homeomorphic to a compact subset of ℓ^2.*

Proof. If d is the metric on X and $\{x_n\}_{n=0}^{\infty}$ is a countable dense subset of X, then the map $h : X \to \ell^2$ given by $h(x) = \{d(x, x_n)/2^n\}_{n=0}^{\infty}$ is easily seen to be one-to-one and continuous. Since X is compact, h is a homeomorphism. $\square$

By the previous Lemma we also have the following corollary of Theorem 3.4.

Corollary 3.7. *If $T = 2B_\infty$ and $f : X \to X$ is any continuous function on a compact metric space X, then there is an invariant compact set $K \subseteq \ell^2(H_\infty)$ such that $T\mid_K$ is topologically conjugate to f.*

This Corollary says that there is a linear operator, namely twice the Backward Shift of infinite multiplicity, that is universal in the sense that its restriction to an invariant compact set can have the 'same dynamics' as any continuous function on any compact metric space.

CHAPTER 4

Chaos for invariant compact sets of operators

We have seen the importance of compact sets in the previous chapter. In this chapter, we will work with K, an absolutely convex compact set of a Banach space X which is invariant by an operator $T : X \to X$. We will also establish that if $T|_K : K \to K$ is transitive, mixing, weakly mixing or chaotic then $T : Y \to Y$, where $Y = \overline{span(K)} = \overline{\bigcup_{n=1}^{\infty} nK}$, is also transitive, mixing, weak mixing or chaotic.

We will also show some criteria under which we can determine whether if an operator defined on an absolutely convex compact is transitive, mixing, weakly mixing or chaotic.

In a recent article [**SP**], the authors study a specification property which is stronger than Devaney chaos. This property requires the existence of absolutely convex compact sets, but we won't work with this property in this chapter.

Definition 4.1. A subset K is absolutely convex if given $x, y \in K$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $|\lambda_1| + |\lambda_2| \leq 1$, we have that $\lambda_1 x + \lambda_2 y \in K$.

We start with some observations that will be useful in this chapter.

If K is absolutely convex and $n, m \in \mathbb{N}$ with $m < n$ then $mK \subset nK$. Given $x \in mK$ we have $x = mk_1$, $k_1 \in K$. Thus $x = mk_1 = n(\frac{m}{n}k_1) \in nK$, since K is absolutely convex and $m < n$.

$T : K \to K$ is topologically conjugate to $T : nK \to nK$ for all $n \in \mathbb{N}$ via the continuous and exhaustive map $h : K \to nK$ where $h(x) = nx$.

An operator $T\ |_Y\colon Y \to Y$, where Y is a subspace, is quasi-conjugate to $T\ |_{\overline{Y}}\colon \overline{Y} \to \overline{Y}$ via the continuous map $h : Y \to \overline{Y}$, where $h(y) = y$. It is easy to see that this map has clearly dense range.

Theorem 4.2. *Let $T : X \to X$ be an operator, K an absolutely convex compact T-invariant set, and $Y = \overline{span(K)} = \overline{\bigcup_{n=1}^{\infty} nK}$. Then:*

(i) *If $T|_K$ is transitive then $T : Y \to Y$ is transitive.*
(ii) *If $T|_K$ is mixing then $T : Y \to Y$ is mixing.*
(iii) *If $T|_K$ is weakly-mixing then $T : Y \to Y$ is weakly-mixing.*
(iv) *If $T|_K$ is chaotic then $T : Y \to Y$ is chaotic.*

Proof.

(i) By the previous remark it is sufficient to prove that $T \mid_{\bigcup_{n=1}^{\infty} nK}$ is transitive. Let U, V be non-empty open sets of $\bigcup_{n=1}^{\infty} nK$. Then there exist U', V' non-empty open sets of X such that

$$U = U' \cap (\bigcup_{n=1}^{\infty} nK) \neq \emptyset \quad \text{and} \quad V = V' \cap (\bigcup_{n=1}^{\infty} nK) \neq \emptyset.$$

Since U and V are non-empty there exist $n_1, n_2 \in \mathbb{N}$ such that $U' \cap n_1 K \neq \emptyset$ and $V' \cap n_2 K \neq \emptyset$. Without loss of generalization, suppose that $n_1 \leq n_2$. Thus, $n_1 K \subset n_2 K$, and then $U' \cap n_2 K \neq \emptyset$.

Then by our induction hypothesis and the fact that $T \mid_{n_2 K}$ is topologically conjugate to $T \mid_K$, $T \mid_{n_2 K}$ is transitive and there exists an $n \in \mathbb{N}$ such that

$$T^n(U' \cap n_2 K) \cap (V' \cap n_2 K) \neq \emptyset.$$

Since

$$U' \cap n_2 K \subseteq U' \cap (\bigcup_{n=1}^{\infty} nK) \quad \text{and} \quad V' \cap n_2 K \subseteq V' \cap (\bigcup_{n=1}^{\infty} nK),$$

then

$$T^n(U' \cap (\bigcup_{n=1}^{\infty} nK)) \cap (V' \cap (\bigcup_{n=1}^{\infty} nK)) = T^n(U) \cap V \neq \emptyset.$$

(ii) It is sufficient to prove that $T \mid_{\bigcup_{n=1}^{\infty} nK}$ is mixing. Let U, V be non-empty open sets of $\bigcup_{n=1}^{\infty} nK$. Then there exist U', V' non-empty open sets of X such that

$$U = U' \cap (\bigcup_{n=1}^{\infty} nK) \neq \emptyset \quad \text{and} \quad V = V' \cap (\bigcup_{n=1}^{\infty} nK) \neq \emptyset.$$

Since U and V are non-empty there exist $n_1, n_2 \in \mathbb{N}$ such that $U' \cap n_1 K \neq \emptyset$ and $V' \cap n_2 K \neq \emptyset$. W.l.og suppose that $n_1 \leq n_2$ then $n_1 K \subset n_2 K$ and $U' \cap n_2 K \neq \emptyset$.

Then by our induction hypothesis and the fact that $T \mid_{n_2 K}$ is topologically conjugate to $T \mid_K$, $T \mid_{n_2 K}$ is mixing and there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$T^n(U' \cap n_2 K) \cap (V' \cap n_2 K) \neq \emptyset.$$

Since

$$U' \cap n_2 K \subseteq U' \cap (\bigcup_{n=1}^{\infty} nK) \quad \text{and} \quad V' \cap n_2 K \subseteq V' \cap (\bigcup_{n=1}^{\infty} nK),$$

we have

$$T^n(U' \cap (\bigcup_{n=1}^{\infty} nK)) \cap (V' \cap (\bigcup_{n=1}^{\infty} nK)) = T^n(U) \cap V \neq \emptyset \quad \text{for} \quad \text{all} \quad n \geq n_0.$$

(iii) From the previous remark it is sufficient to prove that $T\mid_{\bigcup_{n=1}^{\infty} nK}$ is weakly-mixing. Let U_1, U_2, V_1, V_2 be non-empty open sets of $\bigcup_{n=1}^{\infty} nK$. Then there exist U_1', U_2', V_1', V_2' non-empty open sets of X such that

$$U_i = U_i' \cap (\bigcup_{n=1}^{\infty} nK) \neq \emptyset \quad \text{and} \quad V_i = V_i' \cap (\bigcup_{n=1}^{\infty} nK) \neq \emptyset$$

for $i = 1, 2$.
Since U_i and V_i are non-empty there exist $n_1, n_2, n_3, n_4 \in \mathbb{N}$ such that $U_1' \cap n_1 K \neq \emptyset$ $U_2' \cap n_2 K \neq \emptyset$, $V_1' \cap n_3 K \neq \emptyset$ and $V_2' \cap n_4 K \neq \emptyset$.

Let $n_0 = max\{n_1, n_2, n_3, n_4\}$; so, $n_i K \subset n_0 K$.

Then $U_i' \cap n_0 K \neq \emptyset and V_i \cap n_0 K \neq \emptyset$. By our induction hypothesis and the fact that $T\mid_{n_0 K}$ is topologically conjugate to $T\mid_K$, $T\mid_{n_0 K}$ is weakly-mixing, so there exists an $n \in \mathbb{N}$ such that

$$T^n(U_1' \cap n_0 K) \cap (V_1' \cap n_0 K) \neq \emptyset \quad \text{and} \quad T^n(U_2' \cap n_0 K) \cap (V_2' \cap n_0 K) \neq \emptyset.$$

Then

$$T^n(U_i' \cap (\bigcup_{n=1}^{\infty} nK)) \cap (V_i' \cap (\bigcup_{n=1}^{\infty} nK)) = T^n(U_i) \cap V_i \neq \emptyset$$

for $i = 1, 2$.

(iv) It is sufficient to prove that $T\mid_{\bigcup_{n=1}^{\infty} nK}$ is chaotic. Since $T : K \to K$ is chaotic, it is transitive; then, by part (i), it follows that
$T : \bigcup_{n=1}^{\infty} nK \to \bigcup_{n=1}^{\infty} nK$ is transitive.

On the other hand the set of periodic points $Per(T\mid_K) = \{x_0, x_1 \ldots\}$ is dense in K; that is,

$$K \subseteq \overline{\{x_0, x_1 \ldots\}} \Longrightarrow nK \subseteq \overline{\{nx_0, nx_1 \ldots\}} \Longrightarrow \bigcup_{n=1}^{\infty} nK \subseteq \bigcup_{n=1}^{\infty} \overline{\{nx_0, nx_1 \ldots\}}$$

$$\subseteq \overline{\bigcup_{n=1}^{\infty} \{nx_0, nx_1 \ldots\}} \subset \overline{Per(T\mid_{\bigcup_{n=1}^{\infty} nK})}.$$

Then the set of periodic points $Per(T\mid_{\bigcup_{n=1}^{\infty} nK})$ is dense in $\bigcup_{n=1}^{\infty} nK$, and we have that $T\mid_{\bigcup_{n=1}^{\infty} nK}$ is chaotic.

□

Now we will see an example where the previous theorem can be used.

Example 4.3. Let T be a weighted backward shift on the space ℓ^p given by

$$T(x_1, x_2, x_3, \ldots) = (w_2 x_2, w_3 x_3, w_4 x_4, \ldots)$$

with $\sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} |w_k|^p} < \infty$.

Let K be the following compact subset:

$$K = \{x \in \ell^p \quad ; \quad |x_k| \prod_{j=1}^{k} |w_j| \leq 1, \forall k \geq 1\}$$

where $w_1 = 1$. We will see that K is absolutely convex, T-invariant and $T|_K$ is also chaotic. Since $\overline{span(K)} = \ell^p$ we will deduce from theorem 4.2 that $T : \ell^p \to \ell^p$ is chaotic.

First of all, we will show that K is absolutely convex. Let $x, y \in K$ and λ_1, λ_2 such that $|\lambda_1| + |\lambda_2| \leq 1$. Then for all $k \geq 1$:

$|x_k| \prod_{j=1}^{k} |w_j| \leq 1$ and $|y_k| \prod_{j=1}^{k} |w_j| \leq 1$ so:

$$|\lambda_1 x_k + \lambda_2 y_k| \prod_{j=1}^{k} |w_j| \leq |\lambda_1 x_k| \prod_{j=1}^{k} |w_j| + |\lambda_2 y_k| \prod_{j=1}^{k} |w_j| \leq |\lambda_1| + |\lambda_2| \leq 1.$$

It is easy to show that K is T-invariant because, given $x \in K$, we have that $|x_k| \prod_{j=1}^{k} |w_j| \leq 1$ for all $k \geq 1$ and:

$$T(x) = (w_2 x_2, w_3 x_3, w_4 x_4, \ldots).$$

Thus for all $k \geq 1$ we have :

$$|T(x)_k| \prod_{j=1}^{k} |w_j| = |w_{k+1} x_{k+1}| \prod_{j=1}^{k} |w_j| = |x_{k+1}| \prod_{j=1}^{k+1} |w_j| \leq 1;$$

hence $T(K) \subset K$.

Now let us see that $T|_K$ is chaotic. If x is a periodic point for T, then

$$x = (x_1, x_2, \ldots, x_n, \prod_{j=2}^{n+1} w_j^{-1} x_1, \ldots, \prod_{j=n+1}^{2n} w_j^{-1} x_n, \prod_{j=2}^{2n+1} w_j^{-1} x_1, \ldots, \prod_{j=n+1}^{3n} w_j^{-1} x_n, \ldots).$$

Given $x = (x_1, x_2, \ldots, x_N, 0, \ldots, 0) \in K$ we have that :

$$x' = (x_1, x_2, \ldots, x_N, \prod_{j=2}^{N+1} w_j^{-1} x_1, \ldots, \prod_{j=N+1}^{2N} w_j^{-1} x_N, \prod_{j=2}^{2N+1} w_j^{-1} x_1, \prod_{j=N+1}^{3N} w_j^{-1} x_N, \ldots) \in K$$

is periodic and:

$$||x - x'||_p^p = \sum_{l=1}^{\infty} \sum_{i=1}^{N} \left| \left(\prod_{j=i+1}^{lN+i} w_j^{-1} \right) x_i \right|^p = \sum_{l=1}^{\infty} \sum_{i=1}^{N} \prod_{j=i+1}^{lN+i} |w_j^{-1}|^p |x_i|^p \to 0$$

because

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} |w_k|^p} < \infty.$$

Therefore we have that the set of periodic points is dense in K. Now we will show that $T|_K$ is transitive. Let $U, V \subset X$ be non-empty open sets of K. We can find $x \in U$ and $y \in V$ such that:

$$x = (x_1, x_2, \dots, x_N, 0, \dots, 0) \quad \text{and} \quad y = (y_1, y_2, \dots, y_N, 0, \dots, 0)$$

for some $N \in \mathbb{N}$. Let $n \geq N$ be arbitrary. We define $z \in \ell^p$, such that $z_k = x_k$ if $1 \leq k \leq N$ and $z_k = \prod_{i=k-n+1}^{k} w_i^{-1} y_{k-n}$ if $n+1 \leq k \leq n+N$ and $z_k = 0$ otherwise. We obtain a sequence with :

$$T^n(z) = y \quad \text{and} \quad ||x - z||_p^p = \prod_{i=k-n+1}^{k} |w_i^{-1}|^p ||y||_p^p \to 0.$$

Hence $T|_K$ is chaotic.

Our next goal is to derive some criteria under which an operator defined on a compact convex invariant set is transitive, mixing or weakly mixing. These theorems are similar to the results presented in chapter two which characterize when an operator defined on an arbitrary Banach space is transitive, mixing, or weakly mixing.

Theorem 4.4. *Let T be an operator. Let K be a compact absolutely convex invariant set. If there are dense subsets X_0, $Y_0 \subset K$, an increasing sequence $(n_k)_k$ of positive integers, and a sequence of maps $S_{n_k} : Y_0 \to X, k \in \mathbb{N}$, such that, for any $x \in X_0$, $y \in Y_0$,*

(i) $T^{n_k} x \to 0$,
(ii) $S_{n_k} y \to 0$,
(iii) $T^{n_k} S_{n_k} y \to y$,
(iv) *for all $x \in X_0$ and $y \in Y_0$ there exists a k_0 such that $x + S_{n_k}(y) \in K$ for all $k \geq k_0$,*

then $T|_K$ is weakly mixing and ,in particular, transitive.

Proof. Let U_1, U_2, V_1 and V_2 be non-empty open sets of K. By assumption we can find vectors $x_j \in U_j \cap X_0$ and $y_j \in V_j \cap Y_0$, j=1,2. Then by (iii),

$$T^{n_k}(x_j + S_{n_k} y_j) \to T^{n_k} x_j + y_j, j = 1, 2.$$

It follows from (i) and (ii) that, for sufficiently large k, $x_j + S_{n_k} y_j \in U_j$ and $T^{n_k} x_j + y_j \in V_j$ for $j = 1, 2$. This shows that $T|_K$ is weakly mixing and, in particular, transitive.

□

Example 4.5. The previous theorem provides us another way to show that the linear operator in Example 4.3 is transitive. Previously, we showed directly that T is transitive, however we will now show that it satisfies Theorem 4.4, and is therefore transitive.

We consider the chaotic weighted backward shift on the space ℓ^p and the subset K defined as :

$$K = \{ \quad x \in \ell^p \quad ; \quad |x_k| \prod_{j=1}^{k} |w_j| \leq 1, \forall k \geq 1\}.$$

We have already shown that K is a compact absolutely convex invariant set. Now we will see that T verifies the hypothesis of the previous theorem to deduce that $T|_K$ is transitive.

Let $X_0 = Y_0$ be the space of finite sequences in K. If we consider the weighted forward shift $S : Y_0 \to Y_0$ with

$$S(x_1, x_2, \dots) = (0, w_2^{-1}x_1, w_3^{-1}x_2, \dots),$$

we will take $S_{n_k} = S^{n_k}$. It is clear that $TSy = y$ for all $n \in \mathbb{N}$ and $y \in Y_0$, and that $T^{n_k}x \to 0$, $x \in X_0$. We now must find a suitable increasing sequence $(n_k)_k$ of positive integers so that $S^{n_k}y \to 0$ for each $y \in Y_0$ in order to satisfy condition (iii).

Let $n_k = m_k + k, k \in \mathbb{N}$, and denote by e_n the vector that has 1 in the n^{th} position, and 0 otherwise. Then we have that

$$S^{n_k}e_1 = (0, \dots, 0, \prod_{j=2}^{m_k+k+1} w_j^{-1}, 0, \dots) \to 0,$$

because, by hypothesis, $\sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} |w_k|^p} < \infty$.

Therefore, for any $l \geq 1$,

$$S^{n_k}e_l = S^{n_k}\left(\left(\prod_{j=2}^{l} w_j\right) S^{l-1}e_1\right) = \left(\prod_{j=2}^{l} w_j\right) S^{l-1}(S^{n_k}e_1) \to 0.$$

From this we conclude that $S^{n_k}y \to y$ for each $y \in Y_0$, which was what had to be shown.

Finally, given $x \in X_0$ and $y \in Y_0$, we fix $n_0 \in \mathbb{N}$ such that $x_k = 0$ for all $k \geq n_0$. We have that $(S^n y)_k = \prod_{k-n+1}^{k} w_j^{-1} y_{k-n}$, for $k \geq n$ and 0 otherwise. Then:

$$|x_k + (S^n y)_k| \prod_{j=1}^{k} |w_j| = \prod_{j=1}^{k} |w_j||x_k| + \prod_{j=1}^{k-n} |w_j||y_{k-n}| \leq 1$$

for all $n \geq n_0$. We have that $x + S^n y \in K$ for all $n \geq n_0$. Hence condition (iv) is verified, and according to Theorem 4.4 $T|_K$ is transitive.

Example 4.6. Now we will also illustrate with another subset K' that the first three conditions of Theorem 4.4 are not sufficient for $T|_{K'}$ to be transitive.
Let us define K' as:

$$K' = \{x \in \ell^p \quad ; \quad \sum_{k \geq 1} \left(\prod_{j=1}^{k} |w_j|\right)^p |x_k|^p \leq 1, \forall k \geq 1\},$$

with $w_1 = 1$. It is clear that K' is compact. Let us see that K' is absolutely convex. Let $x, y \in K'$ and λ_1, λ_2 such that $|\lambda_1| + |\lambda_2| \leq 1$. Then for all $k \geq 1$:

$$\sum_{k \geq 1} \left(\prod_{j=1}^{k} |w_j|\right)^p |x_k|^p \leq 1$$

and

$$\sum_{k\geq 1}\left(\prod_{j=1}^{k}|w_j|\right)^p |y_k|^p \leq 1$$

Using the triangular inequality we have:

$$\sum_{k\geq 1}\left(\prod_{j=1}^{k}|w_j|\right)^p |\lambda_1 x_k + \lambda_2 y_k|^p \leq 1.$$

It is easy to show that K' is T-invariant because given $x \in K'$, we have that

$$\sum_{k\geq 1}\left(\prod_{j=1}^{k}|w_j|\right)^p |x_k|^p \leq 1, \text{for all} \quad k \geq 1,$$

and

$$T(x) = (w_2x_2, w_3x_3, w_4x_4, \ldots).$$

Thus for all $k \geq 1$ we have

$$\sum_{k\geq 1}\left(\prod_{j=1}^{k}|w_j|\right)^p |T(x)_k|^p = \sum_{k\geq 1}\left(\prod_{j=1}^{k}|w_j|\right)^p |w_{k+1}x_{k+1}|^p = \sum_{k\geq 1}\left(\prod_{j=1}^{k+1}|w_j|\right)^p |x_{k+1}|^p \leq 1.$$

Hence, $T(K') \subset K'$.

Following the proof of the previous example, it is easy to see that $T|_{K'}$ satisfies conditions $(i), (ii) and (iii)$.

Although the first three conditions are satisfied it is easy to see that $T|_{K'}$ is not transitive. Let $x \in K'$. Since T is chaotic, $lim_{k\to\infty} \prod_{j=1}^{k}|w_j| = \infty$ and there exists $\lambda > 0$ such that $\prod_{j=1}^{k}|w_j| \geq \lambda > 0$ for all $k \in \mathbb{N}$. Then:

$$\lambda^p ||T^n x||^p \leq \sum_{k\geq 1}\left(\prod_{j=1}^{k}|w_j|\right)^p |(T^n x)_k|^p = \sum_{k\geq n}\left(\prod_{j=1}^{k}|w_j|\right)^p |x_k|^p \to 0.$$

This shows that x cannot be an hypercyclic vector.

Theorem 4.7. *Let T be an operator. Let K be a compact absolutely convex invariant set. If there are dense subsets $X_0, Y_0 \subset K$ and a sequence of maps $S_n : Y_0 \to X$, $n \in \mathbb{N}$ such that, for any $x \in X_0$, $y \in Y_0$,*

(i) $T^n x \to 0$,
(ii) $S^n y \to 0$,
(iii) $T^n S_n y \to y$,
(iv) *for all $x \in X_0$ and $y \in Y_0$ there exists $n \in \mathbb{N}$ such that $x + S_m y \in K$ for all $m \geq n$,*

then $T|_K$ is mixing.

Proof. Let U and V be non-empty open sets of K. By assumption we can find vectors $x \in U \cap X_0$ and $y \in V \cap Y_0$. Then by (iii),

$$T^n(x + S_n y) \to T^n x + y.$$

It follows from (i) and (ii) that there exists an n_0 such that for $n \geq n_0$ $x+S_n y \in U$ and $T^n x + y \in V$. This proves that $T|_K$ is mixing. $\square$

Remark 4.8. The same proof used in Example 4.5 could be used to show that $T|_K$ is also mixing.

Theorem 4.9. *Let T be an operator, and let K be a compact absolutely convex invariant set. If $T|_K$ satisfies conditions of Theorem 4.4 and the subset*

$$span\{x \in X; \quad Tx = \lambda x \quad \text{for some} \quad \lambda \in \mathbb{C} \quad \text{with} \quad |\lambda|^n = 1 \quad \text{for some} \quad n \in \mathbb{N}\} \cap K$$

is dense in K, then $T|_K$ is chaotic.

Proof. If $T|_K$ satisfies the conditions of Theorem 4.4, we have that $T|_K$ is transitive. By Proposition 1.29, the set $Per(T)$ is given by

$$span\{x \in X \ ; \ Tx = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ with } \lambda^n = 1 \text{ for some } n \in \mathbb{N}\}.$$

As K is an invariant set and $Per(T) \cap K$ is dense in K, we have that $T|_K$ is chaotic. $\square$

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