The Specification Property for C_0-Semigroups*

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Abstract

We study one of the strongest versions of chaos for continuous dynamical systems, namely the specification property. We extend the definition of specification property for operators on a Banach space to strongly continuous one-parameter semigroups of operators, that is, C_0-semigroups. In addition, we study the relationships of the specification property for C_0-semigroups (SgSP) with other dynamical properties: mixing, Devaney's chaos, distributional chaos and frequent hypercyclicity. Concerning the applications, we provide several examples of semigroups which exhibit the SgSP with particular interest on solution semigroups to certain linear PDEs, which range from the hyperbolic heat equation to the Black-Scholes equation.

1 Introduction

A (continuous) map on a metric space satisfies the specification property (SP) if for any choice of points, one can approximate distinct pieces of orbits by a single periodic orbit with a certain uniformity. It was first introduced by Bowen [16]; since then, this property has attracted the interest of many researchers (see, for instance, the early work [41]). In a few words, the specification property requires that, for a given distance $\delta > 0$, and for any finite family of points, there is always a periodic orbit that traces arbitrary long pieces of the orbits of the family, up to a distance δ, allowing a minimum "jump time" N_δ from one piece of orbit to another one, which only depends on δ.

Definition 1. A continuous map $f : X \to X$ on a compact metric space (X, d) has the specification property if for any $\delta > 0$ there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset X$ and any integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ satisfying $i_{r+1} - j_r \geq N_\delta$ for $r = 1, \dots, s-1$, there is a point $x \in X$ such that the following conditions hold:

$$d(f^i(x), f^i(y_r)) < \delta, \text{ with } i_r \leq i \leq j_r, \text{ for every } r \leq s,$$
$$f^{N_\delta + j_s}(x) = x.$$

This definition must be modified when one treats with bounded linear operators defined on separable Banach spaces which are never compact [6, 7]. Here, we denote the specification property for operators by OSP (see [6, 7] for definitions and properties). A

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continuous map on a metric space is said to be chaotic in the sense of Devaney if it is topologically transitive and the set of periodic points is dense. Although there is no common agreement about what a chaotic map is, the specification property is stronger than Devaney's definition of chaos. Recently, several properties of linear operators with the OSP and the connections of this OSP with other well-known dynamical properties, like mixing, chaos in the sense of Devaney and frequent hypercyclicity have been studied in [7], we will use these results throughout the paper. Other recent works on the specification property are [37, 38, 32].

A family $(T_t)_{t\geq 0}$ of linear and continuous operators on a Banach space X is said to be a C_0-*semigroup* if $T_0 = Id$, $T_tT_s = T_{t+s}$ for all $t, s \geq 0$, and $\lim_{t\to s} T_tx = T_sx$ for all $x \in X$ and $s \geq 0$.

Let $(T_t)_{t\geq 0}$ be an arbitrary C_0-semigroup on X. It can be shown that an operator defined by $Ax := \lim_{t\to 0} \frac{1}{t}(T_tx - x)$ exists on a dense subspace of X; denoted by $D(A)$. Then A, or rather $(A, D(A))$, is called the *(infinitesimal) generator* of the semigroup. It can also be shown that the infinitesimal generator determines the semigroup uniquely. If the generator A is defined on X ($D(A) = X$), the semigroup is expressed as $\{T_t\}_{t\geq 0} = \{e^{tA}\}_{t\geq 0}$.

Given a family of operators $(T_t)_{t\geq 0}$, we say that this family of operators is *transitive* if for every pair of non-empty open sets $U, V \subset X$ there exists some $t > 0$ such that $T_t(U) \cap V \neq \emptyset$. Furthermore, if there is some t_0 such that the condition $T_t(U) \cap V \neq \emptyset$ holds for every $t \geq t_0$ we say that it is *topologically mixing* or mixing.

A family of operators $(T_t)_{t\geq 0}$ is said to be *universal* if there exists some $x \in X$ such that $\{T_tx : t \geq 0\}$ is dense in X. When $(T_t)_{t\geq 0}$ is a C_0-semigroup we particularly refer to it as *hypercyclic*. In this setting, transitivity coincides with universality, but it is strictly weaker than mixing [12].

In addition, two notions of chaos are recalled: *Devaney chaos* and *distributional chaos*. An element $x \in X$ is said to be a *periodic point* of $(T_t)_{t\geq 0}$ if there exists some $t_0 > 0$ such that $T_{t_0}x = x$. A family of operators $(T_t)_{t\geq 0}$ is said to be *chaotic in the sense of Devaney* if it is hypercyclic and there exists a dense set of periodic points in X. On the other hand, it is *distributionally chaotic* if there are an uncountable set $S \subset X$ and $\delta > 0$, so that for each $\varepsilon > 0$ and each pair $x, y \in S$ of distinct points we have

$$\overline{\mathrm{Dens}}\{s \geq 0 : ||T_sx - T_sy|| \geq \delta\} = 1 \text{ and}$$
$$\overline{\mathrm{Dens}}\{s \geq 0 : ||T_sx - T_sy|| < \varepsilon\} = 1,$$

where $\overline{\mathrm{Dens}}(B)$ is the upper density of a Lebesgue measurable subset $B \subset \mathbb{R}_0^+$ defined as

$$\limsup_{t\to\infty} \frac{\mu(B \cap [0, t])}{t},$$

with μ standing for the Lebesgue measure on $\mathbb{R}_0^+$. A vector $x \in X$ is said to be *distributionally irregular* for the C_0-semigroup $(T_t)_{t\geq 0}$ if for every $\varepsilon > 0$ we have

$$\overline{\mathrm{Dens}}\{s \geq 0 : ||T_sx|| \geq \varepsilon^{-1}\} = 1 \text{ and}$$
$$\overline{\mathrm{Dens}}\{s \geq 0 : ||T_sx|| < \varepsilon\} = 1.$$

Such vectors were considered in [11] so as to get a further insight into the phenomenon of distributional chaos, showing the equivalence between a distributionally chaotic operator and an operator having a distributionally irregular vector. This equivalence was later generalized for C_0-semigroups in [1].

Devaney chaos, hypercyclicity and mixing properties have been widely studied for linear operators on Banach and more general spaces [12, 14, 23, 26, 28, 39]. The recent books [10] and [29] contain the basic theory, examples, and many results on chaotic linear dynamics. Particular attention deserves the case of C_0-semigroups of operators, since many of them are originated in the analysis of the asymptotic behaviour of solutions to certain linear partial differential equations and to infinite systems of linear differential equations. Especially, different notions of chaos for C_0-semigroups have experienced a great development in recent years (see, e.g., [1, 2, 5, 18, 20, 22, 31, 40]).

A stronger concept than hypercyclic operators is the notion of frequently hypercyclic operators introduced by Bayart and Grivaux [9] (see [29] and the references therein) trying to quantify the frequency with which an orbit meets an open set. This concept was extended to C_0-semigroups in [3].

A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be *frequently hypercyclic* if there exists $x \in X$ (called frequently hypercyclic vector) such that $\underline{\text{Dens}}(\{t \geq 0 : T_t x \in U\}) > 0$ for every non-empty open subset $U \subset X$, where $\underline{\text{Dens}}(B)$ is the lower density of a Lebesgue measurable subset $B \subset \mathbb{R}_0^+$ defined as

$$\liminf_{t\to\infty} \frac{\mu(B \cap [0,t])}{t}.$$

In [19] it was proved that if $x \in X$ is a frequently hypercyclic vector for $(T_t)_{t\geq 0}$, then x is a frequently hypercyclic vector for every the operator T_t, $t > 0$.

In [15] Bonilla and Grosse-Erdmann, based on a result of Bayart and Grivaux, provided a Frequent Hypercyclicity Criterion for operators. Later, Mangino and Peris [33] obtained a continuous version of the criterion based on Pettis integrals, which is called the Frequent Hypercyclicity Criterion for semigroups.

The aim of this work is to study the specification property for strongly continuous semigroups of operators on Banach spaces, that is, for C_0-semigroups and its relationship with other dynamical properties, like hypercyclicity, mixing, chaos and frequent hypercyclicity; and to provide useful tools ensuring that many natural solution semigroups associated to linear PDEs satisfy the specification property for C_0-semigroups. The paper is structured as follows: In Section 2 we introduce the notion of the specification property for C_0-semigroups, from now on denoted by SgSP. Section 3 is devoted to study the SgSP in connection with other dynamical properties. Finally, in Section 4 we provide several applications of the results in previous sections to solution semigroups of certain linear PDEs, and a characterization of translation semigroups which exhibit the SgSP.

2 Specification property for C_0-semigroups

A first notion of the specification property for a one-parameter family of continuous maps acting on a compact metric space was given in [17]. When trying to study the specification property in the context of semigroups of linear operators defined on separable Banach spaces, the first crucial problem is that these spaces are never compact, therefore, our first task should be to adjust the SP in this context, in the vain of the discrete case, and the following definition can be considered the natural extension in this setting.

Definition 2 (Specification property for semigroups, SgSP)**.** A C_0-semigroup $(T_t)_{t\geq 0}$ on a separable Banach space X has the SgSP if there exists an increasing sequence $(K_n)_n$ of T-invariant sets with $0 \in K_1$ and $\overline{\cup_{n\in\mathbb{N}} K_n} = X$ and there exists a $t_0 > 0$, such that for each $n \in \mathbb{N}$ and for any $\delta > 0$ there is a positive real number $M_{\delta,n} \in \mathbb{R}_+$ such that for any integer $s \geq 2$, any set $\{y_1, \ldots, y_s\} \subset K_n$ and any real numbers: $0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$ satisfying $b_s + M_{\delta,n} \in \mathbb{N} \cdot t_0$ and $a_{r+1} - b_r \geq M_{\delta,n}$ for $r = 1, \ldots, s-1$, there is a point $x \in K_n$ such that, for each $t_r \in [a_r, b_r]$, $r = 1, 2, ..., s$, the following conditions hold:

$$\|T_{t_r}(x) - T_{t_r}(y_r)\| < \delta,$$
$$T_t(x) = x, \quad \text{where } t = M_{\delta,n} + b_s.$$

Analogously to the discrete case, the meaning of this property is that if the semigroup has the SgSP then it is possible to approximate simultaneously several finite pieces of orbits by one periodic orbit. Obviously, parameter intervals for the approximations must be disjoint. In contrast, the periodicity condition SgSP looks more relaxed than the corresponding one in the OSP: Observe that in the SgSP we have the existence of certain $t_0 > 0$ such that the periodic vectors x that trace the orbits are such that $T_t(x) = x$ for some integer multiple t of t_0. The reason to relax the requirements on the periods in the SgSP is that almost all kind of behaviours concerning periods in chaotic C_0-semigroups can occur, as Bayart and Bermúdez showed in [8].

The following result is an immediate consequence of the corresponding definitions if we take into account the local equicontinuity (i.e., equicontinuity on arbitrary compact intervals) of C_0-semigroups.

Proposition 3. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space$X$. Then the following assertions are equivalent:

1. $(T_t)_{t\geq 0}$ has the SgSP,
2. Some operator T_{t_0} has the OSP.

3 SgSP and other dynamical properties for C_0-semigroups

In this section, we study the relation between the specification property and topological mixing, Devaney chaos, distributional chaos and frequent hypercyclicity. The following observations are useful to characterize mixing semigroups (see [29]).

Remark 4. From the definition, the semigroup $(T_t)_{t\geq 0}$ is mixing if and only if for every pair of non-empty open sets $U, V \subset X$, such that the complementary of the return set $R(U, V) := \{t \geq 0 : T_t(U) \cap V \neq \emptyset\}$ is (upper) bounded.

Remark 5. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. The semigroup $(T_t)_{t\geq 0}$ is mixing if and only if for every non-empty open set $U \subset X$ and every open 0-neighbourhood W, the complementary of the return sets $R(U, W)$ and $R(W, U)$ are (upper) bounded.

Proposition 6. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ has the SgSP, then $(T_t)_{t\geq 0}$ is mixing.

Proof. Let us consider a non-empty open set U and a 0-neighbourhood W. We claim that there exists some $t_1 > 0$ such that $t \in R(U, W) \cap R(W, U)$, $\forall t > t_1$, and this implies $(T_t)_{t\geq 0}$ is mixing.

Fix $u \in U$ and $\delta > 0$ such that $B(u, 2\delta) \subset U$ and $B(0, 2\delta) \subset W$. By hypothesis, $(T_t)_{t\geq 0}$ has the SgSP, then there are $t_0 > 0$ and a T_{t_0}-invariant set K such that the restriction of

T_{t_0} to K has the SP and $K \cap B(u,\delta) \neq \emptyset$. From Definition 2, there exists M (depending on K and δ, which we suppose $M \in \mathbb{N} \cdot t_0$) such that if we choose $y_1 \in K \cap B(u,\delta)$, $y_2 = 0$, $s > 0$ with $s \in \mathbb{N} \cdot t_0$, and $0 = a_1 = b_1 < a_2 = M < b_2 = M + s$ then there exists a periodic point $x \in K$ with period $2M + s$ such that

$$\|T_t(x) - T_t(y_1)\| < \delta, \; a_1 \leq t \leq b_1 \,,$$
$$\|T_t(x) - T_t(y_2)\| < \delta, \; a_2 \leq t \leq b_2 \,.$$

This implies $\|x - y_1\| < \delta$, so $\|x - u\| < 2\delta$ and hence $x \in U$. From the second line of the previous equation, we have $T_t(x) \in B(0,\delta) \subset W$ for $M \leq t \leq M + s$. Therefore $t \in R(U,W)$ for any $t \geq M$.

Taking now $t > M$), we select $t' \in [M, M + s]$ such that $t + t' \in \mathbb{N} \cdot (2M + s)$. We have $\|T_{t'}(x)\| < \delta$, hence $T_{t'}(x) \in B(0,\delta) \subset W$. Since x is periodic with period $2M + s$, then $T^t(T^{t'}(x)) = x \in U$. Therefore $t \in R(W,U)$ for any $t > M$.

We have proved that the complementary of $R(U,W) \cap R(W,U)$ is (upper) bounded and this finishes the proof.

□

Proposition 7. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ has the SgSP then $(T_t)_{t\geq 0}$ is Devaney chaotic.

Proof. By Proposition 6, $(T_t)_{t\geq 0}$ is topologically transitive and, by the definition of SgSP, it is clear that any vector in the space may be approximated by a periodic point. □

Proposition 8. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ has the SgSP with respect to an increasing sequence $(K_n)_n$ of invariant compact sets, then $(T_t)_{t\geq 0}$ is distributionally chaotic.

Proof. We first recall that for single maps on compact metric spaces, Oprocha [37] showed that the SP implies distributional chaos in our sense. Since there is $t_0 > 0$ such that T_{t_0} has the OSP, and by hypothesis the associated increasing sequence $(K_n)_n$ of invariant sets consists of compact sets, then $T_{t_0}|_{K_n}$ is distributionally chaotic for every $n \in \mathbb{N}$, thus the operator T_{t_0} is distributionally chaotic. Applying Theorem 3.1 in [1] we obtain that the semigroup is distributionally chaotic. □

It is well-known [29, 19, 33] that a C_0-semigroup is hypercyclic (respectively mixing, respectively Devaney chaotic, frequently hypercyclic) if and only if it admits a hypercyclic (resp. mixing, resp. Devaney chaotic, resp. frequently hypercyclic) discretization $(T_{t_n})_n$. In particular, it is useful for our purposes the following characterization of frequent hypercyclicity for semigroups in terms of the frequent hypercyclicity of some of its operators [19, 33].

Proposition 9 ([19, 33])**.** Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. Then the following conditions are equivalent:

(i) $(T_t)_{t\geq 0}$ is frequently hypercyclic.

(ii) For every $t > 0$ the operator T_t is frequently hypercyclic.

(iii) There exists $t_0 > 0$ such that T_{t_0} is frequently hypercyclic.

The implication (i)→(ii) was proved in [19], (ii)→(iii) is obvious, and (iii)→(i) was observed in [33].

We point out the connection between the frequent hypercyclicity property for semigroups and the specification property SgSP.

Proposition 10. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ has the SgSP, then $(T_t)_{t\geq 0}$ is frequently hypercyclic.

Proof. By Proposition 3, if $(T_t)_{t\geq 0}$ has the SgSP, then there exists $t_0 > 0$ such that the operator T_{t_0} has the OSP, then the operator T_{t_0} is frequently hypercyclic [7] and, therefore, the C_0-semigroup $(T_t)_{t\geq 0}$ is frequently hypercyclic (see Proposition 9 [19, 33]). □

The following result is a partial converse.

Proposition 11. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ satisfies the Frequent Hypercyclicity Criterion for semigroups of [33], then every operator T_t, $t > 0$, has the OSP and, therefore, the semigroup $(T_t)_{t\geq 0}$ has the SgSP.

Proof. If $(T_t)_{t\geq 0}$ satisfies the Frequent Hypercyclicity Criterion for semigroups of [33] then every operator T_t, $t > 0$, satisfies the Frequent Hypercyclicity Criterion for operators of [15]. Using a result from [7] about the OSP on operators which says that if an operator T on a Banach space satisfies the Frequent Hypercyclicity Criterion then it has the OSP, hence the operator T_t has the OSP for every $t > 0$, and finally, by using Proposition 3, we conclude that the semigroup $(T_t)_{t\geq 0}$ has the SgSP. □

Corollary 2.3 in [33] showed that under some conditions, expressed in terms of eigenvector fields for the infinitesimal generator A of the C_0-semigroup $(T_t)_{t\geq 0}$, the semigroup is frequent hypercyclic. More precisely, it was proved in [33] that it satisfies the Frequent Hypercyclicity Criterion. Actually, the conditions on the infinitesimal generator are much easier to verify on precise examples and applications than the Frequent Hypercyclicity Criterion. As a consequence, we also obtain the SgSP under the same conditions.

Proposition 12. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable complex Banach space X and let A be its infinitesimal generator. Assume that there exists a family $(f_j)_{j\in\Gamma}$ of locally bounded measurable maps $f_j : I_j \to X$ such that I_j is an interval in $\mathbb{R}$, $f_j(I_j) \subset D(A)$, where $D(A)$ denotes the domain of the generator, $Af_j(t) = itf_j(t)$ for every $t \in I_j$, $j \in \Gamma$ and $\operatorname{span}\{f_j(t) \ : \ j \in \Gamma, \ t \in I_j\}$ is dense in X. If either

a) $f_j \in C^2(I_j, X)$, $j \in \Gamma$,

or

b) X does not contain c_0 and $\langle \varphi, f_j\rangle \in C^1(I_j)$, $\varphi \in X'$, $j \in \Gamma$,

then the semigroup $(T_t)_{t\geq 0}$ has the SgSP.

Proof. The result directly follows from the Corollary 2.3 in [33] and Proposition 11. □

Remark 13. It was pointed out in Remarks 2.4 in [33] that the spectral criterion for chaos in [24] of C_0-semigroups is stronger than the criterion given in Proposition 12. As a consequence, if a semigroup satisfies the spectral criterion, then it has the SgSP.

4 Applications and examples

In this section we will present several examples of C_0-semigroups exhibiting the specification property, with particular interest in solution semigroups to certain PDEs. A characterization of translation semigroups with the SgSP is also provided.

In the following examples, in order to ensure whether the solution semigroup has the SgSP, we will check the conditions of Proposition 12 (*i.e.*, the conditions of Corollary 2.3 in [33]) or the spectral criterion in [24] for chaos.

Example 14 (The solution semigroup of the hyperbolic heat transfer equation)**.** Let us consider the hyperbolic heat transfer equation (HHTE):

$$\begin{cases} \tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \\ u(0,x) = \varphi_1(x), x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(0,x) = \varphi_2(x), x \in \mathbb{R} \end{cases}$$

where φ_1 and φ_2 represent the initial temperature and the initial variation of temperature, respectively, $\alpha > 0$ is the thermal diffusivity, and $\tau > 0$ is the thermal relaxation time.

The dynamical behaviour presented by the solutions of the classical heat equation was studied by Herzog [30] on certain spaces of analytic functions with certain growth control. Later, the dynamical properties of the solution semigroup for the hyperbolic heat transfer equation were also established in [21, 29].

The HHTE can be expressed as a first-order equation on the product of a certain function space with itself $X \oplus X$. We set $u_1 = u$ and $u_2 = \frac{\partial u}{\partial t}$. Then the associated first-order equation is:

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & \frac{-1}{\tau} I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \\ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, x \in \mathbb{R} \end{cases}$$

We fix $\rho > 0$ and consider the space [30]

$$X_\rho = \{f : \mathbb{R} \to \mathbb{C}; f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, (a_n)_{n\geq 0} \in c_0\}$$

endowed with the norm $||f|| = \sup_{n\geq 0} |a_n|$, where c_0 is the Banach space of complex sequences tending to 0.

Since

$$A := \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & \frac{-1}{\tau} I \end{pmatrix}.$$

is an operator on $X := X_\rho \oplus X_\rho$, we have that $(T_t)_{t\geq 0} = (e^{tA})_{t\geq 0}$ is the C_0-semigroup solution of the HHTE. We know from [21] and [29] that, given α, τ and ρ such that $\alpha\tau\rho > 2$, the solution semigroup $(e^{tA})_{t\geq 0}$ defined on $X_\rho \oplus X_\rho$ is mixing and chaotic since it satisfies the hypothesis of the spectral criterion [24]. Therefore, it satisfies the hypothesis of Proposition 12 and it follows that the solution semigroup of the HHTE has the SgSP.

Remark 15. With minor changes, we can apply the previous argument to the wave equation

$$\left\{ \begin{array}{rll} u_{tt}t & = \alpha u_{xx} & \\ u(0,x) & = \varphi_1(x), & x \in \mathbb{R} \\ u_t(0,x) & = \varphi_2(x), & x \in \mathbb{R} \end{array} \right\}$$

which can be expressed as a first order equation in $X_\rho \oplus X_\rho$ (see [29]), in order to state that its semigroup solution has the SgSP.

Remark 16. This result can be extended to the solution semigroup of an abstract Cauchy problem of the form:

$$\left\{ \begin{array}{ll} u_t & = Au \\ u(0,x) & = \varphi(x) \end{array} \right\},$$

where A is a linear operator on a Banach space X and the generator of the solution semigroup. If A satisfy the conditions of Proposition 12, then the semigroup $(T_t)_{t\geq 0}$ with generator A has the SgSP.

Example 17 (C_0-semigroup solution of the Black-Scholes equation)**.** Black and Scholes proposed in [13] a mathematical model which gives a theoretical estimate of the price of stock options. The model is based on a partial differential equation, called the Black-Scholes equation, which estimates the price of the option over time. They proved that under some assumptions about the market, the value of a stock option $u(x,t)$, as a function of the current value of the underlying asset $x \in \mathbb{R}^+ = [0,\infty)$ and time, satisfies the final value problem:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru & \text{in } \mathbb{R}^+ \times [0,T] \\ u(0,T) = 0 & \text{for } t \in [0,T] \\ u(x,T) = (x-p)^+ & \text{for } x \in \mathbb{R}^+ \end{array} \right.$$

where $p > 0$ represents a given strike price , $\sigma > 0$ is the volatility, $r > 0$ is the interest rate and

$$(x-p)^+ = \left\{ \begin{array}{ll} x-p & \text{if } x > p \\ 0 & \text{if } x \leq p. \end{array} \right.$$

Let $v(x,t) = u(x,T-t)$, then it satisfies the forward Black-Scholes equation, defined for all time $t \in \mathbb{R}^+$ by

$$\left\{ \begin{array}{ll} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv & \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \\ v(0,T) = 0 & \text{for } t \in \mathbb{R}^+ \\ v(x,0) = (x-p)^+ & \text{for } x \in \mathbb{R}^+ \end{array} \right.$$

This problem can be expressed in an abstract form:

$$\left\{ \begin{array}{l} \frac{\partial v}{\partial t} = \mathcal{B}v, \\ v(0,T) = 0, \\ v(x,0) = (x-p)^+ \quad \text{for } x \in \mathbb{R}^+. \end{array} \right.$$

where $\mathcal{B} = (D_\nu)^2 + \gamma(D_\nu) - rI$, being $D_\nu = \nu x \frac{\partial}{\partial x}$ with $\nu = \frac{\sigma}{\sqrt{2}}$ and $\gamma = \frac{r}{\nu} - \nu$.

It was shown that the Black-Scholes equation admits a C_0-semigroup solution which can be represented by $T_t := f(tD_\nu)$, where

$$f(z) = e^{g(z)} \text{ with } g(z) = z^2 + \gamma z - r\,.$$

In [27], a new explicit formula for the solution of the Black-Scholes equation was given in certain spaces of functions $Y^{s,\tau}$ defined by

$$Y^{s,\tau} = \left\{ u \in C((0,\infty))\ ;\ \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0, \quad \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0 \right\}$$

endowed with the norm

$$||u||_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right|.$$

Later, it was proved in [25] that the Black-Scholes semigroup is strongly continuous and chaotic for $s > 1, \tau \geq 0$ with $s\nu > 1$ and it was showed in [36] that it satisfies the spectral criterion in [24] under the same restrictions on the parameters and, therefore, the Black-Scholes semigroup has the SgSP.

There exist other C_0-semigroups related with PDEs which present the SgSP. In fact, the examples given in [36] in the context of strong mixing measures, satisfy either the conditions of Proposition 12 or the spectral criterion in [24] and, therefore, they have the SgSP. The examples provided in [36] include the semigroup generated by a linear perturbation of the one-dimensional Ornstein-Uhlenbeck operator [18], the solution C_0-semigroup of a partial differential equation of population dynamics studied by Rudnicki [40], the solution C_0-semigroup associated to Banasiak, Lachowicz and Moszyński models of *birth-and-death* processes [4, 5].

Let $1 \leq p < \infty$ and let $v : \mathbb{R}_+ \to \mathbb{R}$ be a strictly positive locally integrable function, that is, v is measurable with $\int_0^b v(x)\,dx < \infty$ for all $b > 0$. We consider the space of weighted p-integrable functions defined as

$$X = L^p_v(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{K}\ ;\ f \text{ is measurable and } \|f\| < \infty\},$$

where

$$\|f\| = \Big(\int_0^\infty |f(x)|^p v(x)\,dx\Big)^{1/p}.$$

The *translation semigroup* is then given by

$$T_t f(x) = f(x+t), \quad t, x \geq 0.$$

This defines a C_0-semigroup on $L^p_v(\mathbb{R}_+)$ if and only if there exist $M \geq 1$ and $w \in \mathbb{R}$ such that, for all $t \geq 0$, the following condition

$$v(x) \leq M e^{wt} v(x+t) \ \text{ for almost all } x \geq 0.$$

is satisfied. In that case, v is called an *admissible weight function* and we will assume in the sequel that v belongs to this class of weight functions.

For the translation semigroup defined on $L^p_v(\mathbb{R}_+)$, there was proved in [33] that $(T_t)_{t\geq 0}$ is chaotic if and only if it satisfies the Frequent Hypercyclicity Criterion for semigroups and that $(T_t)_{t\geq 0}$ is chaotic if and only if every operator T_t satisfies the Frequent Hypercyclicity Criterion for operators. A more complete characterization of the frequently hypercyclic criterion for the translation semigroup on $L^p_v(\mathbb{R}_+)$ was given in [34]:

Theorem 18 (Theorem 3.10, [34])**.** Let v be an admissible weight function on R. The following assertions are equivalent:

(1) The translation semigroup $(T_t)_{t\geq 0}$ is frequently hypercyclic on $L^p_v(\mathbb{R}_+)$.

(2) $\sum_{k\in\mathbb{Z}} v(k) < \infty$.

(3) $\int_{-\infty}^{\infty} v(t)dt < \infty$.

(4) $(T_t)_{t\geq 0}$ is chaotic on $L^p_v(\mathbb{R}_+)$.

(4) $(T_t)_{t\geq 0}$ satisfies the Frequently Hypercyclicity Criterion.

This result allows us to give a characterization of the SgSP for the translation semigroup on the space $X = L^p_v(\mathbb{R}_+)$.

Theorem 19. Let us consider the translation semigroup on the space $X = L^p_v(\mathbb{R}_+)$, where $1 \leq p < \infty$ and $v : \mathbb{R}_+ \to \mathbb{R}$ is an admissible weight function. We claim that the following assertions are equivalent:

(i) $\int_0^\infty v(x)\,dx < \infty$.

(ii) $(T_t)_{t\geq 0}$ has SgSP.

(iii) $(T_t)_{t\geq 0}$ is Devaney chaotic.

(iv) $(T_t)_{t\geq 0}$ satisfies the Frequently Hypercyclicity Criterion.

(v) The translation semigroup $(T_t)_{t\geq 0}$ is frequently hypercyclic.

Proof. By Theorem 18 [34] and Propositions 10 and 11, it is obvious that for the translation semigroup the SgSP is equivalent to satisfy the Frequently Hypercyclicity Criterion and the SgSP is equivalent to frequently hypercyclic. □

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