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Impact of cost of substitution and joint replenishment on inventory decisions for joint substitutable and complementary items under asymmetrical substitution

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Abstract

In this paper, impact of cost of substitution and joint replenishment on inventory decisions for joint substitutable and complementary items under asymmetrical substitution has been studied. The phenomenon of substitution is considered in a stock-out situation and when items become out of stock due to demand then unfulfilled demand is asymmetrically substituted by another item. We formulate the inventory model mathematically and derived optimal ordering quantities, optimal total costs and extreme value of substitution rate for all possible cases. Moreover, pseudo-convexity of the total inventory cost function is obtained and the solution procedure is provided. Numerical example and sensitivity analysis have been presented to validate the effectiveness of the inventory model and substantial improvement in total optimal inventory cost with substitution with respect to optimal total inventory cost without substitution is seen.

Keywords: *Inventory decisions; cost of substitution; joint replenishment; substitutable items; complementary items; asymmetrical substitution.*

Introduction

As can be observed that most of the people starts his day with the use of complementary items such as toothbrush and toothpaste, tea and sugar etc. Two or more items which are consumed together to get the overall utility, are called complementary items and the process of consumption of these items are called the phenomenon of completion. Complementary items are used in combination with another product because a single use of any part of complementary items has limited usage. All parts of complementary items affect the market demand of complementary items. The phenomenon of substitution is consumption of another alternate item in place of a preferred item to fulfill the customer's demand and this type of alternate item is called substitutable or substitute item. According to Tang and Yin (2007), "There are three kinds of item substitution: stock-out based substitution, assortment-based substitution and price-based substitution." But, in the direction of substitution, stock out substitution occurs frequently in real life and its role in inventory decisions is very crucial. Stock-out substitution is more important in today's busy life because time is a more sensitive factor for everyone and most of them wants to minimize their purchasing time. There are plentiful examples for complementary and substitutable items separately as well as substitutable items composed with complementary components in our daily life. For examples, toothpaste and toothbrush, DVD players and DVDs, computer hardware and software, pencils and erasers are complementary items and different brands of cold drinks, different brands of smartphones, different brands of laptops, different brands of desktops are substitutable items as well as items including different brands of DVD players and DVDs, items including different brands computer hardware and software, items including different brands of tire and tube, items including different brands of pencils and erasers are substitutable items composed with complementary components. Computer hardware cannot work without software, which indicates computer hardware is useless without computer software. As can be seen that complementary items have negative cross elastic property on demand while substitute items have positive cross elastic property on demand. Now a day's, supermarkets or retails industries arrange substitutable items due to some market motives e.g. to minimize the total inventory cost or to maximize the profit, to advertise the substitutable items, to enhance the availability of items and maintain the good-will, etc. In the competitive environment of retails industries, substitutable items contribute a vital role in the business. for example: generally, customers prefer substitutable items because they do not want to go other shopping stores on facing out of stock of desired items. As a result, goodwill of supermarkets remains maintained. Thus, substitution between items can enhance the efficiency of the inventory system. Due to substitution between items, the proposed inventory model involves an additional cost: cost of substitution for each unit of the substituted item. It depends on the number of units substituted. Some time market motives also force to impose the cost of substitution. As we know that joint replenishment enforce the process of stock out substitution and stock-out substitution is categorized as symmetrical substitution and asymmetrical substitution (Kim and Bell, 2011). This model has been derived under asymmetrical substitution because in practice, asymmetrical substitution happens more than symmetrical substitution because all customers do not prefer substitutable items. So, asymmetrical substitution is a better realistic phenomenon than symmetrical substitution. Other ways to categories the substitution as one-way and two-way substitution. One-way substitution is defined as the substitution of items based on an attribute such as quality or speed of service. On the other hand, two-way substitution is the substitution of items based on the same category. Observing all aspects of business, manufacturing companies now start producing substitutable items composed with complementary components. Introducing substitute items in the market, manufacturing companies face tough competition because customers select the alternate item with very carefully so that it can fulfil the quality as well as demand of preferred items (Taleizadeh et al. 2015). On other hand, observing inventory management, complementary items require more storage space because all parts of complementary components are important. So in this paper, we study the impact of cost of substitution and joint replenishment on inventory decisions for two-way substitutable items with asymmetrical stock-out substitution where both items are composed with two complementary components. The aim of this model is to determine optimal order quantities, the optimal total cost and the extreme value of substitution rate.

The rest of the paper is arranged as follows. Section 2 describes literature review, section 3 involves assumptions and notations and section 4 describes problem description and inventory model formulation. In section 5, solution approach is suggested to determine total optimal cost, optimal order quantities and extreme value of substitution rate. In section 6, theoretical analysis of inventory model is studied. In section 7, numerical example and sensitivity analysis are presented and finally section 8 refers to the conclusion and future work.

Literature Review

The literature related to the key parameter of this inventory model i.e. inventory decisions, substitutable items, complementary items, partial substitution, joint replenishment and cost of substitution can be describe as follows: The fundamental inventory model was studied by Harris (1915) and this inventory model was extended by Wilson (1934) to obtain a formula for the economic order quantity (EOQ). The first inventory model of substitutable items was developed by McGillivray and Silver (1978) by proposing that the substitutable items have equal unit variable cost and shortage penalty. On inventory models for substitutable items, there are many research papers available in literature. So, for a detail study on inventory models for substitutable items, readers may refer to the review paper on inventory models for substitutable items by Sin et al. (2015). To the best of our knowledge, the research papers on complementary and substitutable items concurrently are very few contributions in literature. Most of all research papers on complementary items consist of pricing decisions and research papers consisting of inventory decisions are rarely available in the existing literature. Whereas, this paper consists of inventory decisions and develops an inventory decision model for two items under the phenomenon of substitution and completion, considering partial substitution, substitution cost, and joint replenishment. Many firms have limited storage capacity. So, taking consideration of storage capacity, Ouyang et al. (2007) developed an economic order quantity model with limited storage capacity and in this model the supplier offers cash discount and permissible delay in payments for the retailer. Many firms order a group of items simultaneously instead of individually. Joint replenishment policy is more beneficial in an inventory model of two or more than two items because if two or more than two items are ordered jointly then transportation cost, delivery time, fixed ordering cost can be reduced. For a detail study on joint replenishment, readers may refer to review paper on joint replenishment by Khouja and Goyal (2005), Porras and Dekker (2006), Hong and Kim (2009) and Silva and Gao (2013) studied on joint replenishment. Summary of literature review related to our article in categories of inventory decision, substitutable items, complementary items, cost of substitution and partial substitution are presented in Table 1 as the Researcher's contributions table.

Ernst and Kouvelis (1999) studied demand substitution between individual items and packaged goods in stock-out situations with two-way and full substitution. While, assuming one-way substitution, Drezner et al. (1995) developed an economic order quantity model of two substitutable items considering joint replenishment policy and studied the cases of full substitution, partial substitution and no substitution and investigated that only partial substitution or no substitution may be optimal and full substitution is never optimal and Goyal (1996) studied an inventory model of two substitutable product with full substitution. Further, Gurnani and Drezner (2000) extended the work of Drezner et al. (1995) for multiple quality items with one-way and full substitution and consumers are allowed to switch to higher quality items. Hsu et. al (2005) studied a dynamic lot-size model under one-way item and full substitution where the items are indexed in such a way that a lower-index item may be used to substitute for the demand of a higher-index item while, Tang and Yin (2007) studied joint ordering and pricing strategies for two substitutable items under two-way and full substitution. Yue et al. (2006) studied a duopoly market dealing with complementary items and Wei et al. (2013) studied the pricing decision models with two complementary items in supply chain management consisting of two manufacturers and one common retailer and developed five pricing inventory models: MS-Stackelberg, MS-Bertrand, RS-Stackelberg, RS-Bertrand, and NG models. Salameh et al. (2014) studied the EOQ model for two substitutable items with partial substitution and joint replenishment policy and the work of Salameh et al. (2014) is the extension of the research of Drezner et al. (1995) by taking two-way and partial substitution. Taking only complementary items, Yuhong and Shuya (2015) studied on joint selling of complementary components under brand and retail Competition and Hemmati et al. (2018) developed an integrated two-stage model, which consists of one vendor and one buyer for two complementary products under consignment policy and stock-dependent demand. Under partial and two-way substitution, Krommyda et al. (2015) proposed inventory decision model for two substitutable items with stock-dependent demand and Maddah et al. (2016) developed EOQ model for multiple substitutable items to obtain optimal order quantities under joint replenishment which is extension of the work of Salameh et al. (2014) for multiple items. Giri et al. (2016) proposed a two-echelon supply-chain system, having a competition of selling two substitutable items and one complementary item using a common retailer. Transchel (2017) studied a two-items inventory model with price- and stockout-based substitution. In addition, under two-way and partial substitution Mishra and Shanker (2017) proposed an inventory model of two substitutable items to determine optimal order quantities under joint replenishment with cost of substitution, Mishra (2017) proposed an inventory model of two substitutable deteriorating items under joint replenishment policy to determine optimal ordering quantities and Mishra and Shanker (2017) extend the work of Mishra (2017) by considering cost of substitution. Further, Benkherouf et al. (2017) developed an inventory decision model for the finite horizon problem of substitutable items, taking time-varying demand under one-way and full substitution. Under two-way substitution, Pan et al. (2018) developed an inventory replenishment model for two-inventory based substitutable items with full substitution and obtained the optimal replenishment cycle time and ending inventory levels, Chen et al. (2019) developed an inventory model for Joint replenishment decision-taking shortages, partial demand substitution, and defective items and Mokhtari (2018) developed an EOQ model for two-substitutable items where one item is composed with two complementary components and considered joint ordering policy and full substitution. Taleizadeh et al. (2019) studied the pricing decision of two items where items may be complementary or substitutable and Edalatpour et al. (2019) analyzed simultaneous pricing and inventory decisions for complementary and substitutable items considering nonlinear holding cost. To best of our knowledge, research paper with parameter cost of substitution, joint replenishment, complementary and substitutable items and partial substitution, together is not available in the literature to obtain the optimal ordering quantities, optimal total cost and extreme value of substitution.

If we corelate this paper with existing literature then this paper may be consider is an extension of the work of Mokhtari (2018) in directions such as both items contain complementary components, partial substitution, cost of substitution, analytical derivation of optimal ordering quantities, impact of substitution rate, and critical substitution rate on inventory model, the work of Salameh et al. (2014) and Krommyda et al. (2015) in direction such as complementary item, cost of substitution, analytical derivation of optimal ordering quantities, impact of substitution rate and critical substitution rate on inventory model.

Table 1: Researcher's contributions table related to this article.

Researcher(s)	Inventory decision	Substitute item	Complementary item	Cost of substitution	Partial Substitution
McGillivray and Silvar (1978)	✓	✓			✓
Drezner et al. 1995)	✓	✓			✓
Gurnani and Drezner (2000)	✓	✓			
Yue et al. (2006)			✓		
Tang and Yin 2007)		✓			
Wei et al. (2013)			✓		
Salameh et al. (2014)	✓	✓			✓
Krommyda et al. (2015)	✓	✓			✓
Maddah et al. (2016)	✓	✓			✓
Giri et al. (2016)		✓	✓		✓
Benkherouf et al. (2017)	✓				
Chen. et al. (2019)		✓			✓
Hemmati et al. (2018)			✓		
Mokhtari (2018)	✓	✓	✓		
Pan et al (2018)	✓	✓			
Taleizadeh et al. (2019)	✓	✓	✓		
Edalatpour et al. (2019)	✓	✓	✓		
This model	✓	✓	✓	✓	✓

Assumptions and Notations

In this paper, the following assumptions and notations are used for the mathematical formulation of this inventory model.

Assumptions

1. The inventory system contains two substitutable and complementary items and both items are mutually substitutable on lack of availability of stock. However the rate and cost of substitution may differ.
2. Demand is deterministic, constant and the demand of one item can be partially substituted by another item in case of unavailavility of preferred item.
3. Both items are order jointly in each ordering cycle of inventory.
4. Lead time is zero and replenishment is instantaneous i.e. replenishment rate is infinite.

Notations

The notations are categorized as parameters, Intermediate variables, decision variables, and objective functions.

Parameters

A_1, A_2 Fixed ordering costs of items 1 and 2.

D_1, D_2 Demand rates of items 1 and 2.

h_1, h_2 Holding costs per unit time of items 1 and 2.

a_1, a_2 Usages rates of two complementary components of item 1.

a_3, a_4 Usages rates of two complementary components of item 2.

CS_{12} Unit substitution cost for item 1 when it is substituted by item 2.

CS_{21} Unit substitution cost for item 2 when it is substituted by item 1.

σ_1 , σ_2 Shortage cost per unit for items 1 and 2.

Intermediate variables

T_1, T_2 Time when item 1 and 2 completely depleted.

p_1 Time interval during which substitution occurs in situation (i).

p_2 Time interval during which substitution occurs in situation (ii).

z_1, z_2 Inventory level of two complementary components of item 1 at the time T_1 in situation (i).

z_3, z_4 Inventory level of two complementary components of item 1 at the time T_2 in situation (ii).

Decision variables

q_1, q_2 Ordering quantities of two complementary components $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ of item 1.

q_3 , q_4 Ordering quantities of two complementary components $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ of item 2.

γ_1, γ_2 Rates of substitution of item 1 by item 2 and of item 2 by item 1.

γ_{1e} , γ_{2e} Extreme value of rates of substitution of item 1 by item 2 and of item 2 by item 1.

Objective functions

TCU_1 Total cost per unit time in situation 2 (With substitution).

TCU_2 Total cost per unit time in situation 2 (With substitution).

TCU_W Total cost per unit time in situation 3 (without substitution).

Problem Description and Mathematical Formulation

It is assumed that item 1 is composed with two complementary components $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ with usages rates (consumption rates) $a_1$and a_2 i.e. one unit of item 1 is composed with a_1 units of first complementary component ($\boldsymbol{\alpha_1}$) and a_2 units of second complementary component ($\boldsymbol{\alpha_2}$). Similarly, it is also assumed that item 2 is composed with two complementary components $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ with usages rates a_3 and a_4. So, demand rates of complementary components of item 1 are a_1D_1 and a_2D_1 respectively and of item 2 are a_3D_2 and a_4D_2 respectively. At initial time, inventory levels of complementary components of item 1 are q_1 and q_2 and of item 2 are q_3, and q_4 respectively. In this inventory model both complementary components of item 1 and item 2 are ordered jointly in order to take the advantage of joint replenishment so that inventory cost can be reduced. At each inventory cycle time T_1 and T_2 of both items, stored inventory is used until it depletes and depletion of stored inventory is due to demand.

In this section, firstly, we establish the relation between q_1 and q_2 as well as the relation between q_3 and q_4 and then, we formulate and find the solution for all possible situations. Since both complementary components of item 1 consume together so they deplete simultaneously and also work under joint ordering policy. Inventory cycle for the first complementary component $\boldsymbol{\alpha_1}$ of item 1 and inventory cycle for the second complementary component $\boldsymbol{\alpha_2}$ of item 1 are as follows:

$T_1 = \frac{q_1}{a_1D_1}$, $T_1 = \frac{q_2}{a_2D_1}$. Using this relation, we get

$$q_2 = \left(\frac{a_2}{a_1}\right) q_1 \tag{1}$$

This is a relation between q_1 and q_2 due to joint replenishment policy.

Similarly, we get

$$q_4 = \left(\frac{a_4}{a_3}\right) q_3 \tag{2}$$

This is a relation between q_3 and q_4 due to joint replenishment policy.

In the proposed inventory model, three possible situations occur. Inventory model diagrams for three possible situations are represented by Fig 1, Fig 2, and Fig 3 respectively. In three possible situations, two are with substitution and one is without substitution.

Situation 1: Item 1 depletes before item 2 (as shown by Fig 1) i.e. if item 1 is out of stock, then item 1 is partially substituted by item 2 with a rate of substitution γ_1. (with substitution)

Situation 2: Item 2 depletes before item 1 (as shown by Fig 2) i.e. if item 2 is out of stock, then item 2 is partially substituted by item 1 with a rate of substitution γ_2. (with substitution)

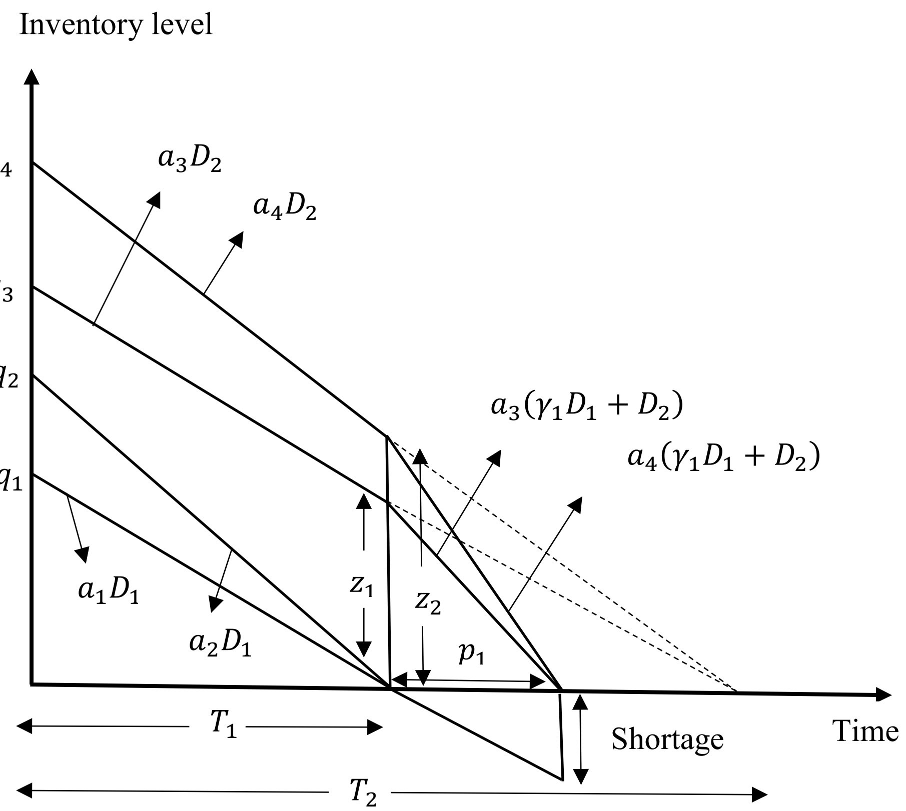
Situation 3: When there is no substitution between items 1 and 2 and both items deplete simultaneously (as shown by Fig 3) i.e. both items reach to zero at the same time. (without substitution)

Now, the mathematical formulation of total inventory cost for all three situations as defined above is described below.

Situation 1: Item 1 depletes before item 2 i.e. ($T_1 < T_2$) (with substitution).

Inventory model diagram is shown in Fig 1 and in this situation, item 1 is consumed within the inventory cycle of item 2. Partial substitution occurs for item 1 by item 2 as a result of which shortage occurs. Demand of item 1 is partially fulfilled by remaining inventory of item 2 with rate γ_1D_1. Indeed, the inventory of both complementary components $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ of item 2 is consumed with consumption rates $a_3(\gamma_1D_1 + D_2)$ and $a_4(\gamma_1D_1 + D_2)$ during the substitution period (p_1)

Fig 1. Inventory model diagram for situation 1 ($T_1 < T_2$)



The total inventory cost per ordering cycle consist of total inventory cost per ordering cycle of item 1, total cost per ordering cycle of item 2, shortage cost and cost of substitution.

Total inventory cost per ordering cycle of item 1 consist of ordering cost and holding cost.

So, total inventory cost of item 1 per ordering cycle (TC_{11}) is

$$TC_{11} = A_1 + h_1\left(\frac{q_1^2}{2a_1D_1} + \frac{q_2^2}{2a_2D_1}\right)$$

$$TC_{11} = A_1 + \frac{h_1q_1^2}{2a_1D_1}\left(1 + \frac{a_2}{a_1}\right) \qquad \text{using eq. (1)} \qquad (3)$$

Total inventory cost per ordering cycle of item 2 consist of ordering cost and holding cost.

So, total inventory cost of item 2 per ordering cycle (TC_{12}) is

$$TC_{12} = A_2 + h_2\begin{pmatrix} \frac{q_1}{2a_1D_1}\left(2q_3 - \frac{q_1a_3}{a_1}\frac{D_2}{D_1}\right) + \frac{1}{2a_3(\gamma_1D_1+D_2)}\left(q_3 - \frac{q_1a_3}{a_1}\frac{D_2}{D_1}\right)^2 \\ + \frac{q_1}{2a_1D_1}\left(2q_4 - \frac{q_1a_4}{a_1}\frac{D_2}{D_1}\right) + \frac{1}{2a_4(\gamma_1D_1+D_2)}\left(q_4 - \frac{q_1a_4}{a_1}\frac{D_2}{D_1}\right)^2 \end{pmatrix}$$

$$TC_{12} = A_2 + h_2\begin{pmatrix} \frac{q_1}{2a_1D_1}\left(2q_3 - \frac{q_1a_3}{a_1}\frac{D_2}{D_1}\right) + \frac{1}{2a_3(\gamma_1D_1+D_2)}\left(q_3 - \frac{q_1a_3}{a_1}\frac{D_2}{D_1}\right)^2 \\ + \frac{q_1}{2a_1D_1}\left(2\frac{a_4}{a_3}q_3 - \frac{q_1a_4}{a_1}\frac{D_2}{D_1}\right) + \frac{1}{2a_4(\gamma_1D_1+D_2)}\left(\frac{a_4}{a_3}q_3 - \frac{q_1a_4}{a_1}\frac{D_2}{D_1}\right)^2 \end{pmatrix} \text{using eq. (2)}$$

(4)

Lastly, we find the cost of substitution and shortage cost.

For this first we find the Inventory levels of complementary components of item 2 when item 1 becomes out of stock and substitution periods. which are as follows:

$$z_1 = q_3 - \frac{q_1 a_3}{a_1}\frac{D_2}{D_1}, \quad z_2 = q_4 - \frac{q_1 a_4}{a_1}\frac{D_2}{D_1} \quad , \quad p_1 = \frac{q_3 - \frac{q_1 a_3}{a_1}\frac{D_2}{D_1}}{a_3(\gamma_1 D_1 + D_2)} = \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3 D_1(\gamma_1 D_1 + D_2)}$$

and length of the inventory cycle is $(T_1 + p_1) = \frac{q_1 a_3 \gamma_1 + a_1 q_3}{a_1 a_3(\gamma_1 D_1 + D_2)}$

Total substituted units of item 1 by item 2 $= \gamma_1 D_1 (a_3 + a_4) p_1 = \gamma_1 (a_3 + a_4) \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)}$

Cost of substitution $= \quad CS_{12}\gamma_1 D_1 (a_3 + a_4) p_1 = CS_{12}\gamma_1 (a_3 + a_4) \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)}$ (5)

Total shortage unit of item 1$= (1-\gamma_1)\,(a_3 + a_4) \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)}$

Shortage cost $= \sigma_1 \,(1-\gamma_1)\,(a_3 + a_4) \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)}$ (6)

Thus, total inventory cost per cycle in situation 1 (TC_1) is

$TC_1 = TC_{11} + TC_{12} +$ cost of substitution $+$ Shortage cost

$$TC_1 = \left\{ \begin{array}{l} A_1 + A_2 + \frac{h_1 q_1^2}{2a_1 D_1}\left(1 + \frac{a_2}{a_1}\right) + h_2 \left(\begin{array}{l} \frac{q_1}{2a_1 D_1}\left(2q_3 - \frac{q_1 a_3}{a_1}\frac{D_2}{D_1}\right) \\ + \frac{1}{2a_3(\gamma_1 D_1 + D_2)}\left(q_3 - \frac{q_1 a_3}{a_1}\frac{D_2}{D_1}\right)^2 \\ + \frac{q_1}{2a_1 D_1}\left(2\frac{a_4}{a_3}q_3 - \frac{q_1 a_4}{a_1}\frac{D_2}{D_1}\right) \\ + \frac{1}{2a_4(\gamma_1 D_1 + D_2)}\left(\frac{a_4}{a_3}q_3 - \frac{q_1 a_4}{a_1}\frac{D_2}{D_1}\right)^2 \end{array} \right) \\ + CS_{12}\gamma_1 (a_3 + a_4) \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)} + \sigma_1\,(1 - \gamma_1)\,(a_3 + a_4) \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)} \end{array} \right\} \quad (7)$$

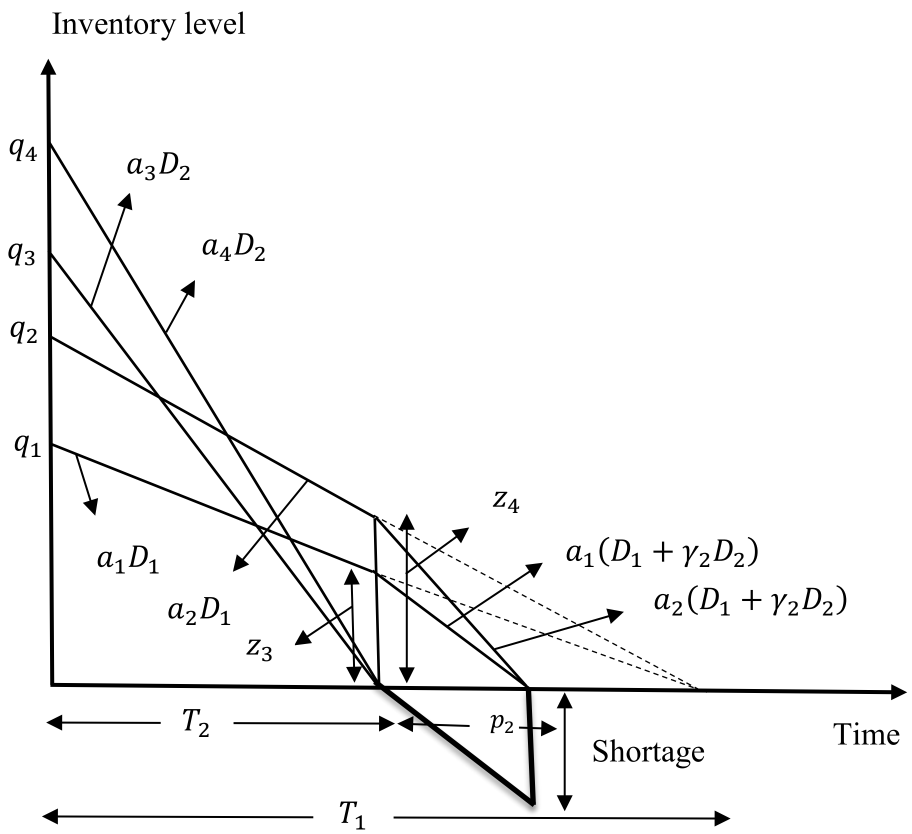
Finally, Total inventory cost per unit time is given by $TCU_1 = TC_1/(T_1 + p_1)$

$$TCU_1 = \left(\frac{a_1 a_3(\gamma_1 D_1 + D_2)}{q_1 a_3 \gamma_1 + a_1 q_3}\right) \left\{ \begin{array}{l} A_1 + A_2 + \frac{h_1 q_1^2}{2a_1 D_1}\left(1 + \frac{a_2}{a_1}\right) \\ + h_2 \left(\begin{array}{l} \frac{q_1}{2a_1 D_1}\left(2q_3 - \frac{q_1 a_3}{a_1}\frac{D_2}{D_1}\right) \\ + \frac{1}{2a_3(\gamma_1 D_1 + D_2)}\left(q_3 - \frac{q_1 a_3}{a_1}\frac{D_2}{D_1}\right)^2 \\ + \frac{q_1}{2a_1 D_1}\left(2\frac{a_4}{a_3}q_3 - \frac{q_1 a_4}{a_1}\frac{D_2}{D_1}\right) \\ + \frac{1}{2a_4(\gamma_1 D_1 + D_2)}\left(\frac{a_4}{a_3}q_3 - \frac{q_1 a_4}{a_1}\frac{D_2}{D_1}\right)^2 \end{array} \right) \\ + CS_{12}\gamma_1 (a_3 + a_4) \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)} \\ + \sigma_1\,(1 - \gamma_1)(a_3 + a_4) \frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)} \end{array} \right\} \quad (8)$$

The condition of the phenomenon for this situation i.e. $T_1 < T_2$ can be re-expressed in terms of q_1 and q_3 as $\frac{q_1}{a_1 D_1} < \frac{q_3}{a_3 D_2}$ which would work as a constraint for the optimization problem of situation 1, discussed in section 5.

Situation 2: Item 2 depletes before item 1 i.e. ($T_1 > T_2$) (with substitution).

Fig 2. Inventory model diagram for situation 2 ($T_1 > T_2$)



Following the analogous approach as the situation 1

The total inventory cost per unit time is given by

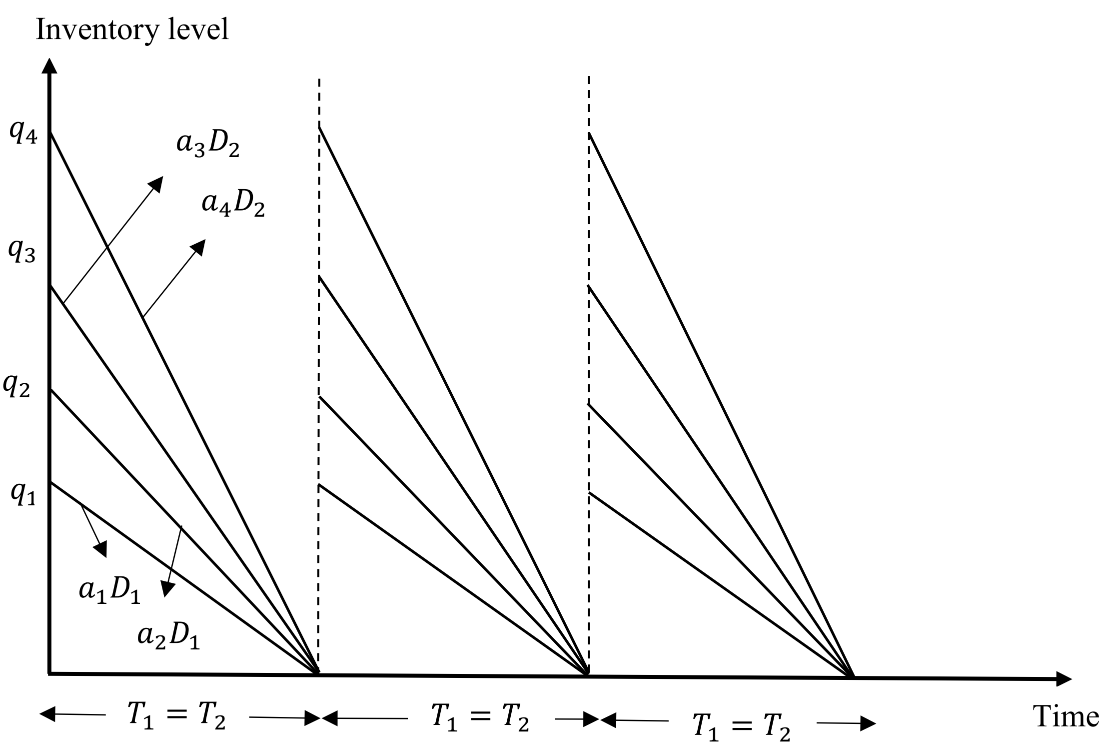
$$TCU_2 = \left(\frac{a_1a_3(D_1+\gamma_2D_2)}{q_3a_1\gamma_2+q_1a_3}\right)\left\{\begin{array}{l} A_1 + A_2 + \frac{h_2q_3^2}{2a_3D_2}\left(1+\frac{a_4}{a_3}\right) \\ + h_1\left(\begin{array}{l} \frac{q_3}{2a_3D_2}\left(2q_1 - \frac{q_3a_1}{a_3}\frac{D_1}{D_2}\right) \\ + \frac{1}{2a_1(D_1+\gamma_2D_2)}\left(q_1 - \frac{q_3a_1}{a_3}\frac{D_1}{D_2}\right)^2 \\ + \frac{q_3}{2a_3D_2}\left(2\frac{a_2}{a_1}q_1 - \frac{q_3a_2}{a_3}\frac{D_1}{D_2}\right) \\ + \frac{1}{2a_2(D_1+\gamma_2D_2)}\left(\frac{a_2}{a_1}q_1 - \frac{q_3a_2}{a_3}\frac{D_1}{D_2}\right)^2 \end{array}\right) \\ +CS_{21}\gamma_2(a_1+a_2)\frac{q_1a_3D_2-q_3a_1D_1}{a_1a_3(D_1+\gamma_2D_2)} \\ +\sigma_2\,(1-\gamma_2)(a_1+a_2)\frac{q_1a_3D_2-q_3a_1D_1}{a_1a_3(D_1+\gamma_2D_2)} \end{array}\right\} \quad (9)$$

The condition of the phenomenon for this situation i.e. $T_1 > T_2$ can be re-expressed in terms of q_1 and q_3 as $\frac{q_1}{a_1 D_1} > \frac{q_3}{a_3 D_2}$ which would work as a constraint for the optimization problem of situation 2, discussed in section 5.

Situation 3: Items 1 and 2 deplete simultaneously $(\boldsymbol{T_1 = T_2})$ $(\boldsymbol{without\ substitution})$

In this situation, ($T_1 = T_2$, as shown in Fig 3), both items become out of stock at the same time.

Fig 3. Inventory model diagram for situation 3. $(\boldsymbol{T_1 = T_2})$



Here, total inventory cost per ordering cycle consists of ordering cost and holding cost.

So, total inventory cost per ordering cycle with no substitution (TC_W) is given by

$$TC_W = A_1 + A_2 + h_1\left(\frac{q_1^2}{2a_1D_1} + \frac{q_2^2}{2a_1D_1}\right) + h_2\left(\frac{q_3^2}{2a_3D_2} + \frac{q_4^2}{2a_4D_2}\right)$$

$$TC_W = A_1 + A_2 + \frac{h_1q_1^2}{2a_1D_1}\left(1 + \frac{a_2}{a_1}\right) + \frac{h_2q_3^2}{2a_3D_2}\left(1 + \frac{a_4}{a_3}\right) \qquad \text{using eq. (3)} \tag{10}$$

Thus, the total inventory cost per unit time is given by $TCU_W = TC_W/T_1$ which simplified as

$$TCU_W = \left(\frac{a_1D_1}{q_1}\right)\left\{A_1 + A_2 + \frac{h_1q_1^2}{2a_1D_1}\left(1 + \frac{a_2}{a_1}\right) + \frac{h_2q_3^2}{2a_3D_2}\left(1 + \frac{a_4}{a_3}\right)\right\} \tag{11}$$

In this situation, the condition of the phenomenon i.e. $T_1 = T_2$ can be re-expressed in terms of q_1 and q_3 as $\frac{q_1}{a_1 D_1} = \frac{q_3}{a_3 D_2}$ which would work as a constraint for the optimization problem of situation 3, discussed in section 5.

Solution Approach

For two situations: situation 1 and situation 2, total inventory cost functions are more complex. So firstly, we analyze the nature of total inventory cost function for situations 1 and 2 with respect to decision variables q_1 and q_3 and then, we apply the method of calculus to obtain optimal values of order quantities. Now, we prove pseudo-convexity for total inventory cost functions subject to certain condition to ensure the unique optimal solution.

For Situation 1

In this situation, the pseudo-convexity, optimal order quantities and total inventory cost are obtained below.

Theorem 1-The total inventory cost function per unit time TCU_1 is pseudo-convex if $h_1(a_1 + a_2) \geq \gamma_1 h_2(a_3 + a_4)$

Proof- See Appendix A.

Due to pseudo-convexity of TCU_1, the unique optimal ordering quantities (q_1^*, q_3^*) are determined by solving the system of equations $\frac{\partial TCU_1}{\partial q_1} = 0$ and $\frac{\partial TCU_1}{\partial q_3} = 0$

Optimal order quantities are given by –

$$q_1^* = \frac{a_1(a_3+a_4)D_1\left(CS_{12}\gamma_1+\sigma_1(1-\gamma_1)\right)}{h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1} \tag{12}$$

$$q_3^* = \frac{a_3\left\{\begin{array}{l}\sqrt{h_2(a_3+a_4)(\gamma_1 D_1+D_2)(h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1)} \\ \times\sqrt{\begin{array}{l}2(A_1+A_2)(h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1) \\ -D_1(a_3+a_4)^2\left(\gamma_1^2 CS_{12}^2+\sigma_1^2(1-\gamma_1)^2+2\sigma_1\gamma_1 CS_{12}(1-\gamma_1)\right)\end{array}}\end{array} -D_1\gamma_1 h_2(a_3+a_4)^2\left(CS_{12}\gamma_1+\sigma_1(1-\gamma_1)\right)\right\}}{h_2(a_3+a_4)(h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1)} \tag{13}$$

Optimal total cost per unit time is given by (from eq. (8))

$TCU_1^* = TCU_1(q_1^*, q_3^*)$ which gives as

$$TCU_1^* = \frac{\sqrt{2(a_3+a_4)}\left\{\begin{array}{l} D_1\left(CS_{12}\gamma_1+\sigma_1(1-\gamma_1)\right)\left(\sqrt{\frac{h_2(a_3+a_4)\left(\begin{array}{c}\sigma_1^2-D_1\left(CS_{12}\gamma_1+\sigma_1(1-\gamma_1)(a_3^2+a_4^2)\right)-2D_1 a_3 a_4\left(CS_{12}\gamma_1+\sigma_1(1-\gamma_1)\right) \\ +2(A_1+A_2)(h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1)\end{array}\right)}{(h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1)(\gamma_1 D_1+D_2)}}\right) \\ +\sqrt{2}h_2(\gamma_1 D_1+D_2)\left(\begin{array}{c}2(A_1+A_2)(h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1) \\ -D_1(a_3+a_4)^2\left(\gamma_1(CS_{12}-\sigma_1)^2-2\gamma_1\sigma_1(CS_{12}-\sigma_1)+\sigma_1^2\right)\end{array}\right)\end{array}\right\}}{\sqrt{h_2(\gamma_1 D_1+D_2)(h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1)}\times\sqrt{\begin{array}{l}2(A_1+A_2)(h_1(a_1+a_2)-h_2(a_3+a_4)\gamma_1)-D_1(a_3+a_4)^2 \\ +\left(2\sigma_1(CS_{12}-\sigma_1)+\gamma_1^2(CS_{12}-\sigma_1)^2+\sigma_1^2\right)\end{array}}} \tag{14}$$

For Situation 2

In this situation, the pseudo-convexity, optimal order quantities and total inventory cost are obtained below.

Theorem 2- The total inventory cost function per unit time (TCU_2) is pseudo-convex if $h_2(a_3+a_3) \geq \gamma_2 h_1(a_1+a_2)$

Proof- See Appendix B.

Due to pseudo-convexity of TCU_1, the unique optimal ordering quantities (q_1^*, q_3^*) are determined by solving the system of equations $\frac{\partial TCU_2}{\partial q_1} = 0$ and $\frac{\partial TCU_2}{\partial q_3} = 0$

Optimal order quantities are given by –

$$q_1^* = \frac{a_3(a_1+a_2)D_2\big(CS_{21}\gamma_2+\sigma_2(1-\gamma_2)\big)}{h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_1} \tag{15}$$

$$q_3^* = \frac{a_1\left(\begin{matrix}\sqrt{h_1(a_1+a_2)(D_1+\gamma_2 D_2)(h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_2)} \\ \times\sqrt{\begin{matrix}2(A_1+A_2)(h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_2) \\ -D_2(a_1+a_2)^2\left(\gamma_2^2 CS_{21}^2+\sigma_2^2(1-\gamma_2)^2+2\sigma_2\gamma_2 CS_{21}(1-\gamma_2)\right)\end{matrix}}\end{matrix} - D_2\gamma_2 h_1(a_1+a_2)^2\big(CS_{21}\gamma_2+\sigma_2(1-\gamma_2)\big)\right)}{h_1(a_1+a_2)(h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_2)} \tag{16}$$

and optimal total cost per unit time is given by (from equation (9))

$TCU_2^* = TCU_2(q_1^*, q_3^*)$ which gives as

$$TCU_2^* = \frac{\sqrt{2(a_1+a_2)}\left\{\begin{matrix} D_2\big(CS_{21}\gamma_2+\sigma_2(1-\gamma_2)\big)\left(\sqrt{\begin{matrix}h_1(a_1+a_2)\left(\begin{matrix}\sigma_2^2-D_2\left(CS_{21}\gamma_2+\sigma_2(1-\gamma_2)(a_1^2+a_2^2)\right)-2D_2 a_1 a_2\big(CS_{21}\gamma_2+\sigma_2(1-\gamma_2)\big) \\ +2(A_1+A_2)(h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_2)\end{matrix}\right) \\ (h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_1)(\gamma_2 D_2+D_1)\end{matrix}}\right) \\ +\sqrt{2}h_1(\gamma_2 D_2+D_1)\left(\begin{matrix}2(A_1+A_2)(h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_2) \\ -D_2(a_1+a_2)^2\left(\gamma_2(CS_{21}-\sigma_2)^2-2\gamma_2\sigma_2(CS_{21}-\sigma_2)+\sigma_2^2\right)\end{matrix}\right)\end{matrix}\right\}}{\sqrt{h_1(\gamma_2 D_2+D_1)(h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_2)}\times\sqrt{\begin{matrix}2(A_1+A_2)(h_2(a_3+a_4)-h_1(a_1+a_2)\gamma_2)-D_2(a_1+a_2)^2 \\ +\left(2\sigma_2(CS_{21}-\sigma_2)+\gamma_2^2(CS_{21}-\sigma_2)^2+\sigma_2^2\right)\end{matrix}}} \tag{17}$$

For Situation 3

In this situation, the optimal order quantities and optimal total cost are calculated by the method of calculus.

The optimal order quantities are as follows

$$q_{1ws}^* = \sqrt{\frac{2a_1^2 D_1^2(A_1+A_2)}{h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)}} \tag{18}$$

$$q_{3ws}^* = \sqrt{\frac{2a_3^2 D_2^2(A_1+A_2)}{h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)}} \tag{19}$$

Optimal total cost with no substitution is given by (from equation (11)).

$$TCU_{Ws}^* = TCU_{Ws}(q_{1ws}^*, q_{3ws}^*)$$

$$TCU_{Ws}^* = \sqrt{2(A_1+A_2)\big(h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)\big)} \tag{20}$$

Derivation of critical substitution rate

For Situation 1

As can be seen from equation (14), optimal total inventory cost (TCU_1^*) is a function of the rate of substitution γ_1. So, we investigate the nature of optimal total inventory cost TCU_1^* with respect to γ_1 and find that value of substitution rate γ_1 which gives the value of TCU_1^* under condition of pseudo-convexity. Such value of substitution rate γ_1 is termed as a critical substitution rate or extreme value of substitution rate.

The critical substitution rate is an indicator of how much substitution should be prearranged so that maximum economic out-put is gained. The critical substitution rate can be also defined with another way. For which, we consider the difference between the optimal total cost with substitution (TCU_1^*) and the optimal total cost without substitution (TCU_{Ws}^*). This cost difference is a function of γ_1. So, that value of substitution rate γ_1 which minimize the above cost difference function under condition $h_1(a_1 + a_2) \geq \gamma_1 h_2(a_3 + a_4)$ is also termed as an extreme value of substitution rate or critical substitution rate.

Now using the second approach, we find the cost difference $C_d(\gamma_1)$ as

$$C_D(\gamma_1) = TCU_{Ws}^* - TCU_1^* \tag{21}$$

Using standard calculus method, extreme value of substitution rate or critical substitution rate for situation 1 is obtained as

$$\gamma_{1e} = \left\{ \frac{\begin{array}{l}\sigma_1(CS_{12}-\sigma_1)(a_3+a_4)\big(h_1D_1(a_1+a_2)+h_2D_2(a_3+a_4)\big)+2h_1h_2(a_1+a_2)(A_1+A_2)\\ \quad +\big(h_1(CS_{12}-\sigma_1)(a_1+a_2)+h_2\sigma_1(a_3+a_4)\big)\sqrt{2(A_1+A_2)\big(h_1D_1(a_1+a_2)+h_2D_2(a_3+a_4)\big)}\end{array}}{\big(2(A_1+A_2)h_2^2-(CS_{12}-\sigma_1)^2\big(h_1D_1(a_1+a_2)+h_2D_2(a_3+a_4)\big)\big)(a_3+a_4)} \right\} \tag{22}$$

For Situation 2

Similarly, the extreme value of substitution rate or critical substitution rate for situation 2 is

$$\gamma_{2e} = \left\{ \frac{\begin{array}{l}\sigma_2(CS_{21}-\sigma_2)(a_1+a_2)\big(h_1D_1(a_1+a_2)+h_2D_2(a_3+a_4)\big)+2h_1h_2(a_3+a_4)(A_1+A_2)\\ \quad +\big(h_2(CS_{21}-\sigma_2)(a_3+a_4)+h_1\sigma_2(a_1+a_2)\big)\sqrt{2(A_1+A_2)\big(h_1D_1(a_1+a_2)+h_2D_2(a_3+a_4)\big)}\end{array}}{\big(2(A_1+A_2)h_1^2-(CS_{21}-\sigma_2)^2\big(h_1D_1(a_1+a_2)+h_2D_2(a_3+a_4)\big)\big)(a_1+a_2)} \right\} \tag{23}$$

Algorithm for obtaining critical substitution rate and optimal ordering quantities

Critical substitution rate, optimal ordering quantities and optimal total cost for situations 1 and 2 as well as optimal ordering quantities and optimal total cost for situation 3 can be obtained using the following algorithm.

Step 1: Initialize all parameters of the inventory model.

Step 2: Determine the critical substitution rate (γ_{1e}) from equation (22).

Step 3: Select appropriate value of the rate of substitution between 0 to critical substitution rate (γ_{1e}).

Step 4: Determine optimal ordering quantities and optimal total inventory cost from equations (12), (13) and (14) respectively under condition $\frac{q_1^*}{a_1D_1} < \frac{q_3^*}{a_3D_2}$ i.e. for situation 1. Optimal ordering quantities for the second component of item 1 is given by $q_2^* = \left(\frac{a_2}{a_1}\right) q_1^*$ and Optimal ordering quantities for the second component of item 2 is given by $q_4^* = \left(\frac{a_4}{a_3}\right) q_3^*$. If situation 1 did not occur then proceed to step 5 to 7 for situation 2.

Step 5: Determine the critical substitution rate (γ_{2e}) from equation (23).

Step 6: Select appropriate value of the rate of substitution between 0 to critical substitution rate (γ_{2e}).

Step 7: Determine optimal ordering quantities and optimal total inventory cost from equations (15), (16). and (17) respectively under condition $\frac{q_1^*}{a_1 D_1} > \frac{q_3^*}{a_3 D_2}$. Optimal ordering quantities for the second component of item 1 is given by $q_2^* = \left(\frac{a_2}{a_1}\right) q_1^*$ and Optimal ordering quantities for the second component of item 2 is given by $q_4^* = \left(\frac{a_4}{a_3}\right) q_3^*$. If situation 1 or 2 did not happen then proceed to step 8 for situation 3.

Step 8: Determine optimal ordering quantities and optimal total inventory cost from equations (18), (19), and (20) under condition $\frac{q_1^*}{a_1 D_1} = \frac{q_3^*}{a_3 D_2}$. i.e. for situation 3. Optimal ordering quantities for the second component of item 1 is given by $q_{2ws}^* = \left(\frac{a_2}{a_1}\right) q_{1ws}^*$ and Optimal ordering quantities for the second component of item 2 is given by $q_{4ws}^* = \left(\frac{a_4}{a_3}\right) q_{3ws}^*$.

Step 8: Exit from the algorithm.

Analysis of inventory model

To get deeper insight into the impact of substitution. Here, the nature of optimal ordering quantities, rates of substitution and optimal total inventory cost are studied. These are presented in form of following theorem.

Theorem 3- In situation 1: at critical substitution rate (extreme value of substitution rate), optimal ordering quantities with substitution and optimal ordering quantities without substitution are the same. i.e. At $\gamma_1 = \gamma_{1e}$, $q_1^* = q_{1ws}^*$ and $q_3^* = q_{3ws}^*$

Proof- See Appendix C.

Theorem 4- The feasible value of substitution rate (γ_1) lies in a closed interval $[0, \gamma_{1e}]$ i.e. $(0 \leq \gamma_1 \leq \gamma_{1e})$

Proof- See Appendix D.

Theorem 5- In situation 2: at critical substitution rate (extreme value of substitution rate), optimal ordering quantities with substitution and optimal ordering quantities without substitution are the same. At $\gamma_2 = \gamma_{2e}$, $q_1^* = q_{1ws}^*$ and $q_3^* = q_{3ws}^*$

Proof- See Appendix E.

Theorem 6- The feasible value of substitution rate (γ_2) lies in a closed interval $[0, \gamma_{2e}]$ i.e. $(0 \leq \gamma_2 \leq \gamma_{2e})$

Proof- See Appendix F.

Numerical Example and Sensitivity Analysis

In this section, numerical example and sensitivity analysis are provided to justify the effectiveness and stability of the proposed inventory model and are discussed using maple mathematical software.

Numerical Example

The values of all parameters used in numerical example in given in Table 1.

Table 2.- Initial parameters

Parameters	Item 1 $(\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2})$ (proper unit)	Item 2 $(\boldsymbol{\beta_1}, \boldsymbol{\beta_2})$ (proper unit)
Demand rates D_1, D_2	100	30
Fixed ordering costs A_1, A_2	200	200
Usages rates $(a_1, a_2), (a_3, a_4)$	1, 2	2, 2
Holding cost h_1, h_2	3	3
Cost of substitution CS_{12}, CS_{21}	2	2
Shortage costs $\sigma_1,\ \sigma_2$	0.35	0.35

Firstly, we consider situation 1. By theorem 1, the total inventory cost function per unit time (TCU_1) is pseudo-convex if $h_1(a_1 + a_2) \geq \gamma_1 h_2(a_3 + a_4)$ which gives as if $\gamma_1 \leq \frac{h_1(a_1+a_2)}{h_2(a_3+a_4)}$

That is if $\gamma_1 \leq 0.75$

Using Algorithm as describe above, we determine critical substitution rate (extreme value of substitution rate, optimal order quantities and optimal total cost for the given numerical example

Critical substitution rate or Extreme value of rate of substitution $(\gamma_{1e}) = 0.3570$

At a critical substitution rate, optimal order quantities are $q_1^* = 79.68$ units, $q_3^* = 47.80$ units and optimal total cost (TCU_1^*) =1003.99 units. Also, $q_2^* = 159.36$ units, $q_4^* = 47.80$ units

Also, optimal order quantities without substitution are $q_{1w}^* = 79.68$ units, $q_{3w}^* = 47.80$ units and optimal total cost (TCU_w^*) =1003.99 units. Also, $q_{2w}^* = 159.36$ units, $q_{4w}^* = 47.80$ units

Further, for different values of the rate of substitution γ_1 (γ_1 lies between 0 and 0.3570) we obtained optimal order quantities and optimal total cost which are given in Table 2.

Table 3.- Optimal values for distinct values of γ_1

Rate of substitution (γ_1) ($0 \le \gamma_1 \le .3570$)	Optimal order quantities and optimal total cost with substitution			Optimal order quantities and optimal total cost without substitution			Percentage improvement in optimal total cost
	q_1^*	q_3^*	TCU_1^*	q_{1w}^*	q_{3w}^*	TCU_w^*	
0.05	20.59	92.37	739.59	79.68	47.80	1003.99	26.33
0.10	26.41	94.42	804.23	79.68	47.80	1003.99	19.89
0.15	33.19	94.01	862.82	79.68	47.80	1003.99	14,06
0.20	41.21	90.59	914.44	79.68	47.80	1003.99	8.91
0.25	50.83	83.32	957.43	79.68	47.80	1003.99	4.63
0.30	62.59	70.92	988.90	79.68	47.80	1003.99	1.50
0.35	77.29	51.34	1003.71	79.68	47.80	1003.99	0.02
0.3570	79.68	47.80	1003.99	79.68	47.80	1003.99	0.00

Table 2 reflects that on increasing the rate of substitution, percentage improvement in optimal total cost decrease and at value of critical substitution rate ($\gamma_{1e} = 0.3570$) it becomes zero.

Graph between cost difference ($C_D(\gamma_1)$) and substitution rate(γ_1), and between total optimal cost (TCU_1^*) and substitution rate(γ_1) are shown in fig 4 and 5.

Fig 4. Cost difference ($C_D(\gamma_1)$) vs. substitution rate(γ_1).

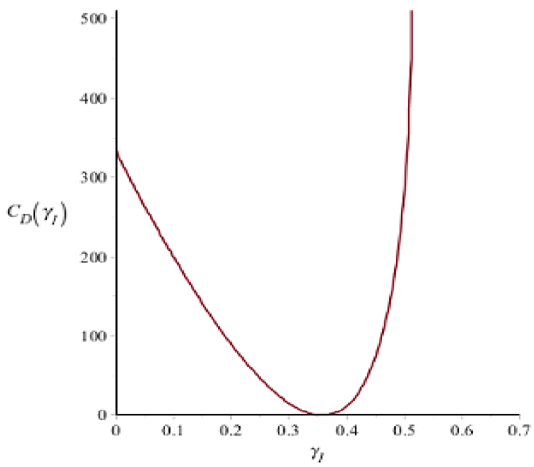


Fig 5. Total optimal cost (TCU_1^*) vs substitution rate(γ_1)

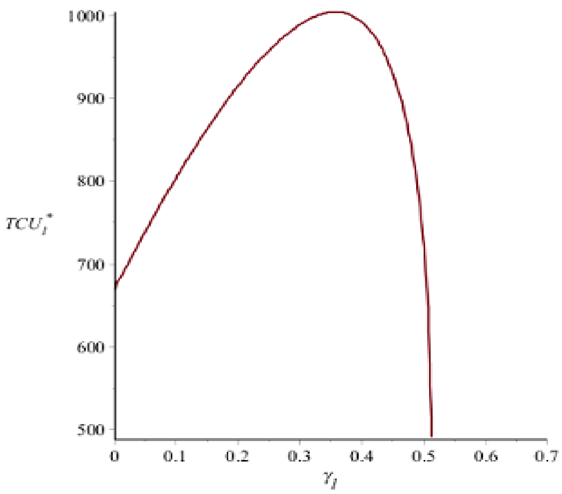


Fig 4 and 5 also verifies the unique critical value of substitution rate (unique extreme value of substitution rate) because it is strictly convex and strictly concave respectively.

Sensitivity Analysis

As we know that sensitivity analysis is termed as a systematic procedure to study the impact of changes in values of parameters of inventory model on its optimal values. In a practical situation, a substantial impact on optimal values of decision variables and the objective function of the inventory model is seen on varying the values of parameters of the inventory model. In this proposed inventory model, we investigate the impact of changes in values of parameters: fixed ordering costs A_1 and A_2, demand rates D_1 and D_2, shortage cost σ_1, σ_2 holding costs h_1 and h_2, usages rates of complementary components of items 1 and 2, a_1, a_2 and a_3, a_4, cost of substitution CS_{12} on optimal total costs and optimal ordering quantities of given numerical example.

Now, changes on the extreme value of substitution rate and optimal solutions by varying the values of parameters are shown in Table 3.

Table 4- Sensitivity analysis with respect to various parameters of the model.

Parameters	Values of Parameters	γ_{1e}	TCU_1^*	q_1^*	q_3^*	TCU_w^*	q_{1w}^*	q_{3w}^*	Percentage improvement in optimal total cost
$A_1 = A_2$	100	0.2747	687.62	41.21	52.78	709.92	56.34	33.80	3.14
	150	0.3232	813.05	41.21	73.69	869.48	69.00	41.40	6.49
	200	0.3570	914.44	41.21	90.59	1003.99	79.68	47.80	8.92
	250	0.3827	1001.89	41.21	105.16	1122.49	89.08	53.45	10.74
	300	0.4032	1079.92	41.21	118.16	1229.63	97.59	58.55	12.18
	40	0.1661	441.53	41.21	11.77	448.99	35.63	21.38	**Not feasible**
D_1	60	0.3955	770.90	24.72	91.39	848.52	56.56	56.56	9.15
	80	0.3748	843.77	32.96	91.17	929.51	68.85	51.63	9.22
	100	0.3570	914.44	41.21	90.59	1003.99	79.68	47.80	8.92
	120	0.3414	983.08	49.45	89.66	1073.31	89.44	44.72	8.41
	140	0.3275	1049.80	57.69	88.42	1138.41	98.38	42.16	7.78
	500	0.1963	1971.26	206.06	19.45	1971.80	202.86	56.56	**Not feasible**
D_2	10	0.3814	769.63	41.21	66.45	903.32	88.56	17.71	14.80
	20	0.3686	846.62	41.21	79.28	954.98	83.77	33.50	11.35
	30	0.3570	914.44	41.21	90.59	1003.99	79.68	47.80	8.92
	40	0.3464	975.76	41.21	100.8	1050.71	76.13	60.91	7.13
	50	0.3366	1032.15	41.21	110.20	1095.44	73.02	73.02	5.78
	350	0.1906	2019.65	41.21	274.79	2019.90	39.60	277.24	**Not feasible**
σ_1	0.15	0.3874	871.82	31.51	98.03	1003.99	79.68	47.80	13.16
	0.25	0.3726	893.93	36.36	94.44	1003.99	79.68	47.80	10.96
	0.35	0.3570	914.44	41.21	90.59	1003.99	79.68	47.80	8.92
	0.45	0.3407	933.27	46.06	86.45	1003.99	79.68	47.80	7.04
	0.55	0.3236	950.30	50.90	82.02	1003.99	79.68	47.80	5.35
	1.20	0.1858	1003.31	82.42	43.58	1003.99	79.68	47.8	**Not feasible**
$h_1 = h_2$	1	0.2260	576.53	123.63	102.81	579.65	138.01	82.80	0.54
	2	0.3092	774.73	61.81	100.95	819.75	97.59	58.55	5.49
	3	0.3570	914.44	41.21	90.59	1003.99	79.68	47.80	8.92
	4	0.3900	1028.79	30.90	82.23	1159.31	69.00	41.40	11.26

	5	0.4148	1128.00	24.72	75.70	1296.14	61.72	37.03	12.97
	0.70	0.1839	483.52	176.62	80.43	484.97	164.95	98.97	**Not feasible**
$a_1 = a_2$	1.5	0.2958	898.86	53.33	83.14	942.33	84.89	50.93	4.61
	1.75	0.3267	907.70	46.49	87.35	973.65	82.16	49.29	6.77
	2	0.3570	914.44	41.21	90.59	1003.99	79.68	47.80	8.92
	2.25	0.3869	919.76	37.00	93.15	1033.44	77.41	46.44	11.00
	2.5	0.4162	924.07	33.58	95.24	1062.07	75.32	45.19	12.99
	0.70	0.1927	833.77	100.74	53.33	834.26	95.89	57.53	**Not feasible**
$a_3 = a_4$	1.5	0.5158	781.91	28.33	87.81	967.47	82.68	37.21	19.18
	1.75	0.4251	851.92	34.49	90.24	985.90	81.14	42.60	13.59
	2	0.3570	914.44	41.21	90.59	1003.99	79.68	47.80	8.92
	2.25	0.3040	969.08	48.57	88.65	1021.76	78.29	52.84	5.16
	2.5	0.2615	1014.90	56.66	84.15	1039.23	76.98	57.73	2.34
	3	0.1976	1073.25	75.55	65.54	1073.31	67.08	74.53	**Not feasible**
CS_{12}	1	0.4745	860.19	29.09	99.72	1003.99	79.68	47.80	14.32
	1.5	0.4075	888.55	35.15	95.36	1003.99	79.68	47.80	11.50
	2	0.3570	914.44	41.21	90.59	1003.99	79.68	47.80	8.92
	2.5	0.3177	937.70	47.27	85.37	1003.99	79.68	47.80	6.60
	3	0.2862	958.1	53.33	79.68	1003.99	79.68	47.80	4.57
	5.50	0.1913	1002.55	83.63	41.63	1003.99	79.68	47.80	**Not feasible**

The summary of results of Table 4 as impact on decision variable with other parameters are summarise in table 5.

The necessity and importance of extreme value of substitution rate can be also seen in the sensitivity analysis. The solution is obtained as a non-feasible solution because taken substitution rate $(\gamma_1 = 0.20)$ greater than the extreme value at that value of each parameter and necessary condition for situation 1 is not fulfilled. As a result of which non-feasible solution is obtained for the taken rate of substitution.

Table 5- Impact on extreme value of substitution rate, optimal total costs and improvement in optimal total cost by varying values of parameters.

Parameters	Variation in values of parameters	γ_{1e}	TCU_1^*	TCU_w^*	Improvement in optimal total cost (%)
A_1, A_2	Increment	Positive	Positive	Positive	Positive
D_1, D_2		Negative	Positive	Positive	Negative
σ_1		Negative	Positive	constant	Negative
h_1, h_2		Positive	Positive	Positive	Positive
a_1, a_2		Positive	Positive	Positive	Positive
a_3, a_3		Negative	Positive	Positive	Negative
CS_{12}		Negative	Positive	constant	Negative

Sensitivity graphs of optimal total cost with substitution, without substitution and percentage improvement in optimal total cost, are shown in below.

Fig 6. Sensitivity with respect to fixed ordering cost

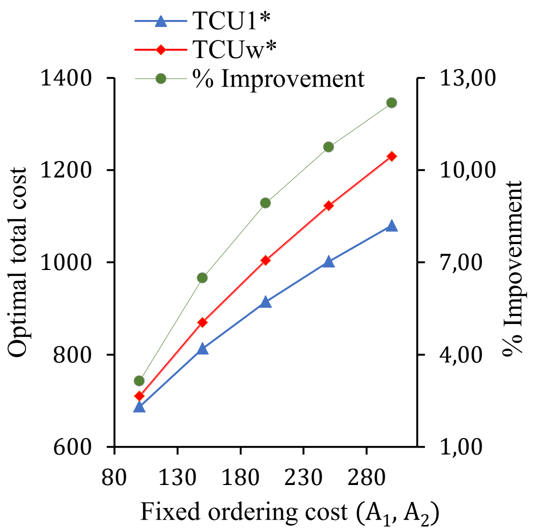


Fig 7. Sensitivity with respect to shortage cost

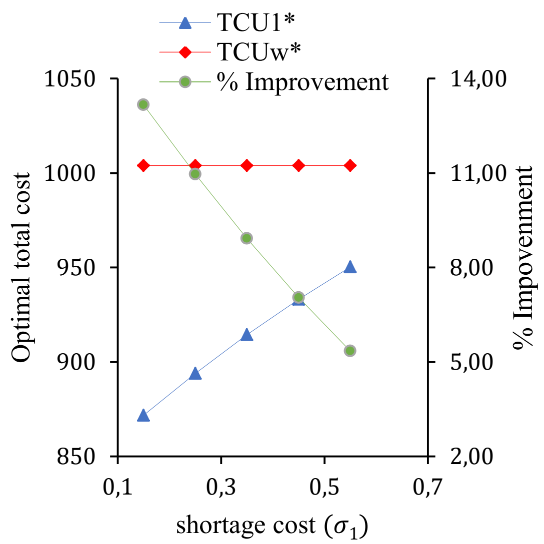


Fig 8. Sensitivity with respect to demand rates

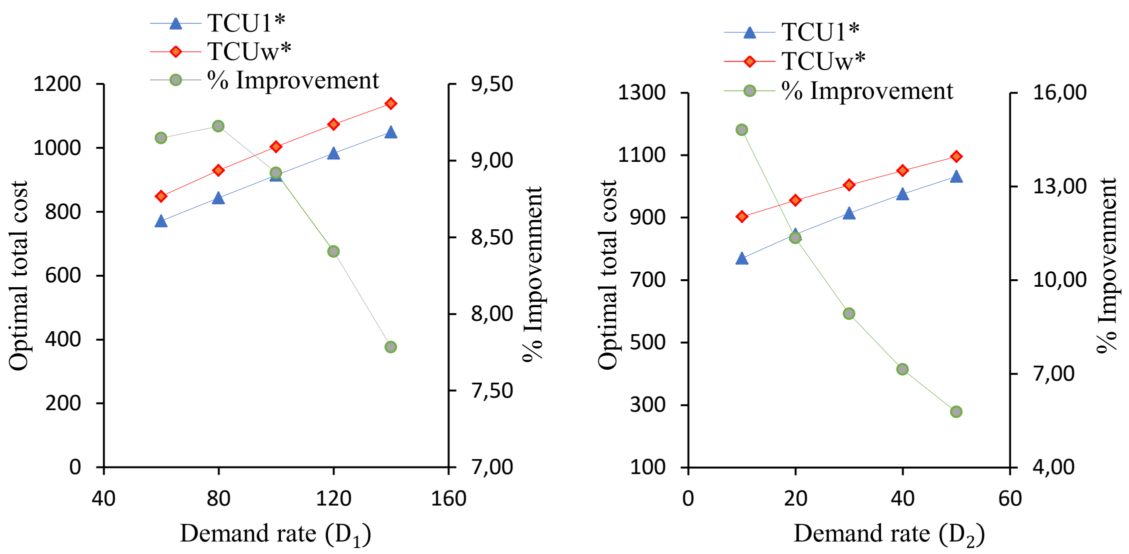


Fig 9. Sensitivity with respect to holding cost

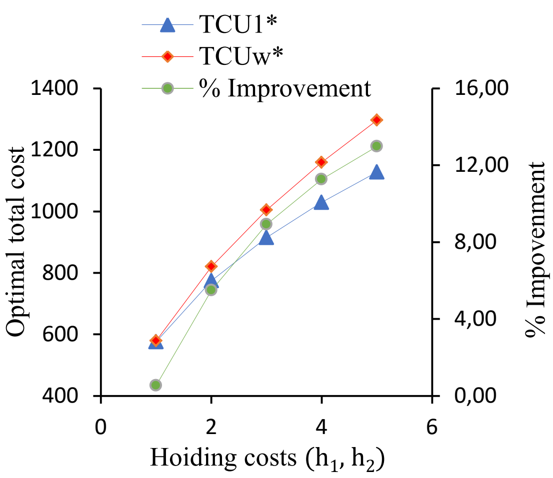


Fig 10. Sensitivity with respect to cost of substitution

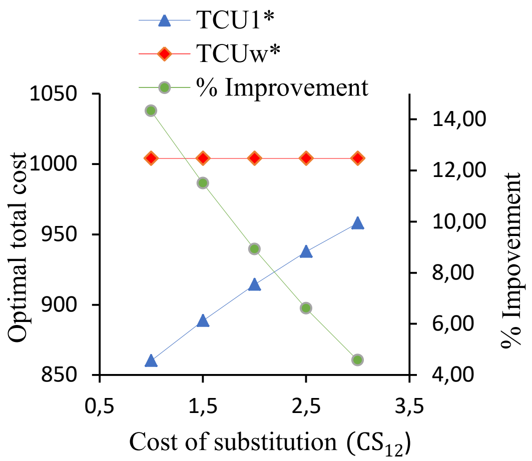
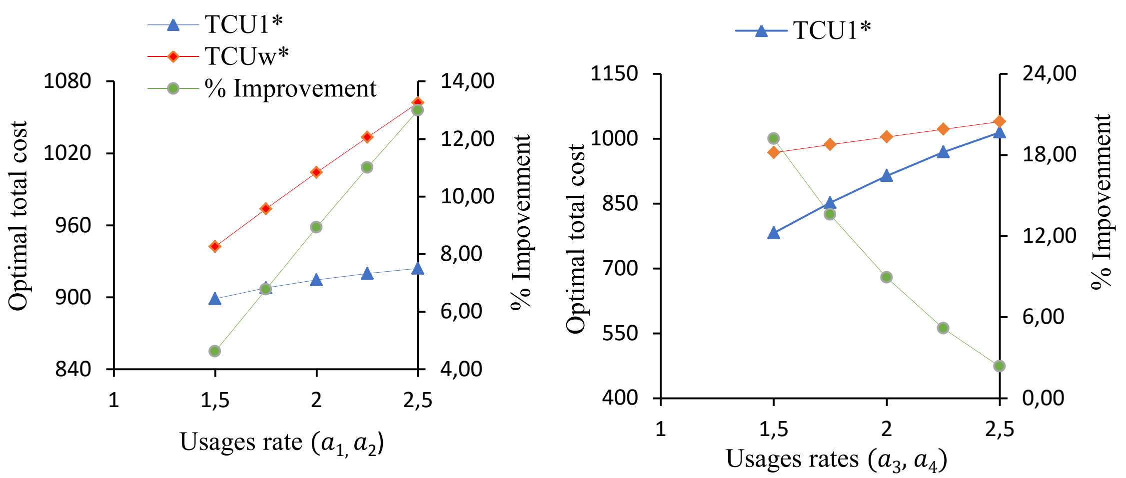


Fig 11. Sensitivity with respect to usages rates of complementary components.



Managerial Implication

This inventory model helps the managers of warehouses to make the decisions for optimal order quantities of items under the category of complementary and substitutable items and also helps the managers of warehouses to make the decisions what value of substitution rate should be considered. Therefore, this inventory model is more applicable than other inventory models existing in this direction.

Conclusion and Future Work

This paper addresses impact of cost of substitution and joint replenishment on an inventory decision for two substitutable items, where both items are formed with two complementary components, by considering stock-out substitution, two-way and partial substitution. The proposed inventory model is applicable to same types of items composed with two complementary items such as different brands of mobile phones and sim cards, different brands of toothbrush and toothpastes, different brands of tea and milk etc. and is also applicable to slightly different items, such as coffee and tea. Three possible situations have been discussed and pseudo-convexity for total inventory cost function has derived. Here, optimal ordering quantities have been obtained analytically. Due to partial substitution, the idea of a critical substitution rate is studied. Analysis of inventory model shows that optimal order quantities with substitution and without substitution are the same at a critical substitution rate and the value of substitution rate beyond the critical substitution rate is not beneficial. Numerical and sensitivity analysis are provided to validate the applicability and performance of the proposed inventory model. A numerical example demonstrates that percentage improvement in optimal total inventory cost decreases when substitution rate tends to critical substitution rate and it is zero at the value of critical substitution rate and substitution is not helpful to consider the value of substitution rate beyond the critical substitution rate.

Further research is needed to generalize this paper for multiple products. In addition, it may be extended for full substitution and deteriorating items. Also, it can be extended in a different direction introducing, stochastic demand rate, stochastic deterioration rate, stochastic lead time, replenishment policies instead of joint replenishment policies, etc.

Appendix A

To prove pseudo-convexity of total cost functions (TCU_1).

Proof of Theorem 1-

In mathematical formulation, the total cost function per unit time in situation 1 is given as

$$TCU_1 = TC_1/(T_1 + p_1)$$

Where from equation (7)

$$TC_1 = \left\{ \begin{array}{l} A_1 + A_2 + \frac{h_1 q_1^2}{2a_1 D_1}\left(1 + \frac{a_2}{a_1}\right) + h_2 \left(\begin{array}{l} \frac{q_1}{2a_1 D_1}\left(2q_3 - \frac{q_1 a_3}{a_1}\frac{D_2}{D_1}\right) \\ + \frac{1}{2a_3(\gamma_1 D_1 + D_2)}\left(q_3 - \frac{q_1 a_3}{a_1}\frac{D_2}{D_1}\right)^2 \\ + \frac{q_1}{2a_1 D_1}\left(2\frac{a_4}{a_3}q_3 - \frac{q_1 a_4}{a_1}\frac{D_2}{D_1}\right) \\ + \frac{1}{2a_4(\gamma_1 D_1 + D_2)}\left(\frac{a_4}{a_3}q_3 - \frac{q_1 a_4}{a_1}\frac{D_2}{D_1}\right)^2 \end{array} \right) \\ + CS_{12}\gamma_1(a_3 + a_4)\frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)} + \sigma_1(1 - \gamma_1)(a_3 + a_4)\frac{q_3 a_1 D_1 - q_1 a_3 D_2}{a_1 a_3(\gamma_1 D_1 + D_2)} \end{array} \right\}$$

$$(T_1 + p_1) = \frac{q_1 a_3 \gamma_1 + a_1 q_3}{a_1 a_3(\gamma_1 D_1 + D_2)}$$

To show, TCU_1 is pseudo-convex. For this firstly we show that TC_1 is convex and use the fact that the ratio of a positive convex function and positive concave function is pseudo-convex (Cambibi and Martein, 2009, Avriel, 2003).

To show convexity of TC_1 , we must prove that its Hessian matrix is positive definite

Hessian matrix of cost function TC_1 is given as $H(q_1, q_3) = \begin{bmatrix} \frac{\partial^2 TC_1}{\partial^2 q_1} & \frac{\partial^2 TC_1}{\partial q_1 \partial q_3} \\ \frac{\partial^2 TC_1}{\partial q_3 \partial q_1} & \frac{\partial^2 TC_1}{\partial^2 q_3} \end{bmatrix}$

For positive definiteness of the Hessian matrix $H(q_1, Q_2)$, we prove that

$\frac{\partial^2 TC_1}{\partial^2 q_1} > 0, \frac{\partial^2 TC_1}{\partial^2 q_3} > 0$ and determinant of the Hessian matrix $|H(q_1, q_3)| \geq 0$ i.e

$$\left(\frac{\partial^2 TC_1}{\partial^2 q_1} * \frac{\partial^2 TC_1}{\partial^2 q_3}\right) - \left(\frac{\partial^2 TC_1}{\partial q_1 \partial q_3}\right)^2 > 0$$

Now

$$\frac{\partial^2 TC_1}{\partial^2 q_1} = \left\{\frac{\left(D_2 h_1(a_1+a_2) - D_2\,\gamma_1 h_2(a_3+a_4)\right) + D_1\,\gamma_1 h_1(a_1+a_2)}{D_1 a_1^2(\gamma_1 D_1 + D_2)}\right\} > 0 \text{ when } h_1(a_1+a_2) \geq \gamma_1 h_2(a_3+a_4)$$

$$\frac{\partial^2 TC_1}{\partial^2 q_3} = \left\{\frac{h_2(a_3+a_4)}{a_3^2(\gamma_1 D_1 + D_2)}\right\} > 0$$

$$\left(\frac{\partial^2 TC_1}{\partial^2 q_1} * \frac{\partial^2 TC_1}{\partial^2 q_3}\right) - \left(\frac{\partial^2 TC_1}{\partial q_1 \partial q_3}\right)^2 = \left\{\frac{h_2(a_3+a_4)\left(h_1(a_1+a_2) - \gamma_1 h_2(a_3+a_4)\right)}{D_1 a_1^2 a_3^2(\gamma_1 D_1 + D_2)}\right\} > 0$$

when $h_1(a_1+a_2) \geq \gamma_1 h_2(a_3+a_4)$.

Thus $\frac{\partial^2 TC_1}{\partial^2 q_1} > 0, \frac{\partial^2 TC_1}{\partial^2 q_3} > 0$ and $\left(\frac{\partial^2 TC_1}{\partial^2 q_1} * \frac{\partial^2 TC_1}{\partial^2 q_3}\right) - \left(\frac{\partial^2 TC_1}{\partial q_1 \partial q_3}\right)^2 > 0$ if $h_1(a_1+a_2) \geq \gamma_1 h_2(a_3+a_4)$

Clearly $(T_1 + p_1) = \frac{q_1 a_3 \gamma_1 + a_1 q_3}{a_1 a_3(\gamma_1 D_1 + D_2)}$ is a positive concave function.

Thus, TCU_1 is pseudo-convex.

This completes the proof of theorem 1.

Appendix B

To prove pseudo-convexity of total cost functions (TCU_2)

Proof of theorem 2. Proof of this theorem is analogous to proof of theorem 1.

Appendix C

To prove $q_1^* = q_{1ws}^*$ and $q_3^* = q_{3ws}^*$ at $\gamma_1 = \gamma_{1e}$

Proof of Theorem 3-

As we seen,

$$q_1^* = \frac{a_1(a_3+a_4)D_1\left(CS_{12}\gamma_1 + \sigma_1(1-\gamma_1)\right)}{h_1(a_1+a_2) - h_2(a_3+a_4)\gamma_1}$$

$$\gamma_{1e} = \left\{\frac{\begin{array}{c}\sigma_1(CS_{12}-\sigma_1)(a_3+a_4)\left(h_1 D_1(a_1+a_2) + h_2 D_2(a_3+a_4)\right) + 2h_1 h_2(a_1+a_2)(A_1+A_2)\\ +\left(h_1(CS_{12}-\sigma_1)(a_1+a_2) + h_2\sigma_1(a_3+a_4)\right)\sqrt{2(A_1+A_2)\left(h_1 D_1(a_1+a_2) + h_2 D_2(a_3+a_4)\right)}\end{array}}{\left(2(A_1+A_2)h_2^2 - (CS_{12}-\sigma_1)^2\left(h_1 D_1(a_1+a_2) + h_2 D_2(a_3+a_4)\right)\right)(a_3+a_4)}\right\}$$

Putting $\gamma_1 = \gamma_{1e}$ in q_1^* we get,

$$q_1^* = \frac{\left((CS_{12}-\sigma_1)\sqrt{2(A_1+A_2)\left(h_1 D_1(a_1+a_2) + h_2 D_2(a_3+a_4)\right)} + 2h_2(A_1+A_2)\right)a_1 D_1}{h_2\sqrt{2(A_1+A_2)\left(h_1 D_1(a_1+a_2) + h_2 D_2(a_3+a_4)\right)} + (CS_{12}-\sigma_1)\left(h_1 D_1(a_1+a_2) + h_2 D_2(a_3+a_4)\right)}$$

$$q_1^* = \frac{a_1 D_1 \sqrt{2(A_1+A_2)}\left(\sqrt{2(A_1+A_2)}+(CS_{12}-\sigma_1)\sqrt{\left(h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)\right)}\right)}{\sqrt{\left(h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)\right)}\left(\sqrt{2(A_1+A_2)}+(CS_{12}-\sigma_1)\sqrt{\left(h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)\right)}\right)}$$

$$q_1^* = \frac{a_1 D_1 \sqrt{2(A_1+A_2)}}{\sqrt{\left(h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)\right)}}$$

$$q_1^* = \sqrt{\frac{2a_1^2 D_1^2 (A_1+A_2)}{h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)}} = q_{1ws}^*$$

Which gives $q_1^* = q_{1ws}^*$

With a similar way, by putting $\gamma_1 = \gamma_{1e}$ in q_2^* we get,

$$q_2^* = \sqrt{\frac{2a_3^2 D_2^2 (A_1+A_2)}{h_1 D_1(a_1+a_2)+h_2 D_2(a_3+a_4)}} = q_{3ws}^*$$

Which gives $q_2^* = q_{3ws}^*$

This completes the proof of theorem 3.

Appendix D

To prove 0≤ $\gamma_1 \leq \gamma_{1e}$

Proof of Theorem 4- We know that rate of substitution (γ_1) lies between 0 and 1 i.e. γ_1 lies in closed interval $[0, 1]$ (0≤ $\gamma_1 \leq$ 1). Critical substitution rate (extreme value of substitution rate) of item 1 when item 1 is substituted by item 2 is γ_{1e}

So, we get $\gamma_1 \leq \gamma_{1e}$ which gives as 0≤ $\gamma_1 \leq \gamma_{1e}$

This completes the proof of theorem 4.

Appendix E

To prove $q_1^* = q_{1ws}^*$ and $q_3^* = q_{3ws}^*$ at $\gamma_2 = \gamma_{2e}$

Proof of Theorem 5- Proof of this theorem is analogous to proof of theorem 3.

Appendix F

To prove 0≤ $\gamma_2 \leq \gamma_{2e}$

Proof of Theorem 6- Proof of this theorem is analogous to proof of theorem 4.

References

Avgeropoulos, S., Sammut-Bonnici, T., and McGee, J. (2015). Complementary products. Encycl. Manag. John Wiley & Sons.

Avriel, M. (2003). Nonlinear Programming: Analysis and Methods. Dover Publications New York.

Benkherouf, L., Skouri, K., and Konstantaras, I. (2017). Inventory decisions for a finite horizon problem with product substitution options and time-varying demand. *Applied Mathematical Modelling, Vol 51,* pp. 669–685.

Cambini, A., and Martein, L. (2009). Generalized convexity and optimization: Theory and application. Springer-Verlag Berlin Heidelberg, USA.

Chen, Y., Yang, L., Jiang, Y., Wahab, M. I. M., and Yang, J. (2019). Joint replenishment decision considering shortages, partial demand substitution, and defective items. *Computers & Industrial Engineering, Vol 127,* pp. 420-435.

Drezner, Z., Gurnani, H., and Pasternack, B. (1995). An EOQ model with substitutions between products. *Journal of the Operational Research Society, Vol 46,* pp. 887–891.

Edalatpour, M. A., and Mirzapour Al-e-Hashem, S. M. J. (2019). Simultaneous pricing and inventory decisions for substitute and complementary items with nonlinear holding costs. *Production Engineering, Vol 13(3-4),* pp. 305-315.

Ernst, R., and Kouvelis, P. (1999). The effects of selling packaged goods on inventory decisions. *Management Science, Vol 45, No. 8,* pp. 1142–1155.

Giri, R. N., Mondal, S. K., and Maiti, (2016). Analysis of pricing decisions for substitutable and complementary products with a common retailer. *Pacific Science Review A: Natural Science and Engineering, Vol 18, No. 3,* pp. 190-202.

Goyal, S. K. (1996). An EOQ model with substitutions between products. *Journal of the Operational Research Society, Vol 47, No. 3,* pp. 476–477.

Gurnani, H., and Drezner, Z. (2000). Deterministic hierarchical substitution inventory models. *Journal of the Operational Research Society, Vol 51,* pp. 129–133.

Harris, F. W. (1915). Operations and Costs: W. Shaw Company, Chicago, 48–54.

Hemmati, M., Federighi, S. M. T., and Sajadieh, M. S. (2018). Inventory of complementary products with stock-dependent demand under vendor-managed inventory with consignment policy. *Scientia Iranica, Vol 25, No. 4,* pp. 2347-2360.

Hong, S. P., and Kim, Y. H. (2009). A genetic algorithm for joint replenishment based on the exact inventory cost. *Computers & Operations Research, Vol 36, No. 1,* pp. 167-175.

Hsu, V. N., Chung-Lun Li, and Wen-Qiang Xiao, (2005). Dynamic lot size problems with one-way product substitution. *IIE Transactions, Vol 37, No. 3,* pp. 201-215.

Khouja, M., and Goyal, S. K. (2008). A review of the joint replenishment problem literature: 1989-2005. *European Journal of Operational Research, Vol 186, No. 1,* pp. 1–16.

Kim, S. W., and Bell, P. C. (2011). Optimal pricing and production decisions in the presence of symmetrical and asymmetrical substitution. *Omega, Vol 39, No. 5,* pp. 528–538.

Krommyda, I., Skouri, K., and Konstantaras, I. (2015). Optimal ordering quantities for substitutable products with stock-dependent demand. *Applied Mathematical Modelling, Vol 39,* pp. 147–164.

Maddah, B., Kharbeche, M., Pokharel, S., and Ghoniemd, A. (2016). Joint replenishment model for multiple products with substitution, *Applied Mathematical Modelling, Vol 40,* 7678–7688.

McGillivray, A. R., and Silver, E. A. (1978). Some concepts for inventory control under substitutable demand. *Information Systems and Operational Research, Vol 16,* pp. 47–63.

Mishra, V. K. (2017). Optimal ordering quantities for substitutable deteriorating items under joint replenishment with the cost of substitution. *Journal of Industrial Engineering International, Vol 13, No. 3,* pp. 381–391.

Mishra, V. K., and Shanker, K. (2017). Optimal ordering quantities for substitutable items under joint replenishment with the cost of substitution. *Operations Research and Decisions, Vol 27, No. 1,* pp. 77-104.

Mishra, V. K., and Shanker, K. (2017). Optimal ordering quantities for substitutable deteriorating items under joint replenishment. *Control and Cybernetics, Vol 46, No. 1,* pp. 49-70.

Mokhtari, H. (2018). Economic order quantity for joint complementary and substitutable items. *Mathematics and Computers in Simulation, Vol 154,* pp. 34-47.

Ouyang, L.Y., Wu, K.S. and Yang, C.T. (2007). An EOQ model with limited storage capacity under trade credits. *Asia-Pacific Journal of Operational Research, Vol 24, No. 04,* 575-592.

Pan, Q. H., Skouri, X. K., Chen, S. C., and Teng, J. T. (2018). An inventory replenishment system with two inventory-based substitutable products. *International Journal of Production Economics, Vol 204,* pp. 35-147.

Porras, E., and Dekker, R. (2006). An efficient optimal solution method for the joint replenishment problem with minimum order quantities. *European Journal of Operational Research, Vol 174, No. 3,* pp.1595–1615.

Salameh, M. K., Yassine, A. A., Maddah, B., and Ghaddar, L. (2014). Joint replenishment model with substitution. *Applied Mathematical Modelling, Vol 38,* pp.3662–3671.

Shin, H., Park, S., Lee, E., and Benton, W. C. (2015). A classification of the literature on the planning of substitutable products. *European Journal of Operational Research, Vol 246,* pp.686–699.

Silva, F., and Gao, L. (2013). A joint replenishment inventory-location model. *Networks and Spatial Economics, Vol 13, No. 1,* pp.107-122.

Taleizadeh, A. A., Babaei, M. S., Sana, S. S., and Sarkar, B. (2019). Pricing Decision within an Inventory Model for Complementary and Substitutable Products. Mathematics, Vol 7, No. 7, pp.568.

Tang, C. S., and Yin, R. (2007). Joint ordering and pricing strategies for managing substitutable products. *Production and Operations Management, Vol 16,* pp.138–153.

Transchel, S. (2017). Inventory management under price-based and stockout-based substitution. *European Journal of Operational Research, Vol 262, No. 3,* pp.996-1008.

Wei, J., Zhao, J., and Li, Y. (2013). Pricing decisions for complementary products with firms' different market powers. *European Journal of Operational Research, Vol 224,* pp.507–519.

Wilson, R. H. (1934). A scientific routine for stock control. *Harvard Business Review, 13,* pp.116-128.

Yue, X., Mukhopadhyay, S. K., and Zhu, X. (2006). A Bertrand model of pricing of complementary goods under information asymmetry. *Journal of Business Research, Vol 59,* pp.1182–1192.

Yuhong, H., and Shuya, Y. (2015). Joint Selling of Complementary Components Under Brand and Retail Competition. *Manufacturing & Service Operations Management, Vol 17, No. 4,* pp.470-479.