On the continuity and differentiability of the (dual) core inverse in C^*-algebras

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ABSTRACT
The continuity of the core inverse and the dual core inverse is studied in the setting of C^*-algebras. Later, this study is specialized to the case of bounded Hilbert space operators and to complex matrices. In addition, the differentiability of these generalized inverses is studied in the context of C^*-algebras.

KEYWORDS
Core inverse; Dual core inverse; C^*-algebra; Hilbert space; Matrices

1. Introduction

The core inverse and the dual core inverse of a matrix were introduced in [1]. These generalized inverses have been studied by several authors, in particular they have been extended to rings with involution ([2]) and to Hilbert space operators ([3]). It is worth noticing that the inverses under consideration are closely related to the group inverse and the Moore-Penrose inverse; to learn more results concerning these notions, see for example [1–4].

So far the properties of the (dual) core inverse that have been researched are mainly of algebraic nature and the setting has been essentially the one of rings with involution. The objective of the present article is to study the continuity and the differentiability of these inverses in the context of C^*-algebras.

In fact, in Section 3, after having recalled several preliminary results in Section 2, the continuity of the core inverse and of the dual core inverse will be studied. Two main characterizations will be presented. The first one relates the continuity of the aforementioned notions to the continuity of the group inverse and of the Moore-Penrose inverse. The second characterization uses the notion of the gap between subspaces; a similar approach has been used to study the continuity of the Drazin inverse and of the Moore-Penrose inverse, see for example [5–7] and [8, Chapter 4]. In Section 4 results regarding the continuity of the (dual) core inverse of Hilbert space operators and

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matrices will be presented. It is worth noticing that concerning the continuity of the (dual) core inverse, in these two particular contexts the notions of range and null space of a Hilbert space operator and of rank of a matrix can be used to prove more results that can not be stated in the frame of arbitrary C^*-algebras. Finally, in Section 5 the differentiability of the generalized inverses under consideration will be researched. Furthermore, some results concerning the continuity and the differentiability of the group inverse and the Moore-Penrose inverse will be also proved.

It is noteworthy to mention that the core inverse and the dual core inverse are two particular cases of the (b, c)-inverse ([9]), see [2, Theorem 4.4]. Therefore the representations and other results presented in [10, Section 7] can be applied to these generalized inverses.

2. Preliminary Definitions

Since properties of C^*-algebra elements will be studied in what follows, although the main notions considered in this article can be given in the context of rings with involution, all the definition will be presented in the frame of C^*-algebras.

From now on $\mathcal{A}$ will denote a unital C^*-algebra with the unity $\mathbb{1}$. In addition, $\mathcal{A}^{-1}$ will stand for the set of all invertible elements in $\mathcal{A}$. Given $a \in \mathcal{A}$, the *image ideals* and the *null ideals* defined by $a \in \mathcal{A}$ are the following sets:

$$a\mathcal{A} = \{ax : x \in \mathcal{A}\}, \qquad \mathcal{A}a = \{xa : x \in \mathcal{A}\},$$
$$a^\circ = \{x \in \mathcal{A} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathcal{A} : xa = 0\}.$$

Recall that $a \in \mathcal{A}$ is said to be *regular*, if there exists $b \in \mathcal{A}$ such that $a = aba$. In addition, $b \in \mathcal{A}$ is said to be *an outer inverse of* $a \in \mathcal{A}$, if $b = bab$.

The notion of invertible element has been generalized or extended in several ways. One of the most important notions of generalized inverse is the Moore-Penrose inverse. An element $a \in \mathcal{A}$ is said to be *Moore-Penrose invertible*, if there is $x \in \mathcal{A}$ such that the following equations hold:

$$axa = a, \qquad xax = x, \qquad (ax)^* = ax, \qquad (xa)^* = xa.$$

It is well known that if such an x exists, then it is unique, and in this case x, the Moore-Penrose inverse of a, will be denoted by $a^\dagger$. Moreover, the subset of $\mathcal{A}$ composed of all Moore-Penrose invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^\dagger$. It is worth noticing that according to [11, Theorem 6], a necessary and sufficient condition for $a \in \mathcal{A}^\dagger$ is that $a \in \mathcal{A}$ is regular, which in turn is equivalent to $a\mathcal{A}$ is closed ([11, Theorem 8]). Moreover, if $a \in \mathcal{A}^\dagger$, then it is not difficult to prove that $a^\dagger\mathcal{A} = a^*\mathcal{A}$ and $\mathcal{A}a^\dagger = \mathcal{A}a^*$. To learn more properties of the Moore-Penrose inverse in the frame of C^*-algebras, see [8,11–15].

Another generalized inverse which will be central for the purpose of this article is the group inverse. An element $a \in \mathcal{A}$ is said to be *group invertible*, if there is $x \in \mathcal{A}$ such that

$$axa = a, \qquad xax = x, \qquad ax = xa.$$

It can be easily proved that if such x exists, then it is unique. The group inverse of a is customarily denoted by $a^\#$. The subset of $\mathcal{A}$ composed by all group invertible elements in $\mathcal{A}$ will be denoted by $\mathcal{A}^{\#}$.

Next follows one of the main notions of this article (see [2, Definition 2.3], see also [1] for the original definition in the context of matrices).

Definition 2.1. Given a unital C^*-algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ will be said to be *core invertible*, if there exists $x \in \mathcal{A}$ such that the following equalities hold:

$$axa = a, \qquad x\mathcal{A} = a\mathcal{A}, \qquad \mathcal{A}x = \mathcal{A}a^*.$$

According to [2, Theorem 2.14], if such an element x exists, then it is unique. This element will be said to be the *core inverse* of $a \in \mathcal{A}$ and it will be denoted by $a^{\oplus}$. In addition, the set of all core invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{\oplus}$.

Recall that according to [2, Theorem 2.14], when $a^{\oplus}$ exists ($a \in \mathcal{A}$), it is an outer inverse of a, i.e., $a^{\oplus}aa^{\oplus} = a^{\oplus}$. Moreover, in [2, Theorem 2.14], the authors characterized the core invertibility in terms of equalities. This characterization was improved in [4, Theorem 3.1]. Specifically, $a \in \mathcal{A}$ is core invertible if and only if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, \qquad xa^2 = a, \qquad (ax)^* = ax.$$

Furthermore, if such x exists, then $x = a^{\oplus}$.

Another generalized inverse, which is related with the core inverse, was defined in [2].

Definition 2.2. Given $\mathcal{A}$ a unital C^*-algebra, an element $a \in \mathcal{A}$ is said to be *dual core invertible*, if there is $x \in \mathcal{A}$ such that $axa = a$, $x\mathcal{A} = a^*\mathcal{A}$, and $\mathcal{A}x = \mathcal{A}a$.

As for the core inverse, it can be proved that this x is unique, when it exists; thus it will be denoted by $a_{\oplus}$ and $\mathcal{A}_{\oplus}$ will stand for the set of all dual core invertible elements of $\mathcal{A}$. In addition, according to [2, Theorem 2.15], when $a \in \mathcal{A}_{\oplus}$, $a_{\oplus}$ is an outer inverse of a, i.e., $a_{\oplus} = a_{\oplus}aa_{\oplus}$.

Observe that according to Definition 2.1 (respectively Definition 2.2), if $a \in \mathcal{A}$ is core invertible (respectively dual core invertible), then it is regular, and hence a is Moore-Penrose invertible ([11, Theorem 6]). Moreover, if $a \in \mathcal{A}^{\oplus} \cup \mathcal{A}_{\oplus}$, then a is group invertible ([2, Remark 2.16]). To learn more on the properties of the core and dual core inverse, see [1,2,4].

In this paragraph, $\mathcal{X}$ will stand for a Banach space and $\mathcal{L}(\mathcal{X})$ for the algebra of all operators defined on and with values in $\mathcal{X}$. When $A \in \mathcal{L}(\mathcal{X})$, the range and the null space of A will be denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. When $\dim \mathcal{X} < \infty$ and $A \in \mathcal{L}(\mathcal{X})$, the dimension of $\mathcal{R}(A)$ will be denoted with $\operatorname{rk}(A)$. Evidently, if $A \in M_n(\mathbb{C})$, the set of all complex $n \times n$ matrices, by considering that $A \in \mathcal{L}(\mathbb{C}^n)$, the rank of the complex matrix A coincides with the previously defined $\operatorname{rk}(A)$; consequently, the same notation will be used for both notions.

One of most studied generalized inverses is the outer inverse with prescribed range and null space. This generalized inverse will be introduced in the Banach frame. Let $\mathcal{X}$ be a Banach space and consider $A \in \mathcal{L}(\mathcal{X})$ and $\mathcal{T}$, $\mathcal{S}$ two closed subspaces in $\mathcal{X}$. If there exists an operator $B \in \mathcal{L}(\mathcal{X})$ such that $BAB = B$, $\mathcal{N}(B) = \mathcal{S}$, and $\mathcal{R}(B) = \mathcal{T}$, then such B is unique ([8, Theorem 1.1.10]). In this case, B will be said to be the $A^{(2)}_{\mathcal{T},\mathcal{S}}$ *outer inverse of* A.

To prove several results of this article, the definition of the gap between two subspaces needs to be recalled. Let $\mathcal{X}$ be a Banach space and consider $\mathcal{M}$ and $\mathcal{N}$ two closed subspaces in $\mathcal{X}$. If $\mathcal{M} = 0$, then set $\delta(\mathcal{M}, \mathcal{N}) = 0$, otherwise set

$$\delta(\mathcal{M}, \mathcal{N}) = \sup\{\text{dist}(x, \mathcal{N}) : x \in \mathcal{M}, \|x\| = 1\},$$

where $\text{dist}(x, \mathcal{N}) = \inf\{\|x - y\| : y \in \mathcal{N}\}$. The *gap between the closed subspaces* $\mathcal{M}$ *and* $\mathcal{N}$ is

$$\widehat{\delta}\,(\mathcal{M}, \mathcal{N}) = \max\{\delta(\mathcal{M}, \mathcal{N}), \delta(\mathcal{N}, \mathcal{M})\}.$$

See [8,16,17] for a deeper insight of this concept.

Another notion needed to study the continuity of the (dual) core inverse is the following. Let p and q be self-adjoint idempotents in a C^*-algebra $\mathcal{A}$. The *maximal angle* between p and q is the number $\psi(p, q) \in [0, \pi/2]$ such that $\|p - q\| = \sin\psi(p, q)$; see [18, Definition 2.3]. In what follows, given $x \in \mathcal{A}^\dagger$, ψ_x will stand for the maximal angle between $xx^\dagger$ and $x^\dagger x$, i.e., $\psi_x = \psi(xx^\dagger, x^\dagger x)$.

Two known results which will be used many times throughout this paper are the following.

Theorem 2.1. *[11, Theorem 6] If a is regular in a C^*-algebra, then a is Moore-Penrose invertible.*

Theorem 2.2. *[2, Theorem 2.19] Let R be a ring, $a \in R^{\oplus}$, and $n \in \mathbb{N}$. Then*

(i) $a^{\oplus} = a^{\#}aa^{\oplus}$.
(ii) $(a^{\oplus})^2 a = a^{\#}$.
(iii) $(a^{\oplus})^n = (a^n)^{\oplus}$.
(iv) $((a^{\oplus})^{\oplus})^{\oplus} = a^{\oplus}$.
(v) *If a is Moore-Penrose invertible, then $a^{\#} = a^{\oplus}aa_{\oplus}$, $a^\dagger = a_{\oplus}aa^{\oplus}$, $a^{\oplus} = a^{\#}aa^\dagger$, and $a_{\oplus} = a^\dagger aa^{\#}$.*

3. Continuity of the (dual) core inverse

In the first place a preliminary result needs to be presented.

Theorem 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra and consider $a \in \mathcal{A}$. The following statements are equivalent.*

(i) *a is core invertible.*
(ii) *a is dual core invertible.*
(iii) *a^* is core invertible.*
(iv) *a^* is dual core invertible.*
(v) *a is group invertible and Moore-Penrose invertible.*

In particular, $\mathcal{A}^{\oplus} = \mathcal{A}_{\oplus} = \mathcal{A}^{\#} \cap \mathcal{A}^\dagger = \mathcal{A}^{\#}$.

Proof. The equivalence between statements (i) and (iv) and between statements (ii) and (iii) can be derived from Definition 2.1 and Definition 2.2. Note that to conclude the proof, it is enough to prove the last statement of the Theorem. In fact, this statement implies that statement (i) and (ii) are equivalent.

According to [2, Remark 2.16], $\mathcal{A}^{\oplus} \cup \mathcal{A}_{\oplus} \subseteq \mathcal{A}^{\#}$. Moreover, according to Theorem 2.1,

$\mathcal{A}^{\oplus} \cup \mathcal{A}_{\oplus} \subseteq \mathcal{A}^{\dagger}$. Therefore, according to [2, Remark 2.16],

$$\mathcal{A}^{\oplus} \subseteq \mathcal{A}^{\#} \cap \mathcal{A}^{\dagger} = \mathcal{A}^{\oplus} \cap \mathcal{A}_{\oplus} \subseteq \mathcal{A}^{\oplus}, \qquad \mathcal{A}_{\oplus} \subseteq \mathcal{A}^{\#} \cap \mathcal{A}^{\dagger} = \mathcal{A}^{\oplus} \cap \mathcal{A}_{\oplus} \subseteq \mathcal{A}_{\oplus}.$$

Finally, according to Theorem 2.1, $\mathcal{A}^{\#} \cap \mathcal{A}^{\dagger} = \mathcal{A}^{\#}$. □

If $\mathcal{A}$ is a unital C^*-algebra, Theorem 3.1 implies $(\mathcal{A}^{\oplus})^* = \mathcal{A}^{\oplus}$ and $(\mathcal{A}_{\oplus})^* = \mathcal{A}_{\oplus}$, where if $X \subseteq \mathcal{A}$ is a set, X^* stands for the following set: $X^* = \{x^* \colon x \in X\}$. However, in contrast to the case of the group inverse and the Moore-Penrose inverse (when $a^{\#}$ (respectively $a^{\dagger}$) exists, $(a^*)^{\#} = (a^{\#})^*$ (respectively $(a^*)^{\dagger} = (a^{\dagger})^*$), $a \in \mathcal{A}$), recall that to obtain the core inverse (respectively the dual core inverse) of a^* it is necessary to consider the dual core (respectively the core) inverse of a:

$$(a^*)^{\oplus} = (a_{\oplus})^*, \qquad (a^*)_{\oplus} = (a^{\oplus})^*.$$

To prove the first characterization of this section some preparation is needed.

Lemma 3.2. *Let $\mathcal{A}$ be a unital C^*-algebra and consider $a \in \mathcal{A}$.*

(i) *If $a \in \mathcal{A}^{\oplus}$, then $aa^{\dagger}a^{\oplus} = a^{\oplus}$.*
(ii) *If $a \in \mathcal{A}^{\oplus}$, then $(aa^{\dagger} + a^{\dagger}a - \mathbb{1})a^{\oplus} = a^{\dagger}$.*
(iii) *Suppose that $a \in \mathcal{A}$ is regular. The element $aa^{\dagger}+a^{\dagger}a-\mathbb{1}$ is invertible if and only if a is core invertible. Moreover, in this case, $(aa^{\dagger}+a^{\dagger}a-\mathbb{1})^{-1} = a^{\oplus}a+(a^{\oplus}a)^*-\mathbb{1}$.*
(iv) *If $a \in \mathcal{A}_{\oplus}$, then $a^{\#} = a(a_{\oplus})^2$.*

Proof. Recall that according to Theorem 2.1, if $a \in \mathcal{A}^{\oplus}$, then $a^{\dagger}$ exists.

The proof of statement (i) can be derived from the fact that $a^{\oplus} \in a\mathcal{A}$.

To prove statement (ii), recall that according to Theorem 2.2 (v), $a^{\oplus} = a^{\#}aa^{\dagger}$. Therefore,

$$(aa^{\dagger} + a^{\dagger}a - \mathbb{1})a^{\oplus} = aa^{\dagger}a^{\oplus} + a^{\dagger}aa^{\oplus} - a^{\oplus} = a^{\dagger}aa^{\oplus} = a^{\dagger}aa^{\#}aa^{\dagger} = a^{\dagger}.$$

Now statement (iii) will be proved. Note that according to Theorem 2.1, $a^{\dagger}$ exists. Recall that according to [19, Theorem 2.3], $aa^{\dagger}+a^{\dagger}a-\mathbb{1} \in \mathcal{A}^{-1}$ is equivalent to $a \in \mathcal{A}^{\#}$. Thus, according to Theorem 3.1, necessary and sufficient for $aa^{\dagger} + a^{\dagger}a - \mathbb{1} \in \mathcal{A}^{-1}$ is that $a \in \mathcal{A}^{\oplus}$. Next the formula of the inverse of $aa^{\dagger} + a^{\dagger}a - \mathbb{1}$ will be proved. Recall that according to [4, Theorem 3.1], $a^{\oplus}aaa^{\dagger} = aa^{\dagger}$.

$$\begin{aligned}
&[a^{\oplus}a + (a^{\oplus}a)^* - \mathbb{1}][aa^{\dagger} + a^{\dagger}a - \mathbb{1}] \\
&\quad = [a^{\oplus}a + (a^{\oplus}a)^* - \mathbb{1}]aa^{\dagger} + [a^{\oplus}a + (a^{\oplus}a)^* - \mathbb{1}]a^{\dagger}a - [a^{\oplus}a + (a^{\oplus}a)^* - \mathbb{1}] \\
&\quad = aa^{\dagger} + (a^{\oplus}a)^*(aa^{\dagger})^* - aa^{\dagger} + a^{\oplus}a + (a^{\oplus}a)^*(a^{\dagger}a)^* - a^{\dagger}a - a^{\oplus}a - (a^{\oplus}a)^* + \mathbb{1} \\
&\quad = (aa^{\dagger}a^{\oplus}a)^* + (a^{\dagger}aa^{\oplus}a)^* - a^{\dagger}a - (a^{\oplus}a)^* + \mathbb{1} \\
&\quad = (a^{\oplus}a)^* + (a^{\dagger}a)^* - a^{\dagger}a - (a^{\oplus}a)^* + \mathbb{1} \\
&\quad = \mathbb{1}.
\end{aligned}$$

Since $aa^{\dagger} + a^{\dagger}a - \mathbb{1}$ is invertible, $(aa^{\dagger} + a^{\dagger}a - \mathbb{1})^{-1} = a^{\oplus}a + (a^{\oplus}a)^* - \mathbb{1}$.

To prove statement (iv), recall that according Theorem 3.1, $a^* \in \mathcal{A}^{\oplus}$. In addition, according to the paragraph between Theorem 3.1 and the present Lemma, $a_{\oplus} =$

$((a^*)^{\oplus})^*$. However, according to Theorem 2.2, $(a^*)^{\#} = ((a^*)^{\oplus})^2 a^*$. Thus,

$$a^{\#} = a(((a^*)^{\oplus})^*)^2 = a(a_{\oplus})^2.$$

□

Note that given a ring with involution $\mathcal{R}$, Lemma 3.2 holds in such a context provided that $a \in \mathcal{R}$ is Moore-Penrose invertible,

In the next theorem the continuity of the (dual) core inverse will be characterized. It is worth noticing that $a \in \mathcal{A}$ will be not assumed to be core invertible, dual core invertible, group invertible or Moore-Penrose invertible. Note also that the following well known result will be used in the proof of the theorem: given $\mathcal{A}$ a unital Banach algebra, $b \in \mathcal{A}$ and $(b_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{-1}$ a sequence such that $(b_n)_{n\in\mathbb{N}}$ converges to b, if $(b_n^{-1})_{n\in\mathbb{N}}$ is a bounded sequence, then b is invertible and the sequence $(b_n^{-1})_{n\in\mathbb{N}}$ converges to b^{-1}.

Theorem 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra and consider $a \in \mathcal{A}$. Let $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$ be such that $(a_n)_{n\in\mathbb{N}}$ converges to a. The following statements are equivalent.*

(i) *The element $a \in \mathcal{A}^{\oplus}$ and $(a_n^{\oplus})_{n\in\mathbb{N}}$ converges to $a^{\oplus}$.*
(ii) *The element $a \in \mathcal{A}_{\oplus}$ and $(a_{n\oplus})_{n\in\mathbb{N}}$ converges to $a_{\oplus}$.*
(iii) *The element $a \in \mathcal{A}^{\#}$ and $(a_n^{\#})_{n\in\mathbb{N}}$ converges to $a^{\#}$.*
(iv) *The element $a \in \mathcal{A}^{\oplus}$ and $(a_n{}^{\oplus})_{n\in\mathbb{N}}$ is a bounded sequence.*
(v) *The element $a \in \mathcal{A}_{\oplus}$ and $(a_{n\oplus})_{n\in\mathbb{N}}$ is a bounded sequence.*
(vi) *The element $a \in \mathcal{A}^{\dagger}$, $(a_n^{\dagger})_{n\in\mathbb{N}}$ converges to $a^{\dagger}$, and $(a_n{}^{\oplus}a_n)_{n\in\mathbb{N}}$ is a bounded sequence.*
(vii) *The element $a \in \mathcal{A}^{\dagger}$, $(a_n^{\dagger})_{n\in\mathbb{N}}$ converges to $a^{\dagger}$, and $(a_n a_{n\oplus})_{n\in\mathbb{N}}$ is a bounded sequence.*
(viii) *The element $a \in \mathcal{A}^{\dagger}$, $(a_n^{\dagger})_{n\in\mathbb{N}}$ converges to $a^{\dagger}$, and there exists $\psi \in [0, \frac{\pi}{2})$ such that $\psi_n = \psi_{a_n} \leq \psi$ for all $n \in \mathbb{N}$.*

Proof. Note that according to Theorem 3.1, $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\oplus} \cap \mathcal{A}_{\oplus} \cap \mathcal{A}^{\#} \cap \mathcal{A}^{\dagger}$.

First the equivalence between statements (i) and (iii) will be proved. Suppose that statement (i) holds. Then according to Theorem 3.1, $a \in \mathcal{A}^{\#}$. In addition, $(a_n^{\oplus} a_n)_{n\in\mathbb{N}}$ converges to $a^{\oplus}a$. However, according to [2, Remark 2.17], $a^{\#}a = a^{\oplus}a$, and for each $n \in \mathbb{N}$, $a_n^{\#} a_n = a_n^{\oplus} a_n$. Consequently, $(a_n^{\#} a_n)_{n\in\mathbb{N}}$ converges to $a^{\#}a$, which according to [7, Theorem 2.4], implies that $(a_n^{\#})_{n\in\mathbb{N}}$ converges to $a^{\#}$.

Suppose that statement (iii) holds. Note that according to Theorem 2.1, $a \in \mathcal{A}^{\dagger}$. In particular, according to Theorem 3.1, $a \in \mathcal{A}^{\oplus}$. Moreover, according to [19, Corollary 2.1 (ii)] and [18, Equation (2.1)],

$$\|a_n^{\dagger}\| = \|(a_n a_n^{\dagger} + a_n^{\dagger} a_n - \mathbb{1}) a_n^{\#} (a_n a_n^{\dagger} + a_n^{\dagger} a_n - \mathbb{1})\| \leq \|a_n a_n^{\dagger} + a_n^{\dagger} a_n - \mathbb{1}\|^2 \|a_n^{\#}\| \leq \|a_n^{\#}\|.$$

Consequently, $(a_n^{\dagger})_{n\in\mathbb{N}}$ is a bounded sequence.

Now two cases need to be considered. If $a = 0$, then $a^{\oplus} = 0$. However, according to Theorem 2.2, $a_n{}^{\oplus} = a_n^{\#} a_n a_n^{\dagger}$. Since $(a_n)_{n\in\mathbb{N}}$ converges to 0 and $(a_n^{\#})_{n\in\mathbb{N}}$ and $(a_n^{\dagger})_{n\in\mathbb{N}}$ are bounded sequences, $(a_n{}^{\oplus})_{n\in\mathbb{N}}$ converges to 0.

Now suppose that $a \neq 0$. Since $(a_n)_{n\in\mathbb{N}}$ converges to a, there exists and $n_0 \in \mathbb{N}$ such that $a_n \neq 0$, $n \geq n_0$. Without loss of generality, it is possible to assume that $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\oplus} \setminus \{0\}$. Thus, according to [20, Theorem 1.6], $(a_n^{\dagger})_{n\in\mathbb{N}}$ converges to

$a^\dagger$. However, according again to Theorem 2.2, $a^{\oplus} = a^{\#}aa^\dagger$ and for each $n \in \mathbb{N}$, $a_n^{\oplus} = a_n^{\#}a_na_n^\dagger$. Therefore, $(a_n^{\oplus})_{n\in\mathbb{N}}$ converges to $a^{\oplus}$.

To prove the equivalence between statements (ii) and (iii), apply a similar argument to the one used to prove the equivalence between statements (i) and (iii). In particular, use the following identities, which hold for $b \in \mathcal{A}_{\oplus}$: ($\alpha$) $b^{\#}b = bb_{\oplus}$ ([2, Remark 2.17]); (β) $b_{\oplus} = b^\dagger bb^{\#}$ (Theorem 2.2).

It is evident that statement (i) implies statement (iv). Now suppose that statement (iv) holds. It will be proved that statement (iii) holds. According to Theorem 3.1, $a \in \mathcal{A}^{\#}$. In addition, according to Theorem 2.2, for each $n \in \mathbb{N}$, $a_n^{\#} = (a_n{}^{\oplus})^2a_n$. In particular, $(a_n^{\#})_{n\in\mathbb{N}}$ is a bounded sequence. Consequently, according to [7, Theorem 2.4], $(a_n^{\#})_{n\in\mathbb{N}}$ converges to $a^{\#}$, equivalently, statement (iii) holds.

The equivalence between statements (ii) and (v) can be proved applying a similar argument to the one used to prove the equivalence between statements (i) and (iv), using in particular Lemma 3.2 (iv).

Next it will be proved that statement (iv) implies statement (vi). Suppose then that statement (iv) holds. Then, $(a_n{}^{\oplus}a_n)_{n\in\mathbb{N}}$ is a bounded sequence. In addition, according to Theorem 3.1, $a \in \mathcal{A}^\dagger$. Now two cases need to be considered. Suppose first that $a = 0$. Since statement (iv) and (v) are equivalent, $(a_{n\oplus})_{n\in\mathbb{N}}$ is a bounded sequence. According to Theorem 2.2, for each $n \in \mathbb{N}$, $a_n^\dagger = a_{n\oplus}a_na_n{}^{\oplus}$. Since $(a_n{}^{\oplus})_{n\in\mathbb{N}}$ and $(a_{n\oplus})_{n\in\mathbb{N}}$ are bounded sequences, $(a_n^\dagger)_{n\in\mathbb{N}}$ converges to $0 = a^\dagger$.

If $a \neq 0$, as when it was proved that statement (iii) implies statement (i), it is possible to assume that $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A} \setminus \{0\}$. According to Lemma 3.2 (ii),

$$\|a_n^\dagger\| \leq \|a_na_n^\dagger + a_n^\dagger a_n - \mathbb{1}\|\|a_n{}^{\oplus}\| \leq 3\|a_n{}^{\oplus}\|.$$

In particular, $(a_n^\dagger)_{n\in\mathbb{N}}$ is a bounded sequence. However, according to [20, Theorem 1.6], $(a_n^\dagger)_{n\in\mathbb{N}}$ converges to $a^\dagger$.

Suppose that statement (vi) holds. It will be proved that statement (vi) implies statement (iv). According to Theorem 2.2, for each $n \in \mathbb{N}$, $a_n{}^{\oplus} = a_n{}^{\oplus}a_na_n^\dagger$. Since $(a_n^\dagger)_{n\in\mathbb{N}}$ and $(a_n{}^{\oplus}a_n)_{n\in\mathbb{N}}$ are bounded sequences, $(a_n{}^{\oplus})_{n\in\mathbb{N}}$ is a bounded sequence.

Note that according to Lemma 3.2 (iii), for each $n \in \mathbb{N}$, $b_n = a_na_n^\dagger + a_n^\dagger a_n - \mathbb{1}$ is invertible and $b_n^{-1} = a_n{}^{\oplus}a_n + (a_n{}^{\oplus}a_n)^* - \mathbb{1}$. In addition, the sequence $(b_n^{-1})_{n\in\mathbb{N}}$ is bounded. In fact, according to [21, Lemma 2.3], $\|b_n^{-1}\| = \|a_n{}^{\oplus}a_n\|$. Now, since $(b_n)_{n\in\mathbb{N}}$ converges to $b = aa^\dagger + a^\dagger a - \mathbb{1}$, the element b is invertible, which in view of Lemma 3.2 (iii), is equivalent to $a \in \mathcal{A}^{\oplus}$.

According to Theorem 2.2, for each $n \in \mathbb{N}$, $a_n{}^{\oplus}a_n = a_na_{n\oplus}$. Thus, statement (vii) is an equivalent formulation of statement (vi).

Finally, statements (vi) and (viii) will be proved to be equivalent. In fact, note that if $a_n = 0$, then $\psi_n = 0$. In addition, according to Lemma 3.2 (iii), $a_na_n^\dagger + a_n^\dagger a_n - \mathbb{1}$ is invertible, and when $a_n \neq 0$, according to Lemma 3.2 (iii), [18, Theorem 2.4 (iii)] and [21, Lemma 2.3],

$$\frac{1}{\cos\psi_n} = \|(a_na_n^\dagger + a_n^\dagger a_n - \mathbb{1})^{-1}\| = \|a_n{}^{\oplus}a_n + (a_n{}^{\oplus}a_n)^* - \mathbb{1}\| = \|a_n{}^{\oplus}a_n\|.$$

In particular, $(a_n{}^{\oplus}a_n)_{n\in\mathbb{N}}$ is bounded if and only if there exists $\psi \in [0, \frac{\pi}{2})$ such that $\psi_n \leq \psi$ for all $n \in \mathbb{N}$. □

Theorem 3.3 shows that the continuity of the group inverse and of the Moore-

Penrose inverse are central for the continuity of the core inverse and the dual core inverse. To learn more on the continuity of the group inverse and the Moore-Penrose inverse, see for example [6,7,15,18,22] and [5,12,14,15,20], respectively, see also [8, Chapter 4].

Observe that the conditions in statement (vi) of Theorem 3.3, (α) $a \in \mathcal{A}^\dagger$, $a_n^\dagger \to a^\dagger$, and ($\beta$) $\{a_n{}^{\oplus} a_n\}$ is a bounded sequence, are independent from each other, as the following two examples show.

Example 3.4. Consider $\mathbb{C}$ as a C^*-algebra. Let $a_n = 1/n$ and $a = 0$. It is evident that $a_n \to a$, $a_n^\dagger = n$, and $(a_n^\dagger)_{n\in\mathbb{N}}$ does not converge to $a^\dagger = 0$. However, it should be clear that $a_n^{\oplus} = n$. Therefore, $a_n^{\oplus} a_n = 1$, and thus, $(a_n{}^{\oplus} a_n)_{n\in\mathbb{N}}$ is a bounded sequence.

Example 3.5. Consider the set of 2×2 complex matrices as a C^*-algebra. Take the conjugate transpose of the matrix as the involution on this matrix. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence in $(0, \pi/2)$ such that $\psi_n \to \pi/2$ and let

$$A_n = \begin{bmatrix} \cos\psi_n & \sin\psi_n \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is simple prove that

$$A_n^\dagger = \begin{bmatrix} \cos\psi_n & 0 \\ \sin\psi_n & 0 \end{bmatrix}, \qquad A^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad A_n^{\oplus} = \begin{bmatrix} 1/\cos\psi_n & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $(A_n^\dagger)_{n\in\mathbb{N}}$ converges to $A^\dagger$ and

$$A_n^{\oplus} A_n = \begin{bmatrix} 1 & \tan\psi_n \\ 0 & 0 \end{bmatrix},$$

which shows that $(A_n^{\oplus} A_n)_{n\in\mathbb{N}}$ is not bounded. Note also that $(A_n{}^{\oplus})_{n\in\mathbb{N}}$ is not a convergent sequence.

Observe also that if $\mathcal{A}$ is a unital C^*-algebra and $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\oplus}$ is such that $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathcal{A}$, Example 3.4 also shows that the condition $(a_n{}^{\oplus} a_n)_{n\in\mathbb{N}}$ is a convergent sequence does not necessarily imply that $(a_n{}^{\oplus})_{n\in\mathbb{N}}$ is convergent.

It is worth noticing that Example 3.5 also proves that $\mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$ is not in general a closed set. In fact, using the same notation as in Example 3.5, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\oplus}$, $(A_n)_{n\in\mathbb{N}}$ converges to A but $A \notin \mathcal{A}^{\oplus}$ ($A^2 = 0$, $\mathrm{rk}(A^2) = 0 \neq 1 = \mathrm{rk}(A)$, i.e., A is not group invertible).

Next an extension of [18, Theorem 2.7] will be derived from Theorem 3.3.

Corollary 3.6. *Let $\mathcal{A}$ be a unital C^*-algebra and consider $a \in \mathcal{A}$. Suppose that the sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^\#$ is such that $(a_n)_{n\in\mathbb{N}}$ converges to a. Then, the following statements are equivalent.*

(i) *The element $a \in \mathcal{A}^\#$ and $(a_n^\#)_{n\in\mathbb{N}}$ converges to $a^\#$.*
(ii) *The sequence $(a_n^\#)_{n\in\mathbb{N}}$ is bounded.*
(iii) *The element $a \in \mathcal{A}^\dagger$, $(a_n^\dagger)_{n\in\mathbb{N}}$ converges to $a^\dagger$, and the sequence $(a_n^\# a_n)_{n\in\mathbb{N}}$ is bounded.*
(iv) *The element $a \in \mathcal{A}^\dagger$, $(a_n^\dagger)_{n\in\mathbb{N}}$ converges to $a^\dagger$, and there exists $\psi \in [0, \frac{\pi}{2})$ such that $\psi_n = \psi_{a_n} \leq \psi$ for all $n \in \mathbb{N}$.*

Proof. Statement (ii) is a consequence of statement (i).

Suppose that statement (ii) holds. Then, $(a_n^{\#} a_n)_{n\in\mathbb{N}}$ is a bounded sequence. To prove that $a \in \mathcal{A}^\dagger$ and $(a_n^\dagger)_{n\in\mathbb{N}}$ converges to $a^\dagger$, proceed as in the corresponding part of the proof of [18, Theorem 2.7] (see statement (ii) implies statement (iii) in [18, Theorem 2.7]).

Suppose that statement (iii) holds. First note that if $a_n = 0$, then $\psi_n = 0$. In addition, according to [18, Theorem 2.5], if $a_n \neq 0$, then,

$$\|a_n a_n^{\#}\| = \frac{1}{\cos \psi_{a_n}}.$$

Therefore, the sequence $(a_n^{\#} a_n)_{n\in\mathbb{N}}$ is bounded if and only if there exists $\psi \in [0, \frac{\pi}{2})$ such that $\psi_n = \psi_{a_n} \leq \psi$ for all $n \in \mathbb{N}$.

To prove that statement (iv) implies statement (i), apply Theorem 3.3 (equivalence between statements (iii) and (viii)). $\square$

In Theorem 3.3 and Corollary 3.6 the general case has been presented for the sake of completeness. However, the case $a = 0$ is particular and it deserves to be studied. Recall that given a unital C^*-algebra $\mathcal{A}$, if $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{-1}$ is such that $(a_n)_{n\in\mathbb{N}}$ converges to 0, then the sequence $(a_n^{-1})_{n\in\mathbb{N}}$ is unbounded. Next the case of a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\#} = \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus} \subseteq \mathcal{A}^\dagger$ such that it converges to 0 will be studied. Firstly, the Moore-Penrose inverse will be considered.

Remark 1. Let $\mathcal{A}$ be a unital C^*-algebra and consider $a \in \mathcal{A}^\dagger$ and $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^\dagger$ such that $(a_n)_{n\in\mathbb{N}}$ converges to a. Recall that according to [20, Theorem 1.6], the following statements are equivalent.

(i) The sequence $(a_n^\dagger)_{n\in\mathbb{N}}$ converges to $a^\dagger$.
(ii) The sequence $(a_n a_n^\dagger)_{n\in\mathbb{N}}$ converges to $aa^\dagger$.
(iii) The sequence $(a_n^\dagger a_n)_{n\in\mathbb{N}}$ converges to $a^\dagger a$.
(iv) The sequence $(a_n^\dagger)_{n\in\mathbb{N}}$ is bounded.

Now when $a = 0$, according to [20, Theorem 1.4], the following equivalence holds:

(v) A necessary and sufficient condition for $(a_n^\dagger)_{n\in\mathbb{N}}$ to converge to 0 is that the sequence $(a_n^\dagger)_{n\in\mathbb{N}}$ is bounded.

However, concerning the convergence of $(a_n a_n^\dagger)_{n\in\mathbb{N}}$, note that given $n \in \mathbb{N}$, since $a_n a_n^\dagger$ is a self-adjoint idempotent, if $\|a_n a_n^\dagger\| < 1$, then $a_n a_n^\dagger = 0$, which implies that $a_n = 0$; a similar result can be derived for the convergence of $(a_n^\dagger a_n)_{n\in\mathbb{N}}$. Consequently, the following statements are equivalent.

(vi) The sequence $(a_n^\dagger)_{n\in\mathbb{N}}$ converges to 0.
(vii) There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $a_n = 0$.

Therefore, according to statements (v)-(vii), given $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^\dagger$ such that $(a_n)_{n\in\mathbb{N}}$ converges to 0, there are only two possibilities.

(viii) There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $a_n = 0$; or
(ix) the sequence $(a_n^\dagger)_{n\in\mathbb{N}}$ is unbounded.

In the following proposition, sequences of group invertible or (dual) core invertible elements that converge to 0 will be studied.

Proposition 3.7. *Let $\mathcal{A}$ be a unital C^*-algebra and consider a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^\# = \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$ such that $(a_n)_{n\in\mathbb{N}}$ converges to 0. The following statements are equivalent.*

(i) *The sequence $(a_n^\#)_{n\in\mathbb{N}}$ converges to 0.*
(ii) *The sequence $(a_n{}^{\oplus})_{n\in\mathbb{N}}$ converges to 0.*
(iii) *The sequence $(a_{n\oplus})_{n\in\mathbb{N}}$ converges to 0.*
(iv) *The sequence $(a_n^\#)_{n\in\mathbb{N}}$ is bounded.*
(v) *There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $a_n = 0$.*

In addition, there exist only two possibilities for the sequence $(a_n)_{n\in\mathbb{N}}$.

(vi) *There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $a_n = 0$; or*
(vii) *the sequence $(a_n^\#)_{n\in\mathbb{N}}$ is unbounded.*

Moreover, statement (vii) *is equivalent to the following two statements.*

(viii) *the sequence $(a_n{}^{\oplus})_{n\in\mathbb{N}}$ is unbounded.*
(ix) *the sequence $(a_{n\oplus})_{n\in\mathbb{N}}$ is unbounded.*

Proof. According to Theorem 3.3, statements (i)-(iii) are equivalent.

It is evident that statement (i) implies statement (iv).

Suppose that statement (iv) holds. According to Theorem 2.1, $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^\# \subset \mathcal{A}^\dagger$. In addition according to [19, Corollary 2.1 (ii)],

$$\|a_n^\dagger\| \leq \|a_n^\#\|\|a_n a_n^\dagger + a_n^\dagger a_n - \mathbb{1}\|^2 \leq 9\|a_n^\#\|.$$

In particular, the sequence $(a_n^\dagger)_{n\in\mathbb{N}}$ is bounded. Thus, according to Remark 1 (v), $(a_n^\dagger)_{n\in\mathbb{N}}$ converges to 0. However, according to Remark 1 (vi)-(vii), statement (v) holds.

It is evident that statement (v) implies statement (i).

Statements (vi) and (vii) can be derived from what has been proved.

According to Theorem 3.3, statements (vii)-(ix) are equivalent. □

To prove the second characterization of this section some preparation is needed.

Remark 2. Let $\mathcal{A}$ be a unital C^*-algebra and consider $a \in \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$. If $L_a\colon \mathcal{A} \to \mathcal{A}$ and $R_a\colon \mathcal{A} \to \mathcal{A}$ are the left and the right multiplication operators defined by a, i.e., for $x \in \mathcal{A}$, $L_a(x) = ax$, $R_a(x) = xa$, respectively, then according to [2, Theorem 2.14],

$$L_{a^{\oplus}} L_a L_{a^{\oplus}} = L_{a^{\oplus}}, \qquad R_{a_{\oplus}} R_a R_{a_{\oplus}} = R_{a_{\oplus}}.$$

Note also that according to Definition 2.1,

$$\mathcal{R}(L_{a^{\oplus}}) = a\mathcal{A}, \qquad \mathcal{N}(L_{a^{\oplus}}) = (a^*)^\circ.$$
$$\mathcal{R}(R_{a_{\oplus}}) = \mathcal{A}a^*, \qquad \mathcal{N}(R_{a_{\oplus}}) = {}^\circ a.$$

Therefore, $L_{a^{\oplus}} = (L_a)^{(2)}_{a\mathcal{A},(a^*)^\circ}$ and $R_{a_{\oplus}} = (R_a)^{(2)}_{\mathcal{A}a^*,{}^\circ a}$.

In addition, since $L_{aa^{\oplus}} = L_a L_{a^{\oplus}}$, $R_{a_{\oplus}a} = R_{a_{\oplus}} R_a \in \mathcal{L}(\mathcal{A})$ are idempotents, observe that according to Definition 2.1 and [2, Theorem 2.14],

$$\mathcal{R}(L_{aa^{\oplus}}) = a\mathcal{A}, \qquad \mathcal{R}(R_{aa^{\oplus}}) = \mathcal{A}a^*,$$
$$\mathcal{N}(L_{aa^{\oplus}}) = (a^*)^\circ, \qquad \mathcal{N}(R_{aa^{\oplus}}) = {}^\circ a.$$

In fact, the identity $a = aa^{\oplus}a$ (respectively $a^{\oplus} = a^{\oplus}aa^{\oplus}$) implies $aa^{\oplus}\mathcal{A} = a\mathcal{A}$ (respectively $\mathcal{A}aa^{\oplus} = \mathcal{A}a^{\oplus} = \mathcal{A}a^*$). Moreover, it is not difficult to prove that the identity $\mathcal{A}a^{\oplus} = \mathcal{A}a^*$ implies that $(aa^{\oplus})^\circ = (a^*)^\circ$, while from the identity $a = aa^{\oplus}a$ it is possible to deduce that ${}^\circ(aa^{\oplus}) = {}^\circ a$.
Similar arguments prove the following facts: $L_{a_{\oplus}} = (L_a)^{(2)}_{a^*\mathcal{A},a^\circ}$, $R_{a_{\oplus}} = (R_a)^{(2)}_{\mathcal{A}a,{}^\circ(a^*)}$ and

$$\mathcal{R}(L_{a_{\oplus}a}) = a^*\mathcal{A}, \qquad \mathcal{R}(R_{a_{\oplus}a}) = \mathcal{A}a,$$
$$\mathcal{N}(L_{a_{\oplus}a}) = a^\circ, \qquad \mathcal{N}(R_{a_{\oplus}a}) = {}^\circ(a^*).$$

Next follows the second characterization of the continuity of the (dual) core inverse. In this case, the notion of the gap between subspaces will be used.

Theorem 3.8. *Let $\mathcal{A}$ be a unital C^*-algebra and consider $a \in \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$, $a \neq 0$. Consider a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$ such that $(a_n)_{n\in\mathbb{N}}$ converges to a. The following statements are equivalent.*

(i) $(a_n^{\oplus})_{n\in\mathbb{N}}$ *converges to* $a^{\oplus}$.
(ii) $(a_n a_n^{\oplus})_{n\in\mathbb{N}}$ *converges to* $aa^{\oplus}$.
(iii) $(\widehat{\delta}\,(a_n\mathcal{A}, a\mathcal{A}))_{n\in\mathbb{N}}$ *and* $(\widehat{\delta}\,((a_n^*)^\circ, (a^*)^\circ))_{n\in\mathbb{N}}$ *converge to* 0.
(iv) $(\widehat{\delta}\,(\mathcal{A}a_n^*, \mathcal{A}a^*))_{n\in\mathbb{N}}$ *and* $(\widehat{\delta}\,({}^\circ a_n, {}^\circ a))_{n\in\mathbb{N}}$ *converge to* 0.
(v) $(a_{n\oplus})_{n\in\mathbb{N}}$ *converges to* $a_{\oplus}$.
(vi) $(a_{n\oplus}a_n)_{n\in\mathbb{N}}$ *converges to* $a_{\oplus}a$.
(vii) $(\widehat{\delta}\,(a_n^*\mathcal{A}, a^*\mathcal{A}))_{n\in\mathbb{N}}$ *and* $(\widehat{\delta}\,(a_n^\circ, a^\circ))_{n\in\mathbb{N}}$ *converge to* 0.
(viii) $(\widehat{\delta}\,(\mathcal{A}a_n, \mathcal{A}a))_{n\in\mathbb{N}}$ *and* $(\widehat{\delta}\,({}^\circ(a_n^*), {}^\circ(a^*)))_{n\in\mathbb{N}}$ *converge to* 0.

Proof. It is evident that statement (i) implies statement (ii). Suppose that statement (ii) holds. According to Remark 2, $a\mathcal{A} = \mathcal{R}(L_{aa^{\oplus}})$, $(a^*)^\circ = \mathcal{N}(L_{aa^{\oplus}})$, $a_n\mathcal{A} = \mathcal{R}(L_{a_n a_n^{\oplus}})$ and $(a_n^*)^\circ = \mathcal{N}(L_{a_n a_n^{\oplus}})$ ($n \in \mathbb{N}$). However, according to [7, Lemma 3.3], statement (iii) holds.

Suppose that statement (iii) holds. Recall that according to Remark 2,

$$L_{a^{\oplus}} = (L_a)^{(2)}_{a\mathcal{A},(a^*)^\circ}, \qquad L_{a_n^{\oplus}} = (L_{a_n})^{(2)}_{a_n\mathcal{A},(a_n^*)^\circ},$$

for each $n \in \mathbb{N}$. Let $\kappa = \|L_a\|\|L_{a^{\oplus}}\| = \|a\|\|a^{\oplus}\|$ and consider $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$r_n = \widehat{\delta}\left(\mathcal{N}\left((L_{a_n})^{(2)}_{a_n\mathcal{A},(a_n^*)^\circ}\right), \mathcal{N}\left((L_a)^{(2)}_{a\mathcal{A},(a^*)^\circ}\right)\right) = \widehat{\delta}\,((a_n^*)^\circ, (a^*)^\circ) < \frac{1}{3+\kappa},$$

$$s_n = \widehat{\delta}\left(\mathcal{R}\left((L_{a_n})^{(2)}_{a_n\mathcal{A},(a_n^*)^\circ}\right), \mathcal{R}\left((L_a)^{(2)}_{a\mathcal{A},(a^*)^\circ}\right)\right) = \widehat{\delta}\,(a_n\mathcal{A}, a\mathcal{A}) < \frac{1}{(1+\kappa)^2},$$

and

$$t_n = \|L_{a^{\oplus}}\|\|L_a - L_{a_n}\| = \|a^{\oplus}\|\|a - a_n\| < \frac{2\kappa}{(1+\kappa)(4+\kappa)}.$$

Thus, according to [16, Theorem 3.5],

$$\|a_n^{\oplus} - a^{\oplus}\| = \left\| L_{a_n^{\oplus}} - L_{a^{\oplus}} \right\| \le \frac{(1+\kappa)(s_n + r_n) + (1+r_n)t_n}{1 - (1+\kappa)s_n - \kappa r_n - (1+r_n)t_n}\|a^{\oplus}\|,$$

which implies statement (i).

Statements (i), (ii) and (iv) are equivalent. To prove this fact, apply a similar argument to the one used to prove the equivalence among statements (i), (ii) and (iii), using in particular $R_{a^{\oplus}} = (R_a)^{(2)}_{\mathcal{A}a^*, {}^{\circ}a}$, $R_{a_n^{\oplus}} = (R_{a_n})^{(2)}_{\mathcal{A}a_n^*, {}^{\circ}a_n}$, $R_{aa^{\oplus}}$ and $R_{a_n a_n^{\oplus}}$ instead of the respectively left multiplication operators (Remark 2, $n \in \mathbb{N}$).

Statements (i) and (v) are equivalent (Theorem 3.3).

To prove the equivalence among statements (v) and (viii), apply a similar argument to the one used to prove that statements (i)-(iv) are equivalent, using in particular Remark 2 and [16, Theorem 3.5]. □

Next, some bounds for $\|a_n^{\oplus} - a^{\oplus}\|$ will be proved, when $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ converges to $a \in \mathcal{A}$ in a C^*-algebra $\mathcal{A}$. Before, a technical lemma is presented.

Lemma 3.9. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a, b \in \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$. Then*

(i) $b^{\oplus} - a^{\oplus} = b^{\oplus}b(b^{\dagger} - a^{\dagger})(\mathbb{1} - aa^{\oplus}) + b^{\oplus}(a-b)a^{\oplus} + (\mathbb{1} - b^{\oplus}b)(b-a)a^{\dagger}a^{\oplus}$.

(ii) $b_{\oplus} - a_{\oplus} = (\mathbb{1} - a_{\oplus}a)(b^{\dagger} - a^{\dagger})bb_{\oplus} + a_{\oplus}(a-b)b_{\oplus} + a_{\oplus}a^{\dagger}(b-a)(\mathbb{1} - bb_{\oplus})$.

Proof. To prove statement (i), recall that since a and b are core invertible, a and b are Moore-Penrose invertible (Theorem 3.1). In addition, according to [4, Theorem 3.1], $b = b^{\oplus}b^2$. Thus, according to Lemma 3.2 (i),

$$(\mathbb{1} - b^{\oplus}b)(b-a)a^{\dagger}a^{\oplus} = -(\mathbb{1} - b^{\oplus}b)aa^{\dagger}a^{\oplus} = -(\mathbb{1} - b^{\oplus}b)a^{\oplus} = b^{\oplus}ba^{\oplus} - a^{\oplus}.$$

Now, according to Theorem 2.2, $b^{\oplus} = b^{\oplus}bb^{\dagger}$. In addition, $a^*aa^{\oplus} = a^*(aa^{\oplus})^* = (aa^{\oplus}a)^* = a^*$, i.e., $a^*(\mathbb{1} - aa^{\oplus}) = 0$. Moreover, since $a^{\dagger} = a^{\dagger}aa^{\dagger} = a^{\dagger}(aa^{\dagger})^* = a^{\dagger}(a^{\dagger})^*a^*$, $a^{\dagger}(\mathbb{1} - aa^{\oplus}) = 0$. Therefore,

$$b^{\oplus}b(b^{\dagger} - a^{\dagger})(\mathbb{1} - aa^{\oplus}) = b^{\oplus}bb^{\dagger}(\mathbb{1} - aa^{\oplus}) = b^{\oplus}(\mathbb{1} - aa^{\oplus}) = b^{\oplus} - b^{\oplus}aa^{\oplus}.$$

As a result,

$$\begin{aligned} b^{\oplus} - a^{\oplus} &= b^{\oplus} - b^{\oplus}aa^{\oplus} + b^{\oplus}aa^{\oplus} - b^{\oplus}ba^{\oplus} + b^{\oplus}ba^{\oplus} - a^{\oplus} \\ &= b^{\oplus}b(b^{\dagger} - a^{\dagger})(\mathbb{1} - aa^{\oplus}) + b^{\oplus}(a-b)a^{\oplus} + (\mathbb{1} - b^{\oplus}b)(b-a)a^{\dagger}a^{\oplus}. \end{aligned}$$

To prove statement (ii), use that $x_{\oplus} = ((x^*)^{\oplus})^*$ ($x \in \mathcal{A}$), and apply statement (i). □

Next, the aforementioned bounds will be given.

Theorem 3.10. *Let $\mathcal{A}$ be a unital C^*-algebra and consider $a \in \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$. The following statements hold.*

(i) *If* $b \in \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$, $b \neq 0$ *and* $b \notin \mathcal{A}^{-1}$, *then*

$$\|b^{\oplus} - a^{\oplus}\| \leq \frac{\|b^{\dagger} - a^{\dagger}\|}{\cos\psi_b} + \left[\|b^{\oplus}\| + \frac{\|a^{\dagger}\|}{\cos\psi_b}\right]\|a^{\oplus}\|\|a-b\|.$$

(ii) *In addition,*

$$\|b_{\oplus} - a_{\oplus}\| \leq \frac{\|b^{\dagger} - a^{\dagger}\|}{\cos\psi_b} + \left[\|b_{\oplus}\| + \frac{\|a^{\dagger}\|}{\cos\psi_b}\right]\|a_{\oplus}\|\|a-b\|.$$

(iii) *If also* $a \neq 0$ *and* $a \notin \mathcal{A}^{-1}$, *then*

$$\max\{\|b^{\oplus} - a^{\oplus}\|, \|b_{\oplus} - a_{\oplus}\|\} \leq \frac{\|b^{\dagger} - a^{\dagger}\|}{\cos\psi_b} + \frac{\|a^{\dagger}\|\left(\|b^{\dagger}\| + \|a^{\dagger}\|\right)}{\cos\psi_a\cos\psi_b}\|a-b\|.$$

(iv) *In particular, if* $a \in \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$, $a \neq 0$ *and* $a \notin \mathcal{A}^{-1}$, *and* $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$, $a_n \neq 0$ *and* $a_n \notin \mathcal{A}^{-1}$, *for all* $n \in \mathbb{N}$, *then*

$$\max\{\|{a_n}^{\oplus} - a^{\oplus}\|, \|a_{n\oplus} - a_{\oplus}\|\} \leq \frac{\|a_n^{\dagger} - a^{\dagger}\|}{\cos\psi_n} + \frac{\|a^{\dagger}\|\left(\|a_n^{\dagger}\| + \|a^{\dagger}\|\right)}{\cos\psi_a\cos\psi_n}\|a-a_n\|,$$

where $\psi_n = \psi_{a_n}$.

Proof. To prove statement (i), observe that $b \neq 0$ (respectively $b \notin \mathcal{A}^{-1}$) if and only if $b^{\oplus}b \neq 0$ (respectively $b^{\oplus}b \neq \mathbb{1}$). Thus, according to [21, Lemma 2.3], Lemma 3.2 (iii) and [18, Theorem 2.4 (iii)],

$$\|\mathbb{1} - b^{\oplus}b\| = \|b^{\oplus}b\| = \|(b^{\oplus}b) + (b^{\oplus}b)^* - \mathbb{1}\| = \|(bb^{\dagger} + b^{\dagger}b - \mathbb{1})^{-1}\| = \frac{1}{\cos\psi_b}.$$

Note that since $\mathbb{1} - aa^{\oplus}$ is a self-adjoint idempotent, $\|\mathbb{1} - aa^{\oplus}\|$ is either 0 or 1. Hence $\|\mathbb{1} - aa^{\oplus}\| \leq 1$, and according to Lemma 3.9,

$$\begin{aligned}\|b^{\oplus} - a^{\oplus}\| &\leq \|b^{\oplus}b\|\|b^{\dagger} - a^{\dagger}\|\|\mathbb{1} - aa^{\oplus}\| + \left[\|b^{\oplus}\|\|a^{\oplus}\| + \|\mathbb{1} - b^{\oplus}b\|\|a^{\dagger}a^{\oplus}\|\right]\|a-b\| \\ &\leq \frac{\|b^{\dagger} - a^{\dagger}\|}{\cos\psi_b} + \left[\|b^{\oplus}\|\|a^{\oplus}\| + \frac{\|a^{\dagger}a^{\oplus}\|}{\cos\psi_b}\right]\|a-b\| \\ &\leq \frac{\|b^{\dagger} - a^{\dagger}\|}{\cos\psi_b} + \left[\|b^{\oplus}\| + \frac{\|a^{\dagger}\|}{\cos\psi_b}\right]\|a^{\oplus}\|\|a-b\|.\end{aligned}$$

Statement (ii) can be derived from statement (i). In fact, recall that given $x \in \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$, $x_{\oplus} = ((x^*)^{\oplus})^*$. Moreover, if $x \in \mathcal{A}^{\dagger} \setminus \{0\}$, then note that $\psi_{x^*} = \psi_{x^{\dagger}} = \psi_x$. Now apply statement (i) to a^* and b^*.

To prove statement (iii), first observe that if $a = 0$ in statement (i), then $\|b^{\oplus}\| \leq \frac{\|b^{\dagger}\|}{\cos\psi_b}$. Thus, if $a \neq 0$ and $a \notin \mathcal{A}^{-1}$, then statement (i) applied to $b = 0$ and a implies that $\|a^{\oplus}\| \leq \frac{\|a^{\dagger}\|}{\cos\psi_a}$. However, if these inequalities are applied to statement (i), then it is not difficult to prove that

$$\|b^{\oplus} - a^{\oplus}\| \leq \frac{\|b^\dagger - a^\dagger\|}{\cos\psi_b} + \frac{\|a^\dagger\|\left(\|b^\dagger\| + \|a^\dagger\|\right)}{\cos\psi_a \cos\psi_b}\|a-b\|.$$

Using statement (ii), a similar argument proves that $\|a_{\oplus}\| \leq \frac{\|a^\dagger\|}{\cos\psi_a}$, $\|b_{\oplus}\| \leq \frac{\|b^\dagger\|}{\cos\psi_b}$ and

$$\|b_{\oplus} - a_{\oplus}\| \leq \frac{\|b^\dagger - a^\dagger\|}{\cos\psi_b} + \frac{\|a^\dagger\|\left(\|b^\dagger\| + \|a^\dagger\|\right)}{\cos\psi_a \cos\psi_b}\|a-b\|.$$

Statement (iv) can be derived from statement (iii). □

Remark 3. (i) As it was used in the proof of Theorem 3.10, given $a \in \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus}$, $a \neq 0$ and $a \notin \mathcal{A}^{-1}$, Theorem 3.10 (i) (respectively Theorem 3.10 (ii)) gives a relationship between the norm of $a^{\oplus}$ (respectively of $a_{\oplus}$) and the norm of $a^\dagger$: $\|a^{\oplus}\| \leq \frac{\|a^\dagger\|}{\cos\psi_a}$ (respectively $\|a_{\oplus}\| \leq \frac{\|a^\dagger\|}{\cos\psi_a}$).
(ii) In addition, under the same hypotheses of Theorem 3.3 and Theorem 3.10 (iv), the latter result gives an estimate of the convergence of $(a_n{}^{\oplus})_{n\in\mathbb{N}}$ and $(a_{n\oplus})_{n\in\mathbb{N}}$ to $a^{\oplus}$ and $a_{\oplus}$, respectively.
(iii) Moreover, in Theorem 3.10 (i), when $aa^{\oplus} = 1$, equivalently when $a \in \mathcal{A}^{-1}$, according to the proof of this statement, it is not difficult to prove that

$$\|b^{\oplus} - a^{\oplus}\| \leq \left[\|b^{\oplus}\|\|a^{-1}\| + \frac{\|a^{-2}\|}{\cos\psi_b}\right]\|a-b\|.$$

(iv) Similarly, in Theorem 3.10 (ii), when $a \in \mathcal{A}^{-1}$, the following inequality can be proved:

$$\|b_{\oplus} - a_{\oplus}\| \leq \left[\|b_{\oplus}\|\|a^{-1}\| + \frac{\|a^{-2}\|}{\cos\psi_b}\right]\|a-b\|.$$

4. Continuity of (dual) core invertible Hilbert space operators

Let $\mathcal{H}$ be a Hilbert space and consider $A \in \mathcal{L}(\mathcal{H})$. The definition of core invertible Hilbert space operators was given in [3, Definition 3.2]. In fact, $A \in \mathcal{L}(\mathcal{H})$ is said to be *core invertible*, if there exists $X \in \mathcal{L}(\mathcal{H})$ such that

$$A = AXA, \qquad \mathcal{R}(X) = \mathcal{R}(A), \qquad \mathcal{N}(X) = \mathcal{N}(A^*).$$

Thus, when $A \in \mathcal{L}(\mathcal{H})$, two definitions of the core inverse of A has been given: as an element of the C^*-algebra $\mathcal{L}(\mathcal{H})$ and as Hilbert space operator. However, as the following proposition shows, both definitions coincide in the Hilbert space context.

Proposition 4.1. *Let $\mathcal{H}$ be a Hilbert space and consider $A \in \mathcal{L}(\mathcal{H})$. The following statements are equivalent.*

(i) *The core inverse of A exists.*
(ii) *There exists an operator $X \in \mathcal{L}(\mathcal{H})$ such that $AXA = A$, $\mathcal{R}(X) = \mathcal{R}(A)$ and $\mathcal{N}(X) = \mathcal{N}(A^*)$.*

Moreover, in this case $X = A^{\oplus} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}$.

Proof. Suppose that $A^{\oplus}$ exists. Then, $A = AA^{\oplus}A$ and there are operator S, T, U, $V \in \mathcal{L}(\mathcal{H})$ such that

$$A^{\oplus} = AS, \qquad A = A^{\oplus}T, \qquad A^{\oplus} = UA^*, \qquad A^* = VA^{\oplus}.$$

In particular, $\mathcal{R}(A^{\oplus}) = \mathcal{R}(A)$ and $\mathcal{N}(A^{\oplus}) = \mathcal{N}(A^*)$.

Now suppose that statement (ii) holds. Then, there exists $X \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{R}(X) = \mathcal{R}(A)$. According to [23, Theorem 1], there are L, $K \in \mathcal{L}(\mathcal{H})$ such that $A = XL$ and $X = AK$. In particular, $X\mathcal{L}(\mathcal{H}) = A\mathcal{L}(\mathcal{H})$. In addition, since $\mathcal{R}(A)$ is closed, $\mathcal{R}(X)$ is closed, which is equivalent to the fact that X is regular. Now since A^* is regular, according to [24, Remark 6], there exist operators M, $N \in \mathcal{L}(\mathcal{H})$ such that $X = MA^*$ and $A^* = NX$. In particular, $\mathcal{L}(\mathcal{H})X = \mathcal{L}(\mathcal{H})A^*$. Since $A = AXA$ and the core inverse is unique, when it exists ([2, Theorem 2.14]), $X = A^{\oplus}$. Finally, since according again to [2, Theorem 2.14], $A^{\oplus}$ is an outer inverse, according to what has been proved, $A^{\oplus} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}$. □

As for the core inverse case, a definition of dual core invertible Hilbert space operators was given in [3, Definition 3.3]. In the following proposition the equivalence between Definition 2.2 and [3, Definition 3.3] will be considered.

Proposition 4.2. *Let* $\mathcal{H}$ *be a Hilbert space and consider* $A \in \mathcal{L}(\mathcal{H})$. *The following statements are equivalent.*

(i) *The dual core inverse of* A *exists.*
(ii) *There exists an operator* $X \in \mathcal{L}(\mathcal{H})$ *such that* $AXA = A$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) = \mathcal{N}(A)$.

Moreover, in this case $X = A_{\oplus} = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A)}$.

Proof. Apply a similar argument to the one used in Proposition 4.1. □

Note that the relationship between the (dual) core inverse and the outer inverse with prescribed range and null space for the case of square complex matrices was studied in [25, Theorem 1.5] (apply [2, Theorem 4.4]).

Next, the continuity of the (dual) core inverse will be characterized using the gap between subspaces. The next theorem is the Hilbert space version of Theorem 3.8.

Theorem 4.3. *Let* $\mathcal{H}$ *be a Hilbert space and consider* $A \in \mathcal{L}(\mathcal{H})$, $A \neq 0$, *such that* A *is (dual) core invertible. Suppose that there exists a sequence of operators* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ *such that for each* $n \in \mathbb{N}$, A_n *is (dual) core invertible and* $(A_n)_{n\in\mathbb{N}}$ *converges to* A. *Then, the following statements are equivalent.*

(i) *The sequence* $(A_n{}^{\oplus})_{n\in\mathbb{N}}$ *converges to* $A^{\oplus}$.
(ii) *The sequence* $(A_{n\oplus})_{n\in\mathbb{N}}$ *converges to* $A_{\oplus}$.
(iii) *The sequence* $(A_n{}^{\oplus}A_n)_{n\in\mathbb{N}}$ *converges to* $A^{\oplus}A$.
(iv) *The sequence* $(A_nA_{n\oplus})_{n\in\mathbb{N}}$ *converges to* $A^{\oplus}A$.
(v) *The sequence* $(\widehat{\delta}\,(\mathcal{R}(A_n{}^{\oplus}), \mathcal{R}(A^{\oplus})))_{n\in\mathbb{N}}$ *converges to 0.*
(vi) *The sequence* $(\widehat{\delta}\,(\mathcal{R}(A_n), \mathcal{R}(A)))_{n\in\mathbb{N}}$ *converges to 0.*
(vii) *The sequence* $(\widehat{\delta}\,(\mathcal{N}(A_n{}^{\oplus}), \mathcal{N}(A^{\oplus})))_{n\in\mathbb{N}}$ *converges to 0.*
(viii) *The sequence* $(\widehat{\delta}\,(\mathcal{N}(A_n^*), \mathcal{N}(A^*)))_{n\in\mathbb{N}}$ *converges to* 0.

(ix) *The sequence* $(\widehat{\delta}\,(\mathcal{R}(A_{n\oplus}), \mathcal{R}(A_{\oplus})))_{n\in\mathbb{N}}$ *converges to 0.*
(x) *The sequence* $(\widehat{\delta}\,(\mathcal{R}(A_n^*), \mathcal{R}(A^*)))_{n\in\mathbb{N}}$ *converges to 0.*
(xi) *The sequence* $(\widehat{\delta}\,(\mathcal{N}(A_{n\oplus}), \mathcal{N}(A_{\oplus})))_{n\in\mathbb{N}}$ *converges to 0.*
(xii) *The sequence* $(\widehat{\delta}\,(\mathcal{N}(A_n), \mathcal{N}(A)))_{n\in\mathbb{N}}$ *converges to* 0.

Proof. First of all recall that $\mathcal{L}(\mathcal{H})^{\oplus} = \mathcal{L}(\mathcal{H})_{\oplus}$ (Theorem 3.1).

Statements (i)-(iv) are equivalent (Theorem 3.8). According to [7, Lemma 3.3], statement (iii) implies statement (v) and according to Proposition 4.1 and [17, Chapter 4, Section 2, Subsection 3, Theorem 2.9], Statements (v)-(viii) are equivalent.

Now suppose that statement (vi) holds. Thus, according to what has been proved, the sequences $(\widehat{\delta}\,(\mathcal{R}(A_n), \mathcal{R}(A)))_{n\in\mathbb{N}}$ and $(\widehat{\delta}\,(\mathcal{N}(A_n^*), \mathcal{N}(A^*)))_{n\in\mathbb{N}}$ converge to 0 (recall that according to [17, Chapter 4, Section 2, Subsection 3, Theorem 2.9], $\widehat{\delta}\,(\mathcal{R}(A_n), \mathcal{R}(A)) = \widehat{\delta}\,((\mathcal{N}(A_n^*), \mathcal{N}(A^*)),\ n \in \mathbb{N})$. In addition, according to Proposition 4.1, for each $n \in \mathbb{N}$,

$$A_n^{\oplus} = (A_n)^{(2)}_{\mathcal{R}(A_n),\mathcal{N}(A_n^*)}, \qquad A^{\oplus} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}.$$

Let $\kappa = \|A\|\|A^{\oplus}\|$ and consider $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\begin{aligned} w_n &= \widehat{\delta}\left(\mathcal{N}((A_n)^{(2)}_{\mathcal{R}(A_n),\mathcal{N}(A_n^*)}), \mathcal{N}(A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)})\right) = \widehat{\delta}\,((\mathcal{N}(A_n^*), \mathcal{N}(A^*)) \\ &= \widehat{\delta}\,(\mathcal{R}(A_n), \mathcal{R}(A)) = \widehat{\delta}\left(\mathcal{R}((A_n)^{(2)}_{\mathcal{R}(A_n),\mathcal{N}(A_n^*)}), \mathcal{R}(A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)})\right) < \frac{1}{(3+\kappa)^2} \end{aligned}$$

and

$$z_n = \|A^{\oplus}\|\|A - A_n\| < \frac{2\kappa}{(1+\kappa)(4+\kappa)}.$$

Since $\frac{1}{(3+\kappa)^2} \leq \min\{\frac{1}{3+\kappa}, \frac{1}{(1+\kappa)^2}\}$, according to [16, Theorem 3.5],

$$\|A_n^{\oplus} - A^{\oplus}\| \leq \frac{2(1+\kappa)w_n + (1+w_n)z_n}{1-(1+2\kappa)w_n - (1+w_n)z_n}\|A^{\oplus}\|,$$

which implies statement (i).

Now, according to [7, Lemma 3.3], statement (iv) implies statement (xi) and according to Proposition 4.2 and [17, Chapter 4, Section 2, Subsection 3, Theorem 2.9], Statements (ix)-(xii) are equivalent.

Suppose that statement (x) holds. Since then statement (xii) also holds, to prove that statement (ii) holds, it is enough to apply an argument similar to the one used to prove that statement (vi) implies statement (i), interchanging in particular A with A^*, A_n with A_n^*, $A^{\oplus}$ with $A_{\oplus}$, $A_n{}^{\oplus}$ with $A_{n\oplus}$, $(A_n)^{(2)}_{\mathcal{R}(A_n),\mathcal{N}(A_n^*)}$ with $(A_n)^{(2)}_{\mathcal{R}(A_n^*),\mathcal{N}(A_n)}$, $A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}$ with $A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A)}$, and κ with $\kappa' = \|A\|\|A_{\oplus}\|$. □

Next, the continuity of the (dual) core inverse will be studied in a particular case. To this end, two results from [15] need to be extended first.

Proposition 4.4. *Let* $\mathcal{X}$ *be a Banach space and consider* $A \in \mathcal{L}(\mathcal{X})$ *such that* A *is group invertible and the codimension of* $\mathcal{R}(A)$ *is finite. Suppose that there exists a* *sequence of operators* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{X})$ *such that for each* $n \in \mathbb{N}$, A_n *is group invertible and* $(A_n)_{n\in\mathbb{N}}$ *converges to* A. *Then the following statements are equivalent.*

(i) *The sequence* $(A_n^{\#})_{n\in\mathbb{N}}$ *converges to* $A^{\#}$.
(ii) *For all sufficiently large* $n \in \mathbb{N}$, $\operatorname{codim} \mathcal{R}(A_n) = \operatorname{codim} \mathcal{R}(A)$.

Proof. Recall that $A \in \mathcal{L}(\mathcal{X})$ is group invertible if and only if $A^* \in \mathcal{L}(\mathcal{X}^*)$ is group invertible. A similar statement holds for each $A_n \in \mathcal{L}(\mathcal{X})$ $(n \in \mathbb{N})$. In addition, $\dim \mathcal{N}(A^*)$ is finite and $(A_n^*)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{X}^*)$ converges to A^*. Thus, according to [15, Theorem 3], statement (i) is equivalent to the fact that for all sufficiently large $n \in \mathbb{N}$, $\dim \mathcal{N}(A_n^*) = \dim \mathcal{N}(A^*)$, which in turn is equivalent to statement (ii). □

Proposition 4.5. *Let* $\mathcal{H}$ *be a Hilbert space and consider* $A \in \mathcal{L}(\mathcal{H})$ *such that* A *is Moore-Penrose invertible and the codimension of* $\mathcal{R}(A)$ *is finite. Suppose that there exists a sequence of operators* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ *such that for each* $n \in \mathbb{N}$, A_n *is Moore-Penrose invertible and* $(A_n)_{n\in\mathbb{N}}$ *converges to* A. *Then the following statements are equivalent.*

(i) *The sequence* $(A_n^{\dagger})_{n\in\mathbb{N}}$ *converges to* $A^{\dagger}$.
(ii) *For all sufficiently large* $n \in \mathbb{N}$, $\operatorname{codim} \mathcal{R}(A_n) = \operatorname{codim} \mathcal{R}(A)$.

Proof. Apply a similar argument to the one in the proof of Proposition 4.4, using in particular [15, Corollary 10] instead of [15, Theorem 3]. □

Corollary 4.6. *Let* $\mathcal{H}$ *be a Hilbert space and consider* $A \in \mathcal{L}(\mathcal{H})$ *such that* A *is group invertible and either the codimension of* $\mathcal{R}(A)$ *is finite or* $\dim \mathcal{N}(A)$ *is finite. Suppose that there exists a sequence of operators* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ *such that for each* $n \in \mathbb{N}$, A_n *is group invertible and* $(A_n)_{n\in\mathbb{N}}$ *converges to* A. *Then, the following statements are equivalent.*

(i) *The sequence* $(A_n^{\#})_{n\in\mathbb{N}}$ *converges to* $A^{\#}$.
(ii) *The sequence* $(A_n^{\dagger})_{n\in\mathbb{N}}$ *converges to* $A^{\dagger}$.

Proof. Recall that given and operator $S \in \mathcal{L}(\mathcal{H})$ such that S is group invertible, then S is Moore-Penrose invertible (Theorem 2.1). To conclude the proof apply, when $\dim \mathcal{N}(A)$ is finite, [15, Theorem 3] and [15, Corollary 10], and when codimension of $\mathcal{R}(A)$ is finite, Proposition 4.4 and Proposition 4.5. □

Now a characterization of the continuity of the (dual) core inverse for a particular case of Hilbert spaces operators will be presented.

Theorem 4.7. *Let* $\mathcal{H}$ *be a Hilbert space and consider* $A \in \mathcal{L}(\mathcal{H})$ *such that* A *is (dual) core invertible and either the codimension of* $\mathcal{R}(A)$ *is finite or* $\dim \mathcal{N}(A)$ *is finite. Suppose that there exists a sequence of operators* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ *such that for each* $n \in \mathbb{N}$, A_n *is (dual) core invertible and* $(A_n)_{n\in\mathbb{N}}$ *converges to* A. *The following statements are equivalent.*

(i) *The sequence* $(A_n^{\oplus})_{n\in\mathbb{N}}$ *converges to* $A^{\oplus}$.
(ii) *The sequence* $(A_{n\oplus})_{n\in\mathbb{N}}$ *converges to* $A_{\oplus}$.
(iii) *The sequence* $(A_n^{\dagger})_{n\in\mathbb{N}}$ *converges to* $A^{\dagger}$.

When $\dim \mathcal{N}(A)$ *is finite, statements* (i)-(iii) *are equivalent to the following statement.*

(iv) *For all sufficiently large* $n \in \mathbb{N}$, $\dim \mathcal{N}(A_n) = \dim \mathcal{N}(A)$.

When $\operatorname{codim}\mathcal{R}(A)$ *is finite, statements* (i)-(iii) *are equivalent to the following statement.*

(v) *For all sufficiently large* $n \in \mathbb{N}$, $\operatorname{codim}\mathcal{R}(A_n) = \operatorname{codim}\mathcal{R}(A)$.

Proof. Apply Theorem 3.3, Corollary 4.6, [15, Theorem 3] and Proposition 4.4. For the case $A = 0$, apply Remark 1 and Proposition 3.7. □

Now the finite dimensional case will be derived from Theorem 4.7. It is worth noticing that the following corollary also provides a different proof of a well known result concerning the continuity of the Moore-Penrose inverse in the matricial setting, see [26, Theorem 5.2].

Corollary 4.8. *Let* $A \in M_m(\mathbb{C})$ *be a (dual) core invertible matrix. Suppose that exists a sequence* $(A_n)_{n\in\mathbb{N}} \subset M_m(\mathbb{C})$ *of (dual) core invertible matrices such that* $(A_n)_{n\in\mathbb{N}}$ *converges to* A*. The following statements are equivalent.*

(i) *The sequence* $(A_n^{\oplus})_{n\in\mathbb{N}}$ *converges to* $A^{\oplus}$*.*
(ii) *The sequence* $(A_{n\oplus})_{n\in\mathbb{N}}$ *converges to* $A_{\oplus}$*.*
(iii) *The sequence* $(A_n^{\dagger})_{n\in\mathbb{N}}$ *converges to* $A^{\dagger}$*.*
(iv) *There exists* $n_0 \in \mathbb{N}$ *such that* $\operatorname{rk}(A_n) = \operatorname{rk}(A)$*, for* $n \geq n_0$*.*

Proof. Apply Theorem 4.7. □

5. Differentiability of the (dual) core inverse

To prove the main results of this section, some preparation is needed.

Let $U \subseteq \mathbb{R}$ be an open set and consider $\mathbf{a}\colon U \to \mathcal{A}$ a function such that $\mathbf{a}(U) \subseteq \mathcal{A}^{\oplus}$. Since according to Theorem 3.1, $\mathcal{A}^{\oplus} = \mathcal{A}_{\oplus} = \mathcal{A}^{\#} \subset \mathcal{A}^{\dagger}$, it is possible to consider the functions

$$\mathbf{a}^{\oplus}, \mathbf{a}_{\oplus}, \mathbf{a}^{\#}, \mathbf{a}^{\dagger}\colon U \to \mathcal{A},$$

which are defined as follows. Given $u \in U$,

$$\mathbf{a}^{\oplus}(u) = (\mathbf{a}(u))^{\oplus}, \qquad \mathbf{a}_{\oplus}(u) = (\mathbf{a}(u))_{\oplus},$$
$$\mathbf{a}^{\#}(u) = (\mathbf{a}(u))^{\#}, \qquad \mathbf{a}^{\dagger}(u) = (\mathbf{a}(u))^{\dagger}.$$

Since in this section functions instead of sequences will be considered and the notion of continuity will be central in the results concerning differentiability, Theorem 3.3 will be reformulated for functions.

Theorem 5.1. *Let* $\mathcal{A}$ *be a unital* C^**-algebra and consider* $U \subseteq \mathbb{R}$ *an open set and a function* $\mathbf{a}\colon U \to \mathcal{A}$ *such that* $\mathbf{a}(U) \subseteq \mathcal{A}^{\oplus}$ *and* $\mathbf{a}$ *is continuous at* $t_0 \in U$*. The following statements are equivalent.*

(i) *The element* $\mathbf{a}(t_0) \in \mathcal{A}^{\oplus}$ *and the function* $\mathbf{a}^{\oplus}$ *is continuous at* t_0*.*
(ii) *The element* $\mathbf{a}(t_0) \in \mathcal{A}_{\oplus}$ *and the function* $\mathbf{a}_{\oplus}$ *is continuous at* t_0*.*
(iii) *The element* $\mathbf{a}(t_0) \in \mathcal{A}^{\#}$ *and the function* $\mathbf{a}^{\#}$ *is continuous at* t_0*.*
(iv) *The element* $\mathbf{a}(t_0) \in \mathcal{A}^{\oplus}$ *and there exists an open set* $V \subseteq U$ *such that* $t_0 \in V$ *and the function* $\mathbf{a}^{\oplus}$ *is bounded on* V*.*

(v) *The element* $\mathbf{a}(t_0) \in \mathcal{A}_{\oplus}$ *and there exists an open set* $W \subseteq U$ *such that* $t_0 \in W$ *and the function* $\mathbf{a}_{\oplus}$ *is bounded on* W.
(vi) *The element* $\mathbf{a}(t_0) \in \mathcal{A}^{\dagger}$, *the function* $\mathbf{a}^{\dagger}$ *is continuous at* t_0, *and there exists and open set* $I \subseteq U$ *such that* $t_0 \in I$ *and the function* $\mathbf{a}^{\oplus}\mathbf{a}$ *is bounded on* I.
(vii) *The element* $\mathbf{a}(t_0) \in \mathcal{A}^{\dagger}$, *the function* $\mathbf{a}^{\dagger}$ *is continuous at* t_0 , *and there exists and open set* $J \subseteq U$ *such that* $t_0 \in J$ *and the function* $\mathbf{a}\mathbf{a}^{\oplus}$ *is bounded on* J.
(viii) *The element* $\mathbf{a}(t_0) \in \mathcal{A}^{\dagger}$, *the function* $\mathbf{a}^{\dagger}$ *is continuous at* t_0, *and there exist an open set* Z *such that* $t_0 \in Z$*and* $\psi \in [0, \frac{\pi}{2})$ *such that when* $\mathbf{a}(t) \neq 0$ *(*$t \in Z$*),* $\psi_t = \psi_{\mathbf{a}(t)} \leq \psi$.

Proof. Apply Theorem 3.3. □

Remark 4. Note that under the same hypotheses of Theorem 5.1, when $\mathbf{a}(t_0) = 0$, the continuity of the function $\mathbf{a}^{\oplus}$ (respectively $\mathbf{a}_{\oplus}$, $\mathbf{a}^{\#}$, $\mathbf{a}^{\dagger}$) at t_0 is equivalent to the following condition: there exists an open set $K \subseteq U$, $t_0 \in K$ and $\mathbf{a}(t) = 0$, for all $t \in K$ (Remark 1, Proposition 3.7).

To study the differentiability of the (dual) core inverse, the differentiability of the Moore-Penrose inverse need to be considered first.

Remark 5. Let $\mathcal{A}$ be a unital C^*-algebra and consider an open set U and $\mathbf{a}\colon U \to \mathcal{A}$ a function such that $\mathbf{a}(U) \subset \mathcal{A}^{\dagger}$ and there is t_0 such that $\mathbf{a}$ is differentiable at t_0. Thus, a necessary and sufficient condition for $\mathbf{a}^{\dagger}$ to be differentiable at t_0 is that $\mathbf{a}^{\dagger}$ is continuous at t_0. In fact, if $\mathbf{a}(t_0) \neq 0$, there is an open set $V \subseteq U$ such that $t_0 \in V$ and $\mathbf{a}(t) \neq 0$ for $t \in V$, and then according to [20, Theorem 2.1], this equivalence holds. On the other hand, if $\mathbf{a}(t_0) = 0$, according to Remark 1 (vi)-(vii), the function $\mathbf{a}^{\dagger}$ is continuous at t_0 if and only if there exists an open set W such that $t_0 \in W$ and $\mathbf{a}(t) = 0$ for $t \in W$, which implies that $\mathbf{a}^{\dagger}$ is differentiable at t_0. As a result, in [20, Theorem 2.1] it is not necessary to assume that $\mathbf{a}(t) \neq 0$ for t in a neighbourhood of t_0.

In the following theorem the differentiability of the (dual) core inverse will be studied. Note that the following notation will be used. Given a unital C^*-algebra $\mathcal{A}$, if $U \subseteq \mathbb{R}$ is an open set and $\mathbf{b}\colon U \to \mathcal{A}$ is a function, then $\mathbf{b}^*\colon U \to \mathcal{A}$ will denote the function $\mathbf{b}^*(t) = (\mathbf{b}(t))^*$ ($t \in U$). In addition, if $\mathbf{b}\colon U \to \mathcal{A}$ is differentiable at $t_0 \in U$, then $\mathbf{b}'(t_0)$ will stand for the derivative of $\mathbf{b}$ at t_0.

Theorem 5.2. *Let* $\mathcal{A}$ *be a unital* C^**-algebra and consider* $U \subseteq \mathbb{R}$ *an open set and* $\mathbf{a}\colon U \to \mathcal{A}$ *a function that is differentiable at* $t_0 \in U$ *and* $\mathbf{a}(U) \subset \mathcal{A}^{\oplus} = \mathcal{A}_{\oplus} = \mathcal{A}^{\#}$. *The following statements are equivalent.*

(i) *The function* $\mathbf{a}^{\oplus}$ *is continuous at* t_0.
(ii) *The function* $\mathbf{a}^{\oplus}$ *is differentiable at* t_0.
(iii) *The function* $\mathbf{a}_{\oplus}$ *is differentiable at* t_0.
(iv) *The function* $\mathbf{a}^{\#}$ *is differentiable at* t_0.

Furthermore, the following formulas hold.

(v)

$$\begin{aligned}(\mathbf{a}^{\oplus})'(t_0) &= \mathbf{a}^{\oplus}(t_0)\mathbf{a}(t_0)(\mathbf{a}^{\dagger})'(t_0)(\mathbb{1} - \mathbf{a}(t_0)\mathbf{a}^{\oplus}(t_0)) - \mathbf{a}^{\oplus}(t_0)\mathbf{a}'(t_0)\mathbf{a}^{\oplus}(t_0) \\ &\quad + (\mathbb{1} - \mathbf{a}^{\oplus}(t_0)\mathbf{a}(t_0))\mathbf{a}'(t_0)\mathbf{a}^{\dagger}(t_0)\mathbf{a}^{\oplus}(t_0).\end{aligned}$$

(vi)

$$(\mathbf{a}_{\oplus})'(t_0) = (\mathbb{1} - \mathbf{a}_{\oplus}(t_0)\mathbf{a}(t_0))(\mathbf{a}^{\dagger})'(t_0)\mathbf{a}(t_0)\mathbf{a}_{\oplus}(t_0) - \mathbf{a}_{\oplus}(t_0)\mathbf{a}'(t_0)\mathbf{a}_{\oplus}(t_0)$$
$$+ \mathbf{a}_{\oplus}(t_0)\mathbf{a}^{\dagger}(t_0)\mathbf{a}'(t_0)(\mathbb{1} - \mathbf{a}(t_0)\mathbf{a}_{\oplus}(t_0)).$$

(vii)

$$\begin{aligned}(\mathbf{a}^{\#})'(t_0) &= (\mathbf{a}^{\oplus})'(t_0)\mathbf{a}^{\oplus}(t_0)\mathbf{a}(t_0) + \mathbf{a}^{\oplus}(t_0)(\mathbf{a}^{\oplus})'(t_0)\mathbf{a}(t_0) + (\mathbf{a}^{\oplus})^2(t_0)\mathbf{a}'(t_0)\\ &= \mathbf{a}'(t_0)(\mathbf{a}_{\oplus})^2(t_0) + \mathbf{a}(t_0)(\mathbf{a}_{\oplus})'(t_0)\mathbf{a}_{\oplus}(t_0) + \mathbf{a}(t_0)\mathbf{a}_{\oplus}(t_0)(\mathbf{a}_{\oplus})'(t_0)\\ &= (\mathbf{a}^{\oplus})'(t_0)\mathbf{a}(t_0)\mathbf{a}_{\oplus}(t_0) + \mathbf{a}^{\oplus}(t_0)\mathbf{a}'(t_0)\mathbf{a}_{\oplus}(t_0) + \mathbf{a}^{\oplus}(t_0)\mathbf{a}(t_0)(\mathbf{a}_{\oplus})'(t_0).\end{aligned}$$

Proof. According to Lemma 3.9,

$$\begin{aligned}\mathbf{a}^{\oplus}(t) - \mathbf{a}^{\oplus}(t_0) = {}& \mathbf{a}^{\oplus}(t)\mathbf{a}(t)(\mathbf{a}^{\dagger}(t) - \mathbf{a}^{\dagger}(t_0))(\mathbb{1} - \mathbf{a}(t_0)\mathbf{a}^{\oplus}(t_0))\\ &+ \mathbf{a}^{\oplus}(t)(\mathbf{a}(t_0) - \mathbf{a}(t))\mathbf{a}^{\oplus}(t_0)\\ &+ (\mathbb{1} - \mathbf{a}^{\oplus}(t)\mathbf{a}(t))(\mathbf{a}(t) - \mathbf{a}(t_0))\mathbf{a}^{\dagger}(t_0)\mathbf{a}^{\oplus}(t_0).\end{aligned}$$

Now suppose that statement (i) holds. According to Theorem 5.1, the function $\mathbf{a}^{\dagger}$ is continuous at t_0, and according to [20, Theorem 2.1] and Remark 5, the function $\mathbf{a}^{\dagger}$ is differentiable at t_0. Thus,

$$\frac{\mathbf{a}^{\oplus}(t)\mathbf{a}(t)(\mathbf{a}^{\dagger}(t) - \mathbf{a}^{\dagger}(t_0))(\mathbb{1} - \mathbf{a}(t_0)\mathbf{a}^{\oplus}(t_0))}{t - t_0}$$

converges to $\mathbf{a}^{\oplus}(t_0)\mathbf{a}(t_0)(\mathbf{a}^{\dagger})'(t_0)(\mathbb{1} - \mathbf{a}(t_0)\mathbf{a}^{\oplus}(t_0))$. In addition,

$$\frac{\mathbf{a}^{\oplus}(t)(\mathbf{a}(t_0) - \mathbf{a}(t))\mathbf{a}^{\oplus}(t_0)}{t - t_0}$$

converges to $-\mathbf{a}^{\oplus}(t_0)\mathbf{a}'(t_0)\mathbf{a}^{\oplus}(t_0)$, and

$$\frac{(\mathbb{1} - \mathbf{a}^{\oplus}(t)\mathbf{a}(t))(\mathbf{a}(t) - \mathbf{a}(t_0))\mathbf{a}^{\dagger}(t_0)\mathbf{a}^{\oplus}(t_0)}{t - t_0}$$

converges to $(\mathbb{1} - \mathbf{a}^{\oplus}(t_0)\mathbf{a}(t_0))\mathbf{a}'(t_0)\mathbf{a}^{\dagger}(t_0)\mathbf{a}^{\oplus}(t_0)$. Consequently statements (ii) and (v) hold.

It is evident that statement (ii) implies statement (i).

Now observe that the function $\mathbf{a}^*\colon U \to \mathcal{A}$ is differentiable at t_0 and $\mathbf{a}^*(U) \subset \mathcal{A}^{\oplus}$ (Theorem 3.1).

Suppose that statement (i) holds. According to the identity $(\mathbf{a}^*)^{\oplus}(t) = (\mathbf{a}_{\oplus})^*(t)$ and Theorem 5.1, the function $(\mathbf{a}^*)^{\oplus}\colon U \to \mathcal{A}$ is continuous at t_0. Thus, according to what has been proved, the function $(\mathbf{a}^*)^{\oplus}\colon U \to \mathcal{A}$ is differentiable at t_0. Therefore, the function $\mathbf{a}_{\oplus}\colon U \to \mathcal{A}_{\oplus}$ is differentiable at t_0. Consequently, statement (iii) holds. Furthermore, since $(\mathbf{a}_{\oplus})'(t_0) = (((\mathbf{a}^*)^{\oplus})')^*(t_0)$, to prove statement (vi), apply statement (v).

On the other hand, if statement (iii) holds, then the function $\mathbf{a}_{\oplus}$ is continuous at t_0. According to Theorem 5.1, statement (i) holds.

Suppose that statement (i) holds. According to Theorem 2.2 and Lemma 3.2 (iv), the following identities hold.

$$\mathbf{a}^{\#} = (\mathbf{a}^{\oplus})^2\mathbf{a} = \mathbf{a}(\mathbf{a}_{\oplus})^2 = \mathbf{a}^{\oplus}\mathbf{a}\mathbf{a}_{\oplus}.$$

Therefore, according to what has been proved, the function $\mathbf{a}^{\#}$ is differentiable at t_0. Furthermore, from these identities statement (vii) can be derived. In fact, using the formula for the differentiation of a product, if $\mathbf{a}^{\#} = (\mathbf{a}^{\oplus})^2\mathbf{a}$, then, given $t_0 \in U$,

$$(\mathbf{a}^{\#})'(t_0) = (\mathbf{a}^{\oplus})'(t_0)\mathbf{a}^{\oplus}(t_0)\mathbf{a}(t_0) + \mathbf{a}^{\oplus}(t_0)(\mathbf{a}^{\oplus})'(t_0)\mathbf{a}(t_0) + (\mathbf{a}^{\oplus})^2(t_0)\mathbf{a}'(t_0).$$

The other two formulas in statement (vii) can be proved in a similar way using the remaining two identities.

On the other hand, according to Theorem 5.1, statement (iv) implies statement (i). □

Remark 6. Under the same hypotheses of Theorem 5.2, the following facts should be noted.

(i) When $a(t_0) = 0$, according to Remark 4,

$$(\mathbf{a}^{\oplus})'(t_0) = (\mathbf{a}_{\oplus})'(t_0) = (\mathbf{a}^{\#})'(t_0) = (\mathbf{a}^{\dagger})'(t_0) = 0.$$

(ii) Recall that in [20, Theorem 2.1], a formula concerning the derivative of the function $\mathbf{a}^{\dagger}$ at t_0 was given.

(iii) Note that according to Theorem 5.1, a necessary and sufficient condition for the function $\mathbf{a}_{\oplus}$ (respectively $\mathbf{a}^{\#}$) to be differentiable at t_0 is that $\mathbf{a}_{\oplus}$ (respectively $\mathbf{a}^{\#}$) is continous at t_0. In fact, the continuity of one of the functions $\mathbf{a}^{\oplus}$, $\mathbf{a}_{\oplus}$ and $\mathbf{a}^{\#}$ at a point t_0 is equivalent to the continuity and the differentiability of the three functions under consideration at t_0 (Theorem 5.1 and Theorem 5.2).

(iv) According to Theorem 2.2,

$$\mathbf{a}^{\dagger} = \mathbf{a}_{\oplus}\mathbf{a}\mathbf{a}^{\oplus}, \qquad \mathbf{a}^{\oplus} = \mathbf{a}^{\#}\mathbf{a}\mathbf{a}^{\dagger}, \qquad \mathbf{a}_{\oplus} = \mathbf{a}^{\dagger}\mathbf{a}\mathbf{a}^{\#}.$$

Thus, the derivative of $\mathbf{a}^{\dagger}$, $\mathbf{a}^{\oplus}$ and $\mathbf{a}_{\oplus}$ at t_0 can also be computed as follows:

$$(\mathbf{a}^{\dagger})'(t_0) = (\mathbf{a}_{\oplus})'(t_0)\mathbf{a}(t_0)\mathbf{a}^{\oplus}(t_0) + \mathbf{a}_{\oplus}(t_0)\mathbf{a}'(t_0)\mathbf{a}^{\oplus}(t_0) + \mathbf{a}_{\oplus}(t_0)\mathbf{a}(t_0)(\mathbf{a}^{\oplus})'(t_0).$$
$$(\mathbf{a}^{\oplus})'(t_0) = (\mathbf{a}^{\#})'(t_0)\mathbf{a}(t_0)\mathbf{a}^{\dagger}(t_0) + \mathbf{a}^{\#}(t_0)\mathbf{a}'(t_0)\mathbf{a}^{\dagger}(t_0) + \mathbf{a}^{\#}(t_0)\mathbf{a}(t_0)(\mathbf{a}^{\dagger})'(t_0).$$
$$(\mathbf{a}_{\oplus})'(t_0) = (\mathbf{a}^{\dagger})'(t_0)\mathbf{a}(t_0)\mathbf{a}^{\#}(t_0) + \mathbf{a}^{\dagger}(t_0)\mathbf{a}'(t_0)\mathbf{a}^{\#}(t_0) + \mathbf{a}^{\dagger}(t_0)\mathbf{a}(t_0)(\mathbf{a}^{\#})'(t_0).$$

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