Document downloaded from:

http://hdl.handle.net/10251/161160

This paper must be cited as:

Hernández-Verón, MA.; Ibañez, M.; Martínez Molada, E.; Singh, S. (2020). Localization and separation of solutions for Fredholm integral equations. Journal of Mathematical Analysis and Applications. 487(2):1-16. https://doi.org/10.1016/j.jmaa.2020.124008

The final publication is available at

https://doi.org/10.1016/j.jmaa.2020.124008

Copyright Elsevier

Additional Information

Localization and separation of solutions for Fredholm integral equations

M.A. Hernández-Verón[a], María Ibáñez[b] Eulalia Martínez[c], Sukhjit Singh[d]

[a] Dept. of Mathematics and Computation,
University of La Rioja. Logroño, Spain.
[b] Facultat de Ciències Matemàtiques,
Universitat de València,. València, Spain.
[c]Instituto Universitario de Matemática Multidisciplinar,
Universitat Politècnica de València. València, Spain.
[d]Department of Mathematics,
Dr. B.R. Ambedkar National Institute of Technology, Jalandhar ,India
E-mail: mahernan@unirioja.es, mai5@alumni.uv.es, eumarti@mat.upv.es
sukhjitmath@gmail.com

Abstract

In this paper, we establish a qualitative study of nonlinear Fredholm integral equations, where we will carry out a study on the localization and separation of solutions. Moreover, we consider an efficient algorithm to approximate a solution. To do this, we study the semilocal convergence of an efficient third order iterative scheme for solving nonlinear Fredholm integral equations under mild conditions. The novelty of our work lies in the fact that this study involves first order Fréchet derivative and mild conditions. A numerical example involving nonlinear Fredholm integral equations, is solved to show the domains of existence and uniqueness of solutions. The applicability of the iterative scheme considered is also shown.

Keywords: Fredholm integral equation, two-steps Newton iterative scheme, domain of existence of solution, domain of uniqueness of solution, Lipschitz condition.
2010 Mathematics Subject Classification: 45G10, 47H99, 65H10, 65J15, 65G49.

This research was partially supported by Ministerio de Economía y Competitividad under grant MTM2014-52016-C2-1 and 2-P,

1 Introduction

In this paper, we consider the integral equations given by

$$x(s) = f(s) + \lambda \int_a^b K(s,t)[\mathcal{H}(x)](t)dt, \tag{1}$$

where $\mathcal{H}$ is a Nemystkii operator [7], $\mathcal{H} : \Omega \subseteq \mathcal{C}[a,b] \longrightarrow \mathcal{C}[a,b]$, with $[\mathcal{H}(x)](t) = H(x(t))$, being $H : \mathbb{R} \longrightarrow \mathbb{R}$ a derivable scalar function, $f : [a,b] \longrightarrow \mathbb{R}$ a continuous function and $K : [a,b] \times [a,b] \longrightarrow R$ a continuous function in both arguments. $\mathcal{C}[a,b]$ denotes the space of continuous real functions in $[a,b]$.

Notice that nonlinear integral equation (1) is a particular case of Fredholm integral equations [11, 13]. The Fredholm integral equations have strong physical background and arise from the electro-magnetic fluid dynamics. These equations appeared in the 30s of the twentieth century as general models for the study of semi-linear boundary value problems, where the kernel $K(s,t)$ typically arises as the Green function of a differential operator. Also, these equations are applied in the theory of radiative transfer and the theory of neutron transport as well as in the kinetic theory of gases. They also play a very significant role in several applications, as for example, the dynamic models of chemical reactors, which are governed by control equations, justifying then their study and solution.

As the Fredholm integral equations of form (1) cannot be solved exactly, we can use numerical methods to solve them. In fact, different numerical techniques can be applied and some of them mentioned in the references of this work. In particular, iterative schemes based on the homotopy analysis method in [3], adapted Newton-Kantorovich schemes in [8] and schemes based on a combination of the Newton-Kantorovich method and quadrature methods in [14].

If we pay attention to the iterative methods that can be applied for approximating a solution $x^* \in \mathcal{C}[a,b]$ of (1), the method of successive approximations play an important role (see, [1, 2, 15]). This method consists of applying the fixed point theorem to the equation

$$x(s) = F(x)(s), \tag{2}$$

with $F : \Omega \subseteq \mathcal{C}[a,b] \longrightarrow \mathcal{C}[a,b]$, where Ω is a nonempty convex domain in $\mathcal{C}[a,b]$, with

$$F(x)(s) = f(s) + \lambda \int_a^b K(s,t)[\mathcal{H}(x)](t)dt \tag{3}$$

and obtaining a sequence $\{x_{n+1} = F(x_n)\}_{n\in\mathbb{N}}$ that converges to a solution $x^* \in \mathcal{C}[a,b]$ of (1), i. e., a fixed point of F.

Two are the main aims of the paper. In first place, we perform a qualitative study of equation (1) by obtaining a result of existence and uniqueness of a solution for (1). In second place, a solution of (1) is successively approximated. Since the application of Fixed point Theorem is restrictive, F must be contractive from a domain to itself (see,

[12]) and besides the method of successive approximation converges slowly to a solution $x^* \in \mathcal{C}[a,b]$ of (1).

Observe that looking for a fixed point of equation (2) is equivalent to solving $G(x) = 0$, where $G : \Omega \subseteq \mathcal{C}([a,b]) \longrightarrow \mathcal{C}([a,b])$ and

$$G(x)(s) = x(s) - F(x)(s) = [(Id - F)(x)](s). \tag{4}$$

The methods for solving the previous equation are usually iterative schemes. So, starting from one initial approximation of a solution x^* of the equation $G(x) = 0$, a sequence $\{x_n\}$ of approximations is constructed such that the sequence $\{\|x_n - x_{n-1}\|\}$ is decreasing and a better approximation to the solution x^* is then obtained at every step. Obviously, the interest focuses on $\lim_n x_n = x^*$. The choice of an iterative scheme for approximating x^* usually depends on its efficiency [16], which links the speed of convergence (order of convergence) of the method to its computational cost.

In relation to the above, we can obtain the sequence of approximations $\{x_n\}$ by different ways, depending on the iterative schemes applied. Between these, the best-known is Newton's method, whose algorithm is the following:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [G'(x_n)]^{-1}G(x_n), \quad n = 0, 1, 2 \dots \end{cases} \tag{5}$$

If we consider one-point iterative schemes without memory, i.e., $x_{n+1} = \Lambda(x_n)$ with x_0 given in Ω, as Newton's method (5), it is known that their order of convergence ρ is a natural number and, moreover, the algorithm of these methods depend explicitly on the first $\rho - 1$ derivatives of the function involved in the equation. So, if we want consider iterative schemes with third order of convergence, the computational cost increases as it is necessary to evaluate the successive derivatives of the function involved in the algorithm of the method. Then, in this paper, we are interested in a numerical iterative scheme that avoid the expensive computation of the derivatives of the function G at each step, but third order of convergence is reached ([9, 10]). Therefore, in this paper, we consider the 2-steps iterative process with frozen first derivative given by the following algorithm:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ y_n = x_n - [G'(x_n)]^{-1}G(x_n) \\ x_{n+1} = y_n - [G'(x_n)]^{-1}G(y_n), n \geq 0, \end{cases}$$

It is well known that if we compose Newton's method with itself twice, but taking into account the derivative frozen, we obtain a iterative scheme of order three. This is a classical result obtained by Traub, [16]. Moreover, being an iterative scheme of third order, it does not increase the expensive computation of derivatives because this iterative scheme only uses the same first derivative in each step. For this, it is easy to check that this iterative scheme is more efficient than Newton's method [6].

So, in this paper, we consider an iterative scheme of fixed point type for approximating a fixed point of F. The algorithm of this iterative scheme is

$$\begin{cases} x_0 \text{ given in } \Omega, \\ y_n = x_n - [I - F'(x_n)]^{-1}(x_n - F(x_n)) \\ x_{n+1} = y_n - [I - F'(x_n)]^{-1}(y_n - F(y_n)), n \geq 0. \end{cases} \tag{6}$$

Notice that this iterative scheme is the frozen two steps Newton method [6] applied to the equation $G(x)(s) = x(s) - F(x)(s) = 0$.

In this paper, we obtain a semilocal convergence result for the iterative scheme (6) from which we will carry out the qualitative study for equation (1). Notice that a semilocal convergence result requires conditions on the operator involved, on the equation to solve and on the starting point of the iterative scheme, $x_0 \in \mathcal{C}[a, b]$. It provides the results on the existence of solution of the equation that allows us to obtain a domain of existence of solution. Moreover, we obtain a result of uniqueness of solution. On the other hand, for approximating a solution $x^* \in \mathcal{C}[a, b]$, the iterative scheme (6) is applied. As above, we have indicated that this iterative scheme has cubic convergence and is more efficient than Newton's method.

The work is organized as follows. In Section 2.1, we study the existence of the fixed point x^* of equation (4), obtaining recurrence relations for the sequences $\{x_n\}$ and $\{y_n\}$ of (6). In Section 2.2, the uniqueness of the fixed point is established, which implies the uniqueness of the solution of equation (1). Finally, in Section 3, we apply the results to a particular nonlinear Fredholm integral equation, obtaining convergence radii and numerical solutions. We also compare our results with the exact solution, obtaining some error estimates, and analyse results obtained by comparing with other numerical techniques.

2 A qualitative study: existence and uniqueness of solution

In what follows, we consider $F : \Omega \subseteq \mathcal{C}[a, b] \longrightarrow \mathcal{C}[a, b]$, where Ω is a nonempty convex domain in $\mathcal{C}[a, b]$, and the Nemytskii operator $\mathcal{H} : \Omega \subseteq \mathcal{C}[a, b] \longrightarrow \mathcal{C}[a, b]$ such that $\mathcal{H}(x)(s) = H(x(t))$. Obviously, it is a Frechet differentiable operator and then the operator

$$F(x)(s) = f(s) + \lambda \int_a^b K(s, t)H(x(t))dt$$

verifies

$$[F'(x)y](s) = \lambda \int_a^b K(s, t)[\mathcal{H}'(x)y](t)dt = \lambda \int_a^b K(s, t)H'(x(t))y(t)dt.$$

2.1 Existence and location of a solution for (1)

Now, to obtain a semilocal convergence result for (6), we assume that the following conditions are satisfied:

(I) $\Gamma_0 = [I - F^{'}(x_0)]^{-1}$ exists for some $x_0 \in \Omega \subseteq \mathcal{C}[a,b]$, with $\|\Gamma_0\| \leq \beta$, $\|\Gamma_0(x_0 - F(x_0))\| \leq \eta$.

(II) $\mathcal{H}'$ is a ω-Lipschitz continuous operator such that

$$\|\mathcal{H}^{'}(u) - \mathcal{H}^{'}(v)\| \leq \omega(\|u - v\|) \text{ for } u, v \in \Omega, \tag{7}$$

where $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and nondecreasing function satisfying $\omega(\alpha z) \leq \phi(\alpha)\omega(z)$ for $\alpha, z \in [0, +\infty)$ with $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a continuous and nondecreasing function.

As first step, from the previous conditions, we easily obtain the following result for the operator $F^{'}$.

Lemma 1. *$F^{'}$ is a ω-Lipschitz continuous operator in Ω such that*

$$\|F^{'}(u) - F^{'}(v)\| \leq |\lambda| M \omega(\|u - v\|) \ \textit{for} \ u, v \in \Omega,$$

with $M = \max_{s \in [a,b]} \left| \int_a^b K(s,t)dt \right|$.

***Proof*.** As,

$$[(F^{'}(u) - F^{'}(v))x](s) = \lambda \int_a^b K(s,t)[(\mathcal{H}^{'}(u) - \mathcal{H}^{'}(v))x](t)dt, \text{ for } x, u, v \in \Omega \subseteq \mathcal{C}[a,b],$$

thus Lemma 1 follows directly from (**II**). ■

As second step, denoting $\Gamma_n = [I - F^{'}(x_n)]^{-1}$, we prove the existence of these operators for each $n \in \mathbb{N}$ from the Banach Lemma [12].

Lemma 2. *Given $R \in \mathbb{R}_+$, if $x_n \in B(x_0, R) \subseteq \Omega$ and $\beta|\lambda|M\omega(R) < 1$, then $[I - F^{'}(x_n)]^{-1}$ exists and $\|[I - F^{'}(x_n)]^{-1}\| \leq \beta_R$, where*

$$\beta_R = \frac{\beta}{1 - \beta|\lambda|M\omega(R)}.$$

***Proof*.** Consider

$$\begin{aligned} \|I - \Gamma_0[I - F^{'}(x_n)]\| &\leq \|\Gamma_0\|\|F^{'}(x_n) - F^{'}(x_0)\| \\ &\leq \beta|\lambda|M\|\mathcal{H}^{'}(x_n) - \mathcal{H}^{'}(x_0)\| \leq \beta|\lambda|M\omega(R) < 1. \end{aligned}$$

Then, by means of Banach's Lemma, the result is obtained.

■

From now on, we denote $\theta(t) = \dfrac{\beta}{1-\beta|\lambda|M\omega(t)}$, and then $\beta_R = \theta(R)$. In what follows, we tested a technical lemma to subsequently obtain recurrence relations for the sequences $\{x_n\}$ and $\{y_n\}$.

Lemma 3. *If* $x_n, y_n \in B(x_0, R) \subseteq \Omega$, *then*

(i) $\|x_{n+1} - y_n\| \leq \psi_R(\|y_n - x_n\|)\|y_n - x_n\|$,

(ii) $\|x_{n+1} - x_n\| \leq \Big(1 + \psi_R(\|y_n - x_n\|)\Big)\|y_n - x_n\|$,

(iii) $\|x_{n+1} - F(x_{n+1})\| \leq |\lambda|\ M\ \Big(\omega(\|y_n - x_n\|) + Q\ \omega(\|x_{n+1} - y_n\|)\Big)\|x_{n+1} - y_n\|$,

(iv) $\|y_{n+1} - x_{n+1}\| \leq \chi_R(\|y_n - x_n\|, \|x_{n+1} - y_n\|)\|x_{n+1} - y_n\|$,

where $\psi_R(u) = \beta_R\ |\lambda|\ M\ Q\ \omega(u)$, $\chi_R(u,v) = \beta_R\ |\lambda|\ M\ \big(\omega(u) + Q\ \omega(v)\big)$ *and* $Q = \int_0^1 \phi(t)dt$.

Proof. Using the following identity:

$$y_n - F(y_n) = -\int_{x_n}^{y_n} (F'(\xi) - F'(x_n))d\xi,$$

we obtain, from Lemma 1, that

$$\|x_{n+1} - y_n\| \leq \|\Gamma_n\|\|y_n - F(y_n)\| \leq \beta_R\ |\lambda|\ M\ Q\ \omega(\|y_n - x_n\|)\|y_n - x_n\| = \psi_R(\|y_n - x_n\|)\|y_n - x_n\|,$$

so, (**i**) is proved.

On the other hand, to prove (**ii**), it is clear that
$\|x_{n+1} - x_n\| \leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \leq \Big(1 + \psi_R(\|y_n - x_n\|)\Big)\|y_n - x_n\|$.

Now, from identity

$$x_{n+1} - F(y_n) = F'(x_n)\left(x_{n+1} - y_n\right),$$

we consider

$$\begin{aligned} x_{n+1} - F(x_{n+1}) &= x_{n+1} - F(y_n) - (F(x_{n+1}) - F(y_n)) \\ &= F'(x_n)\left(x_{n+1} - y_n\right) - \int_0^1 \Big(F'\left(y_n + t\left(x_{n+1} - y_n\right)\right)\Big)(x_{n+1} - y_n)dt \\ &= \Big(F'(x_n) - F'(y_n)\Big)(x_{n+1} - y_n) \\ &\quad - \int_0^1 \Big(F'\left(y_n + t\left(x_{n+1} - y_n\right)\right) - F'\left(y_n\right)\Big)(x_{n+1} - y_n)dt. \end{aligned}$$

Taking norms on both sides, we get

$$\|x_{n+1} - F(x_{n+1})\| \leq \; |\lambda| \; M \; \Big(\omega(\|y_n - x_n\|) + Q \; \omega(\|x_{n+1} - y_n\|)\Big) \|x_{n+1} - y_n\|,$$

and (**iii**) is then proved.

Obviously, (**iv**) can be derived easily from (**iii**). ■

In what follows, we establish the recurrence relations for the sequences $\{x_n\}$ and $\{y_n\}$ using the lemmas proved previously.

For $n = 0$, condition (**I**) gives

$$\|y_0 - x_0\| = \|\Gamma_0(x_0 - F(x_0))\| \leq \eta.$$

Using Lemma 2 and Lemma 3 for $n = 0$, we get

$$\begin{aligned} \|x_1 - y_0\| &\leq \psi_R(\eta)\eta \\ \|x_1 - x_0\| &\leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq (1 + \psi_R(\eta))\eta \end{aligned}$$

So, we define the scalar parameters

$$\begin{aligned} r_0 &= \eta \\ s_0 &= \psi_R(r_0)r_0 \\ S &= 1 + \psi_R(r_0) \\ T &= \chi_R(r_0, s_0)\psi_R(r_0) \end{aligned}$$

For $n = 0$, Lemma 3 gives

$$\|y_1 - x_1\| \leq \chi_R(r_0, s_0)\psi_R(r_0)r_0 = Tr_0 = r_1.$$

If we take $T < 1$, then $r_1 < r_0$. Since $S > 1$, we have

$$\begin{aligned} \|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \leq Tr_0 + Sr_0 \\ &= (T + S)r_0 < (1 + T)Sr_0. \end{aligned}$$

On the other hand, as

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \tag{8}$$

and

$$\begin{aligned} \|x_2 - x_1\| &\leq \|x_2 - y_1\| + \|y_1 - x_1\| \\ &\leq (1 + \psi_R(\|y_1 - x_1\|)) \, \|y_1 - x_1\| \\ &\leq (1 + \psi_R(r_1)) \, \chi_R(r_0, s_0)\psi_R(r_0)r_0 \\ &\leq (1 + \psi_R(r_0)) \, \chi_R(r_0, s_0)\psi_R(r_0)r_0 = STr_0, \end{aligned} \tag{9}$$

using (8) and (9), we get

$$\|x_2 - x_0\| \le \big(STr_0 + Sr_0\big) = (1+T)Sr_0.$$

From the above results, we define the following scalar sequences:

$$\begin{aligned} r_n &= Tr_{n-1} \\ s_n &= \psi_R(r_n)r_n \end{aligned}$$

Clearly, if $T < 1$, r_n and s_n are decreasing scalar sequences and following results can be established.

Lemma 4. *If the equation*

$$t = \frac{1 + \theta(t)\ |\lambda|\ M\ Q\ \omega(\eta)}{1 - \theta(t)\ |\lambda|\ M\ \big(\omega(\eta) + Q\ \omega(\theta(t)\ |\lambda|\ M\ Q\ \omega(\eta)\eta)\big)\theta(t)\ |\lambda|\ M\ Q\ \omega(\eta)}\eta \tag{10}$$

has at least one positive real root and the smallest positive real root, denoted by R, satisfies $\beta\ |\lambda|\ M\ \omega(R) < 1$ and $B(x_0, R) \subseteq \Omega$, then

(a_n) $\|y_n - x_n\| \le r_n$ *and* $\|y_n - x_0\| \le S\big(1 + T + \ldots + T^n\big)r_0$,

(b_n) $\|x_{n+1} - y_n\| \le s_n$,

(c_n) $\|x_{n+1} - x_n\| \le ST^n r_0$ *and* $\|x_{n+1} - x_0\| \le \big(1 + T + \ldots T^n\big)Sr_0$,

(d_n) $x_n,\ y_n \in B(x_0, R)$.

Proof. In first place, notice that as we consider $R \in \mathbb{R}_+$, and $R = \dfrac{S}{1-T}\eta$, then $T < 1$.

We have already proved $(\mathbf{a_n}), (\mathbf{b_n}), (\mathbf{c_n})$ and $(\mathbf{d_n})$ for $n = 1$. In order to apply mathematical induction, assume that $(\mathbf{a_k}) - (\mathbf{c_k})$ holds for $k = 1, 2, \ldots, n$. Then, for $n + 1$, we have

$$\begin{aligned} \|y_{n+1} - x_{n+1}\| &\le \chi_R(\|y_n - x_n\|, \|x_{n+1} - y_n\|)\|x_{n+1} - y_n\| \\ &\le \chi_R(r_n, s_n)\psi_R(r_n)r_n = Tr_n = r_{n+1} \end{aligned}$$

and

$$\begin{aligned} \|y_{n+1} - x_0\| &\le \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_0\| \le Tr_n + (1 + T + \ldots + T^n)Sr_0 \\ &\le T^{n+1}r_0 + (1 + T + \ldots + T^n)Sr_0 < (1 + T + \ldots + T^{n+1})Sr_0 \\ &< \frac{S}{1-T}r_0 = R. \end{aligned}$$

Thus, $y_{n+1} \in B(x_0, R)$. Using Lemma 3, we get

$$\|x_{n+2} - y_{n+1}\| \leq \psi_R(\|y_{n+1} - x_{n+1}\|)\|y_{n+1} - x_{n+1}\| \leq \psi_R(r_{n+1})r_{n+1} = s_{n+1}.$$

Therefore,

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\leq \|x_{n+2} - y_{n+1}\| + \|y_{n+1} - x_{n+1}\| \leq (1 + \psi_R(r_{n+1}))\, r_{n+1} \\ &\leq (1 + \psi_R(r_0))\, r_{n+1} = S r_{n+1}. \end{aligned}$$

Now,

$$\begin{aligned} \|x_{n+2} - x_0\| &\leq \|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_0\| \\ &\leq S r_{n+1} + (1 + T + \ldots + T^n) S r_0 \\ &\leq (1 + T + \ldots + T^{n+1}) S r_0 \\ &< \frac{S}{1-T} r_0 = R. \end{aligned}$$

Hence $x_{n+2} \in B(x_0, R)$. ∎

Now, we establish the existence of the fixed point x^*.

Theorem 5. *Under the previous notations, let $F : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ be a nonlinear Fréchet differentiable operator given by $[F(x)](s) = f(s) - \lambda \int_a^b K(s,t)\mathcal{H}(x)(t)dt$. If the equation* (10) *has at least one positive real root and the smallest positive real root, denoted by R, satisfies $\beta\ |\lambda|\ M\ \omega(R) < 1$, $B(x_0, R) \subseteq \Omega$ and assumptions* (**I**) *and* (**II**) *hold, then, for the starting point x_0, method* (6) *converges to a fixed point x^* of* (2)*. Moreover, $x_n, y_n, x^* \in \overline{B(x_0, R)}$.*

Proof. To prove the convergence, it is sufficient to prove that $\{x_n\}$ is a Cauchy sequence. Using $T < 1$ and Lemma 3, we get

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \sum_{j=n}^{n+m-1} \|x_{j+1} - x_j\| \leq \sum_{j=n}^{n+m-1} S T^j r_0 \\ &\leq S r_0 \sum_{j=n}^{n+m-1} T^j \leq S r_0 \frac{T^n - T^{n+m}}{1-T}. \end{aligned} \tag{11}$$

Hence $\{x_n\}$ is a Cauchy sequence which converges to x^*. Taking $n = 0$ and $m \to \infty$ in (11), we get $\|x_0 - x^*\| \leq R$, and $x^* \in \overline{B(x_0, R)}$.

Now, we have to prove that x^* is a fixed point of (2). Consider $\|x_n - F(x_n)\| \leq \|I - F'(x_n)\|\|\Gamma_n(x_n - F(x_n))\| = \|I - F'(x_n)\|\|x_{n+1} - x_n\|$ and the operator $\{\|I - F'(x_n)\|\}$ is bounded. Taking $n \to \infty$ and from the continuity of the operator, we get that x^* is a solution of $x - F(x) = 0$ and therefore a fixed point of operator F. ∎

2.2 Uniqueness of solution for (1)

Observe that a fixed point of (2) is a solution of equation (1) and reciprocally. For this, we establish the uniqueness of the fixed point which proves the uniqueness of the solution of (1).

Theorem 6. *The fixed point of* (2) *is unique in* $B(x_0,\overline{R})\cap\Omega$*, with* $\overline{R}$ *being the biggest positive solution of the equation*

$$\beta\ |\lambda|\ M\int_0^1\omega(\tau\ t+(1-\tau)R)\,d\tau=1. \tag{12}$$

Proof. To show the uniqueness, we will proceed by reductio ad absurdum. So, we suppose that $y^*\in B(x_0,\overline{R})$ is another fixed point of (2) with $y^*\neq x^*$. Then

$$0=\Gamma_0(y^*-F(y^*))-\Gamma_0(x^*-F(x^*))=\int_0^1\Gamma_0\left(I-F'(x^*+\tau(y^*-x^*))\right)\,d\tau(y^*-x^*).$$

We are going to prove that, under hypothesis of theorem, A^{-1} exists, where A is a linear operator defined by

$$A=\int_0^1\Gamma_0\left(I-F'(x^*+\tau(y^*-x^*))\right)\,d\tau,$$

then $y^*=x^*$.

For this, notice that, for each $x\in\mathcal{C}[a,b]$ and $t\in[a,b]$, we have

$$(A-I)(x)(t)=\int_0^1\Gamma_0\left(F'(x_0)-F'(x^*+\tau(y^*-x^*))\right)\,x(t)d\tau.$$

Then, we obtain

$$\begin{aligned}\|A-I\| &\leq \|\Gamma_0\|\int_0^1\|F'(x_0)-F'(x^*+\tau(y^*-x^*))\|\,d\tau\\ &\leq \beta\ |\lambda|\ M\int_0^1\omega(\|x_0-x^*-\tau(y^*-x^*)\|)\,d\tau\\ &< \beta\ |\lambda|\ M\int_0^1\omega(\tau\overline{R}+(1-\tau)R)\,d\tau.\end{aligned}$$

So, if (12) holds, the operator $\int_0^1\Gamma_0\left(I-F'(x^*+\tau(y^*-x^*))\right)\,d\tau$ has an inverse and, consequently, $y^*=x^*$, which is an absurd. Then, the proof is complete. ■

Notice that, if

$$\beta\ |\lambda|\ M\int_0^1\omega(\tau\ t_*+(1-\tau)R)\,d\tau\leq 1,$$

for $t_*\in[a,b]$, then the fixed point of (2) is unique in $B(x_0,t_*)\cap\Omega$. So, from the previous reasoning, it is easy to check that $R\leq\overline{R}$.

3 Numerical examples

Next, we present two examples where we illustrate all the above, we calculate the domains of existence and uniqueness of solution for the two nonlinear integral equations considered and, for both equations, we approximate a solution from iterative scheme (6). The two examples arise from the two possibilities that may present kernel $K(s,t)$, since it can be separable or not.

3.1 Example 1

Now, we illustrate all the above-mentioned with an application to the following nonlinear Fredholm integral equation,

$$x(s) = \sin(\pi s) + \lambda \int_0^1 \cos(\pi s)\sin(\pi t)x(t)^3\,dt, \tag{13}$$

with $s \in [0,1]$, that has been used by other authors as numerical test [8, 3, 14, 4]. So, we provide a result of existence and uniqueness of solution for nonlinear integral equation (13).

Observe that solving the equation (13) is equivalent to solving $G(x) = 0$, where $G : \Omega \subseteq \mathcal{C}([a,b]) \longrightarrow \mathcal{C}([a,b])$ and

$$G(x)(s) = x(s) - F(x)(s) \tag{14}$$

$$F(x)(s) = \sin(\pi s) + \lambda \int_0^1 \cos(\pi s)\sin(\pi t)x(t)^3\,dt. \tag{15}$$

We then apply the study of the last section to obtain different results on the existence and uniqueness of solution of equation (13), or equivalently a fixed point of F given by (15), for different values of λ.

First of all, we determine the domain Ω. For this, as F' must be ω-Lipschitz, this fact depends on Ω. Notice that the first derivative of operator (15) is

$$[F'(x)y](s) = 3\lambda\cos(\pi s)\int_0^1 \sin(\pi t)x(t)^2y(t)\,dt \tag{16}$$

and

$$[(F'(x) - F'(y))z](s) = 3\lambda\cos(\pi s)\int_0^1 \sin(\pi t)\left(x(t)^2 - y(t)^2\right)z(t)\,dt.$$

Then, we have

$$\|F'(x) - F'(y)\| \leq 3|\lambda|M(\|x\| + \|y\|)\|x - y\|,$$

where $M = \left|\int\limits_0^1 \sin(\pi t)dt\right| = \dfrac{2}{\pi}$.

As a consequence of the last inequality, to obtain the function ω, the quantities $\|x\|$ and $\|y\|$ must be bounded, for what we need to fix the domain Ω. Besides, as a solution $x^*(s)$ of the equation must be contained in Ω, a previous location of $x^*(s)$ is usually done. For this, from (13), it follows

$$\|x^*\| \leq 1 + |\lambda| M \|x^*\|^3 \leq 1 + \frac{2|\lambda|}{\pi}\|x^*\|^3.$$

So, we consider the scalar equation deduced from the last expression and given by

$$1 + \frac{2|\lambda|}{\pi}t^3 - t = 0, \tag{17}$$

and suppose that equation (17) has at least one positive real solution. We then denote the smallest positive real solution by ρ_1. Obviously, if the last is true, condition (**II**) is satisfied, provided that $\|x^*(s)\| < \rho_1$, so that if integral equation (13) has a solution $x^*(s) \in B(0, \rho_1)$, we can choose

$$\Omega = \{x \in \mathcal{C}([a, b]) : \|x\| < \rho\}, \tag{18}$$

for some $\rho > \rho_1$, since Ω is an open domain. As a consequence,

$$\omega(z) = \frac{12|\lambda|\rho}{\pi}z \tag{19}$$

and, in addition, $\omega(tz) \leq h(t)\omega(z)$ with $h(t) = t$.

As we can see in Tables 1 and 2, taking as starting functions $x_0(s) = 0$ and $x_0(s) = \sin(\pi s)$, respectively, for different λ values, we obtain existence and uniqueness radii for a solution. We can observe that the starting function $x_0(s) = \sin(\pi s)$ provides us smaller existence domains than the starting function $x_0(s) = 0$, so that the solution is better located. However, for the domains of uniqueness of solution, the opposite occurs. This is because the starting function $x_0(s) = \sin(\pi s)$ is closer to the solution and therefore is a better choice as starting function.

Value of λ	ρ_1	ρ	R	$\overline{R}$
$\frac{1}{5}$	1.2467	$1.2467 + \epsilon$	1.0815	15.2796
$\frac{1}{8}$	1.1083	$1.1083 + \epsilon$	1.0232	46.0443
$\frac{1}{10}$	1.0803	$1.0803 + \epsilon$	1.0140	74.4242

Table 1: Results for $x_0(s) = 0$ and $\epsilon = 0.01$

In practice it is not easy to construct iterative scheme (6) for operators defined on infinite dimension spaces. The main difficulties arise for calculating at each step the inverse of the linear operator $I - F'(x_n)$ or, equivalently, in solving the associated linear equation. Next, we approximate a solution of (13).

In this example, we are considering a separable kernel of the form $K(s, t) = g(s)h(t)$, where $g(s) = \cos(\pi s)$ and $h(t) = \sin(\pi t)$.

Value of λ	ρ_1	ρ	R	$\overline{R}$
$\frac{1}{5}$	1.2467	$1.2467 + \epsilon$	0.1032	12.0916
$\frac{1}{8}$	1.1083	$1.1083 + \epsilon$	0.0559	39.52062
$\frac{1}{10}$	1.0803	$1.0803 + \epsilon$	0.0430	65.7901

Table 2: Results for $x_0(s) = \sin(\pi s)$ and $\epsilon = 0.01$

We give an algorithm that defines iterative scheme (6) to solve $x = F(x)$, where

$$[F(x)](s) = f(s) + \lambda g(s) \int_a^b h(t)H(x(t))dt, \quad s \in [a,b],$$

which verifies

$$[F^{'}(x)y](s) = \lambda g(s) \int_a^b h(t)H^{'}(x(t))y(t)dt, \quad s \in [a,b].$$

In this case, it is possible to obtain the analytical expression for the inverse of $I - F'(x)$ for each $x \in \mathcal{C}[a,b]$. So, for each $y \in \mathcal{C}[a,b]$ we have

$$[I - F^{'}(x)]y(s) = y(s) - \lambda g(s) \int_a^b h(t)H^{'}(x(t))y(t)dt, \quad s \in [a,b],$$

and considering $[I - F^{'}(x)]y(s) = \phi(s)$ for $\phi \in \mathcal{C}[a,b]$. Then, if there exists $[I - F'(x)]^{-1}$, we have

$$[I - F'(x)]^{-1}\phi(s) = y(s) = \phi(s) + \lambda g(s) \left(\int_a^b h(t)H'(x(t))y(t)\, dt. \right).$$

If we denote $J = \int_a^b h(t)H'(x(t))y(t)\, dt$, the value of J can be obtained independently from y. For this, we multiply next-to-last equality by $h(s)\ H'(x(s))$ and integrate it between a and b, obtaining

$$J = \frac{\int_a^b h(t)\ \phi(t)\ H'(x(t))\, dt}{1 - \lambda\ \int_a^b g(t)h(t)H'(x(t))\, dt},$$

provided that $\lambda\ \int_a^b g(t)h(t)H'(x(t))\, dt \neq 1$. Consequently,

$$[I - F'(x)]^{-1}\phi(s) = \phi(s) + \lambda g(s) \frac{\int_a^b h(t)\ \phi(t)\ H'(x(t))\, dt}{1 - \lambda \int_a^b g(t)\ h(t)\ H'(x(t))\, dt}.$$

Now, as a consequence of the last equation, condition (**I**), that is required to prove Theorems 5 and 6, can be omitted, provided that

$$\int_0^1 g(t)\ h(t)\ H'(x_0(t))\, dt \neq 1. \tag{20}$$

After that, it is enough to choose some starting point $x_0(s)$ for iterative scheme (6) such that condition (20) is satisfied.

Then, the iterates given by iterative scheme (6) can be calculated in the following way:

1.- **First step**: calculate the integrals

$$A_n = \int_a^b h(t)H(x_n(t))\,dt; \quad B_n = \int_a^b h(t)x_n(t)H'(x_n(t))\,dt;$$

$$C_n = \int_a^b h(t)f(t)H'(x_n(t))\,dt; \quad D_n = \int_a^b h(t)g(t)H'(x_n(t))\,dt.$$

2.- **Second step**: define

$$y_n(s) = f(s) + \lambda\, g(s)\, \frac{A_n - B_n + C_n}{1 - \lambda\, D_n}.$$

3.- **Third step**: calculate the integrals

$$A'_n = \int_a^b h(t)H(y_n(t))\,dt; \quad B'_n = \int_a^b h(t)y_n(t)H'(x_n(t))\,dt;$$

4.- **Fourth step**: define

$$x_{n+1}(s) = f(s) + \lambda\, g(s)\, \frac{A'_n - B'_n + C_n}{1 - \lambda\, D_n}.$$

The results of $x_n(s)$ obtained for the starting points $x_0(s) = \sin(\pi s)$ and $x_0(s) = 0$ are shown on Tables 3 and 5 and Tables 4 and 6, respectively, where condition (20) is verified in both cases.

For solving the iterates given in (6), we have solved the four-step algorithm explained above, where the integrals have been approximated with Simpson's quadrature method. For this method, we have divided the interval [0,1] in 10 subintervals, and in the fourth step of the algorithm, a tolerance $\|x_n(s) - x_{n-1}(s)\| < 10^{-32}$ has been imposed.

n	$x_n(s)$
0	$\sin(\pi s)$
1	7.542187500000000e-2 $\cos(\pi s) + \sin(\pi s)$
2	7.542668890493716e-2 $\cos(\pi s) + \sin(\pi s)$
3	7.542668890493716e-2 $\cos(\pi s) + \sin(\pi s)$
4	7.542668890493716e-2 $\cos(\pi s) + \sin(\pi s)$

Table 3: Solution of $x_n(s)$ for starting point $x_0(s) = \sin(\pi s)$ and tolerance 10^{-32}.

n	$x_n(s)$
0	0
1	7.500000000000000e-2 $\cos(\pi s) + \sin(\pi s)$
2	7.542668890404322e-2 $\cos(\pi s) + \sin(\pi s)$
3	7.542668890493716e-2 $\cos(\pi s) + \sin(\pi s)$
4	7.542668890493716e-2 $\cos(\pi s) + \sin(\pi s)$

Table 4: Solution of $x_n(s)$ for starting point $x_0(s) = 0$ and tolerance 10^{-32}.

Once the solutions for each iteration n have been obtained, we calculate different norms. The first column shows the difference between two consecutive solutions, and the second column shows the norm between the solution obtained and the exact solution: $\psi(s) = \sin(\pi s) + \frac{1}{3}\left(20 - \sqrt{391}\right)\cos(\pi s)$.
Once more, our results are shown for both initial points $x_0(s)$ mentioned above.

n	$\|x_n(s) - x_{n-1}(s)\|$	$\|x_n(s) - \psi(s)\|$
1	1.8475e-01	1.179161e-05
2	1.1792e-05	3.144874e-18
3	3.1449e-18	5.843402e-56
4	5.9307e-56	8.869960e-58

Table 5: Errors for starting point $x_0(s) = \sin(\pi s)$ and tolerance 10^{-32}.

The Table 7 compares the error obtained with our method with four other methods, where **Method 1** is a scheme based on a combination of the Newton-Kantorovich method and quadrature methods [14], **Method 2** is an iterative scheme based on the homotopy analysis method [3], **Method 3** is an adapted Newton-Kantorovich iterative scheme [8] and **Method 4** is Newton's method [4]. Some points of the interval [0,1] have been chosen, and the difference in norm between the exact solution $\psi(s)$ and the approximation obtained with each method is shown. As we can observe from these results, the error obtained with our method (6) is noticeably smaller than the ones obtained with the other four methods, since we obtain an error of order 10^{-58} with only four iterations of our method. For this, only the initial point $x_0(s) = \sin(\pi s)$ has been used.

n	$\|x_n(s) - x_{n-1}(s)\|$	$\|x_n(s) - \psi(s)\|$
1	2.2436e+00	1.045170e-03
2	1.0452e-03	2.189688e-12
3	2.1897e-12	2.013868e-38
4	2.0139e-38	9.920915e-58

Table 6: Errors for starting point $x_0(s) = 0$ and tolerance 10^{-32}.

s	**Method** (6)	**Method 1**	**Method 2**	**Method 3**	**Method 4**
0.0	3.7836e-58	4.98e-02	5.53e-15	5.44e-08	2.87e-30
0.2	3.1862e-58	4.03e-02	4.55e-15	4.40e-08	2.32e-30
0.4	1.5931e-58	1.53e-02	1.77e-15	1.68e-08	8.87e-31
0.6	1.5931e-58	1.53e-02	1.77e-15	1.68e-08	8.87e-31
0.8	3.1862e-58	4.03e-02	4.55e-15	4.40e-08	2.32e-30
1.0	3.7836e-58	4.98e-02	5.53e-15	5.44e-08	2.87e-30

Table 7: Errors $\|x^*(s) - \psi(s)\|$ when different methods are applied to equation (13).

3.2 Example 2

Second, we consider the following nonlinear integral equation of Fredholm,

$$x(s) = s + \frac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{st} x(t)^4\, dt, \ s \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{21}$$

that has been used in [5]. Observe that, in this case, kernel $K(s,t) = \mathrm{e}^{st}$ is nonseparable and operator F defined in (3) is such that $F : \Omega \subseteq \mathcal{C}([-\frac{1}{2}, \frac{1}{2}]) \to \mathcal{C}([-\frac{1}{2}, \frac{1}{2}])$ with

$$[F(x)](s) = s + \frac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{st} x(t)^4\, dt, \ s \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{22}$$

Observe that solving the equation (21) is equivalent to solving $G(x) = 0$, where $G : \Omega \subseteq \mathcal{C}([-\frac{1}{2}, \frac{1}{2}]) \to \mathcal{C}([-\frac{1}{2}, \frac{1}{2}])$ and

$$G(x)(s) = x(s) - F(x)(s).$$

Now, we then apply the iterative scheme (6) to obtain a solution of equation (21), or equivalently a fixed point of F given by (22). As usually for this type of nonlinear integral equations, according to equation (21), $x_0(s) = s$ is a reasonable choice of starting point.

As kernel $K(s,t) = \mathrm{e}^{st}$ is nonseparable, the application of iterative scheme (6) for solving (21) is difficult. Taking into account this fact, we first use Taylor's series to approximate $K(s,t) = \mathrm{e}^{st}$. So,

$$K(s,t) = \mathrm{e}^{st} = \widetilde{K}(s,t) + R(\epsilon, s, t); \quad \widetilde{K}(s,t) = \sum_{i=0}^{\ell-1} \frac{s^i\, t^i}{i!}, \quad R(\epsilon, s, t) = \frac{\mathrm{e}^{s\epsilon}}{\ell!} s^\ell\, t^\ell, \tag{23}$$

where $\epsilon \in (\min\{0,t\}, \max\{0,t\})$, and consider the integral equation

$$x(s) = s + \frac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{K}(s,t)x(t)^4\,dt,\ \ s \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{24}$$

If we denote the solutions of (21) and (24) by $x^*(s)$ and $\widetilde{x}(s)$, respectively, and try to locate them previously in $\mathcal{C}([-\frac{1}{2}, \frac{1}{2}])$, we see that

$$\|x^*(s)\| - \frac{1}{2} - \frac{3M}{4}\|x^*(s)\|^4 \le 0 \qquad \text{and} \qquad \|\widetilde{x}(s)\| - \frac{1}{2} - \frac{3\widetilde{M}}{4}\|\widetilde{x}(s)\|^4 \le 0,$$

where $M = \max_{s\in[-\frac{1}{2},\frac{1}{2}]} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{st}dt = 1.010449\ldots$ and $\widetilde{M} = \max_{s\in[-\frac{1}{2},\frac{1}{2}]} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\widetilde{K}(s,t)\right| dt$. In addition, $x^*(s)$ satisfies $\|x^*(s)\| \le 0.594952\ldots = \sigma^*$, where σ^* is the smallest positive solution of scalar equation $t - \frac{1}{2} - \frac{3M}{4}t^4 = 0$, and $\widetilde{x}(s)$ does $\|\widetilde{x}(s)\| \le \widetilde{\sigma}$, where $\widetilde{\sigma}$ is the smallest positive solution of scalar equation $t - \frac{1}{2} - \frac{3\widetilde{M}}{4}t^4 = 0$. We can obtain this last bound once the value of ℓ is fixed to obtain $\widetilde{K}(s,t)$. So, by using that

$$x^*(s) - \widetilde{x}(s) = \frac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathrm{e}^{st}x^*(t)^4 - \widetilde{K}(s,t)\widetilde{x}(t)^4\right) dt$$

we can obtain the following approximation

$$\|x^*(s) - \widetilde{x}(s)\| \le \frac{\frac{3}{4}Q\sigma^{*4}}{1 - \frac{3}{4}\widetilde{M}\left(\sigma^{*3} + \sigma^{*2}\widetilde{\sigma} + \sigma^*\widetilde{\sigma}^2 + \widetilde{\sigma}^3\right)}$$

where $Q = \max_{\epsilon\in[-\frac{1}{2},\frac{1}{2}]}\left(\max_{s\in[-\frac{1}{2},\frac{1}{2}]}\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mathcal{R}(\epsilon,s,t)|\,dt\right) = 0.417977\ldots\times 10^{-4}$ and provided that

$$\widetilde{M}\left(\sigma^{*3} + \sigma^{*2}\widetilde{\sigma} + \sigma^*\widetilde{\sigma}^2 + \widetilde{\sigma}^3\right) \neq \frac{4}{3}.$$

If, for example, we now want to obtain an approximation of the solution $x^*(s)$ of order 10^{-6}, by choosing $\ell = 4$ in (23), we have $\widetilde{M} = 1.010422\ldots$ and $\widetilde{\sigma} = 0.594943\ldots$ and we obtain $\|x^*(s) - \widetilde{x}(s)\| \le 2.664991\ldots \times 10^{-6}$.

Hence, if we now look for a solution $\widetilde{x}(s)$ of (21) by iterative scheme (6), such as $\|\widetilde{x}(s) - x_n(s)\| \le 10^{-6}$, by running the required number of iterations n, then we can assure that

$$\|x^*(s) - x_n(s)\| \le \|x^*(s) - \widetilde{x}(s)\| + \|\widetilde{x}(s) - x_n(s)\|,$$

it is of order 10^{-6}, so we have obtained the solution of the main problem with the required bound of the error.

Let's do it for the analyzed case with $\ell = 4$, we apply iterative scheme (6) from $x_0(s) = s$ in order to approximate a solution $\widetilde{x}(s)$ of integral equation (24) with the stopping criterion $\|x_n(s) - x_{n-1}(s)\| < 10^{-w}$ with $w >> 6$. Then, for example, after 3 iterations we have the approximation $x_3(s)$ verifies this fact, with $\|x_3(s) - x_2(s)\| \leq 1.33153\ldots \times 10^{-14}$. So, the approximated solution obtained by the algorithm described in section 3.1 is given by

$$x_3(s) = 0.0000107161s^3 + 0.000840845s^2 + 1.00036s + 0.00942279$$

and verifies that

$$\|x^*(s) - x_3(s)\| \leq \|x^*(s) - \widetilde{x}(s)\| + \|\widetilde{x}(s) - x_3(s)\| \approx 2.664991\ldots \times 10^{-6},$$

and as a consequence, $x_3(s)$ is an approximation of the solution $x^*(s)$ of equation (21) of order 10^{-6} as we looked for.

4 Conclusions

In this paper, given a nonlinear Fredholm integral equation, we establish a qualitative study that provides us the localization of a solution as well as the separation of this solution from other possible solutions. These results are obtained from the study of the semilocal convergence of an efficient iterative scheme of third order and a result of uniqueness of solution obtained for the same iterative scheme. In addition, we used the iterative scheme considered to approximate a solution of the nonlinear Fredholm integral equation given, obtaining important results that improve notably those obtained by other numerical methods.

References

[1] I. K. Argyros, S. Hilout, M. A. Tabatabai, Mathematical Modelling with Applications in Biosciences and Engineering. Nova Publishers, New York, 2011.

[2] I. K. Argyros, S. Hilout, Numerical methods in Nonlinear Analysis. World Scientific Publ. Comp. New Jersey. 2013.

[3] F. Awawdeh, A. Adawi, S. Al-Shara, A numerical method for solving nonlinear integral equations. Int. Math. Forum **4** (2009), 805–817.

[4] J. A. Ezquerro, M. A. Hernández-Verón, Some numerical methods for solving nonlinear integral equations of Hammerstein type Communications book of The Congress CEDYA+CMA 2017, Cartagena (Spain).

[5] J. A. Ezquerro, M. A. Hernández-Verón, Newton's method: an updated approach of Kantorovich's theory. Birkhäuser, Cham (2017).

[6] M. A. Hernández-Verón, Eulalia Martínez, Carles Teruel, Semilocal convergence of a k-step iterative process and its application for solving a special kind of conservative problems. Numer. Algor. **76** (2017), 309–331.

[7] J. Matrowski, Functional equations and Nemystkii operators. Funkcial. Ekvac. **25** (1982), 127–132.

[8] M. Nadir, A. Khirani, Adapted Newton-Kantorovich method for nonlinear integral equations. J. Math.Statistics **12** (2016), no. 3, 176–181.

[9] Parhi, S.K., Gupta, D.K. : Convergence of a third order method for fixed points in Banach spaces. Numer. Algor. **60** (2012), 419-434.

[10] S. K. Parhi, D .K. Gupta, Semilocal convergence of a Stirling-like method in Banach spaces. International Journal of Computational Methods. **7** (2010), 215-228.

[11] A. D. Polyanin, A. V. Manzhirov, Handbook of integral equations. CRC Press, Boca Raton. 1998.

[12] L. B. Rall, Computational Solutions of Nonlinear Operator equations. Robert E. Krieger, New York. 1969.

[13] J. Rashidinia, A. Parsa, Analytical-numerical solution for nonlinear integral equations of Hammerstein type. International Journal of Mathematical Modelling and Computations **2** (2012), no. 1, 61–69.

[14] J. Saberi-Nadja, M. Heidari, Solving nonlinear integral equations in the Urysohn form by Newton-Kantorovich-quadrature method. Comput. Math. Appl. **60** (2010), 2018–2065.

[15] S. Singh, D. K Gupta, E. Martínez, J. L. Hueso, Enlarging the convergence domain in local convergence studies for iterative methods in Banach spaces. Mediterr. J. Math. **13** (2016), 4219-4235.

[16] J. F. Traub, Iterative Methods for the Solution of Equations. Prentice-Hall, Englewood Cliffs, New Jersey (1964).