A FAMILY OF OPTIMAL FOURTH ORDER METHODS FOR MULTIPLE ROOTS OF NON-LINEAR EQUATIONS

FIZA ZAFAR, ALICIA CORDERO, AND JUAN R. TORREGROSA

Abstract. Newton-Raphson method has always remained as the widely used method for finding simple as well as multiple roots of nonlinear equations. In the past years, many new methods have been introduced for finding multiple zeroes that involve the use of weight function in the second step, thereby, increasing the order of convergence and giving a flexibility to generate a family of methods satisfying some underlying conditions. However, in almost all the schemes developed over the past, the usual way is to use Newton type method at the first step. In this paper, we present a new two-step optimal fourth-order family of methods for multiple roots ($m > 1$). The proposed iterative family has the flexibility of choice at both steps. The development of the scheme is based on using weight functions. The first step can not only recapture Newton's method for multiple roots as special case but is also capable of defining new choices of first step. We compare our methods with the existing methods of same order with a real life application as well as standard test problems. From the numerical results, we find that our methods can be considered as a better alternate for the exiting methods of same order. %Finally, dynamical study and stability analysis is also given to explain the dynamical behavior of the new methods around the multiple roots.

1. Introduction

Construction of stable and optimal iterative methods for multiple roots having prior knowledge of multiplicity ($m > 1$) is one of the most important and challenging tasks in computational mathematics. Some optimal and non-optimal fourth-order methods have been developed in the recent past proposed by Neta and Johnson [12] in (2008), Li et al. [9] in (2009), Neta [11], Sharma and Sharma [14] and Li et al. [8] in (2010), Zhou et al. [19] in (2011), Sharifi et al. [13] in (2012), Soleymani et al. [15], Soleymani and Babajee [16], Liu and Zhou [10] and Zhou et al. [20] in (2013), Thukral [17] in (2014), Behl et al. [1] and Hueso et al. [5] in (2015). However, it is indeed the need of time to design iterative methods for multiple roots not only in a general, optimal and efficient context but also in terms of deep analysis of their stable regions of convergent of initial estimations. Most recently, Behl et al. [2] in (2016), Lee et al. [7] (2017) and Behl et al. [3] (2018) have constructed and analyzed such families of methods. Moreover, most of these schemes are either the modification or extension of Newton's method or Newton-like methods by involving additional functional evaluations and increasing the amount of substeps of the original methods.

We, in this work, propose an iterative family that has the flexibility of choice at both steps. The development of the scheme is based on using weight functions. The first step can not only recapture Newton's method for multiple roots as special case but is also capable of defining new choices of first substep and hence different iterative schemes in terms of both substeps. We compare our methods with the existing ones of the same order for standard test problems. From the numerical results, we find that our methods can be considered as a better alternative for the exiting methods of the same order. Finally, dynamical study and stability analysis is also given to explain the dynamical behavior of the new methods around the multiple roots.

The contents of the paper is organized as follows: in Section 2, we define a new two-point fourth-order scheme and analyze its convergence analysis. Some special cases are given in the Section 3. In Section 4, comparison of the proposed methods with the existing ones is given.

2. Construction of Optimal Fourth-Order Scheme

Let α be a multiple zero with integer multiplicity $m > 1$, of $f : \mathbb{C} \to \mathbb{C}$ an analytic function in the neighborhood of α. Then, for a given initial guess x_0, we define the following iterative scheme in order to find a approximate zero

2000 *Mathematics Subject Classification.* 65H05, 65H10.

Key words and phrases. Nonlinear Equations, Multiple zeroes, Optimal methods, Weight functions.

of f:

$$\begin{aligned} y_n &= x_n - h(x_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - Q_f(v_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{2.1}$$

where, the weight functions $h : \mathbb{C} \to \mathbb{C}$ and $Q_f : \mathbb{C} \to \mathbb{C}$ are analytic functions in the neighborhoods of α and 0, respectively with $v_n = \left[\frac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{m-1}}$.

We, now, investigate the convergence analysis of the proposed family (2.1) and the find the conditions on weight functions $h(x_n)$ and $Q_f(v_n)$ in the next theorem.

Theorem 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic function in the neighborhood of the required multiple zero α of multiplicity $m \in \mathbb{N} - \{1\}$. In addition, we also consider that $Q : \mathbb{C} \to \mathbb{C}$ and $h : \mathbb{C} \to \mathbb{C}$ are an analytic functions in the neighborhood of origin and multiple zero α, respectively. Then, for an initial guess x_0 sufficiently close to α the family of iteration functions (2.1) has fourth-order convergence when the following conditions hold:*

$$\begin{aligned} h(\alpha) &= m, h'(\alpha) = 0, h''(\alpha) = 0, \\ Q_f(0) &= m, Q_f'(0) = m, Q_f''(0) = \frac{4m^2}{m-1} \end{aligned} \tag{2.2}$$

and $\left|h'''(\alpha)\right| < \infty, \left|Q_f'''(0)\right| < \infty.$

Proof. Let α be a multiple zero of $f(x)$ and $e_n = x_n - \alpha$ be the error at nth iterate. Expanding $f(x_n)$ and $f'(x_n)$ about $x = \alpha$ by using the Taylor's series expansion, we have

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^m\left[1 + c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)\right], \tag{2.3}$$

and

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^{m-1}\left[1 + (m+1)c_1e_n + (m+2)c_2e_n^2 + (m+3)c_3e_n^3 + (m+4)c_4e_n^4 + O(e_n^5)\right], \tag{2.4}$$

where $c_k = \frac{m!}{(m+k)!}\frac{f^{(m+k)}(\alpha)}{f^{(m)}(\alpha)}, k = 1, 2, 3, \ldots$, respectively.

By using expressions (2.3) and (2.4), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{1}{m}e_n - \frac{c_1}{m^2}e_n^2 + \frac{A_1}{m^3}e_n^3 + \frac{A_2}{m^4}e_n^4 + O(e_n^5), \tag{2.5}$$

where $A_1 = (1 + m)c_1^2 - 2mc_2,\ A_2 = -(1 + m)^2c_1^3 + m(4 + 3m)c_1c_2 - 3m^2c_3$.

Expanding the Taylor series of $h(x_n)$ about α up to second-order term, we obtain

$$h(x_n) = h(\alpha) + e_nh'(\alpha) + \frac{1}{2}e_n^2h''(\alpha) + O(e_n^3). \tag{2.6}$$

Thus, by using the expressions (2.5) and (2.6) in the first substep leads to:

$$y_n - \alpha = \left(1 - \frac{h(\alpha)}{m}\right)e_n + \frac{(c_1h(\alpha) - mh'(\alpha))}{m^2}e_n^2 + B_1e_n^3 + O(e_n^4), \tag{2.7}$$

where $B_1 = \frac{1}{2m^3}\left[m(4c_2h(\alpha) - mh''(\alpha)) - 2c_1^2h(\alpha)(m+1) + 2mc_1h'(\alpha)\right]$.

With the help of Taylor series expansion, we further get and

$$f'(y_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^{m-1}\left[\frac{m\left(1 - \frac{h(\alpha)}{m}\right)^{m-1}}{m!} + D_1e_n + D_2e_n^2 + O(e_n^3)\right], \tag{2.8}$$

where

$D_1 = \frac{1}{m!(h(\alpha)-m)^2}\left(1 - \frac{h(\alpha)}{m}\right)^m\left[c_1\left(h^2(\alpha)(m+1) - h(\alpha)m(m+3) + m^2(m+1)\right) - h'(\alpha)(m-1)m^2\right],$

$D_2 = \frac{1}{2mm!(h(\alpha)-m)^3}\left(1-\frac{h(\alpha)}{m}\right)^m \Big[c_1\Big((h^{''}(\alpha)(h(\alpha)-m)+\left(h^{'}(\alpha)\right)^2(m-2))((m-1)m^3-2h^{'}(\alpha)m^2(m^3+m)+ h^2(\alpha)(m+1)-h(\alpha)\left(m^2+4m-1\right)\Big)$
$+c_1^2h(\alpha)m\Big(2h(\alpha)(m+1)+2m(m+1)-h(\alpha)m(m+7)\Big)+2c_2\Big(h^4(\alpha)(m+2)-4h^3(\alpha)m(m+2)+2h^2(\alpha)m^2(2m+ 7)-2h(\alpha)m^3(m+5)+m^4(m+2)\Big).$

By using the expressions (2.3)-(2.5), and (2.8), we further have

$$v_n = \left(1-\frac{h(\alpha)}{m}\right)+\frac{\left(c_1h(\alpha)(mh(\alpha)+h(\alpha)-2m)-h^{'}(\alpha)(m-1)m^2\right)}{(m-1)m^3}e_n+E_1e_n^2+O(e_n^3), \tag{2.9}$$

where
$E_1 = \frac{1}{2m^5(m-1)^2}\Big[-4c_1h(\alpha)m^2(m-1)\Big(h(\alpha)(m+1)-m\Big)-c_1^2h(\alpha)(m+1)\Big(6m^2(m-1)+h(\alpha)m(8-3m- 3m^2)+h^2(\alpha)(m^2-m-2)\Big)$
$+c_2m(m-1)\Big]h^{''}(\alpha)m^3(m-1)+2h(\alpha)(6m^2+h^2(\alpha)(m+2)-3mh(\alpha)(2+m))\Big]\Big].$

It is clear from (2.9) that $v_n-\eta = O(e_n)$, where $\eta = 1-\frac{h(\alpha)}{m}$. Therefore, we can expand the weight function $Q_f(v_n)$ about the point η with the help of Taylor's series expansion upto third-order terms, we have

$$Q_f(v_n) = Q_f(\eta)+Q_f^{'}(\eta)v_n+\frac{1}{2!}Q_f^{''}(\eta)v_n^2+\frac{1}{3!}Q_f^{'''}(\eta)v_n^3+O(e_n^4). \tag{2.10}$$

Now, use the expressions (2.3)–(2.10) in the second substep, we further obtain:

$$\begin{aligned} e_{n+1} &= -\Big[1-\frac{1}{6m^4}6m^3\left(Q_f(\eta)+Q_f^{'}(\eta)\right)-6m^2h(\alpha)Q_f^{'}(\eta) \\ &\quad +3m(h(\alpha)-m)^2Q_f^{''}(\eta)+(m-h(\alpha))^3Q_f^{'''}(\eta)\Big]e_n \\ &\quad +\sum_{i=1}^{3}\Gamma_ie_n^{i+1}+O(e_n^5), \end{aligned} \tag{2.11}$$

where Γ_i is dependent on $h(\alpha), h(\alpha), h(\alpha), Q_f(\eta), Q_f^{'}(\eta), Q_f^{''}(\eta), Q_f^{'''}(\eta)$ and $c_1, c_2, \ldots c_6$.
It is apparent from the above error expression (2.11) that we will obtain at least quadratic convergence if we choose $h(\alpha) = m$ and $Q_f(\eta) = m$. So, by inserting this value $h(\alpha) = m$, $\Gamma_1 = 0$ and $\eta = 0$. Furthermore

$$\frac{\left(h^{'}(\alpha)Q_f^{'}(0)-\left(m-Q_f^{'}(0)\right)c_1\right)}{m^2} = 0 \tag{2.12}$$

which the selection

$$h^{'}(\alpha) = 0, Q_f^{'}(0) = m. \tag{2.13}$$

reduces $\Gamma_2 = 0$ and

$$\frac{h^{''}(\alpha)m^2+(\frac{4m^2}{m-1}-Q_f^{''}(0))}{2m^3} = 0. \tag{2.14}$$

Thus,

$$h^{''}(\alpha) = 0,\ Q_f^{''}(0) = \frac{4m^2}{m-1}, \tag{2.15}$$

$\Gamma_3 = 0$ and (2.11) gives us the following error equation:

$$e_{n+1} = \frac{1}{6(m-1)^2m^4}(\xi_1c_1^3-\xi_2c_1c_2+6\xi_3)e_n^4+O(e_n^5), \tag{2.16}$$

where $\xi_1 = 24m^3+3m^4-m^2(Q_f^{'''}(0)-3)-Q_f^{'''}(0)+2m\left(Q_f^{'''}(0)+3\right)$, $\xi_2 = 6m^3(m-1)$, $\xi_3 = m^3(m-1)^2h^{'''}(\alpha)$.
The above asymptotic error constant in (2.16) reveals that the proposed scheme (2.1) reaches at fourth-order

convergence by using only three functional evaluations (using $f(x_n)$ $f'(x_n)$ and $f^{'}(y_n)$) per iteration. This completes the proof. $\square$

Remark 1. *Proposed family (2.1) has an advantage of making selection at both steps. It is also clear that the first step recaptures Newton's method as special case and it is capable of obtaining first step different from the traditional choice of Newton's method.*

3. Some special cases of our scheme

From Theorem 1, we can obtain several new multiple root finding two-point methods by using different cases for $h(x_n)$ and $Q_f(v_n)$ in the proposed scheme (2.1). Some particular cases of the proposed scheme are given as follows:

(1) By choosing $h(x_n) = m$ and $Q_f(v_n) = m + mv_n + \frac{2m^2}{m-1}v_n^2 + Q_3v_n^3$, we obtain the following new two-point new fourth-order iterative method:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, m > 1, \\ x_{n+1} &= x_n - \psi_1\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{3.1}$$

where $\psi_1 = m + mv_n + \frac{2m^2}{m-1}v_n^2 + Q_3v_n^3$ for $Q_3 = 32.6$.

(2) For $h(x_n) = m$ and $Q_f(v_n) = \frac{m((m-1)(1-a_1v_n)-(m+1)v_n)}{(m-1)(1-a_1v_n+a_1v_n^2)-2mv_n}$, the proposed scheme (2.1) reads as:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, n \geq 0, \\ x_{n+1} &= x_n - \psi_2\frac{f(z_n)}{f'(z_n)}, \end{aligned} \tag{3.2}$$

where, $\psi_2 = \frac{m((m-1)(1-a_1v_n)-(m+1)v_n)}{(m-1)(1-a_1v_n+a_1v_n^2)-2mv_n}$ for $a_1 = 0.94$.

(3) For $h(x_n) = \frac{m+mw_n}{1+w_n+a_2w_n^3}$, we have

$$\begin{aligned} y_n &= x_n - \frac{m+mw_n}{1+w_n+a_2w_n^3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \psi_1\frac{f(x_n)}{f'(x_n)}. \end{aligned} \tag{3.3}$$

where, $w_n = f(x_n)$ and we take $a_2 = -18.6$.

(4) For $h(x_n) = \frac{m+w_n^3}{1+a_3w_n^3}$, we have

$$\begin{aligned} y_n &= x_n - \frac{m+w_n^2}{1+a_3w_n^3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \psi_1\frac{f(x_n)}{f'(x_n)}. \end{aligned} \tag{3.4}$$

where, $w_n = f(x_n)$ and we take $a_3 = -16.3$.

(5) By considering $h(x_n) = m + a_4f(x_n)^m$, we have the following particular case:

$$\begin{aligned} y_n &= x_n - [m + a_4f(x_n)^m]\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \psi_1\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{3.5}$$

for $a_4 = 50$.

4. Numerical results and applications

In this section, we investigate the performance and convergence behavior of our proposed fourth order methods namely (3.1)-(3.5) denoted by NM1, NM2, NM3,NM4 and NM5 respectively by carrying out some test functions involving two standard nonlinear functions and one real life problem. We compare the methods with the recent optimal fourth order methods given by Lee et al. [7] (2017) denoted by LKM.

For numerical tests, all computations have been performed in computer algebra software Maple 16 using 1000 significant digits of minimum number of precision. Tables 1–5 show the numerical errors of approximating real root $|x_n - x_{n-1}|$ and the absolute residual error of the respective function for the first three iterations , where $E(-i)$ denotes $E \times 10^{-i}$ in all the tables. The initial approximation x_0 for each test function, computational order of convergence, $r_c \approx \frac{\log|f(x_{n+1})/f(x_n)|}{\log|f(x_n)/f(x_{n-1})|}$ and asymptotic error constant $a_c = \frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|^4}$, are also included in these tables. The appearance of zeros in the tables represents that the exact zero has been approximated accurately up to 1000 significant digits. Now, we consider the following standard test problems:

Example 1. *Let us consider the following standard non-linear test function:*

$$f_1(x) = \left[\sin\left(\frac{1}{x}\right) - x^3 + 1\right]^3. \tag{4.1}$$

The above function has a multiple zero at $\alpha = 1.20253919024135112296187908278$ *of multiplicity* $m = 3$ *with initial guess* $x_0 = 1.25$.

Example 2. *Assume another non-linear test problem which is given below:*

$$f_2(x) = (x-1)\left(x\ln(x) - \sqrt{x} + x^4\right)^2. \tag{4.2}$$

The function f_3 *has multiplicity* $m = 3$*, multiple zero at* $\alpha = 1$ *and initial guess* $x_0 = 1.1$.

Example 3. *Continuous Stirred Tank Reactor (CSTR)*

Consider the isothermal continuous stirred tank reactor (CSTR). Components A & R are fed to the reactor at rates of Q and q-Q respectively. The following reaction scheme develops in the reactor:

$$\begin{aligned} A+R &\rightarrow B \\ B+R &\rightarrow C \\ C+R &\rightarrow D \\ D+R &\rightarrow E. \end{aligned}$$

The problem was analysed by Douglas [4] *in order to design simple feedback control systems. In the analysis, he gave the following equation for the transfer function of the reactor:*

$$K_C \frac{2.98\,(x+2.25)}{(s+1.45)\,(s+2.85)^2\,(s+4.35)} = -1,$$

where K_C *is the gain of the proportional controller. The control system is stable for values of* K_C *that yields roots of the transfer function having negative real part. If we choose* $K_C = 0$ *we get the poles of the open-loop transfer function as roots of the nonlinear equation:*

$$f_3(x) = x^4 + 11.50x^3 + 47.49x^2 + 86.0325x + 51.23266875 = 0 \tag{4.3}$$

given as:

$$x = -1.45, -2.85, -2.85, -4.35.$$

So, we see that there is one multiple roots with multiplicity 2. We take $m = 2$ *and* $x_0 = -3$.

Table 1: Comparison of multiple root finding methods for $f_1(x)$

$f_1(x) = \left(\sin\left(\frac{1}{x}\right) - x^3 + 1\right)^3, x_0 = 1.25$					
Methods	n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	r_c	a_c
LKM	1	4.745384266(-2)	3.749911402(-14)		
	2	6.967096919(-6)	6.184240718(-60)	3.966126232	1.373935383
	3	3.820695713(-21)	4.575159589(-243)	3.999998725	1.621567542
NM1	1	4.745890969(-2)	7.606120898(-16)		
	2	1.900060722(-6)	3.226081940(-74)	4.410031791	3.745384886(-1)
	3	6.626315657(-26)	1.031678958(-307)	4.000088731	5.083960864(-3)
NM4	1	4.746080422(-2)	1.875993439(-23)		
	2	5.530835722(-9)	1.179732013(-104)	3.895632250	1.090059958(-3)
	3	4.738506100(-36)	1.844915398(-429)	4.000000186	5.063821720(-3)
NM5	1	4.746080809(-2)	5.109598493(-25)		
	2	1.664086562(-9)	6.492241143(-111)	3.833089962	3.279709665(-4)
	3	3.883099658(-38)	1.692082630(-454)	4.000000052	5.063780615(-3)

Table 2: Comparison of multiple root finding methods for $f_2(x)$

$f_2(x) = (x-1)\left(x\ln(x) - \sqrt{x} + x^4\right)^2$, $x_0 = 1.1$					
Methods	n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	r_c	a_c
LKM	1	9.973228391(-2)	3.888577634(-10)		
	2	2677160895(-4)	2.352566232(-40)	3.853170778	2.706022566
	3	2.264847390(-14)	3.175439431(-161)	3.999892196	4.409009687
NM1	1	9.983089756(-2)	9.796949329(-11)		
	2	1.691024370(-4)	2.880566614(-47)	4.327807202	1.702511144
	3	1.124650632(-16)	1.859782372(-193)	4.001739973	1.375365624(-1)
NM4	1	9.979306935(-2)	1.795411740(-10)		
	2	2.069306440(-4)	3.353782987(-46)	4.368837238	2.086523533
	3	2.549000828(-16)	3.417356322(-189)	4.002164322	1.390179047(-1)
NM5	1	9.979300583(-2)	1.797065853(-10)		
	2	2.069941600(-4)	3.366337536(-46)	4.368901372	2.0871692921
	3	2.552177510(-16)	3.468814483(-189)	4.002165039	1.390203910(-1)

Table 3: Comparison of multiple root finding methods for $f_3(x)$

$f_3(x) = x^4 + 11.50x^3 + 47.49x^2 + 86.0325x + 51.23266875$, $x_0 = -3.0$					
Methods	n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	r_c	a_c
LKM	1	1.521916174(-1)	1.008561681(-5)		
	2	2.191856237(-3)	1.197664817(-13)	2.160003501	4.085536881
	3	2.388130175(-7)	7.327948015(-58)	5.578713509	1.034688766(4)
NM1	1	1.500116811(-1)	2.865442731(-10)		
	2	1.168116988(-5)	9.489712144(-44)	4.075138694	2.306672973(-2)
	3	2.125772928(-22)	1.141014571(-177)	4.000006184	1.141751293(-2)
NM4	1	1502676260(-1)	1.504078150(-7)		
	2	2.676260156(-4)	7.279363956(-33)	4.606463395	5.248879797(-1)
	3	5.887583365(-17)	3.950506405(-134)	4.000186954	1.147687774(-2)
NM5	1	1.432777532(-1)	9.492439797(-5)		
	2	6.722245222(-3)	4.948225872(-18)	4.927870715	1.595141817(1)
	3	1.535023789(-9)	8.435094413(-75)	4.273783471	7.517227707(-1)

References

[1] R. Behl, A. Cordero, S.S. Motsa, J.R. Torregrosa, On developing fourth-order optimal families of methods for multiple roots and their dynamics, Appl. Math. Comput. 265(15) (2015) 520–532.

[2] R. Behl, A. Cordero, S.S. Motsa, J.R. Torregrosa, V. Kanwar, An optimal fourth-order family of methods for multiple roots and its dynamics. Numer. Algor. 71(4) (2016) 775–796.

[3] R. Behl, A. Cordero, S.S. Motsa, J.R. Torregrosa, Multiplicity anomalies of an optimal fourth-order class of iterative methods for solving nonlinear equations, Nonlinear Dyn. 91 (2018) 81–112.
[4] J. M. Douglas, Process Dynamics and Control, Vol. 2, Prentice Hall, Englewood Cliffs, NJ, 1972.
[5] J.L. Hueso, E. Martínez, C. Teruel, Determination of multiple roots of nonlinear equations and applications. J. Math. Chem. 53 (2015) 880–892.
[6] Leah Edelstein-Keshet, Differential Calculus for the Life Sciences, Univeristy of British Columbia, 2017.
[7] M. Lee, Y.I. Kim, Á.A. Magreñán, On the dynamics of a triparametric family of optimal fourth-order multiple-zero finders with a weight function of the principal mth root of a function-to function ratio, Appl. Math. Comput. 315 (2017) 564–590.
[8] S.G. Li, L.Z. Cheng, B. Neta, Some fourth-order nonlinear solvers with closed formulae for multiple roots. Comput. Math. Appl. 59 (2010) 126–135.
[9] S. Li, X. Liao, L. Cheng, A new fourth-order iterative method for finding multiple roots of nonlinear equations. Appl. Math. Comput. 215 (2009) 1288–1292.
[10] B. Liu, X. Zhou, A new family of fourth-order methods for multiple roots of nonlinear equations, Non. Anal. Model. Cont. 18(2) (2013) 143–152.
[11] B. Neta, Extension of Murakami's high-order non-linear solver to multiple roots, Int. J. Comput. Math. 87(5) (2010) 1023–1031.
[12] B. Neta, A.N. Johnson, High order nonlinear solver for multiple roots, Comp. Math. Appli. 55(9) (2008) 2012–2017.
[13] M. Sharifi, D.K.R. Babajee, F. Soleymani, Finding the solution of nonlinear equations by a class of optimal methods. Comput. Math. Appl. 63 (2012) 764–774.
[14] J.R. Sharma, R. Sharma, Modified Jarratt method for computing multiple roots, Appl. Math. Comput. 217 (2010) 878–881.
[15] F. Soleymani, D.K.R. Babajee, T. Lofti, On a numerical technique for finding multiple zeros and its dynamic. J. Egypt. Math. Soc. 21 (2013) 346–353.
[16] F. Soleymani, D.K.R. Babajee, Computing multiple zeros using a class of quartically convergent methods. Alex. Eng. J. 52 (2013) 531–541.
[17] R. Thukral, A new family of fourth-order iterative methods for solving nonlinear equations with multiple roots, J. Numer. Math. Stoch. 6 (1) (2014) 37–44.
[18] Joseph L. Zachary, Introduction to Scientific Programming: Computational Problem Solving Using Maple and C, Springer, New York, 2012.
[19] X. Zhou, X. Chen, Y. Song, Constructing higher-order methods for obtaining the muliplte roots of nonlinear equations. J. Comput. Math. Appl. 235 (2011) 4199–4206.
[20] X. Zhou, X. Chen, Y. Song, Families of third and fourth order methods for multiple roots of nonlinear equations. Appl. Math. Comput. 219 (2013) 6030–6038.

Instituto Universitario de Matematica Multidisciplinar, Universitat Politecnica de Valencia, Valencia 46022, Spain, Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60800, Pakistan
E-mail address: `fizazafar@gmail.com`

E-mail address: `acordero@mat.upv.es`

E-mail address: `jrtorre@mat.upv.es`