The semi-analytical method for time-dependent wave problems with uncertainties

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This paper provides a constructive procedure for the computation of approximate solutions of random time-dependent hyperbolic mean square partial differential problems. Based on the theoretical representation of the solution as an infinite random improper integral, obtained via the random Fourier transform method, a double approximation process is implemented. Firstly, a random Gauss-Hermite quadrature is applied, and then, the evaluations at the nodes of the integrand are approximated by using a random Störmer numerical method. Numerical results are illustrated with examples.

KEYWORDS

Mean square random calculus, partial differential equations with randomness, problems involving randomness, random Fourier integral transform, random time-dependent hyperbolic problem

MSC CLASSIFICATION

35R60; 60H15; 60H35; 68U20

1 | INTRODUCTION

The integral transform method for solving deterministic mixed problems involving constant coefficient partial differential equations is a well-known fertile method.[1,2] The success of such approach is based on the exact solution of the ordinary differential equation (ODE) for the transformed of the unknown of the original problem. Then, by applying the inverse integral transform, the solution of the original problem is recovered. This classic approach has been extended for some deterministic variable coefficients problems, even when the exact solution of the transformed problem is not available, but representing such solution in terms of a theoretical fundamental set of solutions of the transformed ODE, and further using numerical integration quadrature formulae to approximate improper integrals.[3] In this variable coefficients case, the success of the aforementioned approach relies upon the use of numerical integration methods for approximating the evaluations of the inverse transform integrand at the points where the solution requires to be computed. Hyperbolic and advection partial differential problems appear in many different fields related to engineering being the wave equation its main model.[4-8] Random behaviour appears in problems such as civil and electromechanical structures and structure damage due to carbonation effects,[9,10] earthquake risk analysis in geology and seismology,[11] etc. In microwave drying processes,[12] electrical and chemical flows in soil,[13,14] cardiology,[15] optics,[16] circuit systems with varying parameters.[17] In microwave propagation in ferrite materials,[18] the evaluation of microwave heating processes via the constant model often leads to misleading results due to the complexity of the field distribution within the oven and the variation in dielectric properties of the material with temperature, moisture content, density, and other parameters. Such problems also appear in electromagnetic processing of materials at high power densities and in the analysis of multimode microwave applications.[19-21] The complexity of some of these problems and the uncertainties derived from measurement errors and the appearance of material impurities suggest the consideration of random models where these uncertainties are regarded providing a more realistic mathematical representation of the physical phenomena. Probability theory and stochastic processes (s.p.) are finding an increasing of applications in their interaction with other branches of mathematics.[22-32]

Here, we solve the following random time-dependent hyperbolic problem on an infinite spatial domain

$$u_{tt}(x,t) = c(t)\, u_{xx}(x,t), -\infty < x < +\infty, \quad t > 0, \tag{1}$$

$$u(x,0) = f(x), \quad -\infty < x < +\infty, \tag{2}$$

$$u_t(x,0) = g(x), \quad -\infty < x < +\infty, \tag{3}$$

where $c(t) \equiv c(t;\omega) :]0,+\infty[\times\Omega \to \mathbb{R}$ is a s.p. verifying, in probability, the following positive condition

$$c(t) \geq \delta > 0, \quad \text{almost surely (a.s.)}, \tag{4}$$

and $f(x) \equiv f(x;\omega) : \mathbb{R}\times\Omega \to \mathbb{R}$ and $g(x) \equiv g(x;\omega) : \mathbb{R}\times\Omega \to \mathbb{R}$ are s.p.'s. These random functions, $c(t), f(x)$, and $g(x)$, are defined in a complete probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying certain hypotheses that will be specified later. Furthermore, we consider that $c(t), f(x)$ and $g(x)$ depend on a finite degree of randomness (see Soong[33], p37). For the sake of clarity in the notation, and taking into account that the same results are available only with more complicated notation, we will assume that $c(t), f(x)$, and $g(x)$ depend on a single random variable (r.v.), say $A_1 = A_1(\omega)$, $A_2 = A_2(\omega)$, and $A_3 = A_3(\omega)$, respectively, all of them defined in $(\Omega, \mathcal{F}, \mathbb{P})$, that is,

$$c(t) = c(t;A_1), \quad f(x) = f(x;A_2), \quad g(x) = g(x;A_3).$$

Hereinafter, we will assume that A_1, A_2, and A_3 are independent r.v.'s. Our approach, based on $L_p(\Omega)$ random calculus, allows us to consider a wide class of uncertainty in the random inputs, $c(t), f(x)$, and $g(x)$, involved in the problem (1)-(3). Indeed, most of contributions dealing with randomization of classical PDEs problems introduce uncertainty via particular families of s.p.'s whose sample trajectories are very irregular (like the important Wiener process, which is Gaussian).[34] This treatment of uncertainties requires a special stochastic calculus, usually termed Itô calculus, that restricts the kind of uncertainties to Gaussian ones. Although this approach is interesting, it does not permit to consider other types of uncertainties (like Poisson, beta, and gamma), which may be crucial in practical applications and modelling. Our approach allows us consider all these latter types of randomness including those Gaussian processes possessing regular sample behaviour. This issue has been illustrated in the examples, where a variety of distinctive probabilistic distributions have been assigned to the random inputs in problem (1)-(3). From a practical standpoint, our contribution proposes a double approximation, based on the Störmer numerical scheme together with the Gauss-Hermite quadrature rule to construct an approximation of the solution s.p. This double approximation turns out very useful to compute reliable approximations, by means of Proposition 2.1 (see Section 2), for both the mean and the variance of the solution. To the best of our knowledge, this approach is completely new in the stochastic context.

This paper is organized as follows. Section 2 deals with some random Fourier exponential transform properties, random differential equations, and random improper quadrature formulae of Gauss-Hermite type. Section 3 is addressed to apply the random Fourier exponential method to the problem (1)-(3), including a double-approximation procedure. Firstly, one obtains an abstract improper representation of the solution by successive application of the random exponential Fourier transform and its inverse; see expression (25) below. Secondly, one applies random Gauss-Hermite quadrature formulae, but as the evaluations of the integrand function involve unknown values of a fundamental set of solutions of the random transformed ordinary differential equation, see (27)-(28) below, these theoretical values are approximated using random Störmer-type methods. An algorithm is also included in this section. In Section 4, an illustrative numerical example is included. Finally, in Section 5, conclusions are drawn.

2 | PRELIMINARIES

This section is addressed to introduce some preliminaries, definitions, and results that will be required throughout this paper. Further details about these preliminaries can be checked in Soong and Arnold.[33,34] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, a complex random variable, $\eta : \Omega \to \mathbb{C}$, is said to be of order $p \geq 1$ (in short, p-r.v.), if $\mathbb{E}[|\eta|^p] < +\infty$, being $\mathbb{E}[\cdot]$ the expectation operator. It can be shown that the set of all r.v.'s of order p,

$$\mathrm{L}_p^{\mathrm{RV}}(\Omega) = \{\eta : \Omega \to \mathbb{C} / \mathbb{E}[|\eta|^p] < +\infty\}, \quad 1 \le p < +\infty,$$

endowed with the norm

$$\|\eta\|_{p,\mathrm{RV}} = \left(\mathbb{E}[|\eta|^p]\right)^{1/p} < +\infty,$$

is a Banach space.[34], p9 The convergence inferred by the $\|\cdot\|_{p,\mathrm{RV}}$-norm is usually referred to as the pth mean convergence. More precisely, a sequence of r.v.'s $\{\eta_n : n \ge 0\}$ in $\mathrm{L}_p^{\mathrm{RV}}(\Omega)$ is pth mean convergent to the r.v. $\eta \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$, and it is denoted as $\eta_n \xrightarrow[n\to+\infty]{\|\cdot\|_{p,\mathrm{RV}}} \eta$, if and only if $\|\eta_n - \eta\|_{p,\mathrm{RV}} = (\mathbb{E}[|\eta_n - \eta|^p])^{1/p} \xrightarrow[n\to+\infty]{} 0$. A s.p. $\{x(t) : t \in \mathcal{T} \subset \mathbb{R}\}$ is said to be a p-s.p., or a s.p. of order p, if $\mathbb{E}[|x(t)|^p] < +\infty$ for all $t \in \mathcal{T}$, ie, if for each $t \in \mathcal{T}$ fixed, the r.v. $x(t)$ is a p-r.v. From this norm, one infers the concepts of p-continuity, p-differentiability, and p-integrability of a p-s.p. in a natural manner. The cases $p = 2$ and $p = 4$ corresponding to the so-called mean square and mean fourth convergence, respectively, play a major role in the study of random differential equations.[24,33,35]

A key goal in dealing with random PDEs is to compute the main statistical properties of the solution s.p. via the expectation (mean) and the variance. However, in general, the exact solution s.p. of random problem (1)-(3) is not available, and we must rely on approximations. In this regard, our approach, based on $L_p^{RV}(\Omega)$-random calculus, does have the following advantageous property:

Proposition 2.1. *(Soong[33], Theorem 4.2.1) If $\{\eta_n : n \ge 0\}$ is a sequence of r.v.'s in $L_p^{RV}(\Omega)$ such that is $\|\cdot\|_{p,RV}$–convergent to $\eta \in L_p^{RV}(\Omega)$, then*

$$\mathbb{E}[\eta_n] \xrightarrow[n\to+\infty]{\|\cdot\|_{p,RV}} \mathbb{E}[\eta], \quad \mathrm{Var}[\eta_n] \xrightarrow[n\to+\infty]{\|\cdot\|_{p,RV}} \mathrm{Var}[\eta].$$

This property allows us to guarantee that the expectation and the variance of our approximations will converge to the exact ones. This is a distinctive property of the $L_p^{RV}(\Omega)$ convergence that do not have other types of stochastic convergence, unless restrictive conditions are imposed. In Example 2 (see Section 4), we have taken advantage of this crucial property to compute reliable approximations for both the mean and the variance of the solution stochastic process, since in that example, a closed-form solution is not available.

The following result allows us to obtain the mean square derivative of the product of two mean fourth differentiable s.p.'s, and it will play a key role later

Proposition 2.2. *(Soong[35], Lemma 3.14) Let $\{w(t) : t \in \mathcal{T}\}$ and $\{z(t) : t \in \mathcal{T}\}$ be 4-s.p.'s having fourth derivatives $\frac{\mathrm{d}w(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}z(t)}{\mathrm{d}t}$, respectively. Then $w(t)z(t)$ is m.s. differentiable at $t \in \mathcal{T}$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(t)z(t)) = \frac{\mathrm{d}w(t)}{\mathrm{d}t} z(t) + w(t)\frac{\mathrm{d}z(t)}{\mathrm{d}t}.$$

Additionally to the definition of $\|\cdot\|_{p,\mathrm{RV}}$-integrable s.p. $y(v)$ defined in the space $\mathrm{L}_p^{\mathrm{RV}}(\Omega)$, we will use the concept of $\|\cdot\|_{p,\mathrm{RV}}$ absolutely integrable s.p. Namely, a s.p. $y(v) \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$ is said to be $\|\cdot\|_{p,\mathrm{RV}}$ absolutely integrable s.p. if the following deterministic integral

$$\int_{-\infty}^{+\infty} \|y(v)\|_{p,\mathrm{RV}} \,\mathrm{d}v \tag{5}$$

exists and is finite. If $y(v) \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$ is $\|\cdot\|_{p,\mathrm{RV}}$ absolutely integrable s.p., then its random exponential $\|\cdot\|_{p,\mathrm{RV}}$ Fourier transform is defined by

$$Y(\xi) := \mathfrak{F}[y(v)](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y(v)\exp(-\mathrm{i}\xi v)\,\mathrm{d}v, \quad \xi \in \mathbb{R}, \quad \mathrm{i} = +\sqrt{-1},$$

where this random integral defines a s.p. $\{Y(\xi) : \xi \in \mathbb{R}\}$ in the Banach space $(\mathrm{L}_p^{\mathrm{RV}}(\Omega), \|\cdot\|_{p,\mathrm{RV}})$. If $y(v)$ is $\|\cdot\|_{p,\mathrm{RV}}$ absolutely integrable s.p., it is clear that it admits a random $\|\cdot\|_{p,\mathrm{RV}}$ Fourier transform since

$$\int_{-\infty}^{+\infty} \|y(v)\exp(-\mathrm{i}\xi v)\|_{p,\mathrm{RV}}\,\mathrm{d}v \le \int_{-\infty}^{+\infty} \|y(v)\|_{p,\mathrm{RV}}\,|\exp(-\mathrm{i}\xi v)|\,\mathrm{d}v = \int_{-\infty}^{+\infty} \|y(v)\|_{p,\mathrm{RV}}\,\mathrm{d}v < +\infty,$$

where we have used that $|\exp(-i\xi v)| = 1$ and that $y(v)$ is $\|\cdot\|_{p,\mathrm{RV}}$ absolutely integrable s.p.; hence, by (5), the last integral is finite. In Casabán[25], p5926, it is proved the extension the following well-known properties of the Fourier transform

$$\mathfrak{F}[y'(v)](\xi) = \mathrm{i}\xi \quad \mathfrak{F}[y(v)](\xi), \quad \mathfrak{F}[y''(v)](\xi) = -\xi^2 \quad \mathfrak{F}[y(v)](\xi), \tag{6}$$

to the random framework provided that the involved random $\|\cdot\|_{p,\mathrm{RV}}$ derivatives exist and $y(v)$, $y'(v)$ and $y''(v)$ are $\|\cdot\|_{p,\mathrm{RV}}$ absolutely integrable s.p.'s. These properties will be used later.

In order to formalize our study, besides the above Banach space of complex random variables having absolute moments of order p, $(\mathrm{L}_p^{\mathrm{RV}}(\Omega), \|\cdot\|_{p,\mathrm{RV}})$, we will also need the following Banach space, $(\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R}\times\Omega), \|\cdot\|_{p,\mathrm{SP}})$ where

$$\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R}\times\Omega) = \left\{ f : \mathbb{R}\times\Omega \to \mathbb{C} / \int_{-\infty}^{+\infty} \left(\mathbb{E}\left[|f(v)|^p\right]\right)^{1/p} \mathrm{d}v < +\infty \right\} = \left\{ f : \mathbb{R}\times\Omega \to \mathbb{C} / \int_{-\infty}^{+\infty} \|f(v)\|_{p,\mathrm{RV}}\, \mathrm{d}v < +\infty \right\}, \tag{7}$$

and

$$\|f\|_{p,\mathrm{SP}} = +\left(\int_{-\infty}^{+\infty} \|f(v)\|_{p,\mathrm{RV}}\, \mathrm{d}v\right)^{1/p}, \quad 1 \le p < +\infty.$$

Notice that the elements of $\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R}\times\Omega)$ are $\|\cdot\|_{p,\mathrm{RV}}$ absolutely integrable s.p.'s (see (5)). Observe that if $f \in \mathrm{L}_p^{\mathrm{SP}}(\mathbb{R}\times\Omega)$, then the expectation $\mathbb{E}[|f(v)|^p]$ exists and is finite for every $v \in \mathbb{R}$ fixed (otherwise would not make sense the definition of the space $\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R}\times\Omega)$ given in (7)). Hence, for every $v \in \mathbb{R}$ fixed, $f(v)$ is a r.v. of the space $\mathrm{L}_p^{\mathrm{RV}}(\Omega)$.

Lemma 2.3. *Let us consider the matrix s.p.*

$$W(t) = \begin{bmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{bmatrix}, \quad t \in \mathcal{T},$$

where $\mathcal{T} \subset \mathbb{R}$ and entries $w_{ij}(t)$, $1 \le i,j \le 2$, are assumed to be continuous in the $\|\cdot\|_{2p,RV}$ norm and satisfying the following condition

$$\exists\, \epsilon > 0 \text{ such that } \quad \mathbb{E}\left[(w_{i,j}(s))^{2p}\right] < +\infty, \quad \forall s \in]t-\epsilon, t+\epsilon[, \epsilon > 0, \quad i,j : 1 \le i,j \le 2, \quad p \ge 1, \forall t \in \mathcal{T}. \tag{8}$$

Let

$$Z(t) = \begin{bmatrix} z_{11}(t) & z_{12}(t) \\ z_{21}(t) & z_{22}(t) \end{bmatrix}, \quad t \in \mathcal{T},$$

be a matrix s.p. satisfying

$$Z(t)W(t) = W(t)Z(t) = I_2, \quad \forall t \in]t_0-\delta, t_0+\delta[, \quad t_0 \in \mathcal{T}, \quad \delta > 0, \tag{9}$$

where I_2 denotes the identity matrix of size 2. Then the entries of $Z(t)$, that is, the entries of the inverse matrix of $W(t)$ in a neighbourhood of t_0, are given by

$$\begin{aligned} z_{11}(t) &= \frac{\begin{vmatrix} 1 & w_{12}(t) \\ 0 & w_{22}(t) \end{vmatrix}}{\begin{vmatrix} w_{21}(t) & w_{22}(t) \end{vmatrix}}, \quad z_{12}(t) = \frac{\begin{vmatrix} 0 & w_{12}(t) \\ 1 & w_{22}(t) \end{vmatrix}}{\begin{vmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{vmatrix}}, \\ z_{21}(t) &= \frac{\begin{vmatrix} w_{11}(t) & 1 \\ w_{21}(t) & 0 \end{vmatrix}}{\begin{vmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{vmatrix}}, \quad z_{22}(t) = \frac{\begin{vmatrix} w_{11}(t) & 0 \\ w_{21}(t) & 1 \end{vmatrix}}{\begin{vmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{vmatrix}}, \end{aligned} \tag{10}$$

Proof. As $w_{ij}(t)$, $1 \le i,j \le 2$, are continuous in the $2p$-norm and verifying condition (8), then by proposition 3 of Casabán et al,[26] it is guaranteed that the determinant of matrix s.p. $W(t)$ is continuous in the p norm. Hence, there exists $t_0 \in \mathcal{T}$ and $\delta > 0$, such that

$$\det(W(t)) \neq 0, \quad \forall t \in]t_0-\delta, t_0+\delta[.$$

Therefore, $W(t)$ is invertible in $]t_0-\delta, t_0+\delta[$, for each $\omega \in \Omega$ fixed. Let us denote $Z(t) = (W(t))^{-1}$, $t \in]t_0-\delta, t_0+\delta[$ arbitrary but fixed satisfying (9), ie,

$$\left.\begin{array}{l} w_{11}(t)(\omega) \quad z_{11}(t)(\omega) + w_{12}(t)(\omega) \quad z_{21}(t)(\omega) = 1\,, \\ w_{21}(t)(\omega) \quad z_{11}(t)(\omega) + w_{22}(t)(\omega) \quad z_{21}(t)(\omega) = 0\,, \end{array}\right\} \tag{11}$$

$$\left.\begin{array}{l} w_{11}(t)(\omega) \quad z_{12}(t)(\omega) + w_{12}(t)(\omega) \quad z_{22}(t)(\omega) = 0\,, \\ w_{21}(t)(\omega) \quad z_{12}(t)(\omega) + w_{22}(t)(\omega) \quad z_{22}(t)(\omega) = 1\,, \end{array}\right\} \tag{12}$$

Applying the Cramer's rule for each realization $\omega \in \Omega$ of systems (11)-(12), one gets (10). □

2.1 | Random differential equations and random improper integrals

For the sake of clarity in the presentation of results of the next sections and in order to help the reader, in this section, we summarize and adapt some results related to the analytic and numerical solutions of random differential equations and integrals.

Consider the random second-order initial value problem

$$\left.\begin{array}{rl} x''(t) + a(t)x(t) \;=\; 0\,, & t > 0\,, \\ x(0) \;=\; x_0\,, & \\ x'(0) \;=\; x_1\,, & \end{array}\right\} \tag{13}$$

where $a(t)$ is a s.p. and x_0, x_1 are r.v.'s with properties to be determined later. It is known that problem (13) is equivalent to the linear first-order vector problem

$$\left.\begin{array}{rl} Y'(t) \;=\; L(t)Y(t)\,,\; t > 0\,, \\ Y(0) \;=\; Y_0\,, \end{array}\right\} \tag{14}$$

where

$$Y(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}, \quad L(t) = \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix}, \quad Y(0) = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}. \tag{15}$$

If the system (14)-(15) is p regular (with $p = 2$) in the sense of definition 3 of Casabán et al,[27] then the random fundamental matrix solution $\Phi_L(t;0)$ exists, is invertible, and it and its inverse $\Phi_L^{-1}(t;0)$ both lie in $\mathrm{L}_2^{2\times 2}(\Omega)$. Moreover, they are two-differentiable (ie, mean square differentiable). Hence, let $\{\varphi_1, \varphi_2\}$ be the fundamental set of solutions of scalar random problem (13) satisfying

$$\begin{array}{l} \varphi_1(0) = 1\,,\; \varphi_1'(0) = 0\,, \\ \varphi_2(0) = 0\,,\; \varphi_2'(0) = 1\,. \end{array}$$

Then the unique solution of problem (13) can be written in the form

$$x(t) = \varphi_1(t)x_0 + \varphi_2(t)x_1\,. \tag{16}$$

In terms of data $a(t)$, the property of two-regularity of problem (14)-(15) is satisfied if

$$a(t) \quad \text{is a} \quad \text{4-s.p. differentiable, and there exists} \quad \delta > 0 \quad \text{such that} \quad a(t) \geq \delta > 0 \quad \text{a.s.} \tag{17}$$

Dealing with random numerical solutions, note that a way to approximate (16) is to approximate numerically $\varphi_1(t)$ and $\varphi_2(t)$ using some random multistep methods.[24] The structure of Equation (13) suggests the use of a random Störmer-type method for solving initial value problems of the form

$$\varphi_1''(t) + a(t)\varphi_1(t) = 0, \quad \varphi_1(0) = 1\,, \quad \varphi_1'(0) = 0\,,$$

$$\varphi_2''(t) + a(t)\varphi_2(t) = 0, \quad \varphi_2(0) = 0\,, \quad \varphi_2'(0) = 1\,.$$

For the sake of completeness, we recall the following Störmer's two-step formula

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f_{n+1}, \quad n \geq 0,$$

where

$$f_{n+1} = f(t_{n+1}, y_{n+1}), \quad t_{n+1} = (n+1)h, \quad h > 0,$$

for a second-order differential equations of the type

$$y''(t) = f(t, y(t)), \quad y(0) = \eta, \quad y'(0) = \eta', \quad 0 \le t \le \mathcal{T},$$

see Henrici[36], p. 291 and Jódar and Pérez.[3]

It is well known (see, for instance, Farlow[1] and Jódar and Pérez[3]), that when using a integral transform approach for solving partial differential equations models, the solution is expressed via improper integrals related to the corresponding inversion formulae linked to the integral transform used in the corresponding problem. Thus, in the random framework, it is suitable to have such a tool to take advantage of its key properties such as the good approximation with very few evaluations of the integrand function using appropriate quadrature rules; see Davis and Rabinowitz[37] and Delves and Mohamed.[38]

Let $\mathfrak{h}(\xi)$ be s.p. lying in $\mathrm{L}_2^{\mathrm{SP}}(\mathbb{R} \times \Omega)$, then

$$J = J[\mathfrak{h}] = \int_{-\infty}^{+\infty} \mathfrak{h}(\xi) \exp\left(-\xi^2\right) \mathrm{d}\xi < \infty, \tag{18}$$

i.e., stochastic integral J is m.s. convergent. Note that taking an event $\omega \in \Omega$, the sampled integral associated to (18)

$$J(\omega) = J[\mathfrak{h}](\omega) = \int_{-\infty}^{+\infty} \mathfrak{h}(\xi; \omega) \exp\left(-\xi^2\right) \mathrm{d}\xi, \quad \omega \in \Omega,$$

is well-defined and convergent for all $\omega \in \Omega$, see appendix 1 of Soong.[33] Now, we can use Gauss-Hermite quadrature formula of degree N,[37,38] and then we consider the approximation

$$J_N^{\text{G-H}}[\mathfrak{h}](\omega) = \sum_{j=1}^{N} \rho_j \; \mathfrak{h}(\xi_{j,H}; \omega), \quad \rho_j = \frac{2^{N+1} N! \sqrt{\pi}}{\left(H_N'(\xi_{j,H})\right)^2}, \quad 1 \le j \le N, \quad \omega \in \Omega, \tag{19}$$

where $\xi_{j,H}$ are the roots of the deterministic Hermite polynomial, H_N, of degree N.

3 | RANDOM INTEGRAL-DIFFERENTIAL NUMERICAL APPROXIMATIONS

In this section, we proceed to generate the approximations of the random time-dependent partial differential initial value problem (1)-(3). Let us assume that problem (1)-(3) admits a solution s.p. $u(\cdot, t)$ such that it and its partial derivatives $u_x(\cdot, t)$, $u_t(\cdot, t)$, $u_{xx}(\cdot, t)$, and $u_{tt}(\cdot, t)$, regarded as functions of the active variable x, lie in $\mathrm{L}_p^{\mathrm{SP}}(\mathbb{R} \times \Omega)$.

Let $t > 0$ be fixed and let

$$U(t)(\xi) = \mathfrak{F}\left[u(\cdot, t)\right](\xi).$$

By applying Fourier transform to both sides of Equation (1) and using the properties of Fourier transform and derivatives, see (6), one gets

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(U(t)(\xi)\right) + \xi^2 c(t) U(t)(\xi) = 0, \quad t > 0, \tag{20}$$

$$U(0)(\xi) = F(\xi), \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(U(0)\right)(\xi) = G(\xi). \tag{21}$$

The associated linear first-order vector problem of (20)-(21) is given by (14)-(15), where

$$Y(t) = \begin{bmatrix} U(t)(\xi) \\ U'(t)(\xi) \end{bmatrix}, \quad L(t) = \begin{bmatrix} 0 & 1 \\ -\xi^2 c(t) & 0 \end{bmatrix}, \quad Y(0) = \begin{bmatrix} F(\xi) \\ G(\xi) \end{bmatrix}. \tag{22}$$

As we shown in Section 2, the random fundamental matrix solution of problem (14) via (22) exists and hereinafter we will denote it by

$$\Phi_L(t; 0)(\xi) = \begin{bmatrix} \varphi_1(t, \xi) & \varphi_2(t, \xi) \\ \varphi_1'(t, \xi) & \varphi_2'(t, \xi) \end{bmatrix},$$

where $\{\varphi_1(t,\xi), \varphi_2(t,\xi)\}$ are the fundamental set of solutions of differential equation of (20) verifying

$$\begin{aligned} \varphi_1(0,\xi) = 1\,,\ \varphi_1'(0,\xi) = 0\,, \\ \varphi_2(0,\xi) = 0\,,\ \varphi_2'(0,\xi) = 1\,. \end{aligned} \tag{23}$$

Due to $\Phi_L(0;0)(\xi) = I_2$, there is a neighbourhood of $t = 0$, denoted by $\mathcal{N}(0)$, where there exists the inverse of $\Phi_L(t;0)(\xi)$ given by

$$\Phi_L^{-1}(t;0)(\xi) = \begin{bmatrix} z_{11}(t,\xi) & z_{12}(t,\xi) \\ z_{21}(t,\xi) & z_{22}(t,\xi) \end{bmatrix}.$$

The entries of $\Phi_L^{-1}(t;0)(\xi)$, $t \in \mathcal{N}(0)$, are the solutions of the algebraic system

$$\Phi_L(t;0)(\xi)\,\Phi_L^{-1}(t;0)(\xi) = I_2\,.$$

By Lemma 2.3, the entries of $\Phi_L^{-1}(t;0)(\xi)$ are given by

$$z_{11}(t,\xi) = \frac{\varphi_2'(t,\xi)}{D(t,\xi)}, \quad z_{12}(t,\xi) = \frac{-\varphi_2(t,\xi)}{D(t,\xi)}, \quad z_{21}(t,\xi) = \frac{-\varphi_1'(t,\xi)}{D(t,\xi)}, \quad z_{22}(t,\xi) = \frac{\varphi_1(t,\xi)}{D(t,\xi)},$$

where

$$D(t,\xi) = \varphi_1(t,\xi)\,\varphi_2'(t,\xi) - \varphi_1'(t,\xi)\,\varphi_2(t,\xi), \quad t \in \mathcal{N}(0).$$

Let us assume $c(t)$ verifying condition (17), as $\varphi_1(t,\xi)$, $\varphi_2(t,\xi)$ are twice mean square differentiable, then the solutions of (20), (23) also lie in $\mathrm{L}_4^{\mathrm{RV}}(\Omega)$ (see Prop. 2.2), and thus, $D(t,\xi)$ lies in $\mathrm{L}_2^{\mathrm{RV}}(\Omega)$. By the properties of a fundamental set of solutions of Equation (20) (see Calbo et al[39]), it follows that

$$U(t)(\xi) = \varphi_1(t,\xi)F(\xi) + \varphi_2(t,\xi)\,G(\xi). \tag{24}$$

By the random inverse Fourier transform and (24), it follows that the formal solution s.p. of problem (1)-(3) is given by

$$\begin{aligned} u(x,t) &= \mathrm{Re}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} U(t)(\xi)\,\exp(\mathrm{i}\xi x)\,\mathrm{d}\xi\right] = \frac{1}{\sqrt{2\pi}}\mathrm{Re}\left[\int_{-\infty}^{+\infty}\left\{\varphi_{1,j}(t,\xi)F(\xi) + \varphi_{2,j}(t,\xi)\,G(\xi)\right\}\exp(\mathrm{i}\xi x)\,\mathrm{d}\xi\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left\{\varphi_{1,j}(t,\xi)F(\xi) + \varphi_{2,j}(t,\xi)\,G(\xi)\right\}\cos(\xi x)\,\mathrm{d}\xi\,, \end{aligned} \tag{25}$$

for $t \in \mathcal{N}(0)$ and $x \in \mathbb{R}$, and being $F(\xi)$ and $G(\xi)$ the respective Fourier transforms of initial conditions $f(x)$ and $g(x)$ of problem (1)-(3). Notice that we have introduced the subindex j in (25) for the fundamental set of solutions $\{\varphi_{1,j}(t,\xi), \varphi_{2,j}(t,\xi)\}$. The first approximation solution process turns out by applying the Gauss-Hermite quadrature formula of degree N, see end of Section 2,

$$J_N^{\text{G-H}}\left(u(x,t)\right) = \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{N}\rho_j\,\left\{\varphi_{1,j}(t,\xi_{H_j})F(\xi_{H_j}) + \varphi_{2,j}(t,\xi_{H_j})\,G(\xi_{H_j})\right\}\cos(x\,\xi_{H_j})\exp(\xi_{H_j}^2), \tag{26}$$

where ρ_j are defined by (19) and ξ_{H_j} are the roots of the deterministic Hermite polynomial, H_N, of degree N. Note that $\varphi_{1,j}(\cdot,\xi_{H_j})$ and $\varphi_{2,j}(\cdot,\xi_{H_j})$ are the exact solutions of problems

$$\varphi_{1,j}''(t) + \xi_{H_j}^2\,c(t)\,\varphi_{1,j}(t) = 0, \quad \varphi_{1,j}(0) = 1, \quad \varphi_{1,j}'(0) = 0, \quad j = 1, \ldots, N, \tag{27}$$

$$\varphi_{2,j}''(t) + \xi_{H_j}^2\,c(t)\,\varphi_{2,j}(t) = 0, \quad \varphi_{2,j}(0) = 0, \quad \varphi_{2,j}'(0) = 1, \quad j = 1, \ldots, N. \tag{28}$$

As (27) and (28) are non-autonomous problems (ie, $c(t)$ changes with time), their corresponding solutions $\varphi_{1,j}(\cdot,\xi_{H_j})$ and $\varphi_{2,j}(\cdot,\xi_{H_j})$, respectively, are not known, in general. Thus, we use our second approximation strategy by solving numerically problems (27)-(28) with a random Störmer-type method. We denote by

$$S_1(t,h,j) = S\left[\varphi_{1,j}(t,\xi_{H_j})\right], \quad S_2(t,h,j) = S\left[\varphi_{2,j}(t,\xi_{H_j})\right], \quad \forall j,\, 1 \le j \le N, \tag{29}$$

the approximate values of $\left\{\varphi_{1,j}(t,\xi_{H_j})\right\}_{j=1}^{N}$ and $\left\{\varphi_{2,j}(t,\xi_{H_j})\right\}_{j=1}^{N}$, respectively, at time point t using a Störmer method with stepsize h.

3.1 | Algorithm for computing the random double approximation and its expectation and standard deviation

The following approximation procedure can we written for the random double numerical approximation of $u(x,t)$, given by (25), at a fixed point (x,t).

Step 1. Select the degree N of the Hermite polynomial $H_N(\cdot)$ and obtain the j roots, ξ_{H_j}, of $H_N(\cdot)$ for $1 \le j \le N$. Compute the weights ρ_j using (19).
Step 2. Evaluate $\cos\left(x\xi_{H_j}\right)$ for $1 \le j \le N$.
Step 3. Select a stepsize $h > 0$ and compute the approximations $S_1(t,h,j)$ and $S_2(t,h,j)$ for all $1 \le j \le N$, defined in (29), using the random Störmer method, described in Section 2.1, for random IVP's (27) and (28), respectively.
Step 4. Evaluate the initial conditions, $F(\xi_{H_j})$ and $G(\xi_{H_j})$ for $1 \le j \le N$, given in (21).
Step 5. Compute the numerical approximation of the solution s.p. (25), which for convenience will be denoted by $u_{N,h}(x,t)$, using (26)-(29),

$$u_{N,h}(x,t) = \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{N}\rho_j\left\{S_1(t,h,j)F(\xi_{H_j}) + S_2(t,h,j)G(\xi_{H_j})\right\}\cos(x\xi_{H_j})\exp(\xi_{H_j}^2). \tag{30}$$

Step 6. Assuming that input data s.p.'s $c(t)$, $f(x)$ and $g(x)$ of the problem (1)-(3) are independent, to compute the expectation of the approximation solution s.p. (30) using to the following expression

$$\mathbb{E}\left[u_{N,h}(x,t)\right] = \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{N}\rho_j\left\{\mathbb{E}\left[S_1(t,h,j)\right]\mathbb{E}\left[F(\xi_{H_j})\right] + \mathbb{E}\left[S_2(t,h,j)\right]\mathbb{E}\left[G(\xi_{H_j})\right]\right\}\cos(x\xi_{H_j})\exp(\xi_{H_j}^2). \tag{31}$$

Step 7. Compute the standard deviation of the approximation solution s.p. (30) using the following expression

$$\sqrt{\mathrm{Var}\left[u_{N,h}(x,t)\right]} = \sqrt{\mathbb{E}\left[\left(u_{N,h}(x,t)\right)^2\right] - \left(\mathbb{E}\left[u_{N,h}(x,t)\right]\right)^2}, \tag{32}$$

where

$$\begin{aligned}\mathbb{E}\left[\left(u_{N,h}(x,t)\right)^2\right] = \frac{1}{2\pi}\sum_{j=1}^{N}\sum_{k=1}^{N}\rho_j\,\rho_k\,\Big\{&\mathbb{E}\left[S_1(t,h,j)S_1(t,h,k)\right]\mathbb{E}\left[F(\xi_{H_j})F(\xi_{H_k})\right] + \\ &\mathbb{E}\left[S_2(t,h,j)S_2(t,h,k)\right]\mathbb{E}\left[G(\xi_{H_j})G(\xi_{H_k})\right]\Big\} \\ \times\quad &\cos(x\xi_{H_j})\cos(x\xi_{H_k})\exp(\xi_{H_j}^2)\exp(\xi_{H_k}^2).\end{aligned} \tag{33}$$

4 | NUMERICAL EXAMPLES

This section is devoted to validate the theoretical results previously established by means of two examples. The first one is a test problem, where exact expressions for both the mean and the standard deviation of the solution s.p. are available. We are going to compute these two statistical moments, using our analytic-numerical double approximation, and then checking the numerical values are close to the corresponding exact ones. In this test example, the diffusion coefficient $c(t)$ is constant, ie, a r.v., while in the second example, $c(t)$ is a s.p. In this latter case, the convergence of approximations for the mean and the standard deviation is assessed via relative errors of consecutive approximations since no exact solution is available.

4.1 | Example 1

Let us consider the following particular problem of (1)-(3)

$$\left.\begin{aligned} u_{tt}(x,t) &= c\,u_{xx}(x,t)\,, && -\infty < x < +\infty\,,\ t>0\,,\\ u(x,0) &= A\exp\left(-\tfrac{x^2}{2}\right), && -\infty < x < +\infty\,,\\ u_t(x,0) &= 0\,, && -\infty < x < +\infty\,, \end{aligned}\right\} \tag{34}$$

being $c = c(\omega)$, $\omega \in \Omega$, a r.v. with a beta distribution of parameters $(3;5)$, $c \sim \text{Beta}(3;5)$, and $A = A(\omega)$, $\omega \in \Omega$, an exponential r.v. of parameter $\lambda = 2$, $A \sim \text{Exp}(2)$. Hereinafter, we will assume that c and A are independent r.v.'s. It is known that the solution of problem (34), called d'Alembert solution when both A and c are deterministic, is given by

$$u(x,t) = \frac{1}{2}A\left\{\exp\left(-\frac{(x-ct)^2}{2}\right) + \exp\left(-\frac{(x+ct)^2}{2}\right)\right\}; \tag{35}$$

see Myint-U and Debnath.[2] In our context, both c and A are r.v.'s, and expression (35) must be interpreted as a s.p. Using the independence between r.v.'s c and A, it can be seen that the expectation and the standard deviation of s.p. (35) are, respectively, given by

$$\mathbb{E}\left[u(x,t)\right] = \frac{1}{2}\mathbb{E}\left[A\right]\left\{\mathbb{E}\left[\exp\left(-\frac{(x-ct)^2}{2}\right)\right] + \mathbb{E}\left[\exp\left(-\frac{(x+ct)^2}{2}\right)\right]\right\}, \quad t \geq 0\,, \tag{36}$$

$$\sqrt{\text{Var}\left[u(x,t)\right]} = \sqrt{\mathbb{E}\left[(u(x,t))^2\right] - \left(\mathbb{E}\left[u(x,t)\right]\right)^2}, \quad t \geq 0\,, \tag{37}$$

being

$$\mathbb{E}\left[(u(x,t))^2\right] = \frac{1}{4}\mathbb{E}\left[A^2\right]\left\{\mathbb{E}\left[\exp\left(-(x-ct)^2\right)\right] + \mathbb{E}\left[\exp\left(-(x+ct)^2\right)\right] + 2\mathbb{E}\left[\exp\left(-(x^2+c^2t^2)\right)\right]\right\}, \quad t \geq 0\,. \tag{38}$$

In Figure 1, we have plotted the exact mean (plot (a)) and the exact standard deviation (plot (b)) over the domain $(x,t) \in [-5,\ 5] \times [0,\ 0.5]$. These statistical moments have been computed using expressions (36)-(38). From these plots, we observe that the mean and the standard deviation behave similarly. They are symmetric with respect to the origin of coordinates and outside the spatial domain $[-5,\ 5]$ both tend to zero.

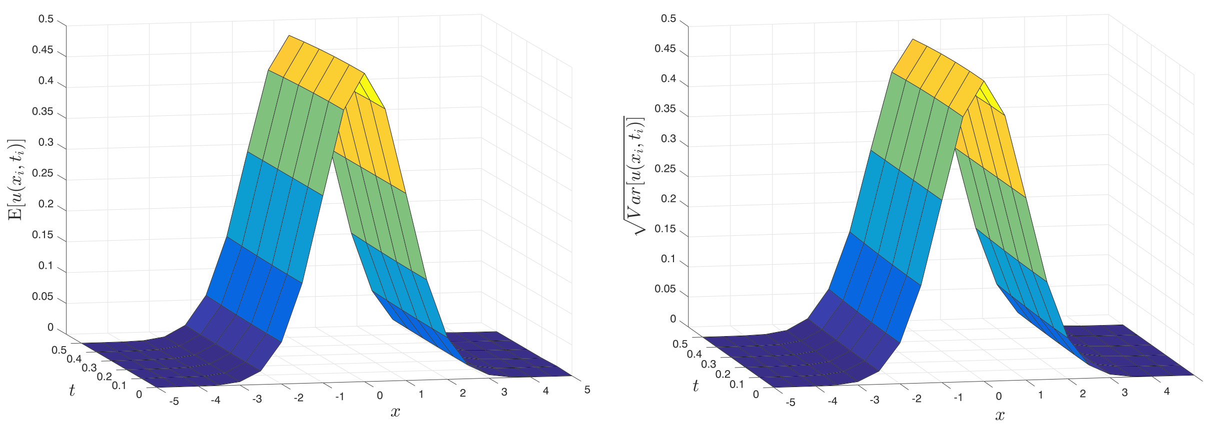

FIGURE 1 Plot (left): Surface of the expectation, $\mathbb{E}[u(x,t)]$, computed according to (36). Plot (right): Surface of the standard deviation, $\sqrt{\text{Var}[u(x,t)]}$, computed according to (37)-(38). Both statistical moment functions correspond to the exact solution s.p. (35) of the problem (34) in Example 1, on the domain $(x,t) \in [-5,\ 5] \times [0,\ 0.5]$ and considering $c \sim \text{Beta}(3;5)$ and $A \sim \text{Exp}(2)$ [Colour figure can be viewed at wileyonlinelibrary.com]

In order to compute the first and second statistical moments of the approximation s.p. (30), we are going to follow steps 1 to 7 of the algorithm shown in Section 3.1. Let us fix a degree N of the Hermite polynomial, $H_N(\cdot)$, and let us obtain its roots $\left\{\xi_{H_j} \quad : \quad 1 \le j \le N\right\}$ and its weights ρ_j, defined by (19). Now, we consider $t = T > 0$ fixed, and we choose the time-step $h > 0$ such that the intermediate time instants are obtained by $t_m = mh$, $0 \le m \le M$, being M an integer. Observe that $T = Mh$. Then we are going to compute the approximations $\{S_1(t,h,j)\}_{j=1}^{N}$ and $\{S_2(t,h,j)\}_{j=1}^{N}$ of problems (27) and (28), being $c(t) = c$, $c \sim$ Beta(3; 5), by applying the Störmer's two-step formula; see Section 2.1. However, in this case, due to the zero value of the initial condition of problem (34), $u_t(x,0) = 0$, it does $G(\xi_{H_j}) = 0$ in (30). Then it will not be required to compute the approximations $\{S_2(t,h,j)\}_{j=1}^{N}$ of (28). The Störmer's two-step formula applied to the N-problems of (27) requires two initial discrete conditions for initializing each of them. With this end, we write each one the N-problems of (27) as a linear system of first-order

$$\Phi_j'(t) = \mathcal{H}_j\,\Phi_j(t), \quad \Phi_j(t_0) = \Phi_j(0) = \begin{bmatrix}1\\0\end{bmatrix}, \quad 1 \le j \le N, \quad \text{with } \mathcal{H}_j = \begin{bmatrix}0 & 1\\ -\xi_{H_j}^2\,c & 0\end{bmatrix}, \quad \Phi_j(t) = \begin{bmatrix}\varphi_{1,j}(t)\\ \varphi_{1,j}'(t)\end{bmatrix}. \tag{39}$$

Applying Euler method to (39), one obtains the following approximations, $\tilde{\Phi}_j(t_{n+1})$, of (39) for each j,

$$\tilde{\Phi}_j(t_{n+1}) = \tilde{\Phi}_j(t_n) + k\mathcal{H}_j\,\tilde{\Phi}_j(t_n), \quad t_n = nk,\ n \ge 0 \text{ integer}, \quad k > 0, \qquad 1 \le j \le N,$$

where taking $n = 0$ one gets

$$\tilde{\Phi}_j(t_1) = \begin{bmatrix}\tilde{\varphi}_{1,j}(t_1)\\ \tilde{\varphi}_{1,j}'(t_1)\end{bmatrix} = \tilde{\Phi}_j(t_0) + k\mathcal{H}_j\,\tilde{\Phi}_j(t_0) = \begin{bmatrix}1\\0\end{bmatrix} + k\begin{bmatrix}0 & 1\\ -\xi_{H_j}^2\,c & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\ -k\,\xi_{H_j}^2\,c\end{bmatrix}, \quad 1 \le j \le N.$$

In order to initialize the Störmer's two-step formula applied to (27) for $1 \le j \le N$, we take the approximate value $\tilde{\varphi}_{1,j}(t_1) = 1$. Then we solve the following N-discrete problems

$$\begin{cases} \tilde{\varphi}_{1,j}(t_{m+2}) - 2\tilde{\varphi}_{1,j}(t_{m+1}) + \tilde{\varphi}_{1,j}(t_m) = -h^2\,\xi_{H_j}^2\,c \quad \tilde{\varphi}_{1,j}(t_{m+1}), \quad 0 \le m \le M-2, \quad t_m = mh, \quad 1 \le j \le N,\\ \tilde{\varphi}_{1,j}(t_0) = \tilde{\varphi}_{1,j}(0) = 1,\\ \tilde{\varphi}_{1,j}(t_1) = 1. \end{cases} \tag{40}$$

For each j, $1 \le j \le N$, discrete problem (40) admits the following explicit solution

$$\begin{aligned} \tilde{\varphi}_{1,j}(t_m) = {} & \frac{1}{\sqrt{-4\cdot 10^6 + c\quad \xi_{H_j}^2}}\,10^{6-m}\left\{-\sqrt{c}\,\xi_{H_j}\left(2\cdot 10^6 - c\xi_{H_j}^2 - \sqrt{c}\,\xi_{H_j}\sqrt{-4\cdot 10^6 + c\xi_{H_j}^2}\right)^m\right.\\ & + \sqrt{-4\cdot 10^6 + c\xi_{H_j}^2}\left(2\cdot 10^6 - c\xi_{H_j}^2 - \sqrt{c}\xi_{H_j}\sqrt{-4\cdot 10^6 + c\xi_{H_j}^2}\right)^m\\ & + \sqrt{c}\,\xi_{H_j}\left(2\cdot 10^6 - c\xi_{H_j}^2 + \sqrt{c}\,\xi_{H_j}\sqrt{-4\cdot 10^6 + c\xi_{H_j}^2}\right)^m\\ & \left. + \sqrt{-4\cdot 10^6 + c\xi_{H_j}^2}\left(2\cdot 10^6 - c\xi_{H_j}^2 + \sqrt{c}\xi_{H_j}\sqrt{-4\cdot 10^6 + c\xi_{H_j}^2}\right)^m\right\}, \quad \forall m \ge 0. \end{aligned} \tag{41}$$

Note that, in the general context described in (29), $S_1(t,h,j) = S\left[\varphi_{1,j}(t,\xi_{H_j})\right]$ corresponds with $\tilde{\varphi}_{1,j}(t_m)$ of (4.1). Expression (4.1) allows us direct computation of any term for the sequence of approximations $\tilde{\varphi}_{1,j}(t_2)$, $\tilde{\varphi}_{1,j}(t_3)$, …. In our case, for computing the solution in $t = T = 0.25$ considering, for example, $h = 0.05$, it is sufficient to take $m = 5$ in (4.1); that is, $M = 7$. This calculation must be carried out for each value of j, $1 \le j \le N$.

Using expressions in steps 6 and 7 of the algorithm of Section 3.1, in Figures 2 and 3, we have plotted the first and second moments of the numerical solution s.p. (30), $\mathbb{E}[u_{N,h}(x_i,t)]$ and $\sqrt{\mathrm{Var}[u_{N,h}(x_i,t)]}$, respectively. These moments have been compared with the corresponding exact ones, $\mathbb{E}[u(x_i,t)]$ and $\sqrt{\mathrm{Var}[u(x_i,t)]}$, respectively, at $t = T = 0.25$, for different degrees N of the Hermite polynomials: $N = \{3,5,8\}$. It is observed that the approximations improve as the degree N increases. Computations have been carried out by Mathematica software version 11.0.0.0 for Mac OS X x86 (32-bit, 64-bit

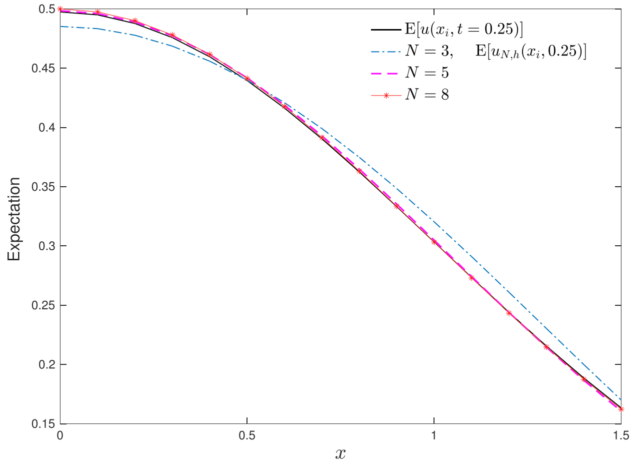

FIGURE 2 Comparative graphics of the expectations in the final time $t = T = 0.25$ on the spatial interval $0 \le x \le 1.5$ in Example 2: The expectation of the exact solution s.p., $\mathbb{E}[u(x_i, t)]$, (36), vs the expectations of the approximation solution s.p., $\mathbb{E}[u_{N,h}(x_i, t)]$, (31), using several degrees N of the Hermite polynomial, $N = \{3, 5, 8\}$ for the stepsize $h = 0.05$ in Equation (40) [Colour figure can be viewed at wileyonlinelibrary.com]

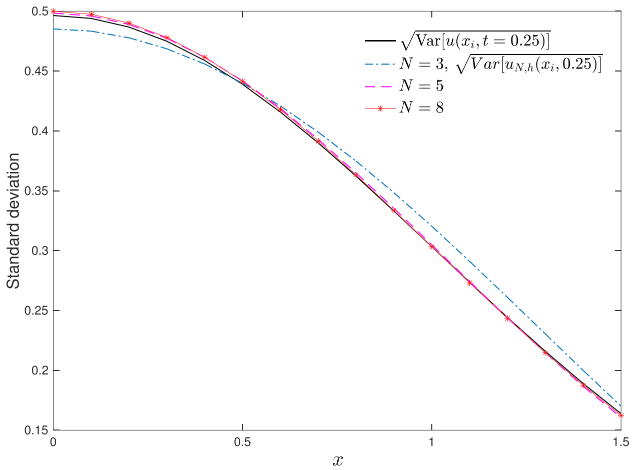

FIGURE 3 Comparative graphics of the standard deviations in the final time $t = T = 0.25$ on the spatial interval $0 \le x \le 1.5$ in Example 1: The standard deviation of the exact solution s.p., $\sqrt{\mathrm{Var}[u(x_i, t)]}$, (37)-(38), vs the standard deviations of the approximation solution s.p., $\sqrt{\mathrm{Var}[u_{N,h}(x_i, t)]}$, (32)-(33), using several degrees N of the Hermite polynomial, $N = \{3, 5, 8\}$, for the stepsize $h = 0.05$ in Equation (40). [Colour figure can be viewed at wileyonlinelibrary.com]

Kernel) Intel Core i7, 2.8-GHz 4 kernels. Regarding computational time for the approximations of the mean and the standard deviation have been 0.48982 s and 108.923045 s, respectively. These timings correspond to the most expensive scenario; that is, $T = 0.5$, $h = 0.05$, and $N = 8$. Therefore, the proposed method is computationally cheap.

In Tables 1 and 2, we show, for $T = 0.25$ and $T = 0.5$, respectively, the numerical values of the relative errors for the approximate expectation (31), $\mathrm{RelErr}\left[\mathbb{E}[u_{N,h}(x,t)]\right]$, and the approximate standard deviation (32)-(33), $\mathrm{RelErr}\left[\sqrt{\mathrm{Var}[u_{N,h}(x,t)]}\right]$, using the following expressions

$$\mathrm{RelErr}\left[\mathbb{E}[u_{N,h}(x,t)]\right] = \left|\frac{\mathbb{E}[u(x,t)] - \mathbb{E}\left[u_{N,h}(x,t)\right]}{\mathbb{E}[u(x,t)]}\right|, \quad \mathrm{RelErr}\left[\sqrt{\mathrm{Var}[u_{N,h}(x,t)]}\right] = \left|\frac{\sqrt{\mathrm{Var}[u(x,t)]} - \sqrt{\mathrm{Var}[u_{N,h}(x,t)]}}{\sqrt{\mathrm{Var}[u(x,t)]}}\right|. \tag{42}$$

These values have been computed using (40) for some stepsize, $h = \{0.025, 0.05, 0.1, 0.25, 0.5\}$, and considering different values for the degree of the Hermite polynomials $N = \{3, 5, 8\}$. Because the symmetry with respect to the origin of coordinates and the fast trend to zero of first and second moments shown in Figure 1, we have chosen the spatial points $x_i = \{0.6, 0.7, 0.8, 0.9, 1\}$ in order to compare exact and approximate values via relative errors. In Table 1, we have collected the relative errors for the mean and the standard deviation, and we can observe that approximations are quite good as N increases and h decreases. In Figures 2 and 3, it can be observed this behaviour for the stepsize $h = 0.05$. In Table 2, it is shown a similar behaviour for $t = 0.5$ and stepsizes $h = \{0.05, 0.1, 0.5\}$.

TABLE 1 Relative errors of the expectations and the standard deviations of the approximation solution s.p., $\mathbb{E}[u_{N,h}(x_i,t)]$, (31), and $\sqrt{\mathrm{Var}[u_{N,h}(x_i,t)]}$, (32)-(33), respectively

		$\mathrm{RelErr}\left[\mathbb{E}[u_{N,h}(x_i,t=0.25)]\right]$			$\mathrm{RelErr}\left[\sqrt{\mathrm{Var}[u_{N,h}(x_i,t=0.25)]}\right]$		
x_i	N	$h=0.125$	$h=0.05$	$h=0.025$	$h=0.125$	$h=0.05$	$h=0.025$
	3	1.09596e−02	1.09571e−02	1.09483e−02	1.22932e−02	1.22907e−02	1.22819e−02
0.6	5	5.83520e−03	5.83269e−03	5.82387e−03	7.16208e−03	7.15955e−03	7.15074e−03
	8	3.47312e−03	3.47072e−03	3.46224e−03	4.79689e−03	4.79446e−03	4.78610e−03
	3	2.22172e−02	2.22149e−02	2.22066e−02	2.32968e−02	2.32945e−02	2.32862e−02
0.7	5	6.44359e−03	6.44150e−03	6.43417e−03	7.50653e−03	7.50443e−03	7.49710e−03
	8	2.81679e−03	2.81486e−03	2.80810e−03	3.87589e−03	3.87396e−03	3.86720e−03
	3	3.40072e−02	3.40051e−02	3.39975e−02	3.47844e−02	3.47822e−02	3.47747e−02
0.8	5	6.65947e−03	6.65788e−03	6.65233e−03	7.41611e−03	7.41452e−03	7.40897e−03
	8	2.02556e−03	2.02419e−03	2.01942e−03	2.77872e−03	2.77735e−03	2.77257e−03
	3	4.56537e−02	4.56518e−02	4.56452e−02	4.60776e−02	4.60757e−02	4.60690e−02
0.9	5	6.29458e−03	6.29360e−03	6.29016e−03	6.70245e−03	6.70147e−03	6.69803e−03
	8	1.09496e−03	1.09424e−03	1.09171e−03	1.50073e−03	1.50000e−03	1.49747e−03
	3	5.62723e−02	5.62708e−02	5.62653e−02	5.62900e−02	5.62884e−02	5.62829e−02
1	5	5.15620e−03	5.15592e−03	5.15494e−03	5.17296e−03	5.17268e−03	5.17170e−03
	8	2.60781e−05	2.60747e−05	2.60628e−05	4.27526e−05	4.27492e−05	4.27373e−05

Note. They have been computed by expressions (42) at $t = 0.25$ on the spatial points $x_i = \{0.6, 0.7, 0.8, 0.9, 1\}$. Different degrees N of the Hermite polynomial $N = \{3, 5, 8\}$ and stepsizes $h = \{0.025, 0.05, 0.125\}$ in Equation (40) have been considered.

TABLE 2 Relative errors of the expectations and the standard deviations of the approximation solution s.p., $\mathbb{E}[u_{N,h}(x_i,t)]$, (31), and $\sqrt{\mathrm{Var}[u_{N,h}(x_i,t)]}$, (32)-(33), respectively

		$\mathrm{RelErr}\left[\mathbb{E}[u_{N,h}(x_i,t=0.5)]\right]$			$\mathrm{RelErr}\left[\sqrt{\mathrm{Var}[u_{N,h}(x_i,t=0.5)]}\right]$		
x_i	N	$h=0.25$	$h=0.1$	$h=0.05$	$h=0.25$	$h=0.1$	$h=0.05$
	3	2.10946e−02	2.10921e−02	2.10832e−02	2.63078e−02	2.63053e−02	2.62964e−02
0.6	5	1.59189e−02	1.59163e−02	1.59074e−02	2.11057e−02	2.11031e−02	2.10942e−02
	8	1.35331e−02	1.35307e−02	1.35221e−02	1.87077e−02	1.87053e−02	1.86967e−02
	3	3.04145e−02	3.04122e−02	3.04039e−02	3.46852e−02	3.46828e−02	3.46745e−02
0.7	5	1.45144e−02	1.45123e−02	1.45049e−02	1.87191e−02	1.87170e−02	1.87096e−02
	8	1.08585e−02	1.08566e−02	1.08498e−02	1.50481e−02	1.50461e−02	1.50393e−02
	3	3.99147e−02	3.99125e−02	3.99049e−02	4.30559e−02	4.30537e−02	4.30461e−02
0.8	5	1.24107e−02	1.24091e−02	1.24035e−02	1.54688e−02	1.54672e−02	1.54616e−02
	8	7.75031e−03	7.74894e−03	7.74413e−03	1.07945e−02	1.07930e−02	1.07882e−02
	3	4.89067e−02	4.89048e−02	4.88981e−02	5.07188e−02	5.07169e−02	5.07102e−02
0.9	5	9.42510e−03	9.42411e−03	9.42066e−03	1.11690e−02	1.11680e−02	1.11645e−02
	8	4.20930e−03	4.20858e−03	4.20604e−03	5.94419e−03	5.94346e−03	5.94092e−03
	3	5.65010e−02	5.64994e−02	5.64939e−02	5.67745e−02	5.67729e−02	5.67674e−02
1	5	5.37380e−03	5.37352e−03	5.37254e−03	5.63404e−03	5.63376e−03	5.63278e−03
	8	2.42565e−04	2.42562e−04	2.42550e−04	5.01473e−04	5.01469e−04	5.01457e−04

Note. They have been computed by expressions (42) at $t = 0.5$ on the spatial points $x_i = \{0.6, 0.7, 0.8, 0.9, 1\}$. Different degrees N of the Hermite polynomial $N = \{3, 5, 8\}$ and stepsizes $h = \{0.05, 0.1, 0.25\}$ in Equation 40 have been considered.

4.2 | Example 2

Let us consider the following random wave equation of the type (1)-(3)

$$\left.\begin{array}{rl} u_{tt}(x,t) = & c(t)\,u_{xx}(x,t), \quad -\infty < x < +\infty,\ t > 0, \\ u(x,0) = & A\exp\left(-\frac{x^2}{2}\right), \ -\infty < x < +\infty, \\ u_t(x,0) = & B\exp\left(-\frac{x^2}{2}\right), \ -\infty < x < +\infty, \end{array}\right\} \tag{43}$$

being $c(t) = C(\omega)\, t = C\, t$, $\omega \in \Omega$, a s.p. with C a r.v. following a normal distribution of parameters $(1; 0.1)$ and truncated in $[0.5, 1.5]$; that is, $C \sim \mathrm{N}_{[0.5,1.5]}(1; 0.5)$. Note that C satisfies condition (4). The random parameters $A = A(\omega)$ and $B = B(\omega)$, $\omega \in \Omega$ are r.v.'s following an exponential distribution of parameter $\lambda = 1, A \sim \mathrm{Exp}(1)$, and a beta distribution of parameters $(2; 3)$, $B \sim \mathrm{Beta}(2; 3)$, respectively. Hereinafter, we will assume that A, B, and C are independent r.v.'s.

To obtain the expectation and the standard deviation of the approximation to the solution s.p. of problem (43), we follow steps 1-7 of the algorithm from Section 3.1. Since, in this example, the second random initial condition of (43) is nonzero, namely, $u_t(x, 0) = B \exp\left(-\frac{x^2}{2}\right)$, we will have to compute the approximations $\{S_1(t, h, j)\}_{j=1}^{N}$ and $\{S_2(t, h, j)\}_{j=1}^{N}$, corresponding to the two random IVP's (27) and (28), respectively, using the random Störmer method described in Section 2.1 for parameters $t = T$, h, and N fixed. The two initial conditions required for initializing the Störmer's two-step formula can be computed in an analogous way as the one developed in Example 1. Then we will solve the following $2N$-discrete problems

$$\begin{cases} \tilde{\varphi}_{1,j}(t_{m+2}) - 2\,\tilde{\varphi}_{1,j}(t_{m+1}) + \tilde{\varphi}_{1,j}(t_m) = -h^3\, \xi_{H_j}^2\, C \quad (m+1)\, \tilde{\varphi}_{1,j}(t_{m+1}), \quad 0 \le m \le M-2, \quad t_m = mh, \quad 1 \le j \le N, \\ \tilde{\varphi}_{1,j}(0) = 1, \\ \tilde{\varphi}_{1,j}(t_1) = 1, \end{cases} \tag{44}$$

and

$$\begin{cases} \tilde{\varphi}_{2,j}(t_{m+2}) - 2\,\tilde{\varphi}_{2,j}(t_{m+1}) + \tilde{\varphi}_{2,j}(t_m) = -h^3\, \xi_{H_j}^2\, C \quad (m+1)\, \tilde{\varphi}_{2,j}(t_{m+1}), \quad 0 \le m \le M-2, \quad t_m = mh, \quad 1 \le j \le N, \\ \tilde{\varphi}_{1,j}(0) = 0, \\ \tilde{\varphi}_{2,j}(t_1) = h, \end{cases} \tag{45}$$

where s.p., $c(t)$ in (43) has been discretized in t as $c(t_m) = C\, t_m = C\, mh$. Unlike what happened in Example 1, now, there is not available explicit solutions for discrete problems (44) and (45). Instead, fixed the time step h and the final time instant (fixed station) $t = T$, we have symbolically computed for each j, the approximations $\{S_1(T, h, j)\}$ and $\{S_2(T, h, j)\}$ by a loop for the integer $m : 0 \le m \le M - 2$, so that $T = Mh$. Then the N values of the roots of the Hermite polynomial, $\left\{\xi_{H_j} \quad : \quad 1 \le j \le N\right\}$, and its corresponding weights, ρ_j, both defined by (19), are substituted into the symbolic expressions $\{S_1(T, h, j)\}$ and $\{S_2(T, h, j)\}$. Afterwards, we symbolically compute the approximations of both statistical moments, the mean and the standard deviation, using (32)-(33). In Figure 4, we show the approximations of the mean and standard deviation at the final time instant $T = 0.5$ on the spatial domain $0 \le x \le 1.5$ because of the symmetry w.r.t. $x = 0$. Computations have been carried out taking the time stepsize $h = 0.1$ in (44)-(45) and increasing the degree of Hermite polynomials from $N = 3$ to $N = 6$. In both plots, we observe convergence as N increases. Timing was similar for the same

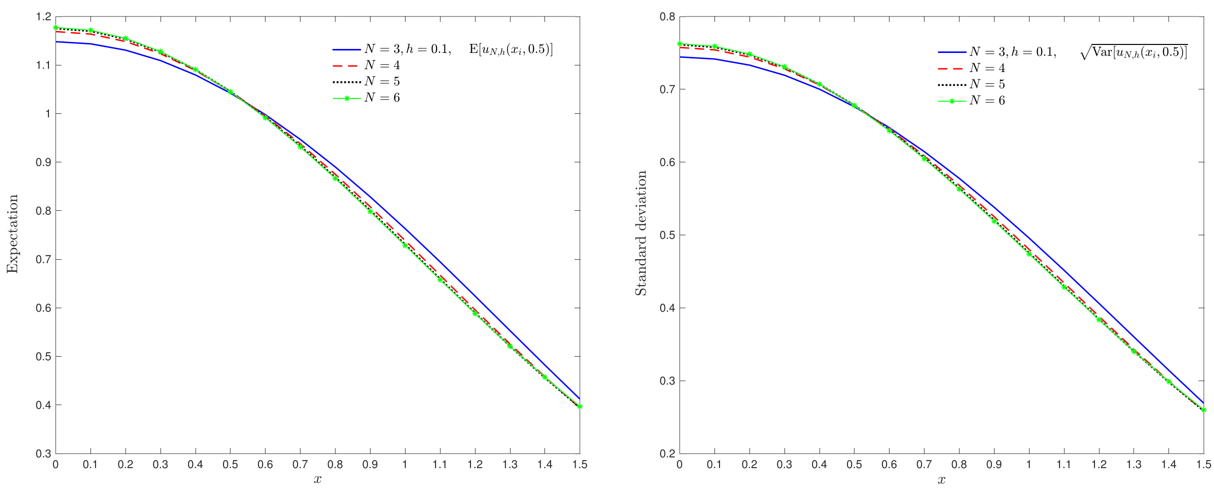

FIGURE 4 Plot (left): Comparative graphics of the expectations of the approximation solution s.p., $\mathbb{E}[u_{N,h}(x_i, T = 0.5)]$, obtained by (31). Plot (right): Comparative graphics of the standard deviations of the approximation solution s.p., $\sqrt{\mathrm{Var}[u_{N,h}(x_i, T = 0.5)]}$, obtained by (32)-(33). Both statistical moment functions correspond to Example 2 for the final time $T = 0.5$ on the spatial interval $0 \le x \le 1.5$. $N = \{3, 4, 5, 6\}$ denote the degrees of the Hermite polynomial and $h = 0.1$ is the time stepsize [Colour figure can be viewed at wileyonlinelibrary.com]

computations shown in Example 1 for the choice $T = 0.5$, $h = 0.05$, and $N = 8$, specifically the approximations to the expectation required 0.706975 s and to the standard deviation 106.273044 s. Since in this example the exact solution s.p. is not available, in order to evidence the convergence of approximations, we will compute the relative errors for consecutive approximations of the mean and the standard deviation using the expressions (46)-(47)

$$\mathrm{RelErr}\left[\mathbb{E}[u_{N_iN_{i+1},h}(x,t)]\right] = \left|\frac{\mathbb{E}\left[u_{N_i,h}(x,t)\right] - \mathbb{E}\left[u_{N_{i+1},h}(x,t)\right]}{\mathbb{E}\left[u_{N_{i+1},h}(x,t)\right]}\right|, \tag{46}$$

$$\mathrm{RelErr}\left[\sqrt{\mathrm{Var}[u_{N_iN_{i+1},h}(x,t)]}\right] = \left|\frac{\sqrt{\mathrm{Var}[u_{N_i,h}(x,t)]} - \sqrt{\mathrm{Var}[u_{N_{i+1},h}(x,t)]}}{\sqrt{\mathrm{Var}[u_{N_{i+1},h}(x,t)]}}\right|. \tag{47}$$

In Figure 5, we have plotted both relative errors taking $h = 0.1$ and using different consecutive values of N, from 3 to 8. One can observe that errors decrease as N increases all over the spatial domain $0 \leq x \leq 1.5$. This graphical behaviour is in full agreement with the results shown in Figure 4. To complete our numerical study, in Table 3, we show the figures corresponding to the infinite norm on the all spatial domain at the time instant $T = 0.5$ by refining the time stepsize h, from 0.1 to 0.05. We observe that the order of the relative errors does not change at the expense of decrease h.

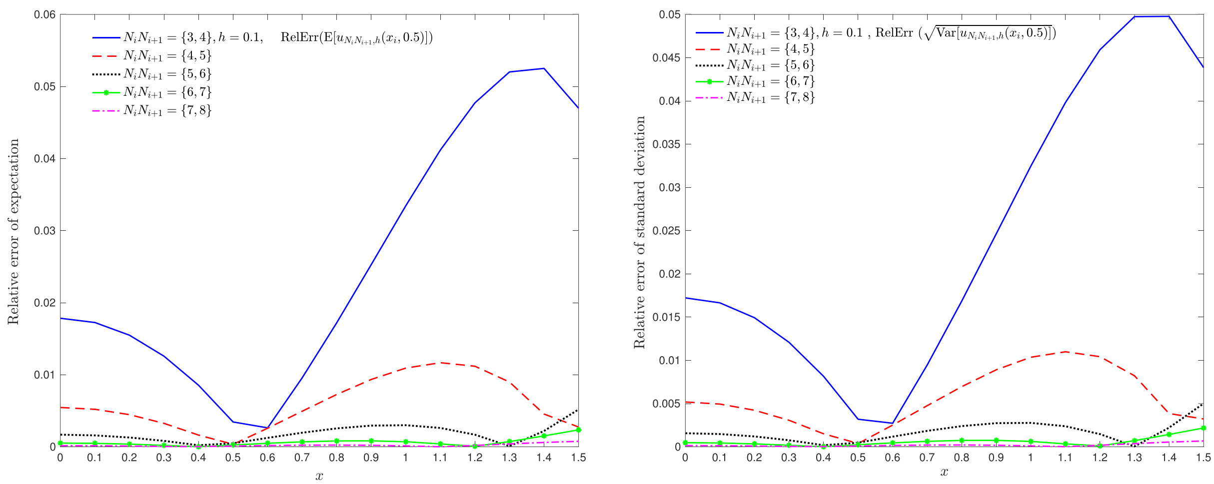

FIGURE 5 Plot (left): Comparative graphics of the relative errors of consecutive approximate expectations (see (31)), $\mathrm{RelErr}\left(\mathbb{E}[u_{N_iN_{i+1},h}(x_i, T=0.5)]\right)$, defined by (46). Plot (right): Comparative graphics of the relative errors of consecutive approximate standard deviations (see (37)-(38)), $\mathrm{RelErr}\left(\sqrt{\mathrm{Var}[u_{N_iN_{i+1},h}(x_i, T=0.5)]}\right)$, defined by (47). Both graphics correspond to Example 2 for the final time $T = 0.5$ on the spatial interval $0 \leq x \leq 1.5$. The integers N_i, N_{i+1}, denote the consecutive degrees of the Hermite polynomial taking values in the subset $N = \{3, 4, 5, 6, 7, 8\}$ and $h = 0.1$ denotes the time stepsize [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3 Values of the infinite norm, $\|\cdot\|_\infty$, of the relative errors of consecutive approximations for both the expectation and the standard deviation, of the approximation solution s.p., defined by (46) and (47), respectively, at the final time instant $T = 0.5$ and the spatial domain $0 \leq x \leq 1.5$

h	N_iN_{i+1}	$\|\mathrm{RelErr}\left[\mathbb{E}[u_{N_iN_{i+1},h}(x_i,0.5)]\right]\|_\infty$	$\|\mathrm{RelErr}\left[\sqrt{\mathrm{Var}\left(u_{N_iN_{i+1},h}(x_i,0.5)\right)}\right]\|_\infty$
0.1	$\{3,4\}$	5.25122e−02	4.97772e−02
	$\{4,5\}$	1.16710e−02	1.09916e−02
	$\{5,6\}$	5.18969e−03	5.01280e−03
	$\{6,7\}$	2.35322e−03	2.18048e−03
	$\{7,8\}$	7.62877e−04	6.78741e−04
0.05	$\{3,4\}$	5.22090e−02	4.94321e−02
	$\{4,5\}$	1.15993e−02	1.09020e−02
	$\{5,6\}$	5.16786e−03	4.98553e−03
	$\{6,7\}$	2.33510e−03	2.15811e−03
	$\{7,8\}$	7.54824e−04	6.68877e−04

Note. The time stepsizes $h = 0.1$ and $h = 0.05$ are considered and the consecutive degrees of Hermite polynomials lie from 3 to 8.

5 | CONCLUSIONS

In this paper, simple and efficient numerical methods for approximating time-dependent random hyperbolic partial differential problems are introduced and applied. Once the numerical methods for solving numerical solutions of random ordinary differential equations were introduced in Cortés et al[40] and the random Gauss-Hermite quadrature formulae are introduced here, the approximation of time-dependent partial differential problems proposed in Jódar and Pérez[3] for the deterministic case can be extended in the random mean square sense. This fact opens a way to be extended to other type of problems even those of higher dimensions with the natural changes of considering multidimensional Fourier transformations[41] and multidimensional quadrature formulae.[37,42] In these problems, the integral transform method is applicable to. In spite of the inherent computational complexity of random problems, the approach is easy to be applied as it also happens in the study of deterministic problems. We close this section by highlighting an important advantage of our approach from a practical standpoint. The simple finite sum approximate stochastic process makes manageable the computational cost of statistical moments versus alternatives methods by taking advantage of quadrature rules.

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CONFLICT OF INTEREST

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