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Domain of existence for the solution of some IVP's and BVP's by using an efficient ninth order iterative method

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Abstract

In this paper, we consider the problem of solving Initial Value Problems and Boundary Value Problems through the point of view of its continuous form. It is well known that in most cases these types of problems are solved numerically by performing a discretization and applying the finite difference technique to approximate the derivatives, transforming the equation into a finite-dimensional nonlinear system of equations. However, we would like to focus on the continuous problem, and therefore we try to set the domain of existence and uniqueness for its analytic solution. For this purpose, we study the semilocal convergence of a Newton-type method with frozen first derivative in Banach spaces. We impose only the assumption that the Fréchet derivative satisfies the Lipschitz continuity condition and that it is bounded in the whole domain in order to obtain appropriate recurrence relations so that we may determine the domains of convergence and uniqueness for the solution.
Our final aim is to apply these theoretical results to solve applied problems that come from integral equations, ordinary differential equations and boundary value problems.

KEYWORDS:
Nonlinear equations, order of convergence, iterative methods, semilocal convergence, computational efficiency.

1 | INTRODUCTION

The problem of solving a nonlinear equation of the form $F(x) = 0$ typically appears when we encounter some kind of differential equation, integral equation, or system of nonlinear equations. One can cite many areas of application, among them being chemical reaction problems, electrical circuits, heat conduction, and signal transmission. These differential and integral equations, in most cases, cannot be solved analytically, thus necessitating a need for proper numerical methods in order to transform the problem into a nonlinear system and to subsequently solve it via iterative techniques.

In the last decade researchers in numerical analysis have designed a great variety of iterative methods to solve nonlinear systems. Most of these invoke the philosophy of the well-known Newton's method, which reaches quadratic convergence and is defined by the following algorithm:

$$x_0 \text{ given in } \Omega, \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2 \dots$$

All of these methods must be compared in terms of the efficiency, a combined effect relating both the speed of convergence and the computational cost. When we choose an iterative method to solve $F(x) = 0$, we must carefully apply the measurements of the efficiency, introduced by Traub[17] and Ostrowski[14]: the efficiency index (EI) and the computational efficiency (CE), which are respectively defined by

$$EI = \rho^{1/d} \qquad \text{and} \qquad CE = \rho^{1/p},$$

where ρ is the order of convergence of the method, d represents the number of functional evaluations necessary to apply the method and p is the number of operations (products and divisions) that are needed to compute each iteration of the method.

In[1] the authors propose a multistep iterative method to solve $F(x) = 0$, the functional iteration of this method referred to as (NMIM), which is defined by:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ z_n &= y_n - \Gamma_n F(y_n), \\ w_n &= z_n - \Gamma_n F(z_n), \\ u_1^n &= \Gamma_n F(w_n), \\ u_i^n &= \Gamma_n F'(w_n) u_{i-1}^n, \qquad i = 2, \cdots, 5 \\ x_{n+1} &= w_n - \frac{21}{4} u_1^n + 11 u_2^n - \frac{23}{2} u_3^n + 6 u_4^n - \frac{5}{4} u_5^n, \end{aligned} \tag{1}$$

with $x_n \in \Omega$, $n \geqslant 0$, where $\Gamma_n = F'(x_n)^{-1}$. This method reaches ninth-order convergence, making it a very efficient method since it uses a frozen Jacobian in all steps where solving a linear system is required. Therefore the cost for obtaining the L and U factors of $F'(x_n)$, which are used to solve the lower and upper triangular systems of equations, is taken into account only once. It should be noted that each iteration of NMIM requires only two Jacobian evaluations, and only one of them is used for solving associated linear systems.

It is well known that one can use iterative methods to obtain an approximation of a root x^* for $F(x) = 0$. In[1] the authors use (1) for approximating the solution of some IVP's and BVP's by using the finite difference technique and obtaining a finite-dimensional nonlinear system. Our aim is to prove the existence of the analytical solution by treating the problem as one between Banach spaces and obtaining the domain of existence and uniqueness for this solution. For this purpose we analyze a semilocal convergence study of the iterative method. We use information about the initial value to start obtaining iterates, and analyze the nonlinear operator which, together with the assumed conditions, guarantees the convergence of the method and gives us the domain of existence and uniqueness to the solution.

In this paper we will devote ourselves to studying the approximation, from the semilocal point of view, to the solution x^* of the equation $F(x) = 0$, where $F : \Omega \subseteq X \to Y$, on an open convex set Ω of a Banach space X with values in a Banach space Y. The paper is organized as follows. First we introduce some preliminary results in section 2.1 that involve the analysis of the first steps and the construction of auxiliary functions in order to perform the semilocal convergence study. We then set the main results in section 2.2, with section 3 devoted to numerical exploration.

2 | SEMILOCAL CONVERGENCE

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear first Fréchet differentiable operator in an open convex domain Ω. For solving the system $F(x) = 0$, we will consider the multistep iterative method with frozen first derivative defined in (1).

We start from an initial estimate $x_0 \in \Omega$ for which $F'(x_0)^{-1}$ exists. We denote $\Gamma_0 = F'(x_0)^{-1}$, and assume that the following conditions are verified:

$(C_1) \|\Gamma_0\| \leq \beta_0$, $\|\Gamma_0 F(x_0)\| \leq \eta_0$,

$(C_2) \|F'(x) - F'(y)\| \leq K\|x - y\|$ for all $x, y \in \Omega$ and $K \geq 0$,

$(C_3) \|F'(x)\| \leq C$ for all $x \in \Omega$ and $C \geq 0$.

Henceforth we define $h_0 = \beta_0 K \eta_0$.

Under these conditions we establish recurrence relations in order to obtain the semilocal convergence for the iterative method mentioned.

The following lemma will be used to guarantee the existence and boundedness of the inverse of a matrix.

Lemma 1. (Banach's Lemma): Let $A \in L(X, X)$ and $\|A\| < 1$ then $I - A$ is an invertible matrix and $\|I - A\|^{-1} < \dfrac{1}{1 - \|A\|}$.

The lemma below provides us an expression of the remainder of the Taylor expansion of the operator F which will be used in subsequent proofs.

Lemma 2. If F has continuous derivatives up to order $k+1$ in a convex open set Ω, for all $x \in \Omega$, then if $x_0 \in \Omega$,

$$F(x) = \sum_{j=0}^{k} \frac{1}{k!} F^{(k)} x_0 (x - x_0)^k + R_{x_0,k}(x - x_0)$$

where the remainder can be expressed as

$$R_{x_0,k}(x - x_0) = \frac{1}{k!} \int_0^1 [F'(x_0 + \tau(x - x_0)) - F'(x_0)] d\tau (x - x_0)$$

The next result guarantees that each iteration in the iterative method under consideration is well defined.

Lemma 3. Under the above mentioned conditions $(C_1) - (C_3)$, let us assume that there exists $1 < R < \frac{1}{h_0}$ for which x_n, y_n, z_n and $x_{n+1} \in B(x_0, R\eta_0)$ with $B(x_0, R\eta_0) \subset \Omega$, for $n \geq 0$ and $n \in N$, then the iterative process (1) is well defined.

Proof. Obviously, $F(x_n)$ is well defined, since $x_n \in B(x_0, R\eta_0) \subset \Omega$, for all $n \in N$ and $n \geq 0$.

On the other hand, it is necessary that $\Gamma_n = [F'(x_n)]^{-1}$ exists for all $n \in N$. For that purpose we obtain:

$$\|I - \Gamma_0 F'(x_n)\| \leq \|\Gamma_0\| \|F'(x_n) - F'(x_0)\| \leq \beta_0 K \|x_n - x_0\| < h_0 R.$$

Then, as $\beta h_0 < 1$ we apply lemma 1 and deduce the existence of Γ_n and it verifies:

$$\|\Gamma_n\| \leq \frac{\beta}{1 - h_0 R}.$$

□

2.1 | Auxiliary functions

In Lemma 3 we have assumed the existence of a value R which gives us a ball where we can obtain the iterates from our model, but now we must obtain this value. For this purpose we must analyze the method step by step and, using the boundary conditions $(C_1)-(C_3)$, we obtain some restrictions that provide us a procedure to obtain R. Since we are analyzing a high order method, we need to establish a high number of parameters and auxiliary functions that are necessary to study our model (1). Let us denote:

$$G_0(s) = 1 + \frac{1}{2}h_0 + \frac{1}{2}h_0^2 s + \frac{1}{2}h_0^3 s^2 P_0 \tag{2}$$

$$Q_0(s) = 1 + P_0 L_0 + \frac{1}{4}h_0^4 s^2 P_0^2 \tag{3}$$

$$g_0(s) = \frac{h_0^3 s^2 Q_0(s)}{2(1 - h_0 s)} \tag{4}$$

$$G_1(t, s) = 1 + \frac{1}{2}t + \frac{1}{2}t^2 s + \frac{1}{2}t^3 s^2 P_1(t, s) \tag{5}$$

$$Q_1(t, s) = 1 + P_1 L_1(s) + \frac{1}{4}t_1^4 s^2 P_1(s)^2 \tag{6}$$

$$g_1(t, s) = \frac{t^3 s^2 Q_1(t, s)}{2} \tag{7}$$

with

$$L_0 = \beta_0 C \tag{8}$$

$$P_0 = \frac{21}{4} + L_0(11 + \frac{23}{2}L_0 + 6L_0^2 + \frac{5}{4}L_0^3) \tag{9}$$

$$L_1(s) = \frac{\beta_0 C}{1 - h_0 s} \tag{10}$$

$$P_1(s) = \frac{21}{4} + L_1(s)[11 + \frac{23}{2}L_1(s) + 6L_1(s)^2 + \frac{5}{4}L_1(s)^3)] \tag{11}$$

where $t \in [0, h_0]$ and $s \in [0, \frac{1}{h_0}[$.

We establish properties of the auxiliary functions mentioned above with the following lemma.

Lemma 4. Under the conditions of Lemma 3 and $(C_1) - (C_3)$ it follows:

(i) Function $g_0(s)$ is increasing and there exists $r_1 \in [0, \frac{1}{h_0}[$ such that $\dfrac{g_0(s)}{1 - h_0 s} \leq 1, \forall s \in [0, r_1]$.

(ii) Function $g_1(t, s)$ is increasing with respect to t, considering fixed s, and for $t = h_0$ there exists $r_2 \in [0, r_1[$ such that $g_1(h_0, s) \leq 1, \forall s \in [0, r_2]$.

(iii) Function $G_1(t, s)$ is increasing with respect to t, with fixed s. Moreover, with fixed $t = h_0$, we have that $G_0(s) \leq G_1(h_0, s) \quad \forall s \in [0, \frac{1}{h_0}[$.

Proof. It is obvious that functions G_0, g_0, G_1 and g_1 are increasing, by construction in their domain. To demonstrate the second part of *(i)*, we define the function $p_1(s) = \dfrac{g_0(s)}{1 - h_0 s} - 1$, where it holds that $p_1(0) = -1$ and $p_1(\frac{1}{h_0}) \to +\infty$ which indicates that there is at least one positive root in $]0, \frac{1}{h_0}[$. We take the smallest one, let it be r_1. Then, $\forall s \in [0, r_1]$ the function $\dfrac{g_0(s)}{1 - h_0 s} \leq 1$.

In case (*ii*), we define the function $p_2(s) = g_1(h_0, s) - 1$, where $p_2(0) = -1$ and $p_2(+\infty) \to +\infty$ and, analogously to the case *(i)*, we find the smallest possible root $r_2 \in]0, +\infty[$. Then, $\forall s \in [0, r_2]$ the function $g_1(h_0, s) \leq 1$.

Now, (*iii*) holds by noting that $P_1(s) > P_0 \quad \forall s \in [0, \frac{1}{h_0}[$ and $L_1(s) > L_0 \quad \forall s \in [0, \frac{1}{h_0}[$. □

2.1.1 | Analyzing the first steps

For $n = 0$, we have already defined η_0, β_0, h_0, through our choice of the starting point x_0.

By using the first step from (1) we have:

$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta_0. \tag{12}$$

Now, by considering the Taylor expansion with remainder given in Lemma 2, for $F(y_0)$ around x_0 and, using first step of (1), we have:

$$F(y_0) = \int_{x_0}^{y_0} [F'(z) - F'(x_0)]dz = \int_0^1 [F'(x_0 + \tau(y_0 - x_0)) - F'(x_0)]d\tau(y_0 - x_0),$$

then, by taking norms and using (C_2) we get:

$$\|F(y_0)\| \leq \frac{1}{2} K\eta_0 \|y_0 - x_0\|,$$

so, for the second step it follows that

$$\|z_0 - y_0\| \leq \|\Gamma_0\| \|F(y_0)\| \leq \frac{1}{2} h_0 \eta_0, \tag{13}$$

and by using the triangle inequality, (12), and (13), we can write

$$\|z_0 - x_0\| \leq \|z_0 - y_0\| + \|y_0 - x_0\| \leq (1 + \frac{1}{2} h_0)\eta_0. \tag{14}$$

In the same way, we bound $F(z_0)$ by a similar process.

$$F(z_0) = \int_{y_0}^{z_0} [F'(z) - F'(x_0)]dz = \int_0^1 [F'(y_0 + \tau(z_0 - y_0)) - F'(x_0)]d\tau(z_0 - y_0),$$

and, by taking norms and using (C_2), we get:

$$\|F(z_0)\| \leq KR\eta_0 \|z_0 - y_0\|, \tag{15}$$

where we have used that $y_0 + \tau(z_0 - y_0) \in B(x_0, R\eta_0)$, since, by the assumption made in Lemma 3, $y_0, z_0 \in B(x_0, R\eta_0)$ and we have applied the convexity property. By using this bound we have in the third step:

$$\|w_0 - z_0\| \leq \|\Gamma_0\|\|F(z_0)\| \leq h_0 R\|z_0 - y_0\| \leq \frac{1}{2}h_0^2 R\eta_0, \tag{16}$$

and, using (14) and (16), we have

$$\|w_0 - x_0\| \leq \|w_0 - z_0\| + \|z_0 - x_0\| \leq \left(1 + \frac{1}{2}h_0 + \frac{1}{2}h_0^2 R\right)\eta_0. \tag{17}$$

We then find the bound of $F(w_0)$, using the previous process, obtaining:

$$F(w_0) = \int_{z_0}^{w_0} [F'(z) - F'(x_0)]dz = \int_0^1 [F'(z_0 + \tau(w_0 - z_0)) - F'(x_0)]d\tau(w_0 - z_0),$$

and, by similarly applying (C_2), we get:

$$\|F(w_0)\| \leq KR\eta_0\|w_0 - z_0\|, \tag{18}$$

where we have used the same conditions as in (15).

From (18) and (C_1) we can obtain $\|u_1^0\| = \|\Gamma_0 F(w_0)\| \leq \frac{1}{2}h_0^3 R^2\eta_0$ and, by the definition of the iterative method (1), $\|u_i^n\| \leq \|\Gamma_n F'(w_n)u_{i-1}^n\|$ for $i = 2, \cdots, 5$ and we then have in the last step of the first iteration:

$$\begin{aligned}\|x_1 - w_0\| &\leq \|\frac{21}{4}u_1^0 + 11u_2^0 + \frac{23}{2}u_3^0 + 6u_4^0 + \frac{5}{4}u_5^0\| \\ &\leq \frac{21}{4}\|u_1^0\| + 11L_0\|u_1^0\| + \frac{23}{2}L_0^2\|u_1^0\| + 6L_0^3\|u_1^0\| + \frac{5}{4}L_0^4\|u_1^0\| \\ &\leq P_0\|u_1^0\| = \frac{1}{2}h_0^3 R^2\eta_0 P_0.\end{aligned} \tag{19}$$

By using the previous bounds (17), (19), and the defined auxiliary functions, one gets:

$$\begin{aligned}\|x_1 - x_0\| &\leq \|x_1 - w_0\| + \|w_0 - x_0\| \\ &\leq \left(1 + \frac{1}{2}h_0 + \frac{1}{2}h_0^2 R + \frac{1}{2}h_0^3 R^2 P_0\right)\eta_0 = G_0(R)\eta_0.\end{aligned} \tag{20}$$

Now, using the Taylor expansion of $F(x_1)$ around w_0, we have:

$$\begin{aligned}F(x_1) &= F(w_0) + F'(w_0)(x_1 - w_0) + \int_{w_0}^{x_1} [F'(z) - F'(w_0)]dz \\ &= F(w_0) + F'(w_0)(x_1 - w_0) + \int_0^1 [F'(w_0 + \tau(x_1 - w_0)) \\ &\quad - F'(w_0)]d\tau(x_1 - w_0),\end{aligned}$$

and, by taking norms and applying (18), (19), $(C_2) - (C_3)$, and the definition of L_0, we get:

$$\begin{aligned}\|F(x_1)\| &\leq \|F(w_0)\| + \|F'(w_0)(x_1 - w_0)\| + \frac{1}{2}K\|x_1 - w_0\|^2 \\ &\leq \|F(w_0)\| + \|F'(w_0)\|P_0\|u_1^0\| + \frac{1}{8}Kh_0^6 R^4\eta_0^2 P_0^2 \\ &\leq \|F(w_0)\|(1 + P_0 L_0) + \frac{1}{8}Kh_0^6 R^4\eta_0^2 P_0^2 \\ &\leq \frac{1}{2}Kh_0^2 R^2\eta_0^2(1 + P_0 L_0) + \frac{1}{8}Kh_0^6 R^4\eta_0^2 P_0^2 \\ &\leq \frac{K}{2}h_0^2 R^2(1 + P_0 L_0 + \frac{1}{4}h_0^4 R^2 P_0^2)\eta_0^2 = \frac{K}{2}h_0^2 R^2 Q_0(R)\eta_0^2.\end{aligned} \tag{21}$$

Next we analyze the second iteration of the method (1), that is we set $n = 1$. First of all, from Lemma (3), we have the existence of $\Gamma_1 = F'(x_1)^{-1}$ and denote

$$\|\Gamma_1\| \leq \frac{\beta}{1 - h_0 R} \equiv \beta_1. \tag{22}$$

Using (21), (22), and the value of β_1 we have:

$$\begin{aligned} \|y_1 - x_1\| &\leq \|\Gamma_1 F(x_1)\| \\ &\leq \beta_1 \frac{K}{2} h_0^2 R^2 Q_0(R) \eta_0^2 = \frac{h_0^3 R^2 Q_0(R)}{2(1 - h_0 R)} \eta_0 = \eta_1, \end{aligned} \tag{23}$$

obtaining a new value of the recurrence relations, $\eta_1 = g_0(R)\eta_0$.

Later, by using the triangle inequality, (20), and (23), we can write

$$\begin{aligned} \|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\|, \\ &\leq (g_0(R) + G_0(R))\eta_0. \end{aligned} \tag{24}$$

Then, once parameters β_1 and η_1 have been obtained, and by following with the previous notation, we have analogous conditions as those in the previous step; that is:

$$\begin{cases} \|\Gamma_1\| \leq \beta_1, \\ \|\Gamma_1 F(x_1)\| \leq \eta_1, \\ h_1 = \beta_1 K \eta_1. \end{cases} \tag{25}$$

Therefore, by the same reasoning as before, we establish for the first step of (1) the following:

$$F(y_1) = \int_{x_1}^{y_1} [F'(z) - F'(y_1)]\mathrm{dz} = \int_0^1 [F'(x_1 + \tau(y_1 - x_1)) - F'(x_1)]\mathrm{d\tau}(y_1 - x_1).$$

Then by taking norms and using (C_2) we get:

$$\|F(y_1)\| \leq \frac{1}{2} K \eta_1 \|y_1 - x_1\|,$$

so that, for the second step, we have:

$$\|z_1 - y_1\| \leq \|\Gamma_1\| \|F(y_1)\| \leq \frac{1}{2} h_1 \eta_1. \tag{26}$$

Applying (24) and (26), we see that

$$\|z_1 - x_0\| \leq \|z_1 - y_1\| + \|y_1 - x_0\| \leq [(1 + \frac{1}{2} h_1) g_0(R) + G_0(R)]\eta_0. \tag{27}$$

We bound $F(z_1)$ by a similar process:

$$F(z_1) = \int_{y_1}^{z_1} [F'(z) - F'(x_1)]dz = \int_0^1 [F'(y_1 + \tau(z_1 - y_1)) - F'(x_1)]d\tau(z_1 - y_1),$$

and, by taking norms, using (C_2), and assuming (as we will prove after determining R) now that

$$y_1, z_1 \in B(x_1, R\eta_1) \tag{28}$$

and, by its convexity property, we have that $y_1 + \tau(z_1 - y_1) \in B(x_1, R\eta_1)$, so that we have:

$$\|F(z_1)\| \leq \frac{1}{2} K R h_1 \eta_1^2,$$

By using this bound we have in the third step:

$$\|w_1 - z_1\| \leq \|\Gamma_1\| \|F(z_1)\| \leq \frac{1}{2} h_1^2 R \eta_1, \tag{29}$$

from which we deduce:

$$\|w_1 - x_0\| \leq \|w_1 - z_1\| + \|z_1 - x_0\| \leq \left[\left(1 + \frac{1}{2} h_1 + \frac{1}{2} h_1^2 R\right) g_0(R) + G_0(R)\right] \eta_0, \tag{30}$$

and so we find for the third step a bound on $F(w_1)$, using the previous process, showing that

$$\|F(w_1)\| \leq KR\eta_1\|w_1 - z_1\| \leq (\frac{1}{2}Kh_1^2R^2\eta_1)\eta_1. \tag{31}$$

Now from (31) and (C_2), we can obtain $\|u_1^1\| = \|\Gamma_1 F(w_1)\| \leq \frac{1}{2}h_1^3R^2\eta_1$ with $\|u_i^n\| \leq \|\Gamma_n F'(w_n)u_{i-1}^n\|$ for $i = 2, \cdots, 5$, and we have in the last step of the second iteration of (1):

$$\begin{aligned}
\|x_2 - w_1\| &\leq \|\frac{21}{4}u_1^1 + 11u_2^1 + \frac{23}{2}u_3^1 + 6u_4^1 + \frac{5}{4}u_5^1\| \\
&\leq \frac{21}{4}\|u_1^1\| + 11L_1(s)\|u_1^1\| + \frac{23}{2}L_1(s)^2\|u_1^1\| + 6L_1(s)^3\|u_1^1\| + \frac{5}{4}L_1(s)^4\|u_1^1\| \\
&\leq \frac{1}{2}h_1^3R^2\eta_1 P_1(R),
\end{aligned} \tag{32}$$

Using (30) and (32), we have

$$\|x_2 - x_0\| \leq \|x_2 - w_1\| + \|w_1 - x_0\| \leq (G_1(h_1, R)g_0(R) + G_0(R))\eta_0. \tag{33}$$

We now use the Taylor expansion of $F(x_2)$ around w_1. For the beginning step of the third iteration it holds that:

$$\begin{aligned}
F(x_2) &= F(w_1) + F'(w_1)(x_2 - w_1) + \int_{w_1}^{x_2} [F'(z) - F'(w_1)]\mathrm{dz} \\
&= F(w_1) + F'(w_1)(x_2 - w_1) + \int_0^1 [F'(w_1 + \tau(x_2 - w_1)) \\
&- F'(w_1)]\mathrm{d}\tau(x_2 - w_1),
\end{aligned}$$

and, by taking norms and using (31), (32), $(C_2) - (C_3)$ and the definition of $L_1(s)$, we get:

$$\begin{aligned}
\|F(x_2)\| &\leq \|F(w_1)\| + \|F'(w_1)(x_2 - w_1)\| + \frac{1}{2}K\|x_2 - w_1\|^2 \\
&\leq \|F(w_1)\| + P_1\|u_1\|\|F'(w_1)\| + \frac{1}{8}Kh_1^6R^4\eta_1^2P_1^2 \\
&\leq \|F(w_1)\|(1 + P_1L_1(s)) + \frac{1}{8}Kh_1^6R^4\eta_1^2P_1^2 \\
&\leq \frac{K}{2}h_1^2R^2(1 + P_1L_1(s) + \frac{1}{4}h_1^4R^2P_1^2)\eta_1^2 = \frac{K}{2}h_1^2R^2Q_1(h_1, R)\eta_1^2,
\end{aligned} \tag{34}$$

hence

$$\|\Gamma_2 F(x_2)\| \leq \beta_2\frac{K}{2}h_1^2R^2Q_1(h_1, R)\eta_1^2 \leq \frac{h_1^3R^2}{2}Q_1(h_1, R)\eta_1 = \eta_2$$

thus we get $\eta_2 = g_1(h_1, R)\eta_1$.

Therefore we have similar conditions as in the previous step; that is:

$$\begin{cases}
\|\Gamma_2\| \leq \beta_2, \\
\|\Gamma_2 F(x_2)\| \leq \eta_2, \\
h_2 = \beta_2 K\eta_2.
\end{cases} \tag{35}$$

2.1.2 | Recurrence relations

As a consequence of the previous study we are now in a position to define the recurrence relations necessary to prove the semilocal convergence of iterative process (1). We work under conditions $(C_1) - (C_3)$ with parameters and auxiliary functions already defined.

Due to the above analysis we can declare the following system of recurrence relations:

$$\begin{aligned} \beta_n &= \beta_1 \quad n \geq 1 \\ \eta_n &= g_1(h_{n-1}, R)\eta_{n-1} \quad n \geq 2 \\ h_n &= \beta_n K \eta_n \quad n \geq 2, \end{aligned} \tag{36}$$

With the following lemma, we obtain a basic property for the last scalar sequence defined in the recurrence relations.

Lemma 5. Sequence $\{h_n\}$ generated by (36) with h_0 and h_1 defined previously by (22) is decreasing.

Proof. We prove this result by an induction process. For $k = 1$, using (22) and Lemma 4 *(i)* we have

$$h_1 = \beta_1 K \eta_1 = \frac{h_0^3 R^2 Q_0(R)}{2(1 - h_0 R)^2} h_0 = \frac{g_0(R)}{1 - h_0 R} h_0 \leq h_0.$$

For $k = 2$, using (35) and Lemma 4 *(ii)* one gets:

$$h_2 = \beta_2 K \eta_2 = \frac{h_1^3 R^2 Q_1(h_1, R)}{2} h_1 = g_1(h_1, R) h_1 \leq h_1,$$

therefore $h_2 \leq h_1 \leq h_0$.

Now by the induction hypothesis assume that $h_0 \geq h_1 \geq h_2 \geq \cdots \geq h_{n-2} \geq h_{n-1}$. Using that $g_1(t, R)$ is an increasing function in t, by Lemma 4 *(ii)* we have:

$$h_n = g_1(h_{n-1}, R) h_{n-1} \leq g_1(h_1, R) h_{n-1} \leq h_{n-1} \tag{37}$$

This completes the induction process. □

Lemma 6. Under the conditions assumed in Lemma 3, and after analyzing the first steps of the iterative method defined by (1), we establish the following inequalities $\forall n \in N, n \geq 1$.

$$\begin{aligned}
a)&\quad \|y_n - x_n\| \leq \eta_n \\
b)&\quad \|y_n - x_0\| \leq \eta_n + \sum_{i=1}^{n-1} G_1(h_i, R)\eta_i + G_0(R)\eta_0 \\
c)&\quad \|z_n - y_n\| \leq \frac{1}{2} h_n \eta_n \\
d)&\quad \|z_n - x_0\| \leq (1 + \frac{1}{2} h_n)\eta_n + \sum_{i=1}^{n-1} G_1(h_i, R)\eta_i + G_0(R)\eta_0 \\
e)&\quad \|w_n - z_n\| \leq \frac{1}{2} h_n^2 R \eta_n \\
f)&\quad \|w_n - x_0\| \leq (1 + \frac{1}{2} h_n + \frac{1}{2} h_n^2 R)\eta_n + \sum_{i=1}^{n-1} G_1(h_i, R)\eta_i + G_0(R)\eta_0 \\
h)&\quad \|x_{n+1} - w_n\| \leq \frac{1}{2} h_n^3 R^2 \eta_n P_1(R) \\
i)&\quad \|x_{n+1} - x_n\| \leq (1 + \frac{1}{2} h_n + \frac{1}{2} h_n^2 R + \frac{1}{2} h_n^3 R^2 P_1)\eta_n = G_1(h_n, R)\eta_n \\
j)&\quad \|x_{n+1} - x_0\| \leq \sum_{i=1}^{n} G_1(h_i, R)\eta_i + G_0(R)\eta_0.
\end{aligned}$$

Proof. The proof follows by an induction procedure. We have verified these conditions for $k = 0, 1$. We assume that the inequalities follow for $k = n - 1$, then by the reasoning made in first steps one can obtain the inequalities for $k = n$.

□

2.2 Main result

After obtaining the recurrence relation system and the bounds for all the steps and successive iterations, we are in a position to establish our main result. That is, for completing the semilocal convergence study we have to prove the assumed assertions that we have made in our previous dissertation, (see Lemma 3), by defining the parameter R.

Theorem 1. Let F be a nonlinear operator, $F : \Omega \subseteq X \longrightarrow Y$, defined on a nonempty open convex domain Ω of a Banach space X with values in a Banach space Y. Suppose that conditions (C_1)–(C_3) are satisfied and take into consideration the functions G_0, Q_0, g_0, G_1, Q_1 and g_1 defined in (2.1) and values r_1, r_2 defined in Lemma 4. Assume that the equation

$$s = G_1(h_0, s)\left(1 + g_0(s)\frac{1}{1 - g_1(h_0, s)}\right), \tag{38}$$

has at least one positive real root, denote the smallest one by r_3, and take $R = min\{r_1, r_2, r_3\}$ if $R > 1$ and $B(x_0, R\eta_0) \subset \Omega$. Then the iterative process given by (1), starting at x_0, satisfies $y_n, z_n, w_n, x_{n+1} \in B(x_0, R\eta_0)$ $\forall n \in N$, converges to a solution x^* of the equation $F(x) = 0$, and $x^* \in \overline{B(x_0, R\eta_0)}$. Moreover the solution is unique in $B(x_0, \frac{2}{K\beta} - R\eta_0) \cap \Omega$.

Proof. First we need to develop the next summation, for which, we use the fact that g_1 is increasing in its domain and the sequence h_n is decreasing, as we stated in Lemmas 4 and 5 respectively, so we get:

$$\begin{aligned}
\sum_{i=0}^{n} \eta_i &= \eta_0 + g_0(R)\eta_0 + g_1(h_1, R)\eta_1 + g_1(h_2, R)\eta_2 + \cdots + g_1(h_{n-1}, R)\eta_{n-1} \\
&\le [\eta_0 + g_0(R)\eta_0 + g_1(h_0, R)\eta_1 + g_1(h_0, R)^2\eta_1 + g_1(h_0, R)^3\eta_1 \\
&+ \cdots + g_1(h_0, R)^{n-1}\eta_1] \\
&= \eta_0 + g_0(R)\eta_0[1 + g_1(h_0, R) + g_1(h_0, R)^2 + g_1(h_0, R)^3 \\
&+ \cdots + g_1(h_0, R)^{n-1}] \\
&\le \left[1 + g_0(R)\frac{1}{1 - g_1(h_0, R)}\right]\eta_0,
\end{aligned}$$

where in the last inequality we recognize a geometric progression with common ratio $g_1(h_0, R) < 1$.

Now, in order to prove $y_n, z_n, w_n, x_{n+1} \in B(x_0, R\eta_0)$, we observe that, for being $R > 1$ by assumption in Lemma 3 we have that

$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \le \eta_0 < R\eta$$

and for $n \ge 1$, and by Lemma 6 it follows:

$$\begin{aligned}
\|y_n - x_0\| &\le \eta_n + \sum_{i=1}^{n-1} G_1(h_i, R)\eta_i + G_0(R)\eta_0 \\
&\le G_1(h_0, R)\eta_n + \sum_{i=1}^{n-1} G_1(h_0, R)\eta_i + G_1(h_0, R)\eta_0 \\
&\le G_1(h_0, R)\sum_{i=0}^{n} \eta_i \le G_1(h_0, R)\left[1 + g_0(R)\frac{1}{1 - g_1(h_0, R)}\right]\eta_0 = r_3\eta \le R\eta
\end{aligned}$$

where in the first inequality we have used that $G_1(h_0, R) > 1$ by construction and $G_0(R) < G_1(h_0, R)$ by (iii) of Lemma 4, and in the last inequality we use the definition of R given by equation (38).

By using similar reasoning we have that:

$$\begin{aligned}
\|z_n - x_0\| &\le (1 + \frac{1}{2}h_n)\eta_n + \sum_{i=1}^{n-1} G_1(h_i, R)\eta_i + G_0(R)\eta_0 \\
&\le G_1(h_n, R)\eta_n + \sum_{i=1}^{n-1} G_1(h_0, R)\eta_i + G_1(h_0, R)\eta_0 \\
&\le G_1(h_0, R)\sum_{i=0}^{n} \eta_i \le G_1(h_0, R)\left[1 + g_0(R)\frac{1}{1 - g_1(h_0, R)}\right]\eta_0 = r_3\eta \le R\eta
\end{aligned}$$

and that it similarly holds for $w_n, x_{n+1} \in B(x_0, R\eta_0)$.

We have proven that the iterates remains in the ball centered at the starting iterate x_0, that is, the sequence given by (1) is well-defined. We now have to prove that it is a Cauchy sequence.

By using Lemmas 4, 5, 6, (36), and the sum of a finite geometric progression with common ratio $g_1(h_0, R)$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \sum_{j=1}^{m} \|x_{n+j} - x_{n+j-1}\| \leq \sum_{j=1}^{m} G_1(h_{n+j-1}, R)\eta_{n+j-1}\\ &\leq G_1(h_0, R)\sum_{j=1}^{m} \eta_{n+j-1} \leq G_1(h_0, R)\sum_{p=0}^{m-1} \eta_{n+p}\\ &\leq G_1(h_0, R)\sum_{p=0}^{m-1}[g_1(h_0, R)]^{n+p}\eta_0\\ &\leq G_1(h_0, R)\frac{[g_1(h_0, R)]^n - [g_1(h_0, R)]^{n+m}}{1 - g_1(h_0, R)}\eta_0\end{aligned}$$

As the last term tends to zero we have that $\{x_n\}$ is a Cauchy sequence and is therefore convergent. Now, if $\lim_{n\to\infty} x_n = x^*$, it follows that $F(x^*) = 0$ from the continuity of the operator F, since

$$\begin{aligned}\|F(x_n)\| &\leq \|F(w_{n-1})\| + \|F(w_{n-1})(x_n - w_{n-1})\| + \frac{1}{2}K\|x_n - w_{n-1}\|\\ &\leq \frac{1}{2}Kh_{n-1}^2 R^2 Q_1(h_{n-1}, R)\eta_{n-1}^2 \leq \frac{1}{2}Kh_0^2 R^2 Q_1(h_0, R)[(g_1(h_0, R))^{n-1}\eta_0]^2,\end{aligned}$$

and $(g_1(h_0, R))^{n-1} \to 0$ when letting $n \to \infty$.

To prove the uniqueness, let us assume some other solution z^* of $F(x) = 0$ in $B(x_0, \frac{2}{K\beta} - R\eta) \cap \Omega$. From the Taylor approximation

$$F(z^*) - F(x^*) = \int_0^1 F'(x^* + \tau(z^* - x^*))d\tau(z^* - x^*) = 0$$

we have to prove that the operator $\int_0^1 F'(x^* + \tau(z^* - x^*))d\tau$ is invertible and therefore $z^* = x^*$. We check it by applying the Banach Lemma:

$$\begin{aligned}\|\Gamma_0 \int_0^1 F'(x^* + \tau(z^* - x^*))d\tau - I\| &\leq \|\Gamma_0\| \int_0^1 \|F'(x^* + \tau(z^* - x^*)) - F'(x_0)\|d\tau\\ \leq K\beta \int_0^1 \|x^* + \tau(z^* - x^*) - x_0\|d\tau &\leq K\beta \int_0^1 \left((1 - t)\|x^* - x_0\| + \tau\|z^* - x^*\|\right) d\tau < 1,\end{aligned}$$

and so, it follows that $[\int_0^1 F'(x^* + \tau(z^* - x^*))d\tau]^{-1}$ exists. □

3 | NUMERICAL RESULTS

In this section we complete our study by demonstrating the effectiveness of the semilocal convergence analysis with some numerical examples. As we discuss in the introduction our aim is to prove the existence of the solution of some IVP's and BVP's in their analytical form by treating the problem as one between Banach spaces. We also compare the radius obtained with our method and the radius obtained by[23] in exercise number four. The numerical solution of the problem produced by using the finite difference technique and obtaining a finite-dimensional nonlinear system has been performed in[1].

In each example we have performed some calculations for expressing the problem as an equivalent integral equation, see[22].

Example 1. Consider the Lane-Emden Equation, presented in[1] :

$$x''(t) + \frac{2}{t}x'(t) + x(t)^p = 0$$
$$x(0) = 1$$
$$x'(0) = 0$$

over the time interval [0, 1] and for $p = 3$. The solution can be written in terms of the integral equation

$$x(t) = \int_0^t \left(\frac{s^2}{t} - s\right) x^p(s)ds$$

with $x(t) \in C([0,1])$. We define the nonlinear operator F between Banach spaces $\Omega = B(0,1) \subseteq C([0,1])$ as follows:

$$F(x) = x(t) - \int_0^t \left(\frac{s^2}{t} - s\right) x^p(s)ds.$$

The Frechet derivative is the linear operator defined by

$$[F'(x)]v = v(t) - \int_0^t \left(\frac{s^2}{t} - s\right) px^{p-1}v(s)ds.$$

Example 2. Consider the Bratu Problem, presented in[1] :

$$x''(t) + \alpha e^{x(t)} = 0$$
$$x(0) = 0$$
$$x(1) = 0$$

over the time interval [0, 1] with $\alpha = 1$. The solution can be written in terms of the integral equation

$$x(t) = \int_0^1 -\alpha G(s,t)e^{x(s)}ds,$$

where

$$G(s,t) = \begin{cases} s(t-1), & 0 \leq s < t \leq 1 \\ t(s-1), & 0 \leq t < s \leq 1. \end{cases}$$

We define the nonlinear operator F between Banach spaces $\Omega = B(0,1) \subseteq C([0,1])$ as follows:

$$F(x) = x(t) + \int_0^1 \alpha G(s,t)e^{x(s)}ds.$$

The Frechet derivative is the linear operator defined by

$$[F'(x)]v = v(t) + \int_0^1 \alpha G(s,t)e^{x(s)}v(s)ds.$$

Example 3. Consider the Frank-Kamenetskii Problem, presented in[1] :

$$x''(t) + \frac{1}{t}x'(t) + \alpha e^{x(t)} = 0$$
$$x'(0) = 0$$
$$x(1) = 0$$

over the time interval [0, 1] with $\alpha = 0.2$. The solution can be written in terms of the integral equation

$$x(t) = \int_0^1 -\alpha s e^{x(s)}G(s,t)ds,$$

where

$$G(s,t) = \begin{cases} \ln(t), & 0 \le s < t \le 1 \\ \ln(s), & 0 \le t < s \le 1. \end{cases}$$

We define the nonlinear operator F between Banach spaces $\Omega = B(0,1) \subseteq C([0,1])$ as follows:

$$F(x) = x(t) + \int_0^1 \alpha s e^{x(s)} G(s,t) ds.$$

The Frechet derivative is the linear operator defined by

$$[F'(x)]v = v(t) + \int_0^1 \alpha s e^{x(s)} G(s,t) v(s) ds.$$

We can obtain the bounds verifying conditions $(C_1) - (C_3)$ for each case, which can be see in Table 1. In Table 2 we can see the different values for the restrictions of the radius for the domain of existence of the solution.

Ex	x_0	β_0	K	η_0	C
F_1	0.100	1.0027	1.5000	0.1004	1.2739
	0.010	1.0000	1.5000	0.0100	1.2739
	0.125	1.0043	1.5000	0.1259	1.2739
F_2	0.250	1.1356	0.0428	0.4662	1.1738
	0.125	1.1264	0.0378	0.3003	1.1738
	0.500	1.1565	0.0550	0.8166	1.1738
F_3	0.050	1.1696	0.5437	0.3044	1.2332
	0.000	1.1647	0.5437	0.2329	1.2332
	0.010	1.1657	0.5437	0.2417	1.2332

TABLE 1 Bounding constants.

Ex	x_0	R_1	R_2	R_3	R	r
F_1	0.750	1.9928	1.9303	1.7720	1.7720	0.1780
	1.000	38.7280	36.5890	1.0078	1.0078	0.0101
	1.125	1.4587	1.4166	1.3075	1.3075	0.1646
F_2	0.250	22.6696	21.4800	1.0124	1.0124	0.4720
	0.500	45.9738	43.3455	1.0066	1.0066	0.3023
	0.750	7.7876	7.4461	1.0394	1.0394	0.8488
F_3	0.150	1.2462	1.2105	1.1178	1.1178	0.3402
	0.200	1.8226	1.7648	1.6185	1.6185	0.3770
	0.250	1.6768	1.6248	1.4924	1.4924	0.3688

TABLE 2 Semilocal convergence radius.

In order to compare our study with other semilocal convergence studies of high order we perform this example:

Example 4. Consider the following example, presented in[23] :

$$x(s) = 1 + \frac{1}{3}\int_0^1 G(s,t)x(t)^4 dt, \tag{39}$$

where $x \in X$. Here $X = C[0,1]$ is the space of continuous functions on $[0,1]$ with the norm

$$\| x \| = \max_{s\in[0,1]} | x(s) | .$$

The Kernel G is the Green's function

$$G(s,t) = \begin{cases} t(1-s), & t \le s \\ s(1-t), & s \le t. \end{cases}$$

Solving (39) is equivalent to solving $F(x) = 0$, where $F : \Omega \subseteq C[0,1] \to C[0,1]$ defined by

$$[F(x)](s) = x(s) - 1 - \frac{1}{3}\int_0^1 G(s,t)x(t)^4 dt, \qquad s \in [0,1],$$

where Ω is a suitable nonempty open convex domain. The integral equation is a Hammerstein integral equation of the second kind[24]. The first derivative of the operator F is given by

$$[F'(x)]v = v(s) - \frac{4}{3}\int_0^1 G(s,t)x(t)^3 v(t)dt.$$

We consider $\Omega = B(0,2) \subseteq X$ as an open convex nonempty domain and choose $x_0(s) = 1$, thereby obtaining the bounds verifying conditions $(C_1) - (C_3)$, which for this example are as follows: $\beta = \frac{6}{5}$, $\eta = \frac{1}{20}$, $K = 2$, $C = \frac{7}{3}$.

For the method given in (1), these parameters produce radii of $R_1 = 0.9775$, $R_2 = 0.9545$, $R_3 = 2.0903$, and hence we take the final radius $R = min\{R_1, R_2, R_3\} = 0.9545$. Alternately, using the method given in[23], we get a radius of $R = 1.2240$, which is a bit bigger than ours, because our method (1) is of higher order and this usually causes the radius of convergence to decrease.

4 | CONCLUSION

The semilocal convergence of a ninth-order method used for solving nonlinear equations in Banach spaces is established by using recurrence relations under the assumption that the first Fréchet derivative satisfies the Lipschitz continuity. The existence domain for the solution is established for multiple examples including both differential and integral equations, and convergence balls for each of them are derived.

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