# Solutions of Fractional Gas Dynamics Equation by a New Technique

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**Abstract**

In this paper, a novel technique is formed to obtain the solution of a fractional gas dynamics equation. Some reproducing kernel Hilbert spaces are defined. Reproducing kernel functions of these spaces have been found. Some numerical examples are shown to confirm the efficiency of the reproducing kernel Hilbert space method. The accurate pulchritude of the paper is arisen in its strong implementation of Caputo fractional order time derivative on the classical equations with the success of the highly accurate solutions by the series solutions. Reproducing kernel Hilbert space method is actually capable of reducing the size of the numerical work. Numerical results for different particular cases of the equations are given in the numerical section.

**KEYWORDS:**
Fractional gas dynamics equation, Hilbert space, operators

## 1 | INTRODUCTION

Fractional calculus is an area of applied mathematics that takes an interest in derivatives and integrals of arbitrary orders. The most valuable benefit of utilizing fractional differential equations is the non-local feature [1]. Fractional differential equations perform an important order to model many problems. There are many fractional differential equations discovered in the literature [2].

The fractional models appear in the fields of Science and Engineering and many researchers applied different techniques to deduce nonlinear differential equations of integer and fractional orders. Recently, many numerical methods have been implemented to solve fractional differential equations. These techniques are Adomian decomposition method [3], homotopy analysis method [4], homotopy perturbation method [5] and variational iteration method [6]. Singh et al. [7] investigated the fractional partial differential equations showing up in biological populations. Generalized differential transform method has been implemented to search the solution of time-fractional reaction diffusion equations [8]. Singh et al. [9] have investigated many nonlinear fractional differential equations concurred in biological populations. Bhrawy et al. [10] have applied the operational matrix concept for solving the fractional diffusion equations. Machado et al. [11] and Yang et al. [12] have implemented local fractional series method to investigate fractional differential equations.

We construct reproducing kernel Hilbert space method to find the solution of non homogeneous fractional gas dynamics equation as [13]:

$$\frac{d^{\gamma}\zeta}{dt^{\gamma}} + \frac{1}{2}(\zeta^2)_x + M(\zeta, x, t) = B(x), \quad 0 < \gamma \leq 1, \tag{1}$$

with initial condition $\zeta(x,0) = A(x)$, where M is a nonlinear function of $\zeta$, $x$ and $t$. In case of $M(\zeta, x, t) = -\zeta(1-\zeta)$, $B(x) = 0$ and $\gamma = 1$, Eq. (1) becomes the classical homogeneous gas dynamics equation as [14]:

$$\frac{d\zeta}{dt} + \frac{1}{2}(\zeta^2)_x - \zeta(1-\zeta) = 0. \tag{2}$$

Reproducing kernels were applied for the first time at the beginning of the twentieth century [15, 16]. Wang et al. [17] have investigated the numerical solution of integro-differential equations of high-order Fredholm by the simplified reproducing kernel method. Gumah et al. [18] have implemented of reproducing kernel method for solving second-order fuzzy Volterra integro-differential equations. Al-Smadi [19] has studied simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation. Kashkari et al. [20] have applied reproducing kernel method for solving nonlinear fractional Fredholm integro-differential equation. Fore more details see [21–29, 29–37].

We construct our work as: In Section 2, the basic definitions are presented. In Section 3, reproducing kernel Hilbert spaces are demonstrated. In Section 4, reproducing kernel Hilbert space method for solving fractional gas dynamics equation is presented. Numerical experiments have been given in Section 5. Conclusion is discussed in Section 6.

## 2 | PRELIMINARIES

We give some definitions of fractional calculus which are further needed in this paper. For the main definitions and features see [14].

**Definition 1.** A real function $\zeta(z),\ z > 0$, is said to be in the space $C_\tau, \tau \in R$ if there exists a real number $p > \tau$, such that $\zeta(z) = z^p \zeta_1(z)$, where $\zeta_1(z) \in C[0, \infty]$. Clearly $C_\tau < C_\beta$ if $\beta < \tau$.

**Definition 2.** A function $\zeta(z),\ z > 0$, is said to be in the space $C_\tau^m, m \in N_0 = N \cup \{0\}$ if $\zeta^{(m)} \in C_\tau$.

**Definition 3.** The fractional derivative of $\zeta(z)$ in the Caputo sense is given as:

$$D^\gamma \zeta(z) = \frac{1}{\Gamma(n-\gamma)} \int_0^z (z-\xi)^{n-\gamma-1} \zeta^{(n)}(\xi) d\xi,$$

where $n-1 < \gamma \le n,\ n \in N,\ z > 0,\ \zeta \in C_{-1}^n$.

For $\zeta \in C_\tau, \quad \tau \ge -1, \quad \gamma, \beta \ge 0, \quad \gamma \ge -1$ and $C$ a real constant, we have:

$$\begin{aligned} &(i) && D^\gamma D^\beta \zeta(z) = D^{\gamma+\beta}\zeta(z), \\ &(ii) && D^\gamma C = 0, \\ &(iii) && D^\gamma z^\alpha = \begin{cases} 0, & \text{if } \alpha \in N_0 \text{ and } \alpha < \gamma, \\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\gamma)} x^{\alpha-\gamma}, & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover, the Caputo fractional derivative is a linear operator

$$D^\gamma \left( \sum_{i=1}^n c_i \zeta_i(z) \right) = \sum_{i=1}^n c_i D^\gamma \zeta_i(z),$$

where $\{c_i\}_{i=1}^n$ are real constants.

**Definition 4.** For $m$ being the smallest integer that exceeds $\gamma$, the Caputo time-fractional derivative operator of order $\gamma > 0$ is given as:

$$D_t^\gamma \zeta(x,t) = \begin{cases} \frac{1}{\Gamma(m-\gamma)} \int_0^t \left[ (t-s)^{m-\gamma-1} \frac{d^m}{ds^m} \zeta(x,s) \right] ds, & m-1 < \gamma < m, \\ \frac{d^m \zeta(x,t)}{dt^m}, & \gamma = m \in N. \end{cases}$$

## 3 | SOME SPECIAL HILBERT SPACES

In this section, we construct the following reproducing kernel Hilbert spaces to get the approximate solutions of Eq. (1). We obtain some important reproducing kernel functions to get accurate numerical results by reproducing kernel Hilbert space method.

**Definition 5.** First of all, we need reproducing kernel Hilbert space $J_2^1[0,1]$ for range:

$$J_2^1[0,1] = \{\zeta \in AC[0,1] : \ \zeta' \in L^2[0,1]\},$$

where AC denotes the absolutely continuous functions. The inner product of this reproducing kernel Hilbert space is given as:

$$\langle \zeta, \eta \rangle_{J_2^1} = \zeta(0)\eta(0) + \int_0^1 \zeta'(\tau)\eta'(\tau)d\tau.$$

We obtain the reproducing kernel function $f_\xi$ of this reproducing kernel Hilbert space as ( [29], pp. 10 and 17):

$$f_\xi(\tau) = \begin{cases} 1+\tau, & 0 \le \tau \le \xi \le 1, \\ 1+\xi, & 0 \le \xi < \tau \le 1. \end{cases} \tag{3}$$

**Definition 6.** The other special Hilbert space that we need for domain is $J_2^2[0,1]$.

$$J_2^2[0,1] = \{\zeta \in AC[0,1] : \ \zeta' \in AC[0,1], \ \zeta'' \in L^2[0,1]\}.$$

We get the inner product of this reproducing kernel Hilbert space as:

$$\langle \zeta, \eta \rangle_{J_2^2[0,1]} = \zeta(0)\eta(0) + \zeta'(0)\eta'(0) + \int_0^1 \zeta''(\tau)\eta''(\tau)\mathrm{d}\tau, \quad \zeta, \eta \in \ J_2^2[0,1].$$

The norm of this reproducing kernel Hilbert space is defined as:

$$\|\zeta\|_{J_2^2[0,1]} = \sqrt{\langle \zeta, \zeta \rangle_{J_2^2[0,1]}}, \quad \zeta \in J_2^2[0,1].$$

Reproducing kernel function $h_\phi$ of reproducing kernel Hilbert space $J_2^2[0,1]$ is obtained as:

$$h_\phi(\sigma) = \begin{cases} 1+\sigma\phi + \frac{1}{2}\phi\sigma^2 - \frac{\sigma^3}{6}, & 0 \le \sigma \le \phi \le 0, \\ 1+\sigma\phi + \frac{1}{2}\phi^2\sigma - \frac{\phi^3}{6}, & 0 \le \phi < \sigma \le 1. \end{cases} \tag{4}$$

**Definition 7.** The third reproducing kernel Hilbert space that we need for domain is ${}^0J_2^2[0,1]$.

$${}^0J_2^2[0,1] = \{\zeta \in AC[0,1] : \ \zeta' \in AC[0,1], \ \zeta'' \in L^2[0,1], \ \zeta(0) = 0\}.$$

We have the inner product of this reproducing kernel Hilbert space as:

$$\langle \zeta, \eta \rangle_{J_2^2[0,1]} = \zeta(0)\eta(0) + \zeta'(0)\eta'(0) + \int_0^1 \zeta''(\tau)\eta''(\tau)\mathrm{d}\tau, \quad \zeta, \eta \in \ {}^0J_2^2[0,1].$$

The norm of this reproducing kernel Hilbert space is defined as:

$$\|\zeta\|_{{}^0J_2^2[0,1]} = \sqrt{\langle \zeta, \zeta \rangle_{{}^0J_2^2[0,1]}}, \quad \zeta \in \ {}^0J_2^2[0,1].$$

**Theorem 1.** We obtain reproducing kernel function $g_\xi$ of reproducing kernel Hilbert space ${}^0J_2^2[0,1]$ as:

$$g_\xi(\tau) = \begin{cases} \tau\xi + \frac{1}{2}\xi\tau^2 - \frac{\tau^3}{6}, & 0 \le \tau \le \xi \le 0, \\ \tau\xi + \frac{1}{2}\xi^2\tau - \frac{\xi^3}{6}, & 0 \le \xi < \tau \le 1. \end{cases} \tag{5}$$

*Proof.* We obtain

$$g_\xi(\tau) = \begin{cases} \sum_{i=1}^4 m_i(\xi)\tau^{i-1}, & 0 \le \tau \le \xi \le 0, \\ \sum_{i=1}^4 n_i(\xi)\tau^{i-1}, & 0 \le \xi < \tau \le 1, \end{cases} \tag{6}$$

by definition of reproducing kernel Hilbert space ${}^{0}J_2^2[0,1]$ and Dirac-Delta function. Then, we get

$$\begin{aligned}\left\langle \zeta, g_\xi(\tau)\right\rangle_{{}^{0}J_2^2[0,1]} &= \zeta(0)g_\xi(0)+\zeta'(0)g'_\xi(0)+\int_0^1 \zeta''(\tau)\frac{\partial'' g_\xi(\tau)}{\partial\tau''}\mathrm{d}\tau \\ &= \zeta'(1)\frac{\partial'' g_\xi(1)}{\partial\tau''}-\zeta'(0)\frac{\partial'' g_\xi(0)}{\partial\tau''}-\int_0^1 \zeta'(\tau)\frac{\partial^3 g_\xi(\tau)}{\partial\tau^3}\mathrm{d}\tau,\end{aligned}$$

by inner product of reproducing kernel Hilbert space ${}^{0}J_2^2[0,1]$ and integration by parts. If we simplify the above equation, we will have

$$\left\langle \zeta, g_\xi(\tau)\right\rangle_{{}^{0}J_2^2[0,1]} = \zeta(\xi).$$

This is the reproducing property. □

**Definition 8.** We introduce the binary reproducing kernel Hilbert space $J(\Upsilon)$, where $\Upsilon=[0,1]\times[0,1]$, as:

$$W(\Upsilon)=\left\{\zeta :\ \frac{\partial^2\zeta}{\partial x\partial t}\in CC(\Upsilon),\quad \frac{\partial^4\zeta}{\partial x^2\partial t^2}\in L^2(\Upsilon),\ \zeta(x,0)=0\right\},$$

where $CC$ means the space of completely continuous functions. The inner product and the norm of binary reproducing kernel Hilbert space $J(\Upsilon)$ are given as:

$$\begin{aligned}\langle\zeta,\eta\rangle_{J(\Upsilon)} &= \sum_{i=0}^{1}\int_0^1\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}\zeta(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}\eta(0,t)\right]\mathrm{d}t \\ &+\sum_{j=0}^{1}\left\langle\frac{\partial^j}{\partial t^j}\zeta(\cdot,0),\frac{\partial^j}{\partial t^j}\eta(\cdot,0)\right\rangle_{J_2^2[0,1]} \\ &+\int_0^1\int_0^1\left[\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial t^2}\zeta(x,t)\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial t^2}\eta(x,t)\right]\mathrm{d}t\mathrm{d}x,\quad \zeta,\eta\in J(\Upsilon),\end{aligned}$$

and

$$\|\zeta\|_{J(\Upsilon)}=\sqrt{\langle\zeta,\zeta\rangle_{J(\Upsilon)}},\quad \zeta\in J(\Upsilon).$$

**Lemma 1** (See [29, page 148])**.** The space $J(\Upsilon)$ is a binary reproducing kernel Hilbert space. Reproducing kernel function $P_{(\phi,\xi)}$ of this space is obtained as:

$$P_{(\phi,\xi)}=h_\phi(\sigma)g_\xi(\tau).$$

**Definition 9.** We describe the binary reproducing kernel Hilbert space $\widehat{J}(\Upsilon)$ as:

$$\widehat{J}(\Upsilon)=\left\{\zeta\in CC(\Omega) :\quad \frac{\partial^2\zeta}{\partial x\partial t}\in L^2(\Omega)\right\}.$$

The inner product and the norm of the binary reproducing kernel Hilbert space $\widehat{J}(\Upsilon)$ are presented as:

$$\begin{aligned}\langle\zeta,\eta\rangle_{\widehat{J}(\Upsilon)} &= \int_0^1\left[\frac{\partial}{\partial t}\zeta(0,t)\frac{\partial}{\partial t}\eta(0,t)\right]\mathrm{d}t+\langle\zeta(\cdot,0),\eta(\cdot,0)\rangle_{J_2^1[0,1]} \\ &+\int_0^1\int_0^1\left[\frac{\partial}{\partial x}\frac{\partial}{\partial t}\zeta(x,t)\frac{\partial}{\partial x}\frac{\partial}{\partial t}\eta(x,t)\right]\mathrm{d}t\mathrm{d}x,\quad \zeta,\eta\in\widehat{J}(\Upsilon)\end{aligned}$$

and

$$\|\zeta\|_{\widehat{J}(\Upsilon)}=\sqrt{\langle\zeta,\zeta\rangle_{\widehat{J}(\Upsilon)}},\quad \zeta\in\widehat{J}(\Upsilon).$$

**Lemma 2** (See [29, page 23])**.** Reproducing kernel function $D_{(\phi,\xi)}$ of the binary reproducing kernel Hilbert space $\widehat{J}(\Upsilon)$ is obtained as:

$$D_{(\phi,\xi)}=f_\phi(\sigma)f_\xi(\tau).$$

## 4 | APPLICATION OF THE METHOD

now, we obtain the solution of Eq.(1) in the reproducing kernel Hilbert space $J(\Upsilon)$ We define the bounded linear operator

$$X : J(\Upsilon)\to\widehat{J}(\Upsilon)$$

as

$$X\zeta = \frac{d^{\gamma}\zeta}{dt^{\gamma}} + M(\zeta, x, t), \quad 0 < \gamma \leq 1. \tag{7}$$

Then, we use the following alternation to homogenize the initial condition,

$$y(x,t) = \zeta(x,t) - A(x).$$

Therefore, Eq. (1) is transformed as:

$$Xy = \frac{d^{\gamma}y}{dt^{\gamma}} + W(y, x, t) = S(y, x, t), \quad 0 < \gamma \leq 1. \tag{8}$$

We select a countable dense subset $\{(\phi_1, \xi_1), (\phi_2, \xi_2), \dots\}$ in $\Upsilon$ and describe

$$\varrho_i = D_{(\phi_i, \xi_i)}, \quad \vartheta_i = X^* \varrho_i,$$

where $X^*$ is the adjoint operator of $X$. The orthonormal system $\{\widehat{\vartheta}_i\}_{i=1}^{\infty}$ of $J(\Upsilon)$ can be attained by the operation of Gram–Schmidt orthogonalization of $\{\vartheta_i\}_{i=1}^{\infty}$ by:

$$\widehat{\vartheta}_i = \sum_{k=1}^{i} \beta_{ik} \vartheta_k.$$

**Theorem 2.** If $\{(\phi_i, \xi_i)\}_{i=1}^{\infty}$ is dense in $\Upsilon$, then the solution of the Eq. (8) has been obtained by reproducing kernel Hilbert space method as:

$$y = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} S(y_k, x_k, t_k) \widehat{\vartheta}_i. \tag{9}$$

*Proof.* We choose $y$ as solution of Eq. (8). We know that $\{\vartheta_i\}_{i=1}^{\infty}$ is a complete system in $J(\Upsilon)$. Therefore, we have:

$$y = \sum_{i=1}^{\infty} \left\langle y, \widehat{\vartheta}_i \right\rangle_{J(\Upsilon)} \widehat{\vartheta}_i = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle y, \vartheta_k \right\rangle_{J(\Upsilon)} \widehat{\vartheta}_i.$$

By using the property of the adjoint operator $X^*$ and obtain:

$$y = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle y, X^* \varrho_k \right\rangle_{J(\Upsilon)} \widehat{\vartheta}_i = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle Xy, \varrho_k \right\rangle_{\widehat{J}(\Upsilon)} \widehat{\vartheta}_i.$$

Then, we apply the reproducing property and acquire:

$$y = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle Xy, D_{(x_k, t_k)} \right\rangle_{\widehat{J(\Upsilon)}} \widehat{\vartheta}_i = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} Xy(x_k, t_k) \widehat{\vartheta}_i.$$

Therefore, we obtain the desired result as:

$$y = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} S(y_k, x_k, t_k) \widehat{\vartheta}_i.$$

□

So, the approximate solution $y_n$ can be constructed by the $n$-term intercept of the exact solution $y$ as:

$$y_n = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} S(y_k, x_k, t_k) \widehat{\vartheta}_i. \tag{10}$$

## 5 | NUMERICAL RESULTS

We consider the following problem by reproducing kernel Hilbert space method in this section.

$$\frac{d^{\gamma}\zeta}{dt^{\gamma}} + \frac{1}{2}(\zeta^2)_x + (1+t)^2 \zeta^2 = x^2, \tag{11}$$

| $x/t$ | $m = 49\ \gamma = 0.25$ | $m = 64\ \gamma = 0.25$ | $m = 49\ \gamma = 0.75$ | $m = 36\ \gamma = 0.75$ |
|---|---|---|---|---|
| 0.0 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.0128059121 | 0.0109412485 | 0.0014313396 | 0.0011349900 |
| 0.2 | 0.0163783274 | 0.0150291295 | 0.0059949369 | 0.0022135496 |
| 0.3 | 0.0234995080 | 0.0258640784 | 0.0086000026 | 0.0055056684 |
| 0.4 | 0.0421179496 | 0.0425838299 | 0.0106332177 | 0.0098200207 |
| 0.5 | 0.0536127447 | 0.0453950185 | 0.0160457585 | 0.0125826690 |
| 0.6 | 0.0524076140 | 0.0510117569 | 0.0198362139 | 0.0116379613 |
| 0.7 | 0.0537372869 | 0.0636812207 | 0.0225650340 | 0.0062413198 |
| 0.8 | 0.0572240855 | 0.0548875939 | 0.0238044796 | 0.0041551731 |
| 0.9 | 0.0427787475 | 0.0527101455 | 0.0227372308 | 0.0199795798 |
| 1.0 | 0.0360241487 | 0.0415249620 | 0.0202224021 | 0.0415693288 |

**TABLE 1** Absolute errors in Example **??** using the $J(\Upsilon)$ kernel and different points near the origin in $[0,1]$ for $\gamma = 0.25$ and $\gamma = 0.75$.

with initial condition $\zeta(x,0) = x$. We need to homogenize the initial condition to apply the proposed technique. Therefore, we use the following transformation.

$$y(x,t) = \zeta(x,t) - x. \tag{12}$$

Then, we have

$$\zeta(x,t) = y(x,t) + x \tag{13}$$

$$\frac{d^{\gamma}\zeta(x,t)}{dt^{\gamma}} = \frac{d^{\gamma}y(x,t)}{dt^{\gamma}} \tag{14}$$

$$\zeta_x(x,t) = y_x(x,t) + 1. \tag{15}$$

If we use these equations in Eq. (11), then we obtain

$$\frac{d^{\gamma}y}{dt^{\gamma}} + (x + 1 + (1+t^2)2x)y = -y^2(2+t^2) + t^2x^2 - x, \tag{16}$$

with the initial condition $y(x,0) = 0$.

We have solved the above example by reproducing kernel Hilbert space method and we show our results by the following tables for different values of $\gamma$. The exact solution of the problem is

$$u(x,t) = \frac{x}{1+t^{\gamma}}.$$

We choose different dense points to get accurate results. We changed the density for $\gamma = 0.75$ in Table 1. Therefore, the results for 36 dense points are better than the results for 49 points. This is related to the density.

## 6 | CONCLUSIONS

In this work, we used reproducing kernel Hilbert space method to solve fractional gas dynamics equation. We tested the method for different values of $\gamma$ and for different numbers of dense points. We constructed reproducing kernel Hilbert spaces and found the solutions in these spaces. We obtained very useful reproducing kernel functions. We showed the efficiency of the method by means of numerical tests.

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| $x/t$ | $m = 4$ | $m = 9$ | $m = 16$ | $m = 64$ |
|---|---|---|---|---|
| 0.0 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.0137438676 | 0.0118257164 | 0.0087766412 | 0.0045477739 |
| 0.2 | 0.0327159013 | 0.0256936343 | 0.0184285971 | 0.0110891094 |
| 0.3 | 0.0488224239 | 0.0344673480 | 0.0245172792 | 0.0155501025 |
| 0.4 | 0.0596603475 | 0.0378689494 | 0.0281715587 | 0.0246731237 |
| 0.5 | 0.0646659142 | 0.0369981912 | 0.0306518224 | 0.0312734755 |
| 0.6 | 0.0640513658 | 0.0338523357 | 0.0343364114 | 0.0378356827 |
| 0.7 | 0.0578528124 | 0.0308188310 | 0.0411213371 | 0.0428262603 |
| 0.8 | 0.0461427511 | 0.0307207277 | 0.0461094580 | 0.0409868845 |
| 0.9 | 0.0287228530 | 0.0332415450 | 0.0424366559 | 0.0386514274 |
| 1.0 | 0.0049938262 | 0.0361331022 | 0.0298574324 | 0.0318394341 |

**TABLE 2** Absolute errors in Example **??** using the $J(\Upsilon)$ kernel and different points near the origin in [0, 1] for $\gamma = 0.5$.

| $x/t$ | $m = 16$ | $m = 25$ | $m = 36$ | $m = 49$ |
|---|---|---|---|---|
| 0.0 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.0032833260 | 0.0017202261 | 0.0008707394 | 0.0003617891 |
| 0.2 | 0.0010140197 | 0.0012319248 | 0.0018248226 | 0.0018640778 |
| 0.3 | 0.0039382302 | 0.0049101362 | 0.0038628464 | 0.0026430397 |
| 0.4 | 0.0073083853 | 0.0056934723 | 0.0029931647 | 0.0019616493 |
| 0.5 | 0.0070698927 | 0.0036136697 | 0.0022095268 | 0.0023067580 |
| 0.6 | 0.0045266740 | 0.0025094840 | 0.0026436300 | 0.0021775196 |
| 0.7 | 0.0029594541 | 0.0026911891 | 0.0023510295 | 0.0019722889 |
| 0.8 | 0.0027772706 | 0.0023098383 | 0.0022310986 | 0.0016297736 |
| 0.9 | 0.0032558105 | 0.0023467895 | 0.0017182675 | 0.0012550185 |
| 1.0 | 0.0029784370 | 0.0021751200 | 0.0013464010 | 0.0010067620 |

**TABLE 3** Absolute errors in Example **??** using the $J(\Upsilon)$ kernel and different points near the origin in [0, 1] for $\gamma = 1.0$.

## References


[1] Singh J., Kumar D., Kilicman A. Homotopy perturbation method for fractional gas dynamics equation using sumudu transform, *Abstr. Appl. Anal.* 2013; Article ID 934060, 8 pages.

[2] Momani S. Analytic and approximate solution of the space and time fractional telegraph equations, *Appl. Math. Comput.* 2005; 170(2):1126–1134.

[3] Momani S., Shawagfeh N.T. Decomposition method for solving fractional Riccati differential equations, *Appl. Math. Comput.* 2006; 182:1083–1092.

[4] Hashim I., Abdulaziz O., Momani S. Homotopy analysis method for fractional IVPs, *Commun. Nonlinear Sci. Numer. Simul.* 2009; 14:674–684.

[5] Yildirim A. He's homotopy perturbation method for solving the space and time fractional telegraph equations, *Int. J. Comput. Math.* 2010; 87:2998–3006.

[6] Momani S., Odibat Z. Numerical comparison of methods for solving linear differential equations of fractional order, *Chaos Solitons Fractals* 2007; 31(5):1248–1255.

[7] Singh J., Kumar D., Kilicman A. Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations *Abst. Appl. Anal.* 2014; Article ID 535793, 12 pages.

| $x/t$ | $m = 64$ | $m = 81$ | $m = 100$ | $m = 225$ |
|---|---|---|---|---|
| 0.0 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.0000494399 | 0.0001331504 | 0.0002342356 | 0.0002850756 |
| 0.2 | 0.0016849007 | 0.0014218863 | 0.0011485154 | 0.0005530163 |
| 0.3 | 0.0018106764 | 0.0013811103 | 0.0012111720 | 0.0007491910 |
| 0.4 | 0.0018153182 | 0.0016905161 | 0.0014656000 | 0.0007951643 |
| 0.5 | 0.0019836820 | 0.0016881536 | 0.0014521946 | 0.0007566967 |
| 0.6 | 0.0018624294 | 0.0015135600 | 0.0012665836 | 0.0006670620 |
| 0.7 | 0.0015694551 | 0.0012533301 | 0.0010520580 | 0.0005582182 |
| 0.8 | 0.0012531656 | 0.0010098556 | 0.0008580806 | 0.0004568696 |
| 0.9 | 0.0009895226 | 0.0008163385 | 0.0006952695 | 0.0003922673 |
| 1.0 | 0.0008040110 | 0.0006564070 | 0.0005478900 | 0.0003081000 |

**TABLE 4** Absolute errors in Example **??** using the $J(\Upsilon)$ kernel and different points near the origin in $[0, 1]$ for $\gamma = 1.0$.


[8] Rida S.Z., El-Sayed A.M.A., Arafa A.A.M. On the solutions of time-fractional reaction-diffusion equations *Commun. Nonlinear. Sci. Numer. Simul.* 2010; 15(12):3847–3854.

[9] Singh J., Kumar D., Sushila. Homotopy perturbation Sumudu transform method for nonlinear equations, *Adv. Appl. Math. Mech.* 2011; 4:165–175.

[10] Doha E.H., Bhrawy A.H., Ezz-Eldien S.S. An efficient Legendre spectral tau matrix formulation for solving fractional subdiffusion and reaction subdiffusion equations, *J. Comput. Nonlinear Dyn.* 2015; 10(2):021019, 8 pages.

[11] Machado J.A.T., Mata M.E. A fractional perspective to the bond graph modelling of world economies, *Nonlinear. Dyn.* 2015; 80:1839–1852.

[12] Yang A.M., Zhang Y.Z., Cattani C., Xie G.N., Rashidi M.M., Zhou Y.Z., Yang X.J. Application of local fractional series expansion method to solve Klein-Gordon equations on Cantor set, *Abstr. Appl. Anal.* 2014; Article ID 372741, 6 pages.

[13] Biazar J., Mostafa E. Differential transform method for nonlinear fractional gas dynamics equation, *Int. J. Phys. Sci.* 2011; 6(5):1203–1206.

[14] Raja Balachandar S., Krishnaveni K., Kannan K., Venkatesh S.G. Analytical Solution for Fractional Gas Dynamics Equation, *Natl. Acad. Sci. Lett.* 2019; 42(1):51–57.

[15] Zaremba, S. L'équation biharmonique et une classe remarquable de fonctions fondamentales harmoniques, *Bull. Int. Acad. Sci. Cracovie* 1907; 147–196.

[16] Zaremba, S. Sur le calcul numérique des fonctions demandées dan le probléme de dirichlet et le probleme hydrodynamique. *Bull. Int. Acad. Sci. Cracovie* 1908; 125–195.

[17] Wang Y.L., Liu Y., Li Z.Y., Zhang H.L. Numerical solution of integro-differential equations of high-order Fredholm by the simplified reproducing kernel method, *Int. J. Comput. Math.* 2019; 96(3):585–593.

[18] Gumah G.N., Naser M.F.M., Al-Smadi M., Al-Omari S.K. Application of reproducing kernel Hilbert space method for solving second-order fuzzy Volterra integro-differential equations, *Adv. Differ. Equ.* 2018; 2018:475.

[19] Al-Smadi M. Simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation, *Ain Shams Eng. J.* 2018; 9:2517–2525.

[20] Kashkari B.S.H., Syam M.I. Reproducing Kernel Method for Solving Nonlinear Fractional Fredholm Integrodifferential Equation, *Complexity* 2018; Article ID 2304858, 7 pages.

[21] Akgül A., Grow D. Existence of solutions to the telegraph equation in binary reproducing kernel Hilbert spaces, *Differential Equations and Dynamical Systems* 2019; doi:10.1007/s12591-019-00453-3.

[22] Akgül A., Khan Y., Akgül E.K., Baleanu D., Mohamed A.Q.M. Solutions of nonlinear systems by reproducing kernel method, *J. Nonlinear Sci. Appl.* 2017; 10(8):4408–4417.

[23] Akgül E.K. Reproducing kernel Hilbert space method for solutions of a coefficient inverse problem for the kinetic equation, *Int. J. Optim. Control. Theor. Appl. IJOCTA* 2018; 8(2):145–151.

[24] Akgül A., Inc M., Karatas E. Reproducing kernel functions for difference equations, *Discrete Contin. Dyn. Syst. Ser. S* 2015; 8:1055–1064.

[25] Akgül A., Inc M., Karatas E., Baleanu D. Numerical Solutions of Fractional Differential Equations of Lane-Emden Type by an Accurate Technique, *Adv. Differ. Equ.*, 2015; 2015:220.

[26] Aronszajn, N. Theory of reproducing kernels, *Trans. Amer. Math. Soc.* 1950; 68:337–404.

[27] Bergman, S. The kernel function and conformal mapping. *Amer. Math. Soc.* New York, 1950.

[28] Cui M., Zhongxing D. On the best operator of interpolation, *Math. Numer. Sinica* 1986; 8:209–216.

[29] Cui M., Lin Y. Nonlinear Numerical Analysis in the Reproducing Kernel Space, *Nova Sci. Publ. Inc.*, New York, 2009.

[30] Inc M., Akgül A. The reproducing kernel Hilbert space method for solving Troesch's problem, *J. Assoc. Arab Univ. Basic Appl. Sci.* 2013; 14:19–27.

[31] Inc M., Akgül A. Approximate solutions for MHD squeezing fluid flow by a novel method, *Bound. Value Probl.* 2014; 2014:18.

[32] Inc M., Akgül A., Geng F. Reproducing kernel Hilbert space method for solving Bratu's problem, *Bull. Malays. Math. Soc.* 2015; 38:271–287.

[33] Inc M., Akgül A., Kilicman A. Numerical solutions of the second order one-dimensional telegraph equation based on reproducing kernel Hilbert space method. *Abstr. Appl. Anal.* 2013; Article ID 768963, 13 pages.

[34] Inc M., Akgül A., Kilicman A. On solving KdV equation using reproducing kernel Hilbert space method, *Abstr. Appl. Anal.* 2013; Article ID 578942, 11 pages.

[35] Wang Y., Chao L. Using reproducing kernel for solving a class of partial differential equation with variable coefficients. *Applied Mathematics and Mechanics* 2008; 29:129–137.

[36] Wu B.Y., Li X.Y. A new algorithm for a class of linear nonlocal boundary value problems based on the reproducing kernel method, *Appl. Math. Lett.* 2011; 24:156–159.

[37] Yao H., Lin Y. Solving singular boundary value problems of higher even order. *J. Comput. Appl. Math.* 2009; 223:703–713.


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