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SPECIAL ISSUE ARTICLE

On PageRank versatility for Multiplex networks: properties and some useful bounds

Francisco Pedroche*[1] | Regino Criado[2,3,4] | Julio Flores[2,3] | Esther García[2] | Miguel Romance[2,3,4]

[1]Institut de Matemàtica Multidisciplinària, Universitat Politècnica de València, 46022 València, Spain
[2]Departamento de Matemática Aplicada, Ciencia e Ingeniería de los Materiales y Tecnología Elctrónica, ESCET Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain
[3]Center for Computational Simulation, 28223 Pozuelo de Alarcón (Madrid), Spain
[4]Data, Networks and Cybersecurity Research Institute, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain

Correspondence
*Francisco Pedroche, Institut de Matemàtica Multidisciplinària, Universitat Politècnica de València, 46022 València. Email: pedroche@imm.upv.es

Abstract

In this paper, some results concerning the PageRank versatility measure for multiplex networks are given. This measure extends to the multiplex setting the well-known classic PageRank. Particularly we focus on some spectral properties of the Laplacian matrix of the multiplex, and on obtaining boundaries for the ranking value of a given node when some personalization vector is added, as in the classic setting.

KEYWORDS:
Versatility; Laplacian matrices; PageRank; centrality measures; multiplex networks

1 | INTRODUCTION

Determining which are the most relevant elements of a complex system is one of the most important problems dealt with by the so-called Complexity Science[4,8,11,12,13,25,36]. This problem is directly related to understanding the relevance of each element within the structure of a system, which is a first step in understanding its behavior. This problem appears in multiple fields ranging from biological and technological systems to social systems[5,7,11,15,21,36]. A great help in determining the relevance of the nodes of a specific network is provided by centrality measures that allow us to detect the most important nodes by associating a numerical value to each vertex of the system. Centrality measures can be very different in nature, since, for example, they can make use of local parameters (node's degree), metric parameters (betweenness centrality) and other mathematical techniques and tools (eigenvector centrality). Among, them the centrality PageRank[28] is a culminating point since it is the basic ingredient in web information in general and in Google's web search engine in particular. Since its appearance in 1998 to classify web pages to the present day, a large number of refinements and new applications of the PageRank algorithm have emerged in the scientific literature[1,6,9,11,14,17,29,34,35,39,41]. These refinements are very varied in nature, and new methodologies are being developed in the literature to detect both the most relevant nodes and the competing nodes[14]. The use of a personalization vector to modify the ranking obtained (personalized PageRank)[30] and a new vision of this algorithm that allows to extend PageRank to multiplex networks are other advances that have appeared in recent years. The problem becomes more complicated when different types of interactions appear in the system under study, forming interconnected multilayer networks[3,5,11,10]. In this regard, some studies on *multilayered* or *multirelational* networks were unable to satisfactorily describe the behavior of the systems by using classical techniques of monoplex networks (see, e.g,[24] for fails regarding detection of communities,[38] for misunderstandings when mixing different interactions on social networks,[30] for ranking differences when ignoring the multilayered nature of a metro system,

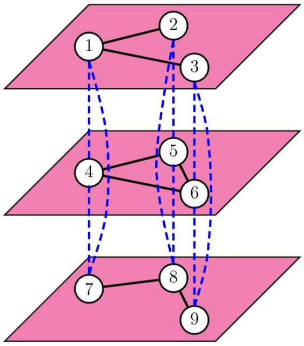


FIGURE 1 A multiplex with three layers and three nodes on each layer. Red dashed lines represent *interlayer* links.

and [31] for an analysis of the transition from a collection of independent networks to a whole multiplex). Therefore, it has been necessary to implement new concepts and techniques to cope with the heterogeneity of links shown by these *complex* networks (see, e.g, [2,3,5,10,32,36,40]). In particular, aggregating information to determine which node is most central is not a simple process and requires in-depth multi-layered analysis. In [11] a suitable structure is introduced to calculate centrality measures adjusted to the context of the interconnected multilayer complex networks, emerging the concept of versatility as a good descriptor of certain dynamic aspects that appear in this type of structures.

As a reference example, consider the situation in [11] where several authors, the nodes, publish a research article together, and the different tasks in the creation process (experiment design, data analysis, writing, etc.) are considered to be the different layers. Then a multi-layered network appears by simply connecting two authors in a given layer when both have contributed to the task represented in that layer. In this setting, determining the most versatile author goes beyond the simple aggregation of the contributions to the different tasks and involves the topology of the multistructure [11].

Thus, in this paper we are interested in multiplex networks [5]: these kind of networks are formed by a number of layers with the same number of nodes such that the only allowed inter-layer links are those corresponding to nodes connected with themselves in all the layers (see an example in Fig. 1).

More precisely, our interest focuses on PageRank versatility [11] in a multiplex network, a concept that extends the well-known classic PageRank to the multistructure. Two aspects are given special importance. The first one refers to the spectral properties of the matrix $\mathbb{T}$ that contains the topology of the multiplex network and is used to construct the "Google matrix" by the convex addition of a some personalization vector $\mathbf{v}$. This is in general a difficult problem which requires to understand the associated Laplacian matrix from the point of view of spectral theory.

The other aspect refers to the PageRank versatility interval of a given node or, in other words, to the set of admissible ranking values that a given node may have in terms of the personalization vector $\mathbf{v}$. Some valuable estimations are obtained. The results are illustrated with an example of a synthetic toy network and one example of a real network (*The Florentine Family Marriage and Business Biplex Network*, see [18],[27],[33]).

PAGERANK VERSATILITY

The concept *PageRank versatility* was introduced in [11], where the authors make extensive use of the tensor notation for multilayer networks developed in [10]. Let us denote by n the number of nodes of each layer, and by k the number of layers.

Formally, a multilayer network is characterized by a *multilayer adjacency tensor* $M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}$, where indices with tilde refer to layers. The tensor $M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}$ can be represented in matrix notation (without explicitly show the indices of the nodes) by a matrix $\mathbb{M}$ of size $nk \times nk$ in the following form

$$M^{i\alpha}_{j\beta} \equiv \mathbb{M} = \sum_{\alpha,\beta=1}^{k} \mathbb{E}(\alpha,\beta) \otimes \mathbb{C}(\alpha,\beta) \tag{1}$$

where $\otimes$ denotes the Kronecker product (see, e.g., [19]) and the matrix $\mathbb{E}(\alpha,\beta) \in \mathbb{R}^{k\times k}$ is given by

$$\mathbb{E}(\alpha,\beta) = \mathbf{e}^k_\alpha \otimes (\mathbf{e}^k_\beta)^T$$

TABLE 1 Relationship between matrix and tensor notation for the PageRank versatility framework.

Matrix notation	Tensor notation in[26] and[11]
$\mathbb{M}$	$M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}$
$\mathbb{T}$	$T^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}$
$\mathbb{G}$	$R^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}$
$\mathbf{e}^{nk}\mathbf{v}^T$	$\frac{1}{nk}u^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}$
Π	$\Omega_{\alpha\tilde{\gamma}}$
π	ω_α

where $\mathbf{e}^k_\alpha$ is the α-th column of the identity matrix of size k, and the superscript T denotes transposition.

The matrices $\mathbb{C}(\alpha,\beta) \in \mathbb{R}^{n\times n}$ represent both the adjacency matrices of the layers and the matrices accounting for the links between layers. In the case of a multiplex network this reduces to the following

$$\mathbb{C}(\alpha,\beta) = \begin{cases} I_n & if \;\; \alpha \neq \beta \\ A_\alpha & if \;\; \alpha = \beta \end{cases}$$

where I_n is the identity matrix of size n and A_α is the adjacency matrix of layer α. In particular, $\mathbb{M}$ is an block-matrix of $k \times k$ blocks, each of them of size $n \times n$, of the form

$$\mathbb{M} = \left(\begin{array}{c|c|c|c} A_1 & I_n & \dots & I_n \\ \hline I_n & A_2 & \dots & I_n \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline I_n & I_n & \dots & A_k \end{array}\right). \tag{2}$$

Once $\mathbb{M}$ is defined, the *PageRank versatility* can be defined following a similar procedure as in the *classic PageRank*[28]. For the sake of clearness we use matrix notation. Note that in a multiplex framework with undirected links, the matrix $\mathbb{M}$ is a symmetric matrix and has no zero rows. Let us denote m_{ij}, $i, j \in \{1, 2, \dots, nk\}$ each element of $\mathbb{M}$. Hence we can define the row stochastic matrix $\mathbb{T}$ with elements t_{ij} as follows

$$t_{ij} = \frac{m_{ij}}{\sum_{j=1}^{nk} m_{ij}}$$

and define a matrix $\mathbb{G} \in \mathbb{R}^{nk\times nk}$ (analogous to the Google matrix[28]) as follows

$$\mathbb{G} = \alpha\mathbb{T} + (1-\alpha)\mathbf{e}^{nk}\mathbf{v}^T$$

where $\mathbf{e}^{nk} \in \mathbb{R}^{nk\times 1}$ is the vector of all ones, and $\mathbf{v}^T = \frac{1}{k}[\mathbf{v_1}^T \, \mathbf{v_2}^T, \dots \mathbf{v_k}^T]$ is a *personalization vector* formed by staking the personalization vector of each layer $\mathbf{v}_i \in \mathbb{R}^{n\times 1}$. Notice that

$$\mathbf{e}^{nk}\mathbf{v}^T = \frac{1}{k}\left(\begin{array}{c|c|c|c} \mathbf{v_1}^T & \mathbf{v_2}^T & \dots & \mathbf{v_k}^T \\ \mathbf{v_1}^T & \mathbf{v_2}^T & & \mathbf{v_k}^T \\ \vdots & \vdots & & \vdots \\ \mathbf{v_1}^T & \mathbf{v_2}^T & & \mathbf{v_k}^T \end{array}\right).$$

Each vector $\mathbf{v_\alpha}^T$ models the probability of teleportation in layer α. We remark that by taking $\mathbf{v}_\alpha = \frac{1}{n}[1, 1, \dots, 1]$ for all α, the matrix $\mathbf{e}^{nk}\mathbf{v}^T$ is $\frac{1}{nk}$ multiplied by a square matrix of size $nk \times nk$ with all its elements equal to one (in Table 1 is shown the correspondence with the tensor notation used in[26] and[11]).

By construction, it is clear that $\mathbb{G}$ has a unique positive left eigenvector $\Pi \in \mathbb{R}^{nk\times 1}$ with norm equal to 1 associated to the dominant eigenvalue of $\mathbb{G}$. This vector can be folded to obtain a vector of size $\mathbb{R}^{n\times 1}$ by doing the following. First, we define

$$p_i = \mathbf{e}^k \otimes \mathbf{e}_i^n = \left(\begin{array}{c} \mathbf{e}_i^n \\ \hline \mathbf{e}_i^n \\ \hline \vdots \\ \hline \mathbf{e}_i^n \end{array}\right), \quad i = 1, 2, \dots, n$$

where $\mathbf{e}^k$ is the vector of all ones in $\mathbb{R}^{k\times 1}$ and $\mathbf{e}_i^n$ is the i-th column of the identity matrix of size n. Second, we define

$$\pi_i = p_i^T \Pi = [(\mathbf{e}_i^n)^T \; (\mathbf{e}_i^n)^T \; \dots \; (\mathbf{e}_i^n)^T] \left(\begin{array}{c} \Pi_1 \\ \hline \Pi_2 \\ \hline \vdots \\ \hline \Pi_k \end{array}\right) =$$

$$= (\mathbf{e}_i^n)^T \sum_{\alpha=1}^{k} \Pi_\alpha = (\sum_{\alpha=1}^{k} (\Pi_\alpha)^T) \mathbf{e}_i^n \in \mathbb{R}$$

and finally, the *personalized PageRank versatility vector* is the vector of $\mathbb{R}^{n\times 1}$ given by

$$\pi = [\pi_1, \pi_2, \dots, \pi_n]^T$$

Note that $\mathbb{G}$ is formally a Google matrix and, therefore, it has well-known *good* properties. In particular, by using Theorem 5.1 in[22] it is known that the spectrum of $\mathbb{G}$ is $\{1, \alpha\lambda_2, \alpha\lambda_3, \dots, \alpha\lambda_{nk}\}$ where $\{1, \lambda_2, \dots, \lambda_{nk}\}$ is the spectrum of the row stochastic matrix $\mathbb{T}$. In general, is difficult to give details about the spectrum of $\mathbb{T}$. We can give, however, some results by looking at the Laplacian matrix of the multiplex (usually called supra-Laplacian[16], [37]). But first, We introduce some notation and known results in the next section.

2 | NOTATION AND KNOWN RESULTS

We recall that given an adjacency matrix $A \in \mathbb{R}^{n\times n}$, with elements a_{ij}, of an undirected network, the Laplacian matrix can be defined as

$$L = D - A$$

where D ia a diagonal matrix with elements $deg(i) = \sum_{j=1}^{n} a_{ij}$.

Analogously, given a multiplex of n nodes and k layers with adjacency matrix $\mathbb{M}$, given by (1), the corresponding supra-Laplacian[1] matrix can be defined by

$$\mathbb{L} = \mathbb{D} - \mathbb{M} \tag{3}$$

where $\mathbb{D}$ is a diagonal matrix with elements $d_{ii} = \sum_{j=1}^{nk} m_{ij}$.

It is straightforward to see that we can write

$$\mathbb{L} = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_k \end{pmatrix} + \begin{pmatrix} (k-1)I_n & -I_n & \cdots & -I_n \\ -I_n & (k-1)I_n & \cdots & -I_n \\ \vdots & \vdots & \ddots & \vdots \\ -I_n & -I_n & \cdots & (k-1)I_n \end{pmatrix}$$

where the first matrix is the direct sum of the Laplacians of each layer (that is, defined by the intralayer links). Let us denote (as in[37])

$$\mathcal{L}^L = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_k \end{pmatrix}, \quad \mathcal{L}^I = \begin{pmatrix} (k-1)I_n & -I_n & \cdots & -I_n \\ -I_n & (k-1)I_n & \cdots & -I_n \\ \vdots & \vdots & \ddots & \vdots \\ -I_n & -I_n & \cdots & (k-1)I_n \end{pmatrix}$$

[1]This matrix is called $\mathcal{L}$ in[37]

Where $\mathcal{L}^L$ is the supra-Laplacian of the independent layers and $\mathcal{L}^I$ is called interlayer supra-Laplacian. The notation can be simplified by introducing a new matrix called interlayer Laplacian in the form:

$$\mathbb{L}^I = \mathbb{S}^I - \mathbb{W}^I \in \mathbb{R}^{k\times k} \tag{4}$$

where

$$\mathbb{S}^I = (k-1)I_k, \quad \mathbb{W}^I = \mathbf{e}^k(\mathbf{e}^k)^T - I_k$$

where $\mathbf{e}^k(\mathbf{e}^k)^T$ is the square matrix of all ones of size k.

With these new matrices we have

$$\mathcal{L}^I = \mathbb{L}^I \otimes I_n \tag{5}$$

expression that will be useful later.

We remark here that we use mostly the notation used in [16] and [37], with the restriction that in our case we do not treat any diffusion problem and therefore all the coefficients denoted as D_α or D_{ij} in [16] and [37] are here treated as 1. This feature will allow us to give some new results on eigenvalues and eigenvectors later.

To summarize, the supra-Laplacian matrix can be written as

$$\mathbb{L} = \mathcal{L}^L + \mathcal{L}^I \tag{6}$$

where $\mathcal{L}^L$ and $\mathcal{L}^I$ are symmetric matrices. In what follows, we will sort the eigenvalues of $\mathcal{L}^L$ and $\mathcal{L}^I$ in increasing order, that is

$$\lambda_1(\mathcal{L}^L) \le \lambda_2(\mathcal{L}^L) \le \ldots \le \lambda_{kn}(\mathcal{L}^L)$$

$$\lambda_1(\mathcal{L}^I) \le \lambda_2(\mathcal{L}^I) \le \ldots \le \lambda_{kn}(\mathcal{L}^I)$$

We remark also that, since $\mathcal{L}^L$ is diagonal by blocks, then its spectrum is the union of the spectrums of matrices L_i. That is,

$$\sigma(\mathcal{L}^L) = \bigcup_{i=1}^{k} \sigma(L_i) \tag{7}$$

Moreover, note that each $L_j \in \mathbb{R}^{n\times n}$ is a symmetric real matrix and therefore there exists a basis of $\mathbb{R}^{n\times n}$ formed by eigenvectors of L_j for each $j = 1, \ldots, k$. The spectrum and the eigenspace of $\mathcal{L}^L$ is readily known as the next result shows.

Proposition 1. Let $\mathbf{w}_i(L_j)$, for $i = 1, \ldots, n$ and for $j = 1, \ldots, k$, be an eigenvector of L_j associated to the eigenvalue $\lambda_i(L_j)$. Then $\mathbf{e}_j^k \otimes \mathbf{w}_i(L_j)$ is an eigenvector of $\mathcal{L}^L$ associated to the eigenvalue $\lambda_i(L_j)$.

Proof. Since $\mathcal{L}^L$ is diagonal by blocks, being L_j each block, for $j = 1, \ldots k$, it is clear that each eigenvalue $\lambda_i(L_j)$, for $i = 1, \ldots, n$, is also an eigenvalue of $\mathcal{L}^L$. Without loss of generality, let $j = 1$ and let us show that $\mathbf{e}_1^k \otimes \mathbf{w}_i(L_1)$ is an eigenvector of $\mathcal{L}^L$ associated to the eigenvalue $\lambda_i(L_1)$. It suffices to see that

$$\begin{aligned}
\mathcal{L}^L \mathbf{e}_1^k \otimes \mathbf{w}_i(L_1) = \mathcal{L}^L \begin{pmatrix} \mathbf{w}_i(L_1) \\ 0_{n\times 1} \\ \vdots \\ 0_{n\times 1} \end{pmatrix} &= \begin{pmatrix} L_1\mathbf{w}_i(L_1) \\ 0_{n\times 1} \\ \vdots \\ 0_{n\times 1} \end{pmatrix} \\
&= \lambda_i(L_1) \begin{pmatrix} \mathbf{w}_i(L_1) \\ 0_{n\times 1} \\ \vdots \\ 0_{n\times 1} \end{pmatrix} \\
&= \lambda_i(L_1)\mathbf{e}_1^k \otimes \mathbf{w}_i(L_1).
\end{aligned}$$

Hence, by repeating this process for any j the proof follows. □

Example 8. For the multiplex depicted in Fig. 1 we have $n = 3$ and $k = 3$. And

$$\mathbb{M} = \begin{pmatrix} A_1 & I_3 & I_3 \\ I_3 & A_2 & I_3 \\ I_3 & I_3 & A_3 \end{pmatrix}$$

with

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and denoting

$$D_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

it is clear that matrix $\mathbb{D}$ can be written as

$$\mathbb{D} = \begin{pmatrix} D_1 + 2I_3 & 0 & 0 \\ 0 & D_2 + 2I_3 & 0 \\ 0 & 0 & D_3 + 2I_3 \end{pmatrix}$$

and therefore

$$\mathbb{L} = \begin{pmatrix} D_1 + 2I_3 - A_1 & -I_3 & -I_3 \\ -I_3 & D_2 + 2I_3 - A_2 & -I_3 \\ -I_3 & -I_3 & D_3 + 2I_3 - A_3 \end{pmatrix}$$

that is

$$\mathbb{L} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix} + \begin{pmatrix} 2I_3 & -I_3 & -I_3 \\ -I_3 & 2I_3 & -I_3 \\ -I_3 & -I_3 & 2I_3 \end{pmatrix} \equiv \mathcal{L}^L + \mathcal{L}^I$$

In this example we also have

$$\mathbb{S}^I = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbb{W}^I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and therefore

$$\mathbb{L}^I = \mathbb{S}^I - \mathbb{W}^I = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Note that $\mathcal{L}^I = \mathbb{L}^I \otimes I_3$, as given by (5).

Let us recall two results that will be useful later. The first one is about the spectrum of the Kronecker product of two matrices.

Theorem 1. [19] Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$. If $\lambda \in \sigma(A)$, $\mu \in \sigma(B)$, $\mathbf{x} \in \mathbb{C}^n$ is a corresponding eigenvector of A, and $\mathbf{y} \in \mathbb{C}^m$ is a corresponding eigenvector of B, then $\lambda\mu \in \sigma(A \otimes B)$ and $\mathbf{x} \otimes \mathbf{y} \in \mathbb{C}^{nm}$ is a corresponding eigenvector of $A \otimes B$. Every eigenvalue of $A \otimes B$ arises as such a product of eigenvalues of A and B. If $\sigma(A) = \{\lambda_i(A); i = 1, \dots, n\}$ and $\sigma(B) = \{\lambda_i(B); j = 1, \dots, m\}$ then $\sigma(A \otimes B) = \{\lambda_i(A)\lambda_j(B); i = 1, \dots, n,\ j = 1, \dots, m\}$ (including algebraic multiplicities in all three cases).

The second result is about symmetric matrices, and it is due to Weyl.

Theorem 2. [20] Let $A, B \in \mathbb{R}^{n\times n}$ be Hermitian and let the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$, and $\lambda_i(A + B)$, be arranged in increasing order. For each $s = 1, 2, \dots n$ we have

$$\lambda_s(A) + \lambda_1(B) \leq \lambda_s(A + B) \leq \lambda_s(A) + \lambda_n(B).$$

3 | SPECTRUM OF THE LAPLACIAN MATRIX

In this section we formalize and extend some results appearing in[37]. In particular, we give an explicit expression for the eigenvalues and eigenvectors of matrices $\mathbb{L}^I$, $\mathcal{L}^I$ and $\mathcal{L}^L$ and we bound the spectrum of the supra-Laplacian matrix $\mathbb{L}$.

We begin by showing a result about the spectrum and the eigenspace of matrix $\mathbb{L}^I$ defined in (4).

Theorem 3. The spectrum of matrix $\mathbb{L}^I$ is given by

$$\lambda_1(\mathbb{L}^I) = 0, \quad \text{and} \quad \lambda_s(\mathbb{L}^I) = k, \quad \text{for} \quad s = 2, \dots k$$

and the corresponding associated eigenvectors are given by

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{e}^k \\ \mathbf{v}_s &= -\mathbf{e}_1^k + \mathbf{e}_s^k, \quad s = 2, \dots, k \end{aligned} \tag{9}$$

where $\mathbf{e}^k$ is the column vector of all ones in $\mathbb{R}^{k\times 1}$ and $\mathbf{e}_i^k$ is the i-th column of the identity matrix of size k. Moreover, the set $\{\mathbf{v}_j; j = 1, \dots, k\}$ is a basis of $\mathbb{R}^{k\times k}$.

Proof. Let us denote by μ_{ij} the elements of $\mathbb{L}^I$. By the definition of $\mathbb{L}^I$ is clear that

$$\mu_{ij} = \begin{cases} k-1 & \text{if} \quad i = j \\ -1 & \text{else} \end{cases}$$

then it is straightforward that

$$\mathbb{L}^I \mathbf{e}^k = [0, \dots, 0]^T = 0\mathbf{e}^k$$

Therefore, is left to show that

$$\mathbb{L}^I \mathbf{v}_s = k\mathbf{v}_s$$

for $s = 2, \dots, k$.

To begin with, let us study the first component of the vector $\mathbb{L}^I \mathbf{v}_s$. That is

$$\begin{aligned} [\mathbb{L}^I \mathbf{v}_s]_1 &= \sum_{j=1}^k \mu_{1j} [\mathbf{v}_s]_j \\ &= \mu_{11}[\mathbf{v}_s]_1 + \mu_{1s}[\mathbf{v}_s]_s + \sum_{j\neq 1, j\neq s}^k \mu_{1j}[\mathbf{v}_s]_j \\ &= (k-1)(-1) + (-1)(1) + 0 = -k = k[\mathbf{v}_s]_1 \end{aligned}$$

where we have used that $[\mathbf{v}_s]_j = \delta_{sj}$ for $j \neq 1$.

The s-th component of the vector $\mathbb{L}^I \mathbf{v}_s$, for $s = 2, \dots, k$, results to be

$$\begin{aligned} [\mathbb{L}^I \mathbf{v}_s]_s &= \sum_{j=1}^k \mu_{sj} [\mathbf{v}_s]_j \\ &= \mu_{ss}[\mathbf{v}_s]_s + \mu_{s1}[\mathbf{v}_s]_1 + \sum_{j\neq 1, j\neq s}^k \mu_{sj}[\mathbf{v}_s]_j \\ &= (k-1)(1) + (-1)(-1) + 0 = k = k[\mathbf{v}_s]_s \end{aligned}$$

And finally, the q-th component of the vector $\mathbb{L}^I \mathbf{v}_s$, for $q \neq 1$ and $q \neq s$, results to be

$$\begin{aligned} [\mathbb{L}^I \mathbf{v}_s]_q &= \sum_{j=1}^k \mu_{qj} [\mathbf{v}_s]_j \\ &= \mu_{qq}[\mathbf{v}_s]_q + \mu_{q1}[\mathbf{v}_s]_1 + \mu_{qs}[\mathbf{v}_s]_s + \sum_{j\neq 1,q,s}^k \mu_{qj}[\mathbf{v}_s]_j \\ &= (k-1)(0) + (-1)(1) + (-1)(-1) + 0 = 0 = k[\mathbf{v}_s]_q \end{aligned}$$

since $[\mathbf{v}_s]_q = 0$ for $q \neq 1$ and $q \neq s$.

Now we show that the eigenvectors $\{\mathbf{v}_j; j = 1, \ldots, k\}$ form a basis of $\mathbb{R}^{k\times 1}$. This can be shown by forming a matrix with the eigenvectors as rows

$$\begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_k^T \end{pmatrix} = \left(\begin{array}{c|c} 1 & (\mathbf{e}^{k-1})^T \\ \hline -\mathbf{e}^{k-1} & I_{k-1} \end{array}\right)$$

and noticing that the determinant of this matrix is

$$\left|\begin{array}{c|c} 1 & (\mathbf{e}^{k-1})^T \\ \hline -\mathbf{e}^{k-1} & I_{k-1} \end{array}\right| = |I_{k-1}||1 + (\mathbf{e}^{k-1})^T I_{k-1}\mathbf{e}^{k-1}| = 1 + k - 1 = k \neq 0.$$

□

It is known (see[37]) that the eigenvalues of $\mathcal{L}^I$ are those of $\mathbb{L}^I$ (multiplying by n the multiplicity of each eigenvalue), and that any eigenvector of $\mathbb{L}^I$ gives raise to an eigenvector of $\mathcal{L}^I$. We now go further, explicitly showing the spectrum and the eigenspaces of $\mathcal{L}^I$.

Corollary 1. The spectrum of $\mathcal{L}^I$ given by (5) is

$$\sigma(\mathcal{L}^I) = \begin{cases} 0 \text{ with algebraic multiplicity } & n \\ k \text{ with algebraic multiplicity } & (k-1)n \end{cases}$$

and the corresponding eigenvectors are

$$\mathbf{e}^k \otimes \mathbf{e}_i^n \quad \text{for} \quad i = 1, \ldots, n, \quad \text{associated to the eigenvalue } 0 \tag{10}$$

$$(-\mathbf{e}_1^k + \mathbf{e}_s^k) \otimes \mathbf{e}_i^n \quad \text{for} \quad s = 2, \ldots, k, \quad \text{and} \quad i = 1, \ldots, n, \tag{11}$$

associated to the eigenvalue k. Moreover, the set of all these eigenvectors is a basis of $\mathbb{R}^{kn\times kn}$.

Proof. By applying Theorem 1 to matrix $\mathcal{L}^I$ given by (5) we have

$$\sigma(\mathcal{L}^I) = \sigma(\mathbb{L}^I \otimes I_n) = \{\lambda_i(\mathbb{L}^I)\lambda_j(I_n); i = 1, \ldots, k,\ j = 1, \ldots, n\} \tag{12}$$

and since each identity matrix I_n has the eigenvalue 1 with algebraic multiplicity n, we conclude that the spectrum of $\mathcal{L}^I$ is formed by the spectrum of $\mathbb{L}^I$ multiplying by n the multiplicity of each eigenvalue. Then, by using Theorem 3, the proof about the spectrum is done. For the eigenvectors, we take into account that I_n has n eigenvectors of the form $\mathbf{e}_i^n$, for $i = 1, \ldots, n$. Then, by using Theorem 1, we have that the eigenvector $\mathbf{v}_1$ of $\mathbb{L}^I$ given by (9) gives raise to an eigenvector of $\mathcal{L}^I$:

$$\mathbf{v}_1 \otimes \mathbf{e}_i^n = \mathbf{e}^k \otimes \mathbf{e}_i^n$$

for each $i = 1 \ldots, n$. We also have that any eigenvector $\mathbf{v}_s$ of $\mathbb{L}^I$, for $s = 2 \ldots, k$, given by (9) gives raise to an eigenvector of $\mathcal{L}^I$:

$$\mathbf{v}_s \otimes \mathbf{e}_i^n = (-\mathbf{e}_1^k + \mathbf{e}^k) \otimes \mathbf{e}_i^n$$

for each $i = 1, 2, \ldots, n$.

Finally, let us consider the following matrices:

$$F_1 = [\mathbf{v}_1 \otimes \mathbf{e}_1^n | \mathbf{v}_1 \otimes \mathbf{e}_2^n | \cdots | \mathbf{v}_1 \otimes \mathbf{e}_n^n] = \mathbf{v}_1 \otimes I_n \in \mathbb{R}^{kn\times n}$$

and, for each $s = 2, \ldots, k$

$$F_s = [\mathbf{v}_s \otimes \mathbf{e}_1^n | \mathbf{v}_s \otimes \mathbf{e}_2^n | \cdots | \mathbf{v}_s \otimes \mathbf{e}_n^n] = \mathbf{v}_s \otimes I_n \in \mathbb{R}^{kn\times n}$$

Now we form the matrix with all the eigenvectors (10) and (11) as column vectors, that is

$$[F_1 F_2 \ldots F_n] = [\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_k] \otimes I_n$$

and since $rank(A \otimes B) = rank(A)rank(B)$ we conclude that $rank([F_1 F_2 \ldots F_n]) = rank([\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_k])n = kn$ since we know from Theorem (3) that the $\mathbf{v}_i$ form a linearly independent set. □

In Proposition 1 we have seen how to obtain the eigenvectors of $\mathcal{L}^L$ knowing the eigenvectors of each L_i for $i = 1, \ldots, k$. In the next result we show that k eigenvectors associated to the zero eigenvalue of $\mathcal{L}^L$ can be constructed by using (9).

Proposition 2. Let $\mathbf{v}_i$, for $i = 1, \dots, k$ be the vectors given by (9). Then the vectors $\mathbf{v}_i \otimes \mathbf{e}^n$, for $i = 1, \dots, k$ form a set of linearly independent eigenvectors of $\mathcal{L}^L$ associated to the eigenvalue 0.

Proof. For $i = 1$ we have

$$\mathcal{L}^L \mathbf{v}_1 \otimes \mathbf{e}^n = \mathcal{L}^L \mathbf{e}^k \otimes \mathbf{e}^n = \mathcal{L}^L \mathbf{e}^{kn} = 0 \in \mathbb{R}^{kn\times 1}$$

since all the rows of $\mathcal{L}^L$ sum up to 0.

For $i = 2, \dots, k$ we have

$$\mathcal{L}^L \mathbf{v}_i \otimes \mathbf{e}^n = \mathcal{L}^L(-\mathbf{e}_1^k + \mathbf{e}_i^k) \otimes \mathbf{e}^n = \begin{pmatrix} -L_1\mathbf{e}^n \\ 0 \\ \vdots \\ L_i\mathbf{e}^n \\ \vdots \\ 0 \end{pmatrix} = 0 \in \mathbb{R}^{kn\times 1}$$

Note that since the the vectors $\mathbf{v}_i$, $i = 1, \dots k$, form a linearly independent set it is straightforward to see (by proceeding analogously as in Corollary 1) that $\mathbf{v}_i \otimes \mathbf{e}^n$, $i = 1, \dots k$, also form a linearly independent set. □

Remark 1. Note that if each layer is connected then $\lambda_1(L_i) = 0$ is simple, for any $i = 1, 2, \dots, k$ and therefore $\mathcal{L}^L$ has the eigenvalue 0 with multiplicity k and the previous proposition gives the corresponding eigenspace.

Proposition 3. The eigenvalues $\lambda = 0$ (simple) and $\lambda = k$ with algebraic multiplicity at least $k - 1$, are eigenvalues of the supra-Laplacian matrix $\mathbb{L}$.

Proof. The supra-Laplacian matrix $\mathbb{L}$ can be considered as a usual Laplacian matrix with adjacency matrix $\mathbb{M}$. Since the multiplex is connected (it has only one component) we have that the eigenvalue 0 of $\mathbb{L}$ must be simple (see, e.g.,[23]). Of course, the corresponding eigenvector is $\mathbf{e}^{kn}$.

Let $\mathbf{v}_i$, for $i = 2, \dots, k$ be the vectors given by (9). Then, by using (6), we have

$$\mathbb{L}\mathbf{v}_i \otimes \mathbf{e}^n = (\mathcal{L}^L + \mathcal{L}^I)\mathbf{v}_i \otimes \mathbf{e}^n = \mathcal{L}^L\mathbf{v}_i \otimes \mathbf{e}^n + \mathcal{L}^I\mathbf{v}_i \otimes \mathbf{e}^n$$

and by Proposition 2 we know that $\mathcal{L}^L\mathbf{v}_i \otimes \mathbf{e}^n = 0$ and from Corollary 1 we have that $\mathcal{L}^I\mathbf{v}_i \otimes \mathbf{e}^n = k\mathbf{v}_i \otimes \mathbf{e}^n$. Therefore, we conclude

$$\mathbb{L}\mathbf{v}_i \otimes \mathbf{e}^n = k\mathbf{v}_i \otimes \mathbf{e}^n.$$

□

Now we present a bound for the eigenvalues of the supra-Laplacian matrix $\mathbb{L}$.

Theorem 4. The eigenvalues of $\mathbb{L}$ are such that $\lambda_1(\mathbb{L}) = 0$ and

$$max[\lambda_s(\mathcal{L}^L), \lambda_s(\mathcal{L}^I)] \le \lambda_s(\mathbb{L}) \le min[\lambda_s(\mathcal{L}^L) + k, \lambda_s(\mathcal{L}^I) + \lambda_{kn}(\mathcal{L}^L)]$$

for $s = 2, \dots, kn$.

Proof. We have seen that $\lambda_1(\mathbb{L}) = 0$ in Proposition 3. By applying Theorem 2 to matrices $A = \mathcal{L}^L$ and $B = \mathcal{L}^I$ we get:

$$\lambda_s(\mathcal{L}^L) + \lambda_1(\mathcal{L}^I) \le \lambda_s(\mathbb{L}) \le \lambda_s(\mathcal{L}^L) + \lambda_{kn}(\mathcal{L}^I), \quad s = 1, 2 \dots, kn \tag{13}$$

and by applying the same Theorem 2 to matrices $A = \mathcal{L}^I$ and $B = \mathcal{L}^L$ we obtain:

$$\lambda_s(\mathcal{L}^I) + \lambda_1(\mathcal{L}^L) \le \lambda_s(\mathbb{L}) \le \lambda_s(\mathcal{L}^I) + \lambda_{kn}(\mathcal{L}^L), \quad s = 1, 2 \dots, kn \tag{14}$$

Now, note that by (7) we have that $\lambda_s(\mathcal{L}^L) = 0$, for $s = 1, \dots, k$ since each L_i (for $i = 1, \dots k$) is a Laplacian matrix. Note also that for Corollary 1 we have that $\lambda_s(\mathcal{L}^I) = 0$ for $s = 1, \dots, n$, and $\lambda_{kn}(\mathcal{L}^I) = k$. Then, by combining the bounds in (13) and (14) the proof follows. □

Example 15. For the matrices associated to the multiplex shown in Fig. 1 we obtain the following

$$\sigma(L_1) = \{0, 1, 3\}, \quad \sigma(L_2) = \{0, 3, 3\}, \quad \sigma(L_3) = \{0, 1, 3\}$$

$$\sigma(\mathcal{L}^L) = \bigcup_{i=1}^{k} \sigma(L_i) = \{0,0,0,1,1,3,3,3,3\}$$

$$\sigma(\mathbb{L}^I) = \{0,3,3\}, \quad \mathbf{v}_1 = (1,1,1)^T,\ \mathbf{v}_2 = (-1,1,0)^T,\ \mathbf{v}_3 = (-1,0,1)^T$$

$$\sigma(\mathcal{L}^I) = \sigma(\mathbb{L}^I \otimes I_3) = \{0(triple), 3(triple), 3(triple)\}, = \{0(triple), 3(alg.multiplicity = 6)\}$$

By using Theorem 4 the eigenvalues of $\lambda_s(\mathbb{L})$ are bouded as

$$\begin{array}{rcl} 0 & \leq \lambda_{2,3}(\mathbb{L}) \leq & 3 \\ 3 & \leq \lambda_{4,5}(\mathbb{L}) \leq & 4 \\ 3 & \leq \lambda_{6,7,8,9}(\mathbb{L}) \leq & 6 \end{array}$$

while computing the eigenvalues we have (rounding to the first decimal place)

$$\sigma(\mathbb{L}) = \{0, 1.7, 2.4, 3, 3, 4.6, 5.3, 6, 6\}$$

Example 16. Let us consider a biplex defined by the Laplacians

$$L_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

Then we obtain

$$\sigma(L_1) = \{0,1,1,4\}, \quad \sigma(L_2) = \{0, 2-\sqrt{2}, 2, 2+\sqrt{2}\}, \quad \sigma(\mathcal{L}^L) = \sigma(L_1) \cup \sigma(L_2)$$

$$\mathbb{L}^I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \sigma(\mathbb{L}^I) = \{0,2\}, \quad \mathbf{v}_1 = (1,1)^T,\ \mathbf{v}_2 = (-1,1)^T,$$

$$\sigma(\mathcal{L}^I) = \{0(alg.multiplicity = 4), 2(alg.multiplicity = 4)\},$$

By using Theorem 4 the eigenvalues of $\lambda_s(\mathbb{L})$ are bouded as

$$\begin{array}{rcl} 0 & \leq \lambda_2(\mathbb{L}) \leq & 2 \\ 0.59 & \leq \lambda_3(\mathbb{L}) \leq & 2.59 \\ 1 & \leq \lambda_4(\mathbb{L}) \leq & 3 \\ 2 & \leq \lambda_5(\mathbb{L}) \leq & 3 \\ 2 & \leq \lambda_6(\mathbb{L}) \leq & 4 \\ 3.41 & \leq \lambda_7(\mathbb{L}) \leq & 5.41 \\ 4 & \leq \lambda_8(\mathbb{L}) \leq & 6 \end{array}$$

while computing the eigenvalues we have (rounding to the second decimal place)

$$\sigma(\mathbb{L}) = \{0, 0.95, 1.55, 2, 2, 3.4, 4.7, 5.4\}$$

4 | BOUNDS FOR THE PAGERANK VERSATILITY

In this section we are going to establish the interval in which each component of the personalized PageRank versatility vector ranges. For each $i = 1, \dots, n$ the corresponding interval is sharp, in the sense that all values in the interval can be achieved as the i^{th}-component of the PageRank versatility for a certain personalization vector.

Theorem 5. Given a multilayer network with n nodes and k layers, let $A_1, \dots, A_k$ be the adjacency matrices of each layer, $\mathbf{v}^T = \frac{1}{k}[\mathbf{v_1}^T\ \mathbf{v_2}^T, \dots \mathbf{v_k}^T]$ a personalization vector of the network, then for each $i = 1, \dots, n$ the set of all possible values of the i^{th}-component of the personalized PageRank versatility vector coincides with the open interval

$$\left(\frac{1}{k}(\min_j (C_1)_{ji} + \cdots + \min_j (C_k)_{ji}),\ \frac{1}{k}(\max_j (C_1)_{ji} + \cdots + \max_j (C_k)_{ji}) \right)$$

where each $C_\beta = \sum_{\alpha=1}^{k} X_{\beta\alpha}$ is the sum of the blocks in $\mathbb{X}$ corresponding to the β^{th}-row in the $k \times k$-decomposition of $\mathbb{X} = (1-\alpha)(I-\alpha\mathbb{T})^{-1}$

$$\mathbb{X} = \left(\begin{array}{c|c|c} X_{11} & \dots & X_{1k} \\ \hline \vdots & \ddots & \\ \hline X_{k1} & \dots & X_{kk} \end{array}\right)$$

and where $\mathbb{T}$ denotes the row-stochastic matrix obtained from

$$\mathbb{M} = \left(\begin{array}{c|c|c|c} A_1 & I_n & \dots & I_n \\ \hline I_n & A_2 & \dots & I_n \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline I_n & I_n & \dots & A_k \end{array}\right)$$

by normalizing each of its rows.

Proof. First we are going to prove that each component of the personalized PageRank versatility vector belongs to the open interval stated in the claim. If we denote the personalized PageRank versatility vector by $\pi = [\pi_1, \pi_2, \dots, \pi_n]^T$, recall that each of its components is calculated as

$$\pi_i = (\sum_{\alpha=1}^{k} (\Pi_\alpha)^T)\mathbf{e}_i^n$$

where Π is the unique positive left eigenvector in $\mathbb{R}^{nk\times 1}$ with norm equal to 1 associated to the dominant eigenvalue of $\mathbb{G} = \alpha\mathbb{T} + (1-\alpha)\mathbf{e}^{nk}\mathbf{v}^T$. Then

$$\Pi^T = \Pi^T\mathbb{G} = \Pi^T(\mathbb{T} + (1-\alpha)\mathbf{e}^{nk}\mathbf{v}^T) = \Pi^T\mathbb{T} + (1-\alpha)\mathbf{v}^T,$$

hence $\Pi^T = \mathbf{v}^T(1-\alpha)(I-\alpha\mathbb{T})^{-1}$ where I denotes the $nk \times nk$-identity matrix. Let us denote $\mathbb{X} = (1-\alpha)(I-\alpha\mathbb{T})^{-1}$, so

$$\Pi^T = \mathbf{v}^T\mathbb{X} = \frac{1}{k}[\mathbf{v}_1^T \dots \mathbf{v}_k^T]\mathbb{X}. \tag{17}$$

If we consider the matrix $\mathbb{X}$ as a $k \times k$-block matrix of the form

$$\mathbb{X} = \left(\begin{array}{c|c|c} X_{11} & \dots & X_{1k} \\ \hline \vdots & \ddots & \\ \hline X_{k1} & \dots & X_{kk} \end{array}\right)$$

we have, from 17, that $\Pi^T = [\Pi_1^T, \Pi_2^T, \dots, \Pi_k^T]$ satisfies

$$\Pi_\alpha^T = \frac{1}{k}\sum_{\beta=1}^{k} \mathbf{v}_\beta^T X_{\beta\alpha}, \qquad \alpha = 1, \dots, k.$$

Define $C_\beta = \sum_{\alpha=1}^{k} X_{\beta\alpha}$ as the sum of the blocks of $\mathbb{X}$ corresponding to the β^{th}-row in the previous $k \times k$-decomposition. Then

$$\pi_i = (\sum_{\alpha=1}^{k} \Pi_\alpha^T)\mathbf{e}_i^n = \frac{1}{k}(\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} \mathbf{v}_\beta^T X_{\beta\alpha})\mathbf{e}_i^n =$$
$$= \frac{1}{k}\sum_{\beta=1}^{k} \mathbf{v}_\beta^T(\sum_{\alpha=1}^{k} X_{\beta\alpha})\mathbf{e}_i^n = \frac{1}{k}\sum_{\beta=1}^{k} \mathbf{v}_\beta^T C_\beta \mathbf{e}_i^n.$$

Since $C_\beta\mathbf{e}_i^n$ is just the i^{th}-column of the matrix C_β and $\mathbf{v}_\beta$ is a (positive) stochastic vector, we have that $\mathbf{v}_\beta^T C_\beta \mathbf{e}_i^n$ is a strict convex combination of the i^{th}-column of C_β, hence

$$\min_j (C_\beta)_{ji} < \mathbf{v}_\beta^T C_\beta \mathbf{e}_i^n < \max_j (C_\beta)_{ji}$$

Therefore each component $\pi_i = \frac{1}{k}\sum_{\beta=1}^{k} \mathbf{v}_\beta^T C_\beta \mathbf{e}_i^n$ of the PageRank versatility vector π satisfies

$$\boxed{\frac{1}{k}(\min_j (C_1)_{ji} + \dots + \min_j (C_k)_{ji}) < \pi_i < \frac{1}{k}(\max_j (C_1)_{ji} + \dots + \max_j (C_k)_{ji})} \tag{18}$$

Conversely, any particular b satisfying

$$\frac{1}{k}(\min_j (C_1)_{ji} + \dots + \min_j (C_k)_{ji}) < b < \frac{1}{k}(\max_j (C_1)_{ji} + \dots + \max_j (C_k)_{ji})$$

can be expressed as $b = \sum_{\beta=1}^{k} b_\beta$ where for every $\beta = 1, \ldots, k$

$$\min_j (C_\beta)_{ji} < k b_\beta < \max_j (C_\beta)_{ji}$$

Using the same argument as in the Proof of Step 2 of[14] Theorem 3.2, there exist (positive) stochastic vectors $\mathbf{v}_\beta$ such that

$$\mathbf{v}_\beta^T C_\beta e_i = k b_\beta.$$

Consider the personalized PageRank versatility vector $\pi = [\pi_1, \ldots, \pi_n]^T$ with personalization vector $\mathbf{v}^T = \frac{1}{k}[\mathbf{v}_1^T \ldots \mathbf{v}_k^T]$ for those precise $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then

$$\pi_i = \frac{1}{k} \sum_{\beta=1}^{k} \mathbf{v}_\beta^T C_\beta e_i = \sum_{\beta=1}^{k} b_\beta = b,$$

i.e., the i^{th}-component of the personalized PageRank versatility vector with personalization vector $\mathbf{v}^T = \frac{1}{k}[\mathbf{v}_1^T \ldots \mathbf{v}_k^T]$ coincides with b. □

5 | EXAMPLES

5.1 | Example 1

In this section we illustrate an application of Theorem 5 to a toy multiplex. To that end, let us consider a multiplex formed by 4 layers and with 4 nodes on each layer. Let the adjacency matrices of the layers be the following

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The matrices C_β referred in Theorem 5 result to be

$$C_1 = \begin{pmatrix} 0.4744 & 0.2625 & 0.1564 & 0.1067 \\ 0.1610 & 0.5068 & 0.1639 & 0.1682 \\ 0.1870 & 0.1932 & 0.4239 & 0.1959 \\ 0.2164 & 0.2040 & 0.1762 & 0.4034 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0.4476 & 0.2375 & 0.1995 & 0.1154 \\ 0.2015 & 0.5127 & 0.1680 & 0.1178 \\ 0.1568 & 0.1305 & 0.4768 & 0.2359 \\ 0.2400 & 0.1610 & 0.1534 & 0.4456 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 0.4453 & 0.2308 & 0.2069 & 0.1170 \\ 0.1890 & 0.4684 & 0.2108 & 0.1317 \\ 0.1535 & 0.1288 & 0.4847 & 0.2330 \\ 0.2209 & 0.1512 & 0.1987 & 0.4292 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0.4726 & 0.2603 & 0.1656 & 0.1016 \\ 0.1507 & 0.4941 & 0.2164 & 0.1388 \\ 0.1576 & 0.1312 & 0.4757 & 0.2354 \\ 0.2443 & 0.1650 & 0.1474 & 0.4432 \end{pmatrix}$$

and according to the cited Theorem the bounds for the personalized PageRank versatility of each node, π_i, are the following (see equation (18))

$$\begin{aligned} \pi_1 &\in (0.1555, 0.4600) \\ \pi_2 &\in (0.1460, 0.4955) \\ \pi_3 &\in (0.1640, 0.4653) \\ \pi_4 &\in (0.1102, 0.4304) \end{aligned} \tag{19}$$

To show that these bounds are sharp, let us compute the PageRank versatility for some particular personalization vectors. For example, when taking the so-called homogeneous personalization vector $\mathbf{v}^T = \frac{1}{16}\mathbf{e}^{16} = \frac{1}{16}[1, 1, 1, 1|1, 1, 1, 1|1, 1, 1, 1|1, 1, 1, 1]$

a computation shows that

$$\begin{aligned}\pi_1 &= 0.2574\\ \pi_2 &= 0.2649\\ \pi_3 &= 0.2515\\ \pi_4 &= 0.2262\end{aligned}$$

and therefore the most important node (as classified by the PageRank versatility) is node number 2.

Note that, as the proof of Theorem 5 suggests, to optimize the PageRank versatility of a node q we must maximize the component j of the personalization vector of each layer i, being j the index of the row where the column q of $\mathbf{C}_i$ takes a maximum value. For example, to obtain the maximum PageRank versatility of node 1 we must use the personalization vector: $\mathbf{w}^T = \frac{1}{4}[1,0,0,0|1,0,0,0|1,0,0,0|1,0,0,0]$. In fact, by using this vector, a computation shows that

$$\begin{aligned}\pi_1 &= 0.4600\\ \pi_2 &= 0.2478\\ \pi_3 &= 0.1821\\ \pi_4 &= 0.1102\end{aligned}$$

and we see that node 1 gets its maximum value of PageRank versatility, according to the bounds shown in (19).

On the contrary, to minimize the PageRank versatility of a node q we must maximize the component j of the personalization vector of each layer i, being j the index of the row where the column q of $\mathbf{C}_i$ takes a minimum value. For example, to obtain the minimum PageRank for node 1, we must use $\mathbf{z}^T = \frac{1}{4}[0,1,0,0|0,0,1,0|0,0,1,0|0,1,0,0]$. In fact, by using this vector, a computation shows that

$$\begin{aligned}\pi_1 &= 0.1555\\ \pi_2 &= 0.3151\\ \pi_3 &= 0.3355\\ \pi_4 &= 0.1940\end{aligned}$$

and we see that node 1 gets its minimum value of PageRank versatility, according to the bounds shown in (19). Note, moreover, that the previous personalization vector $\mathbf{w}^T$ gets the minimum value of the PageRank versatility of node 4 since it gives the maximum bias to the indices where the column 4 of $\mathbf{C}_i$ takes its minimum.

Until now we have shown how Theorem 5 allows to bound the PageRank versatility. One might ask if there is another way of giving a bound for the PageRank in this example. The answer is affirmative: we can use the known bounds corresponding to the classic (monoplex) personalized PageRank that were obtained in [14]. To that end we must construct a monoplex graph to represent the multiplex. In this example, we must construct the adjacency matrix

$$A = \begin{pmatrix} A_1 & I_4 & I_4 & I_4\\ I_4 & A_2 & I_4 & I_4\\ I_4 & I_4 & A_3 & I_4\\ I_4 & I_4 & I_4 & A_4\end{pmatrix}$$

and then we can apply, as it is shown in [14], that the i-th component of the classic PageRank is located in an open interval that depends on the matrix

$$X = (1-\alpha)(I_n - \alpha P_A)^{-1} \tag{20}$$

where P_A is a row stochastic matrix obtained from A by dividing each entry by the sum of each row. More precisely, it holds that the component i of the classic personalized PageRank has the following bound

$$\mathcal{PR}(i) \in (\min_j x_{ji},\ x_{ii}). \tag{21}$$

By using this result we obtain that the bounds for the classic personalized PageRank associated to the adjacency matrix A are the following

$$\begin{aligned}
\mathcal{PR}(1) &\in (0.0337, 0.2084), & \mathcal{PR}(5) &\in (0.0389, 0.2165)\\
\mathcal{PR}(2) &\in (0.0462, 0.2287), & \mathcal{PR}(6) &\in (0.0270, 0.2209)\\
\mathcal{PR}(3) &\in (0.0317, 0.2062), & \mathcal{PR}(7) &\in (0.0322, 0.2081)\\
\mathcal{PR}(4) &\in (0.0323, 0.2121), & \mathcal{PR}(8) &\in (0.0221, 0.2001)
\end{aligned}$$

TABLE 2 Ranking according to the indicated PageRank

PRv	2	1	3	4
$[\Pi(1), \Pi(2), \Pi(3), \Pi(4)]$	2	4	1	3
$[\Pi(5), \Pi(6), \Pi(7), \Pi(8)]$	1	2	3	4
$[\Pi(9), \Pi(10), \Pi(11), \Pi(12)]$	3	1	2	4
$[\Pi(13), \Pi(14), \Pi(15), \Pi(16)]$	2	3	1	4

$$\begin{array}{ll}
\mathcal{PR}(9) \in (0.0376, 0.2132), & \mathcal{PR}(13) \in (0.0292, 0.2040) \\
\mathcal{PR}(10) \in (0.0281, 0.2151), & \mathcal{PR}(14) \in (0.0273, 0.2122) \\
\mathcal{PR}(11) \in (0.0483, 0.2237), & \mathcal{PR}(15) \in (0.0308, 0.2068) \\
\mathcal{PR}(12) \in (0.0250, 0.2076), & \mathcal{PR}(16) \in (0.0219, 0.1988)
\end{array}$$

Now, to translate this PageRank bounds of a monoplex to a 4 layers- 4 nodes multiplex we must add the intervals for each corresponding node of the multiplex, since we are considering that the PageRank of node i is giving by $\pi_i = \mathcal{PR}(i) + \mathcal{PR}(i+4) + \mathcal{PR}(i+8) + \mathcal{PR}(i+12)$. Therefore, we could state the bounds for the multiplex in the following form

$$\begin{array}{l}
\pi_1 \in (0.1394, 0.8420) \\
\pi_2 \in (0.1287, 0.8769) \\
\pi_3 \in (0.1430, 0.8448) \\
\pi_4 \in (0.1013, 0.8185)
\end{array}$$

where we have added the bounds for the nodes $\mathcal{PR}(i + 4(j-1)), j = 1, \dots, 3$.

By comparing these bounds with the bounds given by (19) we see that our result for the bounds of the PageRank versatility is much more sharp than if we simply apply the bounds for the classic (monoplex) PageRank.

To end this example we show in Table 2 the ranking given by the PageRank versatility and the ranking produced in each layer by considering the value of the corresponding components of the PageRank versatility (that is, the corresponding entries of the vector $\Pi \in \mathbb{R}^{16\times 1}$). In this computation we have used the homogeneous personalization vector, that is $\mathbf{v}^T = \frac{1}{16}\mathbf{e}^{16}$.

5.2 | Example 2

In this section we analyse an application of Theorem 5 by using a benchmark network known as *Florentine Family Marriage and Business Ties Data*, see[18],[27],[33]. It can be analysed as a multiplex formed by two layers, with 16 nodes in each layer. One layer is related with the business links and the other one is related with marriage relationships (see Figure 2). In Table 3 we give the numbering of the families.

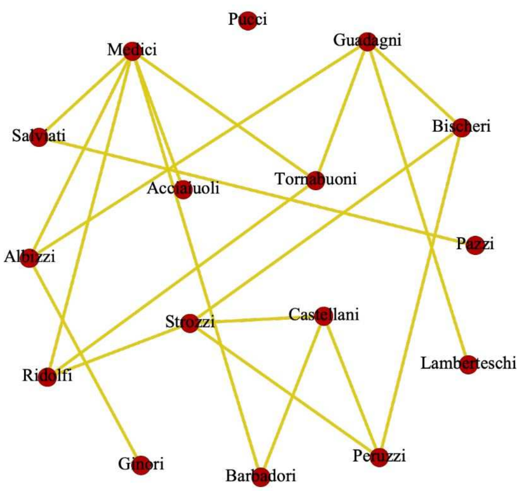
Since we know the adjacency matrices A_1 (business) and A_2 (marriage) we can compute the matrix $\mathbb{M}$ given by (2) and we can apply Theorem 5 to obtain the following bounds for the PageRank versatility of the nodes.

$$\begin{array}{ll}
\pi_1 \in (0, 0.4242), & \pi_9 \in (0, 0.3153) \\
\pi_2 \in (0, 0.3497), & \pi_{10} \in (0, 0.3418) \\
\pi_3 \in (0, 0.2723), & \pi_{11} \in (0, 0.2864) \\
\pi_4 \in (0, 0.2861), & \pi_{12} \in (0, 1.0000) \\
\pi_5 \in (0, 0.2801), & \pi_{13} \in (0, 0.3373) \\
\pi_6 \in (0, 0.2973), & \pi_{14} \in (0, 0.3173) \\
\pi_7 \in (0, 0.2915), & \pi_{15} \in (0, 0.3420) \\
\pi_8 \in (0, 0.3019), & \pi_{16} \in (0, 0.2809)
\end{array}$$

It is worth highlighting that number node 12 corresponds to Pucci family that actually has no links in any of the layers. Despite this feature, the model is capable of assigning a value of the PageRank versatility (and of the bounds by using Theorem 5). We see that node number 12 can achieve any value from 0 to 1 as PageRank versatility.

In the case that we take the usual personalization vector $\mathbf{v}^T = \frac{1}{32}\mathbf{e}^{32}$ a computation shows that the components of the PageRank versatility results to be[2]

[2] These values are used to obtain the ranking in the first column of Table 4 .



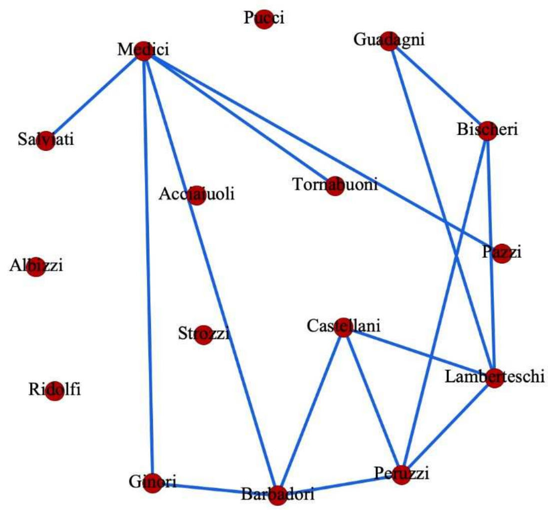


FIGURE 2 Layers of Florentine family business (left) and family marriage.

TABLE 3 Numbering of each family

node	family
1	Acciaiuol
2	Albizzi
3	Barbadori
4	Bischeri
5	Castellan
6	Ginori
7	Guadagni
8	Lambertes
9	Medici
10	Pazzi
11	Peruzzi
12	Pucci
13	Rodolfi
14	Salviatis
15	Strozzi
16	Tornabuon

$$\begin{aligned} \pi_1 &= 0.0416, & \pi_9 &= 0.1199 \\ \pi_2 &= 0.0537, & \pi_{10} &= 0.0454 \\ \pi_3 &= 0.0690, & \pi_{11} &= 0.0736 \\ \pi_4 &= 0.0671, & \pi_{12} &= 0.0625 \\ \pi_5 &= 0.0670, & \pi_{13} &= 0.0522 \\ \pi_6 &= 0.0502, & \pi_{14} &= 0.0529 \\ \pi_7 &= 0.0701, & \pi_{15} &= 0.0575 \\ \pi_8 &= 0.0602, & \pi_{16} &= 0.0573 \end{aligned}$$

Note that the most important node is node number 9, that corresponds to Medici family. We know that by properly chosen the personalization vectors of each layer we can bias the personalized PageRank versatility. Actually, by taking $\mathbf{v_1} = \mathbf{v_2} = \mathbf{e}_{12}^{16}$

TABLE 4 Ranking obtained by using the indicated PageRank

PRv	Π(1 to 16)	Π(17 to 32)
Medici	Medici	Medici
Peruzzi	Strozzi	Barbadori
Guadagni	Guadagni	Lambertes
Barbadori	Albizzi	Peruzzi
Bischeri	Rodolfi	Bischeri
Castellan	Tornabuon	Castellan
Pucci	Castellan	Pucci
Lambertes	Bischeri	Ginori
Strozzi	Peruzzi	Guadagni
Tornabuon	Salviatis	Pazzi
Albizzi	Pucci	Salviatis
Salviatis	Barbadori	Tornabuon
Rodolfi	Acciaiuol	Acciaiuol
Ginori	Pazzi	Albizzi
Pazzi	Ginori	Rodolfi
Acciaiuol	Lambertes	Strozzi

we obtain the personalized PageRank versatility to be $\mathbf{e}_{12}^{16}$. That is, node 12 has the maximum PageRank (1), and the rest have PageRank 0.

In the case that we want to biass the PageRank to node 11 we take the personalization vectors $\mathbf{v_1} = \mathbf{v_2} = \mathbf{e}_{11}^{16}$ and we obtain that the components of the PageRank versatility are

$$\begin{array}{llll} \pi_1 & = 0.0071, & \pi_9 & = 0.0505 \\ \pi_2 & = 0.0183, & \pi_{10} & = 0.0096 \\ \pi_3 & = 0.0776, & \pi_{11} & = 0.2864 \\ \pi_4 & = 0.1164, & \pi_{12} & = 0.0000 \\ \pi_5 & = 0.1187, & \pi_{13} & = 0.0276 \\ \pi_6 & = 0.0232, & \pi_{14} & = 0.0129 \\ \pi_7 & = 0.0641, & \pi_{15} & = 0.0813 \\ \pi_8 & = 0.0824, & \pi_{16} & = 0.0240 \end{array}$$

Note that in this case we obtain the minimum component in node 12.

In Table 4 it is shown the ranking given by the PageRank versatility and the rankings produced in each layer by considering the value of the corresponding components of the PageRank versatility in each layer. We have used the homogeneous personalization vector, that is $\mathbf{v}^T = \frac{1}{32}\mathbf{e}^{32}$. In this Table we see that being Medici Family the most important node in all three methods, the complete ranking is very different when considering the whole network as a multiplex (column 1) or when considering each layer independently.

Since node 12 is not linked to any other family in any layer we have performed a new computation by taking out this node from the whole multiplex. As a result we obtain a multiplex with two layers, with 15 nodes in each layer. The results are shown in Table 5 , by using the homogeneous personalization vector.

In Table 5 we see that the elimination of Pucci family does not change the resulting rankings (note that new node 14 is the old 15, new node 15 is the old 14, etc.). As a result we have shown that the inclusion or not of the Pucci family does not alterer the ranking when we use the ranking given by the PageRank versatility as a whole, or considering only its components on each layer.

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TABLE 5 Ranking obtained by using the indicated PageRank, omitting Pucci Family

PRv	Π(1 to 15)	Π(16 to 30)
Medici	Medici	Medici
Peruzzi	Strozzi	Barbadori
Guadagni	Guadagni	Lambertes
Barbadori	Albizzi	Peruzzi
Bischeri	Ridolfi	Bischeri
Castellan	Tornabuon	Castellan
Lambertes	Castellan	Ginori
Strozzi	Bischeri	Guadagni
Tornabuon	Peruzzi	Pazzi
Albizzi	Salviati	Salviati
Salviati	Barbadori	Tornabuon
Ridolfi	Acciaiuol	Acciaiuol
Ginori	Pazzi	Albizzi
Pazzi	Ginori	Ridolfi
Acciaiuol	Lambertes	Strozzi

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