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Improved Iterative Solution of Linear Fredholm Integral Equations of Second Kind via Inverse-Free Iterative Schemes

José Manuel Gutiérrez [1], Miguel Ángel Hernández-Verón [1,*] and Eulalia Martínez [2]

[1] Department of Mathematics and Computation, University of La Rioja, 26006 Logroño, Spain; jmguti@unirioja.es
[2] Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, 46022 València, Spain; eumarti@mat.upv.es
* Correspondence: mahernan@unirioja.es

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Abstract: This work is devoted to Fredholm integral equations of second kind with non-separable kernels. Our strategy is to approximate the non-separable kernel by using an adequate Taylor's development. Then, we adapt an already known technique used for separable kernels to our case. First, we study the local convergence of the proposed iterative scheme, so we obtain a ball of starting points around the solution. Then, we complete the theoretical study with the semilocal convergence analysis, that allow us to obtain the domain of existence for the solution in terms of the starting point. In this case, the existence of a solution is deduced. Finally, we illustrate this study with some numerical experiments.

Keywords: Fredholm integral equation; iterative processes; Newton's method; separable and non-separable kernels; local and semilocal convergence

MSC: 45B05; 47H09; 47H10; 47H30; 65J15

1. Introduction

In this study, we focus on some particular integral equations known as Fredholm integral equations [1]. These kinds of integral equations appear frequently in many scientific disciplines, such as mathematical physics, engineering, or applied mathematics, due to the fact that a large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations. The potential theory has contributed more than any field to give rise to integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of integral equations. Moreover, Volterra's population growth model, biological species living together, propagation of stocked fishing in a new lake, heat transfer, and the heat radiation are among many areas that are described by integral equations. Many scientific problems give rise to integral equations with logarithmic kernels. Integral equations often arise in electrostatic, low frequency electromagnetic problems, electro magnetic scattering problems, and the propagation of acoustical and elastic waves [2–7].

Usually, these type of equations can not be solved analytically, so different numerical methods have been used to handle these integral equations. For instance, we can mention the successive approximation method, the Adomian decomposition method, or the variational iteration method, amongst many other techniques.

Fredholm integral equations of first kind have the form

$$y(x) = \int_u^v K(x,t)y(t)\,dt, \quad x \in [u,v],$$

and those of second kind can be written as

$$y(x) = h(x) + \lambda \int_u^v K(x,t)y(t)\,dt, \quad x \in [u,v], \quad \lambda \in \mathbb{R}. \tag{1}$$

In both cases, $-\infty < u < v < +\infty$, $\lambda \in \mathbb{R}$, function $h(x) \in \mathcal{C}[u,v]$ is given, the function $K(x,t)$ is a known function in $[u,v] \times [u,v]$, called the kernel of the integral equation, and $y(x) \in \mathcal{C}[u,v]$ is the unknown function to be determined. In addition, if function $h(x)$ is the zero constant function, the integral equation is said to be homogeneous.

For integral Equation (1), we can consider the operator $\mathcal{K} : \mathcal{C}[u,v] \to \mathcal{C}[u,v]$, given by

$$[\mathcal{K}(y)](x) = \int_u^v K(x,t)y(t)\,dt, \; x \in [u,v]. \tag{2}$$

Then, the Equation (1) can be expressed as $(\mathcal{I} - \lambda\mathcal{K})y(x) = h(x)$. Therefore, when there exists $(\mathcal{I} - \lambda\mathcal{K})^{-1}$, a solution of Equation (1) is given by

$$y^*(x) = (\mathcal{I} - \lambda\mathcal{K})^{-1}h(x). \tag{3}$$

Formula (3) provides us with the solution of Equation (1), from a theoretical point of view. However, in many cases, the calculus of the inverse operator $(\mathcal{I} - \lambda\mathcal{K})^{-1}$ could be very complicated. Hence, for practical purposes, it is advisable to look for alternative techniques to approach this inverse operator. One of these techniques is the use of iterative methods (Newton, Chebyshev, and others) to the problem of the calculus of inverse operators. In the case of Fredholm integral equations, the approximation of the inverse $(\mathcal{I} - \lambda\mathcal{K})^{-1}$ by different iterative schemes allows us to approximate the corresponding solution of (1). This is the aim and main target of this work. The rest of the document is organized as follows.

In Section 2, we present a technique for solving Fredholm integral equations with separable kernels. This technique provides us with good starting points for the iterative methods we want to use to approximate the inverse operator that appears in (3). These iterative schemes are developed and analyzed in Section 3. In Section 4, we introduce the main theoretical results, with the study of the local and semilocal convergence of the aforementioned methods. Next, Section 5 is devoted to the numerical experiments. Finally, Section 6 contains the conclusions of our study.

2. Fredholm Integral Equations with Separable Kernels

In some particular cases, it is possible to calculate the exact solution of an integral equation in the form of (1). For example, when the kernel $K(x,t)$ is separable, that is

$$K(x,t) = \sum_{i=1}^{m} a_i(x)b_i(t),$$

there is a known technique to obtain the exact solution of the corresponding integral equation [8,9].

Actually, if we denote $I_j = \int_u^v b_j(t)y^*(t)\,dt$, by (3) we have

$$(\mathcal{I} - \lambda\mathcal{K})y^*(x) = y^*(x) - \lambda \sum_{j=1}^{m} a_j(x)I_j = h(x).$$

Then, if there exists $(\mathcal{I} - \lambda\mathcal{K})^{-1}$, we have

$$(\mathcal{I} - \lambda\mathcal{K})^{-1}h(x) = y^*(x) = h(x) + \lambda \sum_{j=1}^{m} a_j(x) I_j. \tag{4}$$

It is easy to check that the integrals I_j can be calculated independently of y^*. To do this, we multiply the second equality of (4) by $b_i(x)$, and we integrate in the x variable. Hence, we have

$$I_i - \lambda \sum_{j=1}^{m} \left(\int_a^b b_i(x) a_j(x)\, dx \right) I_j = \int_u^v b_i(x) h(x)\, dx.$$

Now, if we denote

$$\alpha_{ij} = \int_u^v b_i(x) a_j(x)\, dx \quad \text{and} \quad \beta_i = \int_u^v b_i(x) h(x)\, dx,$$

we obtain the following linear system of equations

$$I_i - \lambda \sum_{j=1}^{m} \alpha_{ij} I_j = \beta_i, \quad i = 1, \ldots, m. \tag{5}$$

This system has a unique solution if

$$\text{Det} \begin{pmatrix} \alpha_{11} - \frac{1}{\lambda} & \alpha_{12} & a_{13} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} - \frac{1}{\lambda} & \alpha_{23} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & a_{m2} & \alpha_{m3} & \cdots & \alpha_{mm} - \frac{1}{\lambda} \end{pmatrix} \neq 0,$$

so, we assume $\frac{1}{\lambda}$ is not an eigenvalue of the matrix (α_{ij}). Thus, if $I_1, I_2, \ldots, I_m$ is the solution of system (5), we can obtain directly the inverse operator $(\mathcal{I} - \lambda\mathcal{K})^{-1}$, and therefore the exact solution of Equation (1) given by

$$y^*(x) = (\mathcal{I} - \lambda\mathcal{K})^{-1}h(x) = h(x) + \lambda \sum_{j=1}^{m} a_j(x) I_j. \tag{6}$$

Now, we approach the case of non-separable kernels. In this situation, the successive approximations method [10] and Picard's method [11] are usually applied. In both methods, neither inverse operators nor derivative operators are needed. These two facts make them very interesting from a practical point of view. However, they only reach a linear order of convergence, and this aspect could condition their practical application.

3. Iterative Methods to Approximate the Inverse of an Operator

With the aim of reaching quadratic convergence, we can apply Newton's method [10] to approximate the inverse of the operator $L = \mathcal{I} - \lambda\mathcal{K}$ defined from Equation (1) to obtain the solution $y^*(x)$ of Fredholm integral Equation (1). To do this, we first consider

$$\mathcal{G} : GL(\mathcal{C}[u,v], \mathcal{C}[u,v]) \to \mathcal{L}(\mathcal{C}[u,v], \mathcal{C}[u,v]) \quad \text{and} \quad \mathcal{G}(T) = T^{-1} - L,$$

so that L^{-1} is the solution of the equation $\mathcal{G}(T) = 0$, where

$$GL(\mathcal{C}[u,v], \mathcal{C}[u,v]) = \{T \in \mathcal{L}(\mathcal{C}[u,v], \mathcal{C}[u,v]) : T^{-1} \text{ exists}\}$$

and $\mathcal{L}(\mathcal{C}[u,v],\mathcal{C}[u,v])$ is the set of bounded linear operators from the Banach space $\mathcal{C}[u,v]$ into the Banach space $\mathcal{C}[u,v]$. Hence, following [10] or [12], we can apply Newton's method to the nonlinear functional equation $\mathcal{G}(T)=0$ with $\mathcal{G}(T)=T^{-1}-L$ to approximate L^{-1}. Consequently, the following iterative algorithm is obtained

$$\begin{cases} N_0 \text{ given}, \\ N_{k+1}=2N_k-N_kLN_k, \quad k\geq 0. \end{cases} \tag{7}$$

We can use other iterative methods to approximate the solution of the functional equation $\mathcal{G}(T)=0$ with $\mathcal{G}(T)=T^{-1}-L$. For instance, if we use the well-known Chebyshev's method ([13]), we obtain the algorithm

$$\begin{cases} C_0 \text{ given}, \\ C_{k+1}=3C_k-3C_kLC_k+C_kLC_kLC_k, \quad k\geq 0. \end{cases} \tag{8}$$

Note that Chebyshev's method has cubic convergence. Like Newton's method (7), Chebyshev's method (8) does not use inverse operators for approximating the solution of the equation $G(T)=0$ to obtain the inverse operator of L. For this reason, our main aim in this paper is to generalize this idea and to use iterative methods which do not use inverse operators for approaching this inverse operator and, therefore, the solution of the integral Equation (1). In addition, we want to approximate a solution of Equation (1) with any prefixed R-order of convergence $p\geq 2$. It is known [14] that if we find an R-order of convergence p of a sequence $\{y_m\}$, this sequence has the order of convergence of at least p.

Now, we are interested in generalizing the iterative schemes (7) and (8), obtained from Newton's and Chebyshev's methods respectively. Our idea is to construct iterative schemes, with a prefixed order of convergence, that do not use inverse operators for approximating the inverse of an operator L.

For this, we observe that both Newton's and Chebyshev's methods satisfy equalities in the form

$$\mathcal{I}-N_kL=\mathcal{I}-(2N_{k-1}-N_{k-1}LN_{k-1})L=(\mathcal{I}-N_{k-1}L)^2,$$

$$\mathcal{I}-C_kL=\mathcal{I}-(3C_{k-1}-3C_{k-1}LC_{k-1}+C_{k-1}LC_{k-1}LC_{k-1})L=(\mathcal{I}-C_{k-1}L)^3.$$

Therefore, if we want to obtain an iterative scheme with a prefixed R-order of convergence $p\geq 2$, we can consider a sequence $T_k=\phi(T_{k-1})$ such that $\mathcal{I}-T_kL=(\mathcal{I}-T_{k-1}L)^p$. Consequently, we have

$$\begin{aligned} T_kL &= \mathcal{I}-(\mathcal{I}-T_{k-1}L)^p \\ &= \mathcal{I}-\sum_{j=0}^{p}\binom{p}{j}(-1)^j(T_{k-1}L)^j \\ &= \sum_{j=1}^{p}\binom{p}{j}(-1)^{j+1}(T_{k-1}L)^j \\ &= \sum_{j=0}^{p-1}\binom{p}{j+1}(-1)^j(T_{k-1}L)^jT_{k-1}L, \end{aligned}$$

so, we obtain, for approximating L^{-1}, the following general iterative method:

$$\begin{cases} T_0 \text{ given}, \\ T_k=\displaystyle\sum_{j=0}^{p-1}\binom{p}{j+1}(-1)^j(T_{k-1}L)^jT_{k-1}, \quad k\geq 1 \quad \text{and} \quad p\in\mathbb{N}-\{1\}. \end{cases} \tag{9}$$

It is possible to get this family of iterative schemes, by using inverse interpolation, as shown in [15]. In this work it is also proved that for a fixed value $p \geq 2$, this iterative scheme has an R-order of convergence p and therefore an order of convergence of at least p.

Now, from the previous iterative schemes given in (9), we can approximate a solution of Equation (1) by means of the iterative scheme given by the following algorithm

$$\begin{cases} T_0 \text{ given}, \\ y_k = T_k h, \\ T_{k+1} = \displaystyle\sum_{j=0}^{p-1} \binom{p}{j+1} (-1)^j (T_k L)^j T_k, \quad k \geq 1 \quad \text{and} \quad p \in \mathbb{N} - \{1\}. \end{cases} \tag{10}$$

Observe that in this case, the iterative scheme (10) does not use inverse operators for approximating the solution of Equation (1).

4. Convergence Study

We can analyze the convergence of the iterative scheme (10) in two different ways. The first way, called local convergence, assumes that there exists a solution y^* of the equation $(\mathcal{I} - \lambda\mathcal{K})y(x) = h(x)$. Under certain conditions on the operator $L = (\mathcal{I} - \lambda\mathcal{K})$, we obtain a ball $\overline{B(y^*, R_1)}$, called convergence ball, in which the convergence of the iterative scheme is guaranteed by taking any point on the ball as the starting point y_0. The second way, called semilocal convergence, consists in giving conditions on the starting point y_0 and on the operator L, obtaining a ball of existence of the solution of the equation $Ly(x) = h(x)$, $\overline{B(x_0, R_2)}$, called existence ball, and the convergence of the iterative scheme taking y_0 as the starting point.

Next, we establish the local and semilocal convergence of the iterative scheme given by (10).

4.1. Local Convergence Study

As we have previously indicated, to obtain a local convergence result of the iterative scheme given by (10), we assume that there exists a solution y^* of the equation $Ly(x) = h(x)$. To do this, we suppose that there exists L^{-1}, so, there exists $y^* = L^{-1}h$.

Firstly, we obtain a result for the sequence $\{T_m\}$ given in (9).

Lemma 1. *Suppose that there exists L^{-1} and let $T_0 \in B\left(L^{-1}, \frac{\gamma}{\|L\|}\right)$ with $\gamma \in (0,1)$. Then, the sequence $\{T_m\}$ defined by (9) belongs to $B\left(L^{-1}, \frac{\gamma}{\|L\|}\right)$ and*

$$\|L^{-1} - T_m\| \leq \gamma^{p^m - 1} \|L^{-1} - T_0\|.$$

Proof. Taking into account the definition of the sequence (9), we have $\mathcal{I} - T_m L = (\mathcal{I} - T_{m-1}L)^p$. Then, if $T_{m-1} \in B\left(L^{-1}, \frac{\gamma}{\|L\|}\right)$, we have

$$\|L^{-1} - T_m\| \leq \|(\mathcal{I} - T_m L)L^{-1}\| \leq \|L^{-1} - T_{m-1}\|^p \|L\|^{p-1} < \frac{\gamma^p}{\|L\|} < \frac{\gamma}{\|L\|},$$

so, $T_m \in B\left(L^{-1}, \frac{\theta}{\|L\|}\right)$ for all $m \geqslant 1$.

If we apply recursively the previous inequalities, we obtain

$$\|L^{-1} - T_m\| \leq \|L^{-1} - T_{m-1}\|^p \|L\|^{p-1} \leq \cdots \leq \|L^{-1} - T_0\|^{p^m} \|L\|^{p^m - 1},$$

and, therefore, by the hypotheses, we obtain

$$\|L^{-1} - T_m\| \leq \gamma^{p^m - 1} \|L^{-1} - T_0\|.$$

□

Now, we can apply the previous Lemma to obtain a local convergence result for the iterative scheme (10).

Theorem 1. *We suppose that there exists L^{-1} and $T_0 \in \mathcal{L}(\mathcal{C}[u,v], \mathcal{C}[u,v])$ such that $T_0 \in B\left(L^{-1}, \dfrac{\gamma}{\|L\|}\right)$, with $\gamma \in (0,1)$. Then, for each $y_0 \in B\left(y^*, \dfrac{\gamma\|h\|}{\|L\|}\right)$, the sequence $\{y_m\}$ defined by (10) belongs to $B\left(y^*, \dfrac{\gamma\|h\|}{\|L\|}\right)$ and converges with R-order of convergence at least p to y^*, the solution of Equation (1). In addition,*

$$\|y^* - y_m\| \leq \gamma^{p^m - 1} \|L^{-1} - T_0\| \|h\|.$$

Proof. From Lemma 1, the sequence $\{T_m\}$ belongs to $B\left(L^{-1}, \dfrac{\gamma}{\|L\|}\right)$, therefore

$$\|y^* - y_m\| = \|(L^{-1} - T_m)h\| \leq \frac{\gamma\|h\|}{\|L\|},$$

so, we have that $y_m \in B\left(y^*, \dfrac{\gamma\|h\|}{\|L\|}\right)$ for $m \in \mathbb{N}$.

On the other hand, by taking into account Lemma 1 again, we obtain

$$\|y^* - y_m\| \leq \|L^{-1} - T_m\| \|h\| \leq \gamma^{p^m - 1} \|L^{-1} - T_0\| \|h\|,$$

and then $\{y_m\}$ converges with R-order of convergence at least p to y^*, the solution of Equation (1). □

4.2. Semilocal Convergence Study

To obtain a semilocal convergence result, we give conditions on the starting point y_0 and on the operator L. As $y_0 = T_0 h$, it is sufficient to give conditions for the operator T_0. Firstly, we obtain a result for the sequence $\{T_m\}$ defined by (9).

Lemma 2. *Let $T_0 \in \mathcal{L}(\mathcal{C}[u,v], \mathcal{C}[u,v])$ such that $\|\mathcal{I} - T_0 L\| \leq \mu$, with $\mu \in (0,1)$. Then, the sequence $\{T_m\}$ defined by (9) belongs to $B\left(T_0, \dfrac{\mu\|T_0\|}{1-\mu}\right)$. Moreover,*

$$\|T_{m+k} - T_m\| \leq \frac{\mu^{p^m} - \mu^{p^{m+k}}}{1-\mu} \|T_0\|, \ \ k \geq 1.$$

Proof. As $\mathcal{I} - T_k L = (\mathcal{I} - T_{k-1} L)^p$, we have

$$\|\mathcal{I} - T_m L\| \leq \|\mathcal{I} - T_{m-1} L\|^p \leq \cdots \leq \|\mathcal{I} - T_0 L\|^{p^m} \leq \mu^{p^m}. \tag{11}$$

On the other hand, as we can see in [16],

$$\sum_{k=j}^{p-1} \binom{k}{j} = \binom{p}{j+1}$$

so we have

$$\begin{aligned}\left(\sum_{k=0}^{p-1}(\mathcal{I}-T_{m-1}L)^k\right)T_{m-1} &= \left(\sum_{k=0}^{p-1}\left[\sum_{i=0}^{k}\binom{k}{i}(-1)^i(T_{m-1}L)^i\right]\right)T_{m-1}\\ &= \left(\sum_{j=0}^{p-1}\left[\sum_{k=j}^{p-1}\binom{k}{j}(-1)^j(T_{m-1}L)^j\right]\right)T_{m-1}\\ &= \sum_{j=0}^{p-1}\binom{p}{j+1}(-1)^j(T_{m-1}L)^jT_{m-1}\\ &= T_m.\end{aligned}$$

Then, from (11) we obtain

$$\|T_m-T_{m-1}\| \le \sum_{k=1}^{p-1}\|\mathcal{I}-T_{m-1}L\|^k\|T_{m-1}\| \le \sum_{k=1}^{p-1}(\mu^{p^{m-1}})^k\|T_{m-1}\| \le \frac{\mu^{p^{m-1}}-\mu^{p^m}}{1-\mu^{p^{m-1}}}\|T_{m-1}\| \tag{12}$$

and

$$\|T_m\| \le \|\sum_{k=0}^{p-1}\|\mathcal{I}-T_{m-1}L\|^k\|T_{m-1}\| \le \sum_{k=0}^{p-1}(\mu^{p^{m-1}})^k\|T_{m-1}\| \le \frac{1-\mu^{p^m}}{1-\mu^{p^{m-1}}}\|T_{m-1}\| \le \frac{1-\mu^{p^m}}{1-\mu}\|T_0\|. \tag{13}$$

Now, by applying the previous inequalities (12) and (13), we have for $k \ge 1$:

$$\begin{aligned}\|T_{m+k}-T_m\| &\le \|T_{m+k}-T_{m+k-1}\|+\cdots+\|T_{m+1}-T_m\|\\ &\le \sum_{j=0}^{k-1}\|T_{m+j+1}-T_{m+j}\|\\ &\le \left[\sum_{j=0}^{k-1}(\mu^{p^{m+j}}-\mu^{m+j-1})\right]\frac{\|T_0\|}{1-\mu}\\ &\le \frac{\mu^{p^m}-\mu^{p^{m+k}}}{1-\mu}\|T_0\|.\end{aligned}$$

Hence, as $\mu < 1$, for $m = 0$, we conclude

$$\|T_k-T_0\| < \frac{\mu\|T_0\|}{1-\mu},$$

and then $T_k \in B\left(T_0, \dfrac{\mu\|T_0\|}{1-\mu}\right)$ for $k \ge 1$. □

Now, we can use this technical lemma to establish a semilocal convergence theorem for the sequence $\{y_m\}$ defined by (10).

Theorem 2. *Let $T_0 \in \mathcal{L}(\mathcal{C}[u,v],\mathcal{C}[u,v])$ such that $\|\mathcal{I}-T_0L\| \le \mu$, with $\mu \in (0,1)$. Then, the sequence $\{y_m\}$ defined by (10), with $y_0 = T_0h$, belongs to $B\left(y_0, \dfrac{\mu\|T_0\|}{1-\mu}\|h\|\right)$ and converges to y^*, the solution of Equation (1). In addition,*

$$\|y^*-y_m\| \le \gamma^{p^m-1}\|L^{-1}-T_0\|\|h\|.$$

Proof. Taking into account (10) and Lemma 2, we have

$$\|y_m-y_0\| = \|T_mh-T_0h\| \le \|T_m-T_0\|\|h\| < \frac{\mu\|T_0\|}{1-\mu}\|h\|,$$

then $\{y_m\} \in B\left(y_0, \frac{\mu\|T_0\|}{1-\mu}\|h\|\right)$. On the other hand, from Lemma 2, we obtain

$$\|y_{m+k} - y_m\| \leq \|T_{m+k} - T_m\|\|h\| \leq \frac{\mu^{p^m} - \mu^{p^{m+k}}}{1-\mu}\|T_0\|\|h\|,$$

therefore $\{y_m\}$ is a Cauchy sequence in a Banach space and then $\{y_m\}$ converges to $\widetilde{y}$. Now, we must prove that $\widetilde{y} = y^*$, the solution of Equation (1). For this, notice that from Lemma 2 the sequence $\{T_m\}$ verifies

$$\|T_{m+k} - T_m\| \leq \frac{\mu^{p^m} - \mu^{p^{m+k}}}{1-\mu}\|T_0\|, \ \ k \ \geq 1,$$

so, the sequence $\{T_m\}$ is a Cauchy sequence and then converges to $\widetilde{T}$ such that, from (10), $\widetilde{T}L = \mathcal{I}$.

Then, if y^* is the solution of Equation (1), we have $y^* = \widetilde{T}Ly^* = \widetilde{T}h = \widetilde{y}$. □

One of the main practical problems for implementing iterative processes is the localization of starting points. In our case, the location of a starting function y_0 involves locating an operator T_0, that allows us to construct the starting function y_0. At the same time, a suitable choice of y_0 ensures the convergence of the iterative process (10).

As we have seen, the sequence $\{T_m\}$ converges to an inverse operator to the left of the operator L, that is $\widetilde{T}L = \mathcal{I}$. We have already seen that if the kernel $K(x,t)$ is a separable kernel, it is possible to calculate $(\mathcal{I} - \lambda\mathcal{K})^{-1}$ explicitly. On the other hand, if the kernel $K(x,t)$ is non-separable, we can approximate it by a separable kernel $\widetilde{K}(x,t)$, such that

$$K(x,t) = \widetilde{K}(x,t) + \mathcal{R}(\theta, s, t),$$

where $\mathcal{R}$ is the error function. With this approximation, we can calculate the inverse operator $(\mathcal{I} - \lambda\widetilde{\mathcal{K}})^{-1}$, where

$$[\widetilde{\mathcal{K}}(y)](x) = \int_u^v \widetilde{K}(x,t)y(t)\,dt, \ x \in [u,v],$$

that will be an approximation of the inverse operator in the non-separable case. Hence, we can consider $T_0 = (\mathcal{I} - \lambda\widetilde{\mathcal{K}})^{-1}$ as a starting point for the sequence (10). By reducing the error function, $\mathcal{R}$, we will be able to obtain a good starting operator T_0 and, therefore, the starting function y_0. This strategy is developed in the following section.

5. Application to Fredholm Integral Equations

In this section, we perform some numerical problems for approximating the solution of Fredholm integral Equation (1). We consider equations with non-separable kernels. In these examples, the solution belongs to the Banach space of continuous functions in the closed interval $[0,1]$.

In the first example, the exact solution is known. Hence, with this example we are interested in checking our theoretical results and in making comparisons of the corresponding convergence ball obtained with the semilocal convergence theory. In the second example, we center our interests in the calculus of successive approximations.

Example 1. *We consider the following simple Fredholm integral equation,*

$$y(x) = \frac{30\pi x - \sin(\pi x)}{15} + \frac{1}{15}\int_0^1 x\cos(\pi x t^2)y(t)dt, \tag{14}$$

whose exact solution is $y^*(x) = 2\pi x$.

The non-separable kernel $K(x,t) = x\cos(\pi x t^2)$ can be approached, for example, by the separable kernel $\widetilde{K}(x,t)$ defined by

$$\widetilde{K}(x,t) = x - \frac{1}{2}\pi^2 t^4 x^3 + \frac{1}{4!}\pi^4 t^8 x^5 - \frac{1}{6!}\pi^6 t^{12} x^7.$$

Consequently,

$$K(x,t) = \widetilde{K}(x,t) + R(\theta,x,t), \text{ with } R(\theta,x,t) = \frac{\pi^8}{8!}x^9 t^{16}.$$

As we have indicate previously, we denote

$$[\widetilde{\mathcal{K}}(y)](x) = \int_0^1 \widetilde{K}(x,t)y(t)\,dt.\, x \in [0,1],$$

Hence, we consider $T_0 = (I - \frac{1}{15}\widetilde{\mathcal{K}})^{-1}$ and then

$$\|I - T_0 L\| \leq \frac{1}{15}\|T_0\|\|R\| \leq \frac{\pi^8}{15 \cdot 8!}\|T_0\| = 0.0156887\,\|T_0\|.$$

Note that

$$\|\lambda\widetilde{\mathcal{K}}\| < \frac{1}{15}\left(1 + \frac{\pi^2}{2} + \frac{\pi^4}{4!} + \frac{\pi^6}{6!}\right) = 0.755252 < 1,$$

so by the Banach Lemma on inverse operators, there exists $T_0 = (I - \frac{1}{15}\widetilde{\mathcal{K}})^{-1}$ and $\|T_0\| \leq 4.08583$. Consequently,

$$\|I - T_0 L\| \leq 0.0641014 = \mu < 1.$$

Now we take $y_0 = T_0 h$ with $h(x) = \dfrac{30\pi x - \sin(\pi x)}{15}$. Then, from Theorem 2, the sequence $\{y_m\}$ defined by (10) belongs to $B\left(y_0, \dfrac{\mu\|T_0\|}{1-\mu}\|h\|\right) \subseteq B(y_0, 1.75833)$ and converges to y^*, the solution of Equation (14). Actually, as $\widetilde{K}$ is a separable kernel, we can obtain the initial approach $y_0 = (I - \frac{1}{15}\widetilde{\mathcal{K}})^{-1})h$ by following the procedure showed in the Introduction. Thus, if we consider $K(x,t) = \sum_{i=1}^{m} a_i(x)b_i(t)$ and we take $m = 4$, we have the real functions:

$$a_1(x) = x,\ a_2(x) = x^3,\ a_3(x) = x^5,\ a_4(x) = x^7,$$

$$b_1(t) = 1,\ b_2(t) = -\frac{1}{2}\pi^2 t^4,\ b_3(t) = \frac{1}{4!}\pi^4 t^8,\ b_4(t) = -\frac{1}{6!}\pi^6 t^{12}.$$

In this case we have

$$(\alpha_{ij}) = \begin{pmatrix} 0.5 & 0.25 & 0.166667 & 0.125 \\ -0.822467 & -0.61685 & -0.49348 & -0.411234 \\ 0.405871 & 0.338226 & 0.289908 & 0.25367 \\ -0.0953759 & -0.0834539 & -0.0741813 & -0.0667631 \end{pmatrix}$$

and $(\beta_i) = \begin{pmatrix} 3.09915, & -5.13871, & 2.54139, & -0.597789 \end{pmatrix}$.

Hence, the solution of the system (5) is:

$$(I_i) = (3.1411, -5.1659, 2.5490, -0.59896).$$

Then, by using (6) one has the solution of nonlinear integral Equation (14), when the kernel $\widetilde{\mathcal{K}}$ is considered, and we have

$$y_0(x) \quad = \quad h(x) + \lambda\sum_{j=1}^{4} a_j(x)I_j$$

that, now, can be expressed as:

$$y_0(x) \quad = \quad 6.49259x - 0.0666667\sin(3.14159x) - 0.344396x^3 + 0.169936x^5 - 0.0399307x^7.$$

The distance of the initial approximation $y_0(x)$ to the exact solution is

$$\|y_0(x) - 2\pi x\| \leq 0.0119938.$$

Hence, we take $y_0(x)$ as the starting point in order to apply iterative scheme (10). We consider different values of p to obtain next iterate $y_1^p(x)$. We perform the integrals that appears in the process due to the operator (2), by Gauss-Legendre formula with 8 nodes having the approximations given in Table 1. We can check the improvement of each new approximation, the distance to the exact solution is decreasing significantly. Note that these approximations given in Table 1 are the first iterations of the iterative scheme (10) for different values of p.

Table 1. Distance of the first step of iterative methods (10) to the exact solution of integral Equation (14).

p	$\|y_1^p(x) - 2\pi x\|$
1	1.1994×10^{-2}
2	1.0432×10^{-4}
3	1.6138×10^{-6}
4	1.6244×10^{-8}

Example 2. *We consider the following Fredholm integral equation,*

$$y(x) = x^2 - x + 1 + \frac{1}{4}\int_0^1 e^{xt} y(t)dt. \tag{15}$$

The kernel $K(x,t) = e^{xt}$ is non-separable. It satisfies $\|\frac{1}{4}\mathcal{K}\| < 1$, so there exists $L^{-1} = (I - \frac{1}{4}\mathcal{K})^{-1}$. Then, as $\mathcal{I} - T_k L = (\mathcal{I} - T_{k-1}L)^p = \cdots = (\mathcal{I} - T_0 L)^{p^k}$, we have

$$\begin{aligned}
\mathcal{I} - T_k L &= \sum_{j=0}^{p^k} \binom{p^k}{j} (-1)^j (T_0 L)^j \\
&= \mathcal{I} + \sum_{j=1}^{p^k} \binom{p^k}{j} (-1)^j (T_0 L)^j \\
&= \mathcal{I} + \sum_{j=0}^{p^k-1} \binom{p^k}{j+1} (-1)^{j+1} (T_0 L)^j T_0 L,
\end{aligned}$$

and, as there exists L^{-1}, we obtain

$$y_k = T_k h = \sum_{j=0}^{p^k-1} \binom{p^k}{j+1} (-1)^j (T_0 L)^j y_0.$$

Therefore, we can get any iteration y_k through y_0. This fact simplifies considerably the calculus of the successive iterations.

In this problem, the kernel is not separable, so we use Taylor's development to approximate the kernel $K(x,t) = e^{xt}$. In this way, we obtain

$$K(x,t) = e^{xt} = \sum_{i=0}^{\ell} \frac{x^i\, t^i}{i!} + \mathcal{R}(\theta, x, t), \;\; \mathcal{R}(\theta, x, t) = \frac{e^{x\theta}}{3!} x^{\ell+1}\, t^{\ell+1}. \tag{16}$$

Thus, if we consider $K(x,t) = \sum_{i=1}^{m} a_i(x) b_i(t)$ *and we take* $m = 3$*, we have*

$$\widetilde{K}(x,t) = 1 + xt + x^2 \frac{t^2}{2}.$$

Consequently, we have

$$K(x,t) = \widetilde{K}(x,t) + \mathcal{R}(\epsilon, x, t), \text{ with } \widetilde{K}(x,t) = \sum_{i=1}^{3} a_i(x) b_i(t) \text{ and } \mathcal{R}(\theta, x, t) = \frac{e^{x\theta}}{3!} x^3\, t^3,$$

and the real functions:

$$a_1(x) = 1,\ a_2(x) = x,\ a_3(x) = x^2,$$

$$b_1(t) = 1,\ b_2(t) = t,\ b_3(t) = \frac{t^2}{2}.$$

Now, as $\widetilde{K}$ *is a separable kernel, we can obtain the initial approach* $y_0 = (I - \frac{1}{4}\widetilde{\mathcal{K}})^{-1} h$*, with* $h(x) = x^2 - x + 1$*, by following the procedure showed in the Introduction, that is by (5) and (6), we have:*

$$(\alpha_{ij}) = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/6 & 1/8 & 1/10 \end{pmatrix}$$

and $(\beta_i) = \begin{pmatrix} 5/6, & 5/12, & 17/120 \end{pmatrix}$.

Hence, the solution of the system (5) is:

$$I_i = \begin{pmatrix} 1.2419, & 0.63882, & 0.21885 \end{pmatrix}.$$

Then, by using (6) one has the solution of the nonlinear integral Equation (15), when the kernel $\widetilde{\mathcal{K}}$ *is considered, and therefore*

$$y_0(x) \quad = \quad h(x) + \lambda \sum_{j=1}^{3} a_j(x) I_j$$

that, now, can be expressed as:

$$y_0(x) \quad = \quad 1.05471x^2 - 0.840296x + 1.31047.$$

Now, we take $y_0(x)$ *as starting approximation in order to apply iterative scheme (10). We consider different values of* p *to obtain next iterate* $y_1^p(x)$*. We perform the integrals that appears in the process due to the operator (2), by Gauss–Legendre formula with 8 nodes having the following approximations:*

$$\begin{aligned} y_1^2(x) &= x^2 - x + 1 + 0.0536111 \exp(0.591717x) + 0.0163763 \exp(0.0198551x) \\ &+ 0.0503173 \exp(0.762766x) + 0.0343564 \exp(0.101667x) + 0.051828 \exp(0.408283x) \\ &+ 0.0391047 \exp(0.898333x) + 0.0458987 \exp(0.237234x) + 0.0189817 \exp(0.980145x). \end{aligned}$$

$$\begin{aligned} y_1^3(x) &= x^2 - x + 1 + 0.0537599 \exp(0.591717x) + 0.0163763 \exp(0.0198551x) \\ &+ 0.0506036 \exp(0.762766x) + 0.0343568 \exp(0.101667x) + 0.051875 \exp(0.408283x) \\ &+ 0.0394466 \exp(0.898333x) + 0.0459064 \exp(0.237234x) + 0.0191877 \exp(0.980145x). \end{aligned}$$

$$\begin{aligned} y_1^4(x) &= x^2 - x + 1 + 0.053836\exp(0.591717x) + 0.0163897\exp(0.0198551x) \\ &+ 0.0506795\exp(0.762766x) + 0.0343881\exp(0.101667x) + 0.0519405\exp(0.408283x) \\ &+ 0.0395067\exp(0.898333x) + 0.0459557\exp(0.237234x) + 0.019217\exp(0.980145x). \end{aligned}$$

To measure the proximity between the solution of (15) *and our first iteration* y_1^p, *given by* (10) *for different values of* p, *since the solution of Equation* (15) *is not known, we consider the operator* $F : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ *given by*

$$F(y)(x) = y(x) - x^2 + x - 1 - \frac{1}{4}\int_0^1 e^{xt} y(t)dt.$$

Obviously, a solution of (15) *is a solution of equation* $F(y) = 0$. *Therefore, we evaluate the operator* F *in our first approximations* y_1^p *and, as we can see in Table 2, these approximations are close to the solution.*

Table 2. Proximity of the first step of iterative methods (10) to the solution of integral Equation (15), for different values of p.

p	$\|F(y_1^p)\|$
2	1.6490×10^{-2}
3	5.5647×10^{-3}
4	1.8822×10^{-3}

An important aspect of our development is that if we consider higher values of m, *the influence on the operational cost of the procedure that we have followed is reduced to solving a linear system of order* $m \times m$ *for the calculation of the initial approximation* y_0. *Therefore, below, we apply the procedure for different values of* m, *in this case* $m = 6$ *and* $m = 10$.

As we can see in Table 3, the results are excellent. We would like to highlight that we are considering the first iteration of the iterative scheme (10) *for different values of* $p \geq 2$.

Table 3. Proximity of the first step of iterative methods (10) to the solution of integral Equation (15), for different values of m, the number of terms in the considered Taylor's development (16).

m	$\|F(y_1^2)\|$	$\|F(y_1^3)\|$	$\|F(y_1^4)\|$
6	4.6464×10^{-5}	1.5659×10^{-5}	5.2963×10^{-6}
10	3.7927×10^{-9}	1.2770×10^{-9}	4.3191×10^{-10}

6. Conclusions

In this work, we have established a procedure for solving a Fredholm integral equation of second kind by using an inverse problem. Actually, the main idea is to approach the inverse operator that appears in the theoretical exact solution by means of iterative schemes. These algorithms are the generalization of the sequences obtained when iterative methods of different orders of convergence (Newton's or Chebyshev's methods, for instance) are applied to the problem of the calculus of inverse operators. In this way, we obtain inverse-free iterative procedures with a prefixed order of convergence to approximate the solution of the proposed integral equation. We have analyzed the local and semilocal convergence for these iterative schemes.

The procedure can be applied for separable and non-separable kernels. Furthermore, the location of the starting points for the application of the considered iterative schemes, which is the main problem posed by the application of the iterative schemes, has been favorably solved. The theoretical results obtained have been checked with two particular problems. In these test problems, we have gotten hopeful results, so we believe the techniques introduced in this paper are very competitive and can be used for approximating the solution of Fredholm integral equations of second kind.

As a further work, we would like to apply the techniques introduced in this paper in two different ways. First, we would be interested in studying some particular problems, such us Fredholm integral equations with non-singular kernels. Second, we would like to generalize our study to other integral equations, especially to nonlinear integral equations [17,18].

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