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ABSTRACT. Consider a finite measure space (Ω, Σ, μ) and a Banach space $X(\mu)$ consisting of (equivalence classes of) real measurable functions defined on Ω such that $f\chi_A \in X(\mu)$ and $\|f\chi_A\| \leq \|f\|$, $\forall f \in X(\mu)$, $A \in \Sigma$. We prove that if it satisfies the subsequence property, then it is an ideal of measurable functions and has an equivalent norm under which it is a Banach function space. As an application we characterize norms that are equivalent to a Banach function space norm.

1. Introduction and preliminaries

Throughout this paper all vector spaces we consider are real and (Ω, Σ, μ) is a finite measure space. If $X(\mu)$ is a normed (Banach) space consisting of (equivalence classes of) real Σ -measurable functions, then we say that $X(\mu)$ is a *normed (Banach) space of measurable functions*. A Banach space $X(\mu)$ is called a *Banach function space* if it is an *ideal* of Σ -measurable functions and has a *Riesz norm*. This means that if g is a Σ -measurable function, $f \in X(\mu)$ and $|g| \leq |f|$, then we always have $g \in X(\mu)$ and $\|g\| \leq \|f\|$.

In this paper (see Corollary 4.2) we prove that a Banach space of measurable functions $X(\mu)$ admits an equivalent norm $\|\cdot\|_V$ for which $(X(\mu), \|\cdot\|_V)$ is a Banach function space if and only if it satisfies:

(P1) There exists $C > 0$ such that for any $f \in X(\mu)$ we have

$$f\chi_A \in X(\mu), \forall A \in \Sigma, \text{ and } \|f\chi_A\|_{X(\mu)} \leq C\|f\|_{X(\mu)}.$$

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(P2) Subsequence property: any sequence that converges in $X(\mu)$ has a subsequence converging pointwise μ -a.e.

In order to better understand the role of condition (P1) in the ideal property of a Banach function space, we analyze first a particular representation of Banach spaces with Schauder basis. To do this, we associate a Banach sequence space $\mathcal{S}(X)$ to each Banach space X having such a basis (see definitions below). We show that the unconditionality of the basis corresponds, precisely, to condition (P1) (see Theorem 1.1).

The vector space consisting of all real sequences $s = \{a_n\}$ is denoted by \mathcal{S} , and a *Banach sequence space* is simply a Banach space $X \subset \mathcal{S}$. By taking $\Omega := \mathbb{N}$, $\Sigma := 2^{\mathbb{N}}$ and, for example, $\mu(A) := \sum_{n \in A} \frac{1}{n^2}$, $\forall A \subset \mathbb{N}$, we can consider any Banach sequence space as a Banach space of measurable functions with respect to a finite measure.

Given a Banach space X , recall that a sequence $\{y_n\} \in X$ is *unconditionally summable* if the series $\sum_{n=1}^{\infty} y_{\pi(n)}$ converges for each permutation $\pi : \mathbb{N} \rightarrow \mathbb{N}$. A set $\{x_n\} \subset X$ is a *Schauder basis for X* , if for each $x \in X$ there is a unique sequence $\{a_n\} \in \mathcal{S}$ satisfying

$$(1.1) \quad x = \sum_{n=1}^{\infty} a_n x_n.$$

If X is a Banach space and $\{x_n\}$ is a Schauder basis of X , we can define the linear space of scalar sequences $\mathcal{S}(X)$ such that their associated vector valued sequences are norm convergent,

$$\mathcal{S}(X) := \left\{ \{a_n\} \in \mathcal{S} : \lim_N \sum_{n=1}^N a_n x_n \in X \right\}.$$

Thus, $\mathcal{S}(X)$ is formed by the ‘coordinates’ of the elements of X . In this case we can define a linear map $T : \mathcal{S}(X) \rightarrow X$ by

$$T(s) := \sum_{n=1}^{\infty} a_n x_n,$$

which is an isometric isomorphism with the norm $\|s\|_{\mathcal{S}(X)} := \|T(s)\|_X$. In this way any Banach space X with a Schauder basis can be considered as a Banach sequence space, $\mathcal{S}(X)$. Additionally, by means of T the order in $\mathcal{S}(X) \subset \mathcal{S}$ can be translated to X . Namely, for $s, t \in \mathcal{S}(X)$ we define $T(s) \leq T(t)$ if $s \leq t$.

If $\{x_n\}$ is a Schauder basis for X , and for each $x \in X$ the sequence $\{a_n x_n\}$ is unconditionally summable (where $\{a_n\}$ is as in (1.1)), then $\{x_n\}$ is said to be an *unconditional basis for X* . The following properties are well known (see for example [3, p. 5, p. 344], [7, pp. 15–16]). For each $\{a_n\} \in \mathcal{S}(X)$

$$(1.2) \quad \sum_{n=1}^{\infty} c_n a_n x_n \text{ converges for any } \{c_n\} \in \ell^{\infty},$$

and there exists $C > 0$ such that

$$(1.3) \quad \left\| \sum_{n=1}^{\infty} c_n a_n x_n \right\|_X \leq C \|t\|_{\infty} \left\| \sum_{n=1}^{\infty} a_n x_n \right\|_X, \quad \forall t = \{c_n\} \in \ell^{\infty},$$

where $\|\cdot\|_{\infty}$ is the norm of ℓ^{∞} . Note that for $s = \{a_n\} \in \mathcal{S}$ and $A \subset \mathbb{N}$, $s\chi_A := \{b_n\}$, where $b_n := a_n$ when $n \in A$ and $b_n := 0$ when $n \notin A$.

Theorem 1.1. *Let $\{x_n\}$ be a Schauder basis of the Banach space X . The following properties are equivalent:*

- i) $\{x_n\}$ is an unconditional basis for X .
- ii) $\mathcal{S}(X)$ is an ideal of \mathcal{S} and there exists $C > 0$ such that

$$(1.4) \quad \|t\|_{\mathcal{S}(X)} \leq C \|s\|_{\mathcal{S}(X)} \quad \text{if } t, s \in \mathcal{S}(X) \text{ and } |t| \leq |s|.$$

- iii) There exists $C > 0$ such that for any $s \in \mathcal{S}(X)$ we have

$$(1.5) \quad s\chi_A \in \mathcal{S}(X), \quad \forall A \subset \mathbb{N}, \quad \text{and } \|s\chi_A\|_{\mathcal{S}(X)} \leq C \|s\|_{\mathcal{S}(X)}.$$

Proof. Since the other implications are well known, we will only prove that i) implies ii). Take $s = \{a_n\}, t = \{b_n\} \in \mathcal{S}$ such that $|t| \leq |s|$. Let us define

$$c_n = \begin{cases} \frac{b_n}{a_n} & a_n \neq 0, \\ 0 & a_n = 0, \end{cases}$$

and consider $r := \{c_n\}$. Assume that $\{a_n x_n\}$ is unconditionally summable for any $s = \{a_n\} \in \mathcal{S}(X)$. Since $r \in \ell^{\infty}$, applying (1.2) we find that $t = rs \in \mathcal{S}(X)$. This shows that $\mathcal{S}(X)$ is an ideal of \mathcal{S} .

From (1.3), we also have the inequality $\|T(rs)\|_X \leq C \|r\|_{\infty} \|T(s)\|_X$ for the operator T defined above, what gives

$$\|t\|_{\mathcal{S}(X)} = \|rs\|_{\mathcal{S}(X)} \leq C \|r\|_{\infty} \|s\|_{\mathcal{S}(X)} \leq C \|s\|_{\mathcal{S}(X)},$$

and then (1.4) holds. \square

With this result in mind, let us consider the general case of a Banach space of measurable functions, say $X(\mu)$, with its own order structure. Note that in this case, Theorem 1.1 may not make sense. This is for example the case of ℓ^{∞} with its usual order, for which properties ii) and iii) are fulfilled, but it does not have a Schauder basis since it is not separable.

It is therefore a natural question to ask whether conditions ii) and iii) remain equivalent in this more general context. In this paper we address this question. Actually, we prove that ii) is equivalent to condition (P1) —which would correspond to condition iii) in Theorem 1.1— together with the already introduced subsequence property (P2).

The paper is organized as follows: Section 2 collects the preliminary results we will need further. Subsection 2.1 is devoted to study what we call *subsequence property*, namely, condition (P2) in the Introduction. Condition (P1) is studied in Subsection 2.2. We call *rectangular* those spaces $X(\mu)$, or norms,

that satisfy (P1). Note that this property does not explicitly require that $X(\mu)$ be an ideal (see Subsection 2.2).

In Section 3 we prove the main result of the paper, Theorem 3.7. As a consequence, as we note in Section 4, it follows readily that basic properties of Banach function spaces remain valid in the more general context of Banach rectangular spaces with the subsequence property. In Section 4 we also show that rectangular sequence spaces always have the subsequence property (see Proposition 4.5), that is, in this case (P1) implies (P2). This result explains why in the case of sequence spaces, condition ii) in Theorem 1.1 is equivalent to iii) (namely to (P1)). Observe also that our setting includes Banach spaces with unconditional basis.

2. Preliminaries

Let (Ω, Σ, μ) be a finite measure space. The space $L^0(\Sigma)$ consists of all Σ -measurable functions $f : \Omega \rightarrow \mathbb{R}$, $S(\Sigma)$ is the vector space formed by the simple functions $s : \Omega \rightarrow \mathbb{R}$ and $L^0(\mu)$ is the vector space consisting of the equivalence classes of Σ -measurable functions, where two functions are equivalent when $f = g$ μ -a.e. As usual, $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\mu)$.

In $L^0(\mu)$ we consider the canonical order, given by $f \leq g$ if $f \leq g$ μ -a.e. Given $A \subset L^0(\mu)$, then $A^+ := \{f \in A : f \geq 0\}$. The notation $X(\mu)$ will indicate that $X(\mu)$ is a vector subspace of $L^0(\mu)$; we define $S(X(\mu)) := S(\Sigma) \cap X(\mu)$ and $X(\mu)^+$ is the *positive cone* of $X(\mu)$.

A space $X(\mu)$ is a *Riesz space* if $\max\{f, g\} \in X(\mu)$, $\forall f, g \in X(\mu)$ [11]. Since

$$(2.1) \quad \max\{f, g\} = \frac{f + g + |f - g|}{2}, \quad \forall f, g \in L^0(\mu),$$

if the absolute value $|h| \in X(\mu)$ for any $h \in X(\mu)$, then $X(\mu)$ is a Riesz space.

If $f \in X(\mu)$ when $f \in L^0(\mu)$, $g \in X(\mu)$ and $0 \leq |f| \leq |g|$, the vector space $X(\mu)$ is an *ideal* of $L^0(\mu)$. Notice that in this case, given $f \in L^0(\mu)$, we have $f \in X(\mu)$ if and only if $|f| \in X(\mu)$. Hence an ideal is always a Riesz space.

A norm $\|\cdot\|$ on a Riesz space $X(\mu)$ is a *Riesz norm* [8, p. 48, p. 438], (or $X(\mu)$ is a *normed Riesz space*) when $f, g \in X(\mu)$ and $|f| \leq |g|$ imply that $\|f\| \leq \|g\|$. In this case we clearly have $\|f\| = \||f|\|$, $\forall f \in X(\mu)$.

A normed Riesz space $X(\mu)$ is a μ -*normed function space* (μ -n.f.s. or n.f.s. for short), when $X(\mu)$ is an ideal in $L^0(\mu)$. If additionally $X(\mu)$ is complete, then it is a Banach function space (Banach f.s.).

2.1. The subsequence property

In this subsection we study the subsequence property and relate it to other basic concepts connected with the order. Although the subsequence property is a well-known feature of Banach function spaces, there are some current research papers that study natural generalizations for abstract Banach lattices (see for example [2, Proposition 4.1 and Corollary 3.5] and the references in this paper). However, in our case we work in the framework of the spaces of measurable

functions. Hence the underlying structure of measurability allows us to define the order as the a.e. order.

Recall that a normed space $X(\mu)$ has the *subsequence property* if for any sequence $\{f_n\} \subset X(\mu)$ and $f \in X(\mu)$ such that $f_n \rightarrow f$ in the norm, there is a subsequence $\{f_{n(k)}\}$ converging pointwise to f μ -a.e. It is well known that it can be characterized as follows [10, 18.14].

Proposition 2.1. *A normed space $X(\mu)$ has the subsequence property if and only if the inclusion map $X(\mu) \subset L^0(\mu)$ is continuous.*

Given a Riesz space $X(\mu)$, we write as usual $X(\mu)^+$ for its positive cone. Let us consider the function $V : X(\mu) \rightarrow X(\mu)$, defined by $V(f) := |f|$. The next lemma relates these concepts on Riesz spaces having a norm.

Lemma 2.2. *Let $X(\mu)$ be a Riesz space with a norm.*

- i) *If V is continuous, then $X(\mu)^+$ is closed.*
- ii) *If $X(\mu)$ has the subsequence property, then $X(\mu)^+$ is closed.*

Proof. i) Let $\{f_n\} \subset X(\mu)^+$ and $f \in X(\mu)$ be such that $f_n \rightarrow f$. Since V is continuous, this implies $f_n = |f_n| \rightarrow |f|$. Hence $f = |f| \in X(\mu)^+$.

ii) Let $\{f_n\} \subset X(\mu)^+$ and $f \in X(\mu)$ be such that $f_n \rightarrow f$. Next, take a subsequence $\{f_{n(k)}\}$ such that $\{f_{n(k)}\}$ converges to f μ -a.e. Since $f_{n(k)} \geq 0$ μ -a.e., it follows that $f \geq 0$ μ -a.e., that is $f \in X(\mu)^+$. \square

The following normed sequence space is an example of a Riesz space whose positive cone is not closed. Consequently, it does not have the subsequence property and the function V is not continuous on it.

Example 2.3. For each $n \in \mathbb{N}$, let $v_n := e_1 + \cdots + e_n$, where $\{e_n\}$ is the canonical orthonormal basis in $(\ell^2, \|\cdot\|_2)$. We define

$$X(\mu) := \text{span}\{v_n : n \in \mathbb{N}\}, \quad \left\| \sum_{n=1}^N a_n v_n \right\| := \left\| \sum_{n=1}^N \frac{a_n}{n} e_n \right\|_2.$$

Then $\|\cdot\|$ is a norm in $X(\mu)$ and $\|v_n\| = \frac{1}{n} \rightarrow 0$.

Recall that we consider in $X(\mu)$ the natural order between sequences. Then $-e_1 + v_n \in X(\mu)^+$, $\forall n \in \mathbb{N}$, and $-e_1 + v_n \rightarrow -e_1 \notin X(\mu)^+$. This shows that $X(\mu)^+$ is not closed.

Since $|f| \in X(\mu) \forall f \in X(\mu)$, and so $X(\mu)$ is a Riesz space, it follows from i) of Lemma 2.2 that the function V is not continuous.

Even when X is both a Riesz space and a Banach sequence space with the subsequence property, the function V defined by $V(s) := |s|$ can be discontinuous, as an example of D. H. Fremlin shows [5, 2XD].

Let us finish this section by recalling some relations of the subsequence property with other classical and recently introduced Banach lattice properties. We are specially interested in those lattices that are endowed with a natural structure of measurability, as the Banach function spaces. In this paper, a particular

type of monotonicity for the norm of the function spaces involved will be relevant. In the case of Banach function spaces monotonicity is often related to order continuity or Fatou type properties. These properties play a relevant role regarding the relation among convergence in measure and norm convergence. They provide a sort of converse of the subsequence property: *every increasing order bounded sequence converges in the norm*. Thus, several concepts associated to monotonicity properties for Banach lattices are defined in terms of sequences. Hence, in the case of Banach function spaces, they are naturally connected with the convergence in measure —as the subsequence property— (see for example [4, Theorem 2.5]). More results for concrete Banach function spaces can be found in the excellent survey [4]. For current developments on the role of sequential properties and the geometry of Banach function spaces see [1], and [6] for the vector valued case.

2.2. Rectangular function spaces

We present in what follows the main definition of this paper and the fundamental properties of the spaces satisfying it. As the reader may notice, our definition lies in a basic monotonicity property as the ones commented in the previous section for a certain class of functions: an inequality in the almost everywhere order implies an inequality for the corresponding norms.

Definition 2.4. A normed space of measurable functions $X(\mu)$ is a *rectangular function space* (rectangular f.s.) if there exists $C > 0$ such that

$$(2.2) \quad \chi_A f \in X(\mu) \text{ and } \|\chi_A f\| \leq C\|f\|, \quad \forall f \in X(\mu), A \in \Sigma.$$

In this situation, the norm will also be called *rectangular*. When $C = 1$ we will say that $X(\mu)$, and its norm, is *strictly rectangular*. If a rectangular (or strictly rectangular) f.s. is also norm complete, we will say it is a *Banach rectangular* (or *strictly rectangular*) f.s.

Example 2.5. Every normed function space $X(\mu)$ is a strictly rectangular f.s. Even more, if $\|\cdot\|_0$ is any equivalent norm for $X(\mu)$, then $\|\cdot\|_0$ is rectangular.

Proof. Given a n.f.s $X(\mu)$, consider $f \in X(\mu)$, $A \in \Sigma$. Since $|\chi_A f| \leq |f|$, we have $\chi_A f \in X(\mu)$ and

$$\|\chi_A f\| = \|\chi_A |f|\| \leq \| |f| \| = \|f\|.$$

This shows that $X(\mu)$ is a strictly rectangular f.s.

Next, let us take $0 < a < b$ such that $a\|f\|_0 \leq \|f\| \leq b\|f\|_0$. Then, for $A \in \Sigma$ we have $a\|\chi_A f\|_0 \leq \|\chi_A f\| \leq \|f\| \leq b\|f\|_0$. It follows that the norm $\|\cdot\|_0$ is rectangular. \square

If $X(\mu)$ is a rectangular f.s., we next show that $|f| \in X(\mu)$ when $f \in X(\mu)$. By (2.1) it follows that any rectangular f.s. is a Riesz space.

Given $f \in L^0(\mu)$, take $A := \{x \in \Omega : f(x) \geq 0\}$ and $B := \Omega \setminus A$. Then for the positive and negative parts of f we have

$$(2.3) \quad f^+ = f\chi_A = |f|\chi_A, \quad f^- = -f\chi_B = |f|\chi_B.$$

Since $f = f^+ - f^-$ and $|f| = f^+ + f^-$, we obtain immediately the following.

Lemma 2.6. *Let $X(\mu)$ be a rectangular f.s. with constant $C > 0$.*

i) *Let $f \in L^0(\mu)$. Then the following properties are equivalent:*

- a) $f \in X(\mu)$.
- b) $f^+, f^- \in X(\mu)$.
- c) $|f| \in X(\mu)$.

ii) $\frac{1}{2C}\|f\| \leq \||f|\| \leq 2C\|f\|, \forall f \in X(\mu)$.

Our next example shows that already in \mathbb{R}^2 there are rectangular norms that are not Riesz norms.

Example 2.7. Let

$$B := \{(x, y) \in \mathbb{R}^2 : xy \geq 0, x^2 + y^2 \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : xy \leq 0, |y - x| \leq 1\}.$$

Then B is an absorbing closed convex set satisfying $-B = B$. Hence, by means of its Minkowski functional we obtain a norm $\|\cdot\|_B$ for \mathbb{R}^2 , whose closed unit ball is B ([9, p. 25]). The norm $\|\cdot\|_B$ is rectangular. Take now $f := (-1, 1)$. Then $\||f|\|_B = \|(1, 1)\| = \sqrt{2}$. On the other side we have $\|f\| = 2$. Hence $\|f\| > \||f|\|$ and so this norm is not a Riesz norm.

The next result proves that for Banach rectangular f.s., implication ii) in Lemma 2.2 is, indeed, an equivalence.

Lemma 2.8. *Let $X(\mu)$ be a Banach rectangular f.s. Then, the following properties are equivalent:*

- i) $X(\mu)$ has the subsequence property.
- ii) $X(\mu) \subset L^0(\mu)$ continuously.
- iii) $X(\mu)^+$ is closed.

Proof. The equivalence between i) and ii) is already proved in Proposition 2.1. That i) implies iii) is proved in Lemma 2.2. To prove the converse implication, consider $\{f_n\} \subset X(\mu)$ and $f \in X(\mu)$ such that $f_n \rightarrow f$. Given $k \in \mathbb{N}$ let us take $n(k) \in \mathbb{N}$ so that $\|f - f_{n(k)}\| \leq 2^{-k}$ and $n(k) < n(k+1)$. Since $X(\mu)$ is a Banach space, using Lemma 2.6 we conclude that $S := \sum_{k=1}^{\infty} |f - f_{n(k)}| \in X(\mu)$. Define $s_K := \sum_{k=1}^K |f - f_{n(k)}|, K \in \mathbb{N}$. For a fixed $K \in \mathbb{N}$ and $\ell \in \mathbb{N}$, we have $s_{K+\ell} \geq s_K$. Letting now $\ell \rightarrow \infty$ and taking into account that $X(\mu)^+$ is closed, it follows that $S \geq s_K, \forall K \in \mathbb{N}$. From here we can find a set $A \in \Sigma$ such that $\mu(A) = 0$ and $S(x) \geq s_K(x), \forall x \in \Omega \setminus A, K \in \mathbb{N}$. This implies that the series $\sum_{k=1}^{\infty} |f - f_{n(k)}|$ converges in $\Omega \setminus A$ and so $f_{n(k)} \rightarrow f$ in $\Omega \setminus A$. \square

It will follow from Theorem 5.8 that for a Banach rectangular f.s. the continuity of V is also equivalent with the properties considered in the above lemma.

Definition 2.9. The norm in $X(\mu)$ is *monotone* if there exists $C > 0$ such that for $f, g \in X(\mu)^+$ satisfying $f \leq g$, we have $\|f\| \leq C\|g\|$. When $C = 1$, we will say the norm is *strictly monotone*.

Lemma 2.10. *If $X(\mu)$ is a rectangular f.s. with constant $C > 0$ with a monotone norm with constant $C' > 0$, then the function V is continuous. More precisely,*

$$(2.4) \quad \||g| - |h|\| \leq 4C^2C'\|g - h\|, \forall g, h \in X(\mu).$$

Proof. Take $C > 0$ as in Definition 2.9. Let $g, h \in X(\mu)$. Then

$$0 \leq \||g| - |h|\| \leq |g - h|.$$

Using now Lemma 2.6 and the hypothesis we obtain

$$\frac{1}{2C} \||g| - |h|\| \leq \||g| - |h|\| \leq C'\|g - h\| \leq 2CC'\|g - h\|,$$

and the conclusion follows. □

Using Lemmas 2.10, 2.2 and 2.8 we obtain the following well known result.

Corollary 2.11. *Any Banach function space $X(\mu)$ has the subsequence property.*

3. Associated Banach function space norm

In this section we prove the main result of this paper, namely Theorem 3.7. It gives a characterization of when a Banach space consisting of measurable functions can be renormed in order to obtain a Banach function space.

Definition 3.1. Let $X(\mu)$ be a rectangular f.s. Given a function $f \in L^0(\mu)$, let $\Sigma_f := \{A \in \Sigma : f\chi_A \in X(\mu)\}$ and $S(f, X(\mu)) := \text{span}\{f\chi_A : A \in \Sigma_f\} \subset X(\mu)$.

Note that Σ_f is always a ring of subsets, that is, $\phi \in \Sigma_f$ and $A \cup B, A \setminus B \in \Sigma_f$, when $A, B \in \Sigma_f$. We also have $S(\chi_\Omega, X(\mu)) = S(X(\mu))$ and $\Sigma_f = \Sigma, \forall f \in X(\mu)$.

Lemma 3.2. *Let $X(\mu)$ be a rectangular f.s. If $g \in S(f, X(\mu))$, then it can be expressed as $g = \sum_{n=1}^N a_n f\chi_{A_n}$, where $\{A_n\}_{n=1}^N \subset \Sigma_f$ consists of disjoint sets.*

Proof. Let $g \in S(f, X(\mu))$. Then $g = \sum_{m=1}^M b_m f\chi_{B_m}$ where $\{B_m\}_{m=1}^M \subset \Sigma_f$. We will proceed by induction on M . For $M = 1$, the conclusion is clear.

Let us now consider $g = \sum_{m=1}^{M+1} b_m f\chi_{B_m}$ where $\{B_m\}_{m=1}^{M+1} \subset \Sigma_f$. By the induction hypothesis there is a finite set J and a collection of disjoint sets $\{C_j : j \in J\} \subset \Sigma_f$ such that $\sum_{m=1}^M b_m f\chi_{B_m} = \sum_{j \in J} c_j f\chi_{C_j}$. If $B_{M+1} \cap C_j = \phi, \forall j \in J$, or $B_{M+1} = C_j$ for some $j \in J$, the conclusion is clear. Assume this is not so, take $J_0 := \{j \in J : B_{M+1} \cap C_j \neq \phi\}$ and define $D_{M+1} := B_{M+1} \setminus \cup_{j \in J} C_j$. Then

$$(3.1) \quad D_{M+1} \in \Sigma_f, \quad D_{M+1} \cap C_j = \phi, \quad \forall j \in J.$$

Now for each $j \in J_0$, let $D_j := C_j \cap B_{M+1}$ and $E_j := C_j \setminus B_{M+1}$. Then $D_j, E_j \in \Sigma_f$, $D_j \cap E_j = \emptyset$ and $C_j = D_j \cup E_j$. We also have

$$(3.2) \quad B_{M+1} \cap (\cup_{j \in J} C_j) = \cup_{j \in J_0} D_j.$$

Hence

$$\begin{aligned} & \sum_{m=1}^{M+1} b_m f \chi_{B_m} \\ &= \sum_{j \in J} c_j f \chi_{C_j} + b_{M+1} f \chi_{B_{M+1}} \\ &= \sum_{j \in J \setminus J_0} c_j f \chi_{C_j} + \sum_{j \in J_0} c_j f \chi_{C_j} + b_{M+1} \left(\sum_{j \in J_0} f \chi_{D_j} + f \chi_{D_{M+1}} \right) \\ &= \sum_{j \in J \setminus J_0} c_j f \chi_{C_j} + \sum_{j \in J_0} c_j f \chi_{E_j} + \sum_{j \in J_0} (c_j + b_{M+1}) f \chi_{D_j} + b_{M+1} f \chi_{D_{M+1}}, \end{aligned}$$

and the conclusion follows. \square

Lemma 3.3. *Let $X(\mu)$ be a strictly rectangular f.s.. Take $f \in L^0(\mu)^+$ and a finite collection of disjoint sets $\{A_j\}_{j=1}^n \subset \Sigma_f$. If $0 \leq a_j \leq b_j$, $j = 1, \dots, n$, then*

$$\left\| \sum_{j=1}^n a_j \chi_{A_j} f \right\| \leq \left\| \sum_{j=1}^n b_j \chi_{A_j} f \right\|.$$

Proof. Let $n \in \mathbb{N}$. For $n = 1$ the conclusion is clear. So we now consider $n \geq 2$ and first assume that $a_1 = b_1, \dots, a_{n-1} = b_{n-1}$ and $0 \leq a_n < b_n$. Take $x := \sum_{1 \leq j < n} a_j \chi_{A_j} f$ and $A := \cup_{1 \leq j < n} A_j$. Since the norm is rectangular, we have

$$\|x\| = \|(x + b_n \chi_{A_n} f) \chi_A\| \leq \|x + b_n \chi_{A_n} f\|.$$

Then

$$\begin{aligned} \|(x + a_n \chi_{A_n}) f\| &= \left\| \frac{a_n}{b_n} (x + b_n \chi_{A_n} f) + \left(1 - \frac{a_n}{b_n}\right) x \right\| \\ &\leq \frac{a_n}{b_n} \|x + b_n \chi_{A_n} f\| + \left(1 - \frac{a_n}{b_n}\right) \|x\| \\ &\leq \|x + b_n \chi_{A_n} f\|. \end{aligned}$$

The general case is now obtained by applying what we have just proved to

$$\sum_{1 \leq j < n} a_j \chi_{A_j} f + a_n \chi_{A_n} f \quad \text{and} \quad \sum_{1 \leq j < n} a_j \chi_{A_j} f + b_n \chi_{A_n} f,$$

then to $\sum_{1 \leq j < n-1} a_j \chi_{A_j} f + a_{n-1} \chi_{A_{n-1}} f + b_n \chi_{A_n} f$ and $\sum_{1 \leq j < n-1} a_j \chi_{A_j} f + b_{n-1} \chi_{A_{n-1}} f + b_n \chi_{A_n} f$, until we finish with $a_1 \chi_{A_1} f + \sum_{1 < j \leq n} b_j \chi_{A_j} f$ and $b_1 \chi_{A_1} f + \sum_{1 < j \leq n} b_j \chi_{A_j} f$. \square

Proposition 3.4. *The norm in a strictly rectangular f.s. $X(\mu)$ is always strictly monotone on $S(f, X(\mu))$, $\forall f \in L^0(\mu)$, and so*

$$(3.3) \quad \left\| |g| - |h| \right\| \leq 4\|g - h\|, \quad \forall g, h \in S(f, X(\mu)).$$

If $f \in X(\mu)$ we additionally have

$$(3.4) \quad \left\| |sg| \right\| \leq \|s\|_\infty \left\| |g| \right\|, \quad \|sg\| \leq 4\|s\|_\infty \|g\|, \quad \forall s \in S(\Sigma), g \in S(f, X(\mu)).$$

Proof. Consider $g, h \in S(f, X(\mu))$ such that $0 \leq g \leq h$ and let us express $g = \sum_{j=1}^n a_j \chi_{A_j} f$ and $s = \sum_{k=1}^m b_k \chi_{B_k} f$, where $\{A_j\}, \{B_k\} \subset \Sigma$ are finite partitions of Ω . Then $g = \sum_{j,k} a_j \chi_{A_j \cap B_k} f$ and $h = \sum_{j,k} b_k \chi_{A_j \cap B_k} f$. If $A_j \cap B_k$ has positive measure, then $a_j \leq b_k$ and if $A_j \cap B_k$ has measure zero we can go on by eliminating this term. Proceeding in this way we can apply the above lemma to obtain that $\|g\| \leq \|h\|$.

Inequality (3.3) follows now from (2.4) in Lemma 2.10.

Assume now that $f \in X(\mu)$. Let $s \in S(\Sigma)$ and $g \in S(f, X(\mu))$. Since $\chi_A f \in X(\mu), \forall A \in \Sigma$, it follows that $sg \in X$ and $|sg| \leq \|s\|_\infty |g|$. Since the norm is monotone on $S(f, X(\mu))$, we obtain the first inequality in (3.4). Using now Lemma 2.6 we have

$$\|sg\| \leq 2\left\| |sg| \right\| \leq 2\|s\|_\infty \left\| |g| \right\| \leq 4\|s\|_\infty \|g\|. \quad \square$$

In the sequel, we will assume that the spaces are complete. The next example shows that completeness is basic for $X(\mu)$ to be an ideal.

Example 3.5. Let $\Omega \subset \mathbb{R}^n$ be a (Lebesgue) measurable set having positive measure and take μ as Lebesgue’s measure defined on the σ -algebra of measurable subsets of Ω . Consider $X(\mu) := S(\Sigma)$ with the corresponding L^1 -norm. Then $X(\mu)$ is a rectangular f.s. and is not an ideal in $L^0(\mu)$.

Lemma 3.6. *Let $X(\mu)$ be a Banach space of measurable functions with the subsequence property. If $f \in L^0(\mu)$ and $\{f_n\} \subset X(\mu)$ is a Cauchy sequence that has a subsequence $\{f_{n(k)}\}$ which converges pointwise μ -a.e. to f , then $f \in X(\mu)$, and $f_n \rightarrow f$ in $X(\mu)$.*

Proof. Let $g_k := f_{n(k)}, \forall k \in \mathbb{N}$. Then $\{g_k\} \subset X(\mu)$ is also a Cauchy sequence and so there is some $g \in X(\mu)$ such that $g_k \rightarrow g$. To obtain the conclusion we will show that $g = f \mu$ -a.e. Since $X(\mu)$ has the subsequence property, let $\{g_{k(j)}\}_j$ be a subsequence such that $g_{k(j)} \rightarrow g$ when $j \rightarrow \infty$, pointwise μ -a.e. By the hypothesis we also have $g_{k(j)} \rightarrow f$ pointwise μ -a.e. It follows that $f = g \mu$ -a.e. □

Theorem 3.7. *Let $X(\mu)$ be a Banach strictly rectangular f.s. with the subsequence property.*

i) *If $h \in L^\infty(\mu)$ and $f \in X(\mu)$, then $hf \in X(\mu)$ and*

$$\left\| |hf| \right\| \leq \|h\|_\infty \left\| |f| \right\|, \quad \|hf\| \leq 4\|h\|_\infty \|f\|.$$

ii) *$X(\mu)$ is an ideal in $L^0(\mu)$.*

iii) The norm in $X(\mu)$ is monotone, (2.4) holds and the function V is continuous.

iv) The function

$$(3.5) \quad \|f\|_V := \| |f| \|, \quad \forall f \in X(\mu),$$

defines an equivalent norm for $X(\mu)$. Moreover, the norm $\|\cdot\|_V$ coincides with that of $X(\mu)$ in $X(\mu)^+$ and, with the norm $\|\cdot\|_V$, the space $X(\mu)$ is a Banach f.s.

Proof. i) Take $h \in L^\infty(\mu)$, $f \in X(\mu)$ and let $\{s_n\} \subset S(\Sigma)$ be such that $s_n \rightarrow h$ in $L^\infty(\mu)$. Since $f \in S(f, X(\mu))$, it now follows from the second inequality in (3.4) that $\{s_n f\} \subset X(\mu)$ is a Cauchy sequence.

On the other hand, we can find a subsequence $\{s_{n(k)}\}$ such that $s_{n(k)} \rightarrow h$ pointwise μ -a.e. Then $s_{n(k)} f \rightarrow hf$ pointwise μ -a.e. This allows us to apply Lemma 3.6 and conclude that $hf \in X(\mu)$ and $s_n f \rightarrow hf$ in $X(\mu)$. Hence from (3.4) we obtain

$$\|hf\| = \lim \|s_n f\| \leq 4 \lim \|s_n\|_\infty \|f\| = 4\|h\|_\infty \|f\|.$$

Since $s_n \rightarrow h$ in $L^\infty(\mu)$, it follows that $|s_n| \rightarrow |h|$ in $L^\infty(\mu)$. As above this implies that $|s_n| |f| \rightarrow |h| |f|$ in $X(\mu)$. Using now the first inequality in (3.4) we have $\| |hf| \| \leq \| |h| \|_\infty \| |f| \|$.

ii) Take $f \in L^0(\mu)$ and $g \in X(\mu)$ such that $|f| \leq |g|$. By Lemma 2.6 it is enough to show that $|f| \in X(\mu)$. For this, note that we can express $|f| = h|g|$ for some $h \in L^\infty(\mu)$ with $\|h\|_\infty \leq 1$. Since $|g| \in X(\mu)$, applying i) we obtain that $|f| \in X(\mu)$.

iii) Let $f, g \in X(\mu)$ be such that $0 \leq f \leq g$. Then we can express $f = hg$ where $h \in L^\infty(\mu)$ and $\|h\|_{L^\infty(\mu)} \leq 1$. By i) we now have $\|f\| = \|hg\| \leq \|g\|$.

iv) The other properties of a norm being clear, we will only establish the triangle inequality. So, take $f, g \in X(\mu)$. Then $0 \leq |f+g| \leq |f| + |g|$. Since the norm in $X(\mu)$ is monotone we have

$$\| |f+g| \| \leq \| |f| + |g| \| \leq \| |f| \| + \| |g| \|.$$

This shows that $\|f+g\|_V \leq \|f\|_V + \|g\|_V, \forall f, g \in V$.

From ii) in Lemma 2.6 it now follows that $\|\cdot\|_V$ and $\|\cdot\|$ are equivalent norms in $X(\mu)$.

Finally, consider $f, g \in X(\mu)$ such that $|f| \leq |g|$. By the monotonicity of the norm in $X(\mu)$, we obtain $\|f\|_V = \| |f| \| \leq \| |g| \| = \|g\|_V$. \square

4. Some consequences

4.1. Banach rectangular f.s with the subsequence property

Lemma 4.1. *If $X(\mu)$ is a normed rectangular f.s., then $X(\mu)$ has an equivalent strictly rectangular norm.*

Proof. Consider the constant $C > 0$ in the definition of rectangular norm (2.2). We define

$$(4.1) \quad \|f\|_r := \sup \{ \|\chi_A f\| : A \in \Sigma \}, \forall f \in X(\mu).$$

Clearly $\|\cdot\|_r$ is a seminorm. Since

$$(4.2) \quad \|f\| \leq \|f\|_r \leq C\|f\|, \forall f \in X(\mu),$$

it follows that $\|\cdot\|_r$ is an equivalent norm for $X(\mu)$.

Consider now $f \in X(\mu)$ and $A \in \Sigma$. Then

$$\|\chi_B \chi_A f\| = \|\chi_{B \cap A} f\| \leq \|f\|_r, \forall B \in \Sigma.$$

Hence $\|\chi_A f\|_r \leq \|f\|_r$. This shows that the norm $\|\cdot\|_r$ is strictly rectangular. \square

Corollary 4.2. *Let $X(\mu)$ be a Banach space of measurable functions. Then, there is an equivalent norm under which $X(\mu)$ is a Banach f.s. if, and only if, $X(\mu)$ is a Banach rectangular f.s. with the subsequence property.*

Proof. Let us first assume that there is an equivalent norm $\|\cdot\|_B$ in $X(\mu)$ such that $X_B := (X(\mu), \|\cdot\|_B)$ is a Banach f.s. Then Example 2.5 indicates that $X(\mu)$ is a Banach rectangular f.s. and from Corollary 2.11 it follows that $X(\mu)$ has the subsequence property.

The remaining implication follows from Lemma 4.1 and Theorem 3.7. \square

As we will now see, applying Theorem 3.7 we can obtain well known properties of Banach f.s. for Banach rectangular f.s. with the subsequence property.

Definition 4.3. Given a Banach rectangular f.s. $X(\mu)$ with the subsequence property, we denote by $X(\mu)_V$ the Banach f.s. that we obtain when we consider in $X(\mu)$ the norm $\|\cdot\|_V$ in (3.5).

In what follows we assume that $X(\mu)$ and $Y(\mu)$ are Banach rectangular f.s. with the subsequence property. In this context, as in the Banach lattice case, we say that a linear operator $T : X(\mu) \rightarrow Y(\mu)$ is *positive* if $Tf \geq 0$ for every $f \geq 0$. On the other hand, note that by Theorem 3.7, the norms in $X(\mu)$ and $X(\mu)_V$ are equivalent, and $X(\mu)_V$ is indeed a Banach f.s. Consequently, $X(\mu)$ will satisfy also all the properties of $X(\mu)_V$ which only depend on its topology. Next we give a simple example.

Corollary 4.4. i) *If $T : X(\mu) \rightarrow Y(\mu)$ is a positive operator, then T is continuous.*

ii) *If $X(\mu)$ is also a Banach rectangular space under a norm $\|\cdot\|_2$, then $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent norms.*

Proof. i) Clearly $T : X(\mu)_V \rightarrow Y(\mu)_V$ is a positive operator. Being $X(\mu)_V$ and $Y(\mu)_V$ B.f.s., it follows that T is bounded. Since the norm in $X(\mu)_V$ is equivalent to that of $X(\mu)$, and the norm in $Y(\mu)_V$ is equivalent to the one of $Y(\mu)$, this implies that $T : X(\mu) \rightarrow Y(\mu)$ is bounded.

ii) Let $X(\mu)_j$, $j = 1, 2$, be the B.f.s. associated to the Banach rectangular f.s. $X(\mu)$ with the norm $\|\cdot\|_j$. Then the map $I : X(\mu)_1 \rightarrow X(\mu)_2$ given by $I(f) := f$ is a linear isomorphism and both I and its inverse map are positive operators. It follows from i) that both are bounded, and this corresponds to the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. \square

4.2. Normed sequence spaces

We do not know if an arbitrary Banach rectangular f.s. has the subsequence property. However, we will now show that this holds for a rectangular sequence space.

We will say that $X(\mu) = (\Omega, \Sigma, \mu)$ is a *normed sequence space* (Banach sequence space when complete) when $\Omega := \mathbb{N}$, $\Sigma := 2^\Omega$ and μ is a finite measure such that $\mu(\{n\}) > 0, \forall n \in \mathbb{N}$. The specific measure will not be important; we can take, for example, the measure defined by $\mu(A) := \sum_{n \in A} \frac{1}{2^n}$. Then any function $f : \mathbb{N} \rightarrow \mathbb{R}$ is measurable and, since the empty set is the only set in Σ with measure zero, each equivalence class in $L^0(\mu)$ has only one member. So in this context we will identify $L^0(\mu)$ with $L^0(\Sigma)$ and also this last space with the set of real sequences \mathcal{S} . Let $A \subset \mathbb{N}$. Considering the above identification, notice that $\chi_A := \{a_n\}$, where $a_n = 1$ when $n \in A$ and $a_n = 0$ when $n \notin A$.

Let $X(\mu)$ be a normed sequence space. For each $n \in \mathbb{N}$, we define the linear function $\pi_n : X(\mu) \rightarrow \mathbb{R}$ by $\pi_n(f) := f(n)$.

Proposition 4.5. *If $X(\mu)$ is a rectangular sequence space, then each linear functional π_n is continuous. Hence $X(\mu)$ has the subsequence property.*

Proof. Take $C > 0$ as in Definition 2.4. Fix $n \in \mathbb{N}$. If $g(n) = 0 \forall g \in X(\mu)$, the conclusion is clear. So, assume that $g(n) \neq 0$, for some $g \in X(\mu)$. Then $g(n)e_n = g\chi_{\{n\}} \in X(\mu)$. Hence $e_n \in X(\mu)$. Since $\|f(n)e_n\| = \|fe_n\| \leq C\|f\|$, we have

$$|\pi_n(f)| = |f(n)| \leq \frac{C\|f\|}{\|e_n\|}, \forall f \in X(\mu).$$

It follows from the above inequality that $X(\mu)$ has the subsequence property. \square

Using the result above together with Theorem 3.7, we get the following characterization.

Corollary 4.6. *Let $X(\mu) \subset \mathcal{S}$ be a Banach sequence space. Then $X(\mu)$ has an equivalent norm under which it is a Banach f.s. if and only if $X(\mu)$ is a Banach rectangular f.s.*

Finally, let us remark that rectangularity of the norm implies that the canonical sequence defines a basis of the closure of its linear span. Let $X(\mu)$ be a Banach rectangular sequence space and, for simplicity, assume that each canonical sequence e_n belongs to $X(\mu)$ ($e_n(k) := 1$ if $k = n$ and $e_n(k) = 0$ otherwise).

Take $C > 0$ as in Definition 2.4 and consider $1 \leq N < M$ and $A := \{1, \dots, N\}$. Let $a_1, \dots, a_M \in \mathbb{R}$. Taking $s = (a_1, \dots, a_M, \dots)$ we have

$$\left\| \sum_{n=1}^N a_n e_n \right\| = \|\chi_A s\| \leq C \|s\| = C \left\| \sum_{n=1}^M a_n e_n \right\|.$$

By a well known result [3, p. 411], this implies that $\{e_n\}$ is a basic sequence, that is, it is a Schauder basis for the closure of its span, say X_0 .

Proposition 4.7. $\{e_n\}$ is an unconditional basis for X_0 .

Proof. Let $s := \{a_n\} \in \mathcal{S}(X_0)$, so $x := \sum_{n=1}^{\infty} a_n e_n$ converges in X . Take $A \subset \mathbb{N}$. By Theorem 3.7, multiplication by χ_A is a bounded operator on $X(\mu)$. Hence $\chi_A x = \sum_{n=1}^{\infty} \chi_A(n) a_n e_n$ and so $\chi_A s \in S(X_0)$. Since $X(\mu)$ has a rectangular norm, from iii) of Theorem 1.1 we conclude that $\{e_n\}$ is an unconditional basis for X_0 . \square

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