APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 22, no. 1 (2021), 169-181
doi:10.4995/agt.2021.14446

Metric spaces related to Abelian groups

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Communicated by F. Lin

Abstract

When working with a metric space, we are dealing with the additive group $(\mathbb{R}, +)$. *Replacing* $(\mathbb{R}, +)$ *with an Abelian group* $(G, *)$, *offers a new structure of a metric space. We call it a* G-metric space *and the induced topology is called* the G-metric topology. *In this paper, we are studying* G*-metric spaces based on L-groups (i.e., partially ordered groups which are lattices). Some results in* G*-metric spaces are obtained. The* G*-metric topology is defined which is further studied for its topological properties. We prove that if* G *is a densely ordered group or an infinite cyclic group, then every* G*-metric space is Hausdorff. It is shown that if* G *is a Dedekind-complete densely ordered group,* (X, d) *a* G*-metric space,* $A \subseteq X$ *and* d *is bounded, then* $f : X \to G$ *with* $f(x) = d(x, A) := \inf\{d(x, a) : a \in A\}$ *is continuous and further* $x \in cl_X A$ *if and only if* $f(x) = e$ *(the identity element in* G*). Moreover, we show that if* G *is a densely ordered group and further a closed subset of* $\mathbb{R}$, $\mathcal{K}(X)$ *is the family of nonempty compact subsets of* X, $e < g \in G$ *and* d *is bounded, then* $d'(A, B) < g$ *if and only if* $A \subseteq N_d(B, g)$ *and* $B \subseteq N_d(A, g)$, *where* $N_d(A, g) = \{x \in X : d(x, A) < g\}$, $d_B(A) = \sup\{d(a, B) : a \in A\}$ *and* $d'(A, B) = \sup\{d_A(B), d_B(A)\}$.

2010 MSC: *54C40; 06F20; 16H20.*

Keywords: *G-metric space; L-group; Dedekind-complete group; densely ordered group; continuity.*

Received 08 October 2020 – Accepted 29 January 2021

1. Introduction

In this article, a group $(G, *)$ (briefly, G) is an Abelian group and for readability, we use g_1g_2 instead of $g_1 * g_2$. Let X be a set and $\leq$ relation on X, we recall that the pair $(X, \leq)$ is a *partially ordered set* (in brief, a *poset*) if the following conditions hold: $x \leq x$, if $x \leq y$ and $y \leq x$, then $x = y$; if $x \leq y$ and $y \leq z$, then $x \leq z$. In a poset, the symbol $a \vee b$ denotes $\sup\{a, b\}$, i.e., the smallest element c, if one exists, such that $c \geq a$ and $c \geq b$. Likewise, $a \wedge b$ stands for $\inf\{a, b\}$. When both $a \vee b$ and $a \wedge b$ exist, for all $a, b \in A$, then A is called a *lattice*. A subset S is a *sublattice* of A provided that, for all $x, y \in S$, the elements $x \vee y$ and $x \wedge y$ of A belong to S. (Thus, it is not enough that x and y have a supremum and infimum in S.) For instance, $C(X)$, the ring of real-valued continuous functions on the topological space X is a lattice. If $f, g \in C(X)$, then $f \vee g = \frac{f+g+|f-g|}{2} \in C(X)$ (note, $f \wedge g = -(-f \vee -g) = \frac{f+g-|f-g|}{2} \in C(X)$). In fact, $C(X)$ is a sublattice of $\mathbb{R}^X$, the ring of real-valued functions on the set X (note, the partial ordering on $\mathbb{R}^X$ is: $f \leq g$ if and only if $f(x) \leq g(x)$ for all x in X). A poset in which every nonempty subset has both a supremum and an infimum is said to be a *lattice-complete*. For example, $P(X)$, the set of all subsets of X with inclusion is lattice-complete. Union (resp. intersection) of sets is the supremum (resp. the infimum) of them. A *totally (*or *linearly) ordered set* is a poset in which every pair of elements is comparable, i.e., $x \leq y$ or $y \leq x$ for all x and y in X. We use "ordered sets" instead of "totally ordered sets". An ordered set is often referred to as a chain. A lattice need not be an ordered set, necessarily, but the converse is always true. We notice that $C(X)$ and $C_c(X)$, its subalgebra consisting of elements with countable image, are lattices, while they are not ordered sets, also, they are not lattice-complete necessarily. An ordered set is said to be *Dedekind-complete* provided that every nonempty subset with an upper bound has a supremum, or equivalently, every nonempty subset with a lower bound has an infimum. (For example, $\mathbb{R}$, the set of real numbers is Dedekind-complete, but not lattice-complete.) An ordered field $\mathbb{F}$ is said to be *archimedean* if $\mathbb{Z}$, the set of integers is *cofinal*, i.e., for every $x \in \mathbb{F}$, there exists $n \in \mathbb{Z}$ such that $n \geq x$. For instance, $\mathbb{Q}(\sqrt{n}) := \{a + b\sqrt{n} : a, b \in \mathbb{Q}, \sqrt{n} \notin \mathbb{Q}\}$ is an archimedean field.

Theorem 1.1 ([3, Theorem 0.21])**.** *An ordered field is archimedean if and only if it is isomorphic to a subfield of the ordered field* $\mathbb{R}$*.*

A brief outline of this paper is as follows. In Section 2, we introduce the G-metric spaces related to L-groups (i.e., partially ordered groups which are lattices) and study them further. In Section 3, the basic topological properties based on the notion of g-disk are studied. We prove that if G is a densely ordered group or an infinite cyclic group, then every G-metric space is Hausdorff. It is shown that if G is a Dedekind-complete densely ordered group, (X, d) a G-metric space, d is bounded and $A \subseteq X$, then $f : X \to G$ given by $f(x) = d(x, A) := \inf\{d(x, a) : a \in A\}$ is continuous and further $x \in \mathrm{cl}_X A$ if and only if $f(x) = e$ (the identity element of G). Moreover, let

$\mathcal{F}(X)$ be the family of nonempty closed sets in X, $e < g \in G$, $A, B \in \mathcal{F}(X)$ and $d_A(B) = \sup\{d(b, A) : b \in B\}$. Then for the G-metric space $(\mathcal{F}(X), d')$ (note, $d'(A, B) = \sup\{d_A(B), d_B(A)\}$), we have $d'(A, B) \leq g$ if and only if $A \subseteq N_d(B, \bar{g})$ and $B \subseteq N_d(A, \bar{g})$, where $N_d(A, \bar{g}) = \{x \in X : d(x, A) \leq g\}$. Particularly, if G is a densely ordered group and further a closed subset of $\mathbb{R}$, X is a G-metric space and $\mathcal{K}(X)$ is the family of nonempty compact sets in X, then $d'(A, B) < g$ if and only if $A \subseteq N_d(B, g)$ and $B \subseteq N_d(A, g)$, where $N_d(A, g) = \{x \in X : d(x, A) < g\}$.

2. G-metric spaces

Definition 2.1. A group G with a partial ordering relation $\leq$ is called a *partially ordered group* (in brief, a *poset group*) if the binary operation of G preserves the order, i.e.,

$$g_1 \geq g_2 \text{ implies } g_1 g_3 \geq g_2 g_3 \text{ for all } g_1, g_2, g_3 \in G. \tag{R_1}$$

Moreover, if a poset group G is a lattice then G is called an *L-group.*

From the above definition, the following facts are evident: $g_1 \geq g_2$ if and only if $g_1 g_2^{-1} \geq e$; $g \geq e$ if and only if $g^{-1} \leq e$; if $g_1 \geq g_3$ and $g_2 \geq g_4$, then $g_1 g_2 \geq g_3 g_4$. For example, every archimedean field with the addition is an L-group. But $\mathbb{Z}_n$ with the addition of modulo n is not a poset group yet, since this addition does not preserve the order. For an L-group G and $g \in G$, we let $|g| = \sup\{g, g^{-1}\} = g \vee g^{-1} = |g^{-1}|$.

Example 2.2. Consider the group $G := \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$ (k-times) with the usual addition and the identity element $e = (0, 0, \ldots, 0)$. Let $g_1 = (m_1, m_2, \ldots, m_k)$, $g_2 = (n_1, n_2, \ldots, n_k) \in G$. Define

$$g_1 \leq g_2 \text{ if and only if } m_i \leq n_i \text{ for all } i = 1, 2, \ldots, k.$$

We see that $\leq$ is a partial ordering relation on G. Also, the condition (R_1) in the above definition is satisfied, i.e., G is a poset group. Let $z_i = \max\{m_i, n_i\}$ and $z'_i = \min\{m_i, n_i\}$, where $i = 1, 2, \ldots, k$. Let $g_3 = (z_1, z_2, \ldots, z_k)$ and $g_4 = (z'_1, z'_2, \ldots, z'_k)$. Then we obtain $g_1 \vee g_2 = g_3$ and $g_1 \wedge g_2 = g_4$. Hence, G is an L-group.

By an *ordered group*, we mean a poset group which is a totally ordered set by its partial ordering relation. It is clear that an ordered group is an L-group. Finally, by *Dedekind-complete group*, we mean an ordered group which is a Dedekind-complete set with its partial ordering relation, i.e., every nonempty subset with an upper bound has a supremum, or equivalently, every nonempty subset with a lower bound has an infimum. For example, every archimedean field with the addition is a Dedekind-complete group.

Corollary 2.3. *If G is an L-group and $g_1, g_2 \in G$, then $|g_1 g_2| \leq |g_1||g_2|$.*

Proof. We note that $g_1, g_1^{-1} \leq |g_1|$ and $g_2, g_2^{-1} \leq |g_2|$. Definition 2.1 now gives $g_1 g_2 \leq |g_1||g_2|$ and also $g_1^{-1} g_2^{-1} \leq |g_1||g_2|$. So we have that $|g_1 g_2| = \sup\{g_1 g_2, (g_1 g_2)^{-1}\} \leq |g_1 g_2|$, and the result holds. $\square$

Definition 2.4. Let G be a poset group and X a nonempty set. We say the function $d : X \times X \to G$ is a *G-metric* on X, whenever the following conditions hold, for every $x, y, z \in X$.

(i) $d(x, y) \geq e$ (e is the identity element in G),
(ii) $d(x, y) = e$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, y) \leq d(x, z)d(z, y)$ (*triangle inequality*).

The pair (X, d) (briefly, X) is called a *G-metric space*. Evidently, every metric space is a G-metric space, when $(G, *) = (\mathbb{R}, +)$. If all axioms but the second part of Definition 2.4 are satisfied, we call d a *G-pseudometric* and then X a *G-pseudometric space*. Defining $d(x, y) = e$ for all x and y in X, gives a G-pseudometric on X, called the trivial G-pseudometric, in this case, d is a G-metric if and only if X is the singleton set $\{x\}$. Although all the material of this section is developed for G-metric spaces, the basic results remain true for G-pseudometric spaces as well. If (X, d) is a G-metric on X and A is a subset of X, then A inherits a G-metric structure from X in an obvious way, making A a G-metric space.

In the following example, we will present some examples of G-metric spaces.

Before it, let $X = \mathbb{R}^n$, $G_1 = (\mathbb{R}, +), G_2 = ((0, +\infty), .), G_3 = (\mathbb{R} - \{-1\}, *)$ and $G_4 = (\mathbb{Z}_2, \oplus)$, where $+, .$ are usual addition and multiplication, the symbol $\oplus$ is the addition of modulo 2 and $*$ is defined as follows: $x * y = x + y + xy$. In G_3, the identity element is 0 and the inverse of x is $x^{-1} = \frac{-x}{1+x}$. Checking of the associative property of $*$ is easy.
Moreover, let $\varphi_i : G_i \times G_i \to G_i$, where $i = 1, 2, 3, 4$, such that $\varphi_1(x, y) = x - y$, $\varphi_2(x, y) = \frac{x}{y}$, $\varphi_3(x, y) = \frac{-xy}{1+x}$ and $\varphi_4(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$ Then, since each φ_i is continuous, each G_i is a topological group as subspaces of $\mathbb{R}$ with the usual topology.

Example 2.5. Let $X = \mathbb{R}^n$, G_i, where $i = 1, 2, 3, 4$, be as defined in the previous discussion. For $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in X$, $\|x - y\|$ is the usual norm, i.e., $\|x - y\| = \big(\sum_{i=1}^{n}(x_i - y_i)^2\big)^{\frac{1}{2}}$. We claim that each d_i, the functions below, is a G_i-metric and therefore (X, d_i) is a G_i-metric space. We only check that d_2 and d_3 satisfy (iv) of Definition 2.4. Other conditions are routine.

(1) Let $d_1 : X \times X \to G_1$ such that $d_1(x, y) = \|x - y\|$.
(2) Let $d_2 : X \times X \to G_2$ such that $d_2(x, y) = e^{\|x-y\|}$.
(3) Let $d_3 : X \times X \to G_3$ such that $d_3(x, y) = e^{\|x-y\|} - 1$.
(4) Let $d_4 : X \times X \to G_4$ such that $d(x, y) = 0$ if $x = y$; and 1 if $x \neq y$. d_4 is called a *discrete G-metric*.

We notice that the identity elements in G_2 and G_3 are 1 and 0 respectively. Moreover, $d_2(x, y) \geq 1$ and $d_3(x, y) \geq 0$. Now,

$$d_2(x, y) = e^{\|x-y\|} \leq e^{(\|x-z\|+\|z-y\|)} = e^{\|x-z\|}e^{\|z-y\|} = d_2(x, z)d_2(z, y).$$

Also, we have

$$\begin{aligned} d_3(x,y) &= e^{\|x-y\|} - 1 \leq e^{\|x-z\|} e^{\|z-y\|} - 1 \\ &= (e^{\|x-z\|} - 1) + (e^{\|z-y\|} - 1) + (e^{\|x-z\|} - 1)(e^{\|z-y\|} - 1) \\ &= d_3(x,z) + d_3(z,y) + d_3(x,z)d_3(z,y) \\ &= d_3(x,z)d_3(z,y). \end{aligned}$$

So d_2 and d_3 satisfy the triangle inequality of Definition 2.4.

A G-metric d on a set X is called *bounded* if $d(x,y) \leq g_0$, for all $x, y \in X$ and some $g_0 \in G$. Thus, the next result is now immediate.

Corollary 2.6. *Let G be an L-group, d a G-metric on X, $e < g_1$ a fixed element in G and $d_1(x,y) = \inf\{d(x,y), g_1\}$. Then d_1 is a bounded G-metric.*

Lemma 2.7. *Let G be an L-group; A and B are finite subsets of G such that $A \geq e$ (i.e., $a \geq e$, for all $a \in A$) and $B \geq e$. Then*

- (i) *if $A \leq B$ (i.e., for each $a \in A$ there is $b \in B$ such that $a \leq b$) and $e \leq g \in G$, then $\sup(gA) = g \sup A \leq g \sup B = \sup(gB)$, where $gA = \{ga : a \in A\}$.*
- (ii) *if $A \geq B$ (i.e., for each $a \in A$ there is $b \in B$ such that $b \leq a$) and $e \leq g \in G$, then $\inf(gB) = g \inf B \leq g \inf A = \inf(gA)$.*
- (iii) *$\sup(AB) = \sup A \sup B$, and also $\inf(AB) = \inf A \inf B$, where $AB = \{ab : a \in A, b \in B\}$.*

Proof. First, we note that by definition of an L-group, each of the finite sets A, B and AB has a supremum and an infimum in G. The proofs of (i) and (ii) are routine. (iii). Let $\sup A = \alpha$, $\sup B = \beta$ and $\sup(AB) = \gamma$. Since G is an L-group, it is a poset group. So by Definition 2.1, we have $ab \leq \alpha\beta$, for all $a \in A$ and $b \in B$. Evidently, $\gamma \leq \alpha\beta$. Now, we are ready to show that $\gamma = \alpha\beta$. For the reverse inclusion, let $a \in A$ be fixed. Then $ab \leq \gamma$ implies $b \leq a^{-1}\gamma$. Therefore, B is bounded by $a^{-1}\gamma$. So $\beta = \sup B \leq a^{-1}\gamma$, in other words, $a \leq \beta^{-1}\gamma$. Since $a \in A$ is arbitrary, we deduce that A is bounded by $\beta^{-1}\gamma$. Thus, $\alpha \leq \beta^{-1}\gamma$. This yields $\alpha\beta \leq \gamma$, and we are through. The proof of another assertion (infimum) is done similarly. □

Proposition 2.8. *Let G be an ordered group. Then defining*

$$d : G \times G \to G \text{ given by } d(g_1, g_2) = |g_1 g_2^{-1}|,$$

turns G into a G-metric space.

Proof. We claim that d is a G-metric on G. First, we note that since G is an ordered group, it is an L-group and further $g \in G$ gives $g \geq e$ or $g^{-1} \geq e$. So $|g| = |g^{-1}| = \sup\{g, g^{-1}\} = g$ or g^{-1}. Hence, $|g| \geq e$ and therefore conditions (i)-(iii) of Definition 2.4 hold. Moreover, if $g_1, g_2, g_3 \in G$, then Corollary 2.3 implies

$$\begin{aligned} d(g_1, g_2) = |g_1 g_2^{-1}| = |(g_1 g_3^{-1})(g_3 g_2^{-1})| &\leq |g_1 g_3^{-1}||g_3 g_2^{-1}| \\ &= d(g_1, g_3) d(g_3, g_2) \end{aligned}$$

This gives d satisfies the triangle inequality, i.e., it is a G-metric on G and hence G is a G-metric space. $\square$

Corollary 2.9. *Let G be an ordered group, X a nonempty set and $f : X \to G$ a function. Then $d : X \times X \to G$ given by $d(x,y) = |f(x)f^{-1}(y)|$ is a G-pseudometric on X. Moreover, d is a G-metric on X if and only if f is one-one.*

The next proposition generalizes Proposition 2.8.

Proposition 2.10. *Let G be an ordered group. Then each of the following binary operations on G^n (the n product of G), turns it into a G-metric space, where $g = (g_1, g_2, \dots, g_n)$ and $g' = (g'_1, g'_2, \dots, g'_n)$ are arbitrary elements of G^n.*

(i) $d_1 : G^n \times G^n \to G$ *defined by* $d_1(g,g') = |g_1^{-1}g'_1||g_2^{-1}g'_2| \dots |g_n^{-1}g'_n|$.

(ii) $d_2 : G^n \times G^n \to G$ *defined by*

$$d_2(g,g') = \sup\{|g_1^{-1}g'_1|, |g_2^{-1}g'_2|, \dots, |g_n^{-1}g'_n|\}.$$

Proof. We only check that the triangle inequality for d_1 and d_2. Other conditions are routine. (i). Let $g^{''} = (g_1^{''}, g_2^{''}, \dots, g_n^{''}) \in G^n$. Then

$$\begin{aligned}
d_1(g,g^{''}) &= |g_1^{-1}g_1^{''}||g_2^{-1}g_2^{''}| \dots |g_n^{-1}g_n^{''}| \\
&= |(g_1^{-1}g'_1)(g_1'^{-1}g_1^{''})||(g_2^{-1}g'_2)(g_2'^{-1}g_2^{''})| \dots |(g_n^{-1}g'_n)(g_n'^{-1}g_n^{''})| \\
&\le |g_1^{-1}g'_1||g_1'^{-1}g_1^{''}||g_2^{-1}g'_2||g_2'^{-1}g_2^{''}| \dots |g_n^{-1}g'_n||g_n'^{-1}g_n^{''}| \\
&= \big(|g_1^{-1}g'_1|g_2^{-1}g'_2| \dots |g_n^{-1}g'_n|\big)\big(|g_1'^{-1}g_1^{''}||g_2'^{-1}g_2^{''}| \dots |g_n'^{-1}g_n^{''}|\big) \\
&= d_1(g,g')d_1(g',g^{''}).
\end{aligned}$$

Notice that the above inequality is obtained by Corollary 2.3. (ii). Let $g^{''} = (g_1^{''}, g_2^{''}, \dots, g_n^{''}) \in G^n$ and let

$$A = \{|g_1^{-1}g_1^{''}|, |g_2^{-1}g_2^{''}|, \dots, |g_n^{-1}g_n^{''}|\},$$

$$B = \{|g_1^{-1}g'_1||g_1'^{-1}g_1^{''}|, |g_2^{-1}g'_2||g_2'^{-1}g_2^{''}|, \dots, |g_n^{-1}g'_n||g_n'^{-1}g_n^{''}|\},$$

$$\begin{aligned}
B_1 &= \{|g_1^{-1}g'_1|, |g_2^{-1}g'_2|, \dots, |g_n^{-1}g'_n|\}, \text{and} \\
B_2 &= \{|g_1'^{-1}g_1^{''}|, |g_2'^{-1}g_2^{''}|, \dots, |g_n'^{-1}g_n^{''}|\}.
\end{aligned}$$

We notice that G is an L-group. Now, according to Lemma 2.7, we have $A \le B \le B_1B_2$. Therefore,

$$d_2(g,g^{''}) = \sup A \le \sup B \le \sup(B_1B_2) = \sup B_1 \sup B_2 = d_2(g,g')d_2(g',g^{''}),$$

which completes the proof. $\square$

3. Basic topological concepts in G-metric spaces and some related results

We begin with the following definition.

Definition 3.1. Let G be a poset group, (X, d) a G-metric space and x a point of X. Given $e < g \in G$, we let

$$N_d(x, g) = \{y \in X : d(x, y) < g\},$$

and call it the g-disk centered at x. Also, we put $N_d(x, \overline{g}) = \{y \in X, d(x, y) \leq g\}$.

A subset U of X is said to be *open* in X if either $U = \varnothing$ or for every $x \in U$ there is a $g \in G$ such that $N_d(x, g) \subseteq U$. Here, x is called an *interior point* of U. The set of all interior points of U is called the *interior* of U, denoted by U° (or $\mathrm{int}_X U$). Also, a set F is called *closed* if and only if its set-theoretic complement is an open set in X. Evidently, a set F is closed if and if every g-disk centered at x meets F, then $x \in F$.

Corollary 3.2. *Every g-disk $N_d(x, g)$ is an open set in X (and hence $X \setminus N_d(x, g) = \{y \in X : d(x, y) \geq g\}$ is a closed set in X).*

Proof. Let $y \in N_d(x, g)$. Then $g_1 = d(x, y) < g$. We claim that $N_d(y, gg_1^{-1}) \subseteq N_d(x, g)$ (note, $gg_1^{-1} > e$). To see this, assume that $z \in N_d(y, gg_1^{-1})$. Hence, $d(z, x) \leq d(z, y)d(y, x) < gg_1^{-1}g_1 = g$. This yields $z \in N_d(x, g)$, i.e., $N_d(y, gg_1^{-1}) \subseteq N_d(x, g)$, and we are done. □

Definition 3.3. Let X be a G-metric space and $A \subseteq X$. The *closure* of A in X is denoted by $\mathrm{cl}_X A$ (or briefly clA) and defined by the set

$$\mathrm{cl}A = \cap\{F \subseteq X : F \text{ is closed in } X \text{ and } A \subseteq F\}.$$

By the above definition, A is closed if and only if $A = \mathrm{cl}A$.

Corollary 3.4. *Let G be a poset group, (X, d) a G-metric space and $\bar{A} = \{x \in X : N_d(x, g) \cap A \neq \varnothing$ for all $e < g \in G\}$, where $A \subseteq X$. Then*

(1) $\overline{A} = clA$.
(2) *If $x \in X$ and $g \in G$, then $\overline{N_d(x, g)} \subseteq N_d(x, \overline{g})$.*

Proof. (1). Let $x \notin \mathrm{cl}A$. Then $x \notin F$, for some closed set F containing A. Now, since $X \setminus F$ is open, there exists $e < g \in G$, such that $x \in N_d(x, g) \subseteq X \setminus F$. So $x \notin \bar{A}$. Conversely, suppose that $x \notin \bar{A}$. So for some $e < g \in G$, $N_d(x, g) \cap A = \varnothing$. Therefore, the closed set $X \setminus N_d(x, g)$ contains A but not x. This gives $x \notin \mathrm{cl}A$, and we are done. (2). Suppose that $y \notin N_d(x, \overline{g})$. So $d(x, y) > g$ and hence $g_1 = d(x, y)g^{-1} > e$. We claim that $N_d(y, g_1) \cap N_d(x, g) = \varnothing$. Otherwise, for some $z \in N_d(y, g_1) \cap N_d(x, g)$, we have $d(x, y) \leq d(x, z)d(z, y) < gg_1 = d(x, y)$, a contradiction. Hence, $y \notin \overline{N_d(x, g)}$ and we are done. □

Proposition 3.5. *Let G be an ordered group and X a G-metric space. Then the open sets in X have the following properties:*

(i) *X and $\varnothing$ are both open.*

(ii) *Every union of open sets is open.*
(iii) *Every finite intersection of open sets is open.*

Proof. (i) and (ii) are clear. (iii). Let $x \in \bigcap_{i=1}^{n} U_i$, where U_i is an open set in X. Take $g_i \in G$ such that $x \in N_d(x, g_i) \subseteq U_i$. Since G is an ordered group, there exists $g \in G$ such that $g = \inf\{g_i\}_{i=1}^{n}$ (note, the elements g_i form a chain and hence g is one of them). Thus, $x \in N_d(x, g) \subseteq \bigcap_{i=1}^{n} N_d(x, g_i) \subseteq \bigcap_{i=1}^{n} U_i$, which completes the proof. □

By the above proposition, every G-metric d on a set X defines a topology τ_d on X; members of τ_d, or, open subsets of X are unions of g-disks. Clearly, the family of all g-disks is a base for (X, τ_d). We call τ_d *the topology induced by the G-metric d* (or *G-metric topology*).

Remark 3.6. Even if G is a Dedekind-complete group, a countable intersection of open sets in a G-metric space need not be an open set necessarily. To see this, consider $\mathbb{R}$ as a G-metric space, where G is $(\mathbb{R}, +)$ or $((0, \infty), .)$. Also, recall the fact that every point a of $\mathbb{R}$ is a G_δ-set, i.e., $\{a\} = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n})$.

In [2, I.3], an ordered set X is called a *densely ordered set*, if no cut of X is a jump, or equivalently, for every pair x, y of elements of X satisfying $x < y$, there exists a $z \in X$, such that $x < z < y$.

Definition 3.7. An ordered group G is called a *densely ordered group* if it is a densely ordered set with its total ordering relation.

It is clear that densely ordered groups are infinite. For example, every archimedean field like as $\mathbb{R}$ and $\mathbb{Q}(\sqrt{n}) := \{a + b\sqrt{n} : a, b \in \mathbb{Q}, \sqrt{n} \notin \mathbb{Q}\}$ with the addition is a densely ordered group. $\mathbb{Z}$, the group of integers is an ordered group which is not a densely ordered group while $\mathbb{Z}_n$ with the addition of modulo n, is not a poset group yet.

From now on, the group G is assumed to be a densely ordered group.

Proposition 3.8. *Let G be a densely ordered group, (X, d) a G-metric space, $x \in X$ and $g \in G$ be fixed, and $A = \{y \in X : d(x, y) > g\}$. Then A is an open set in X (and hence $N_d(x, \overline{g}) := \{y \in G : d(x, y) \leq g\}$ is closed).*

Proof. Let $y \in A$ be fixed. Then $d(x, y) > g$. We must show that y is an interior point of A. Let $g_1 = d(x, y)g^{-1}$. Then $g_1 > e$ and $d(x, y) = gg_1$. Since G is a densely ordered group, we take $e < g_2 < g_1$ and claim that $N_d(y, g_2)$ is contained in A entirely. To see this, let $z \in N_d(y, g_2)$. Then we have $d(y, z) < g_2$ and so $g_2^{-1} < d^{-1}(y, z)$. Now, the inequality $d(x, y) \leq d(x, z)d(z, y)$ yields

$$g < gg_1g_2^{-1} < d(x, y)d^{-1}(y, z) \leq d(x, z).$$

(Notice that $g < gg_1g_2^{-1}$ if and only if $g_2 < g_1$.) Therefore, $g < d(x, z)$, i.e., y is an interior point of A. So A is an open set in X, and we are done. □

Proposition 3.9. *Let G be an ordered group and (X, d) a G-metric space. Then the following statements hold.*

(i) *If G is a densely ordered group, then X is Hausdorff.*
(ii) *If G is an infinite cyclic group, then X is discrete (and so it is first countable).*

Proof. (i). Let $x, y \in X$ and $d(x, y) = g > e$. By assumption, since G is a densely ordered set, we can take $g_1, g_2 \in G$ such that $e < g_1 < g_2 < g$. Now, we claim that two disks $N_d(x, g_1)$ and $N_d(y, gg_2^{-1})$ are disjoint (note, $gg_2^{-1} > e$). Otherwise, for some $x' \in N_d(x, g_1) \cap N_d(y, gg_2^{-1})$, we have $d(x, x') < g_1$ and $d(x', y) < gg_2^{-1}$. Hence, $g = d(x, y) \leq d(x, x')d(x', y) < g_1gg_2^{-1}$. Therefore, $e < g_1g_2^{-1}$, or equivalently, $g_2 < g_1$, a contradiction. So we are done.

(ii). Let $e < g \in G$ be the generator of G. Then $G = \{g^n : n \in \mathbb{Z}\}$, in fact, we have $G \cong \mathbb{Z}$, the additive group of integers with the generator 1 (or -1). We note that the elements of G form a chain. So we obtain

$$\ldots < g^{-3} < g^{-2} < g^{-1} < e < g < g^2 < g^3 < \ldots$$

Therefore, for each $x \in X$, $N_d(x, g) = \{y \in X : d(x, y) < g\} = \{y \in X : d(x, y) = e\} = \{x\}$. This yields X is discrete (note, in this case G is not a densely ordered group). □

Remark 3.10. By Proposition 3.9(ii), if G is an infinite cyclic group then X is first countable. But the converse of that result may be false, since every metric space is first countable, whereas the additive group $(\mathbb{R}, +)$ is not even countably generated.

In general, the converse of the above proposition does not need to be true. In the next example, we give examples of Hausdorff G-metric spaces such that the group G is neither a densely ordered group nor an infinite cyclic group.

Example 3.11.
(i) Let $d_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $d_1(m, n) = |m - n|$. Then, since $N_{d_1}(m, 1) = \{m\}$, we obtain $\mathbb{Z}$ is a discrete $\mathbb{Z}$-metric space. So it is Hausdorff, whereas $\mathbb{Z}$ is not a densely ordered group. But if $\mathbb{Z}$ is considered as a $\mathbb{Q}$-metric space with the same definition, $d_1(m, n) = |m - n|$, it is a discrete $\mathbb{Q}$-metric space while $\mathbb{Q}$ is a densely ordered group.
(ii) Let $G := \mathbb{Z} \times \mathbb{Z}$ with the identity element $e = (0, 0)$. Define $d_2 : \mathbb{Z} \times \mathbb{Z} \to G$ with $d_2(m, n) = (|m - n|, |m - n|)$. By example 2.2, G is an L-group. It is easy to see that d_2 is a G-metric on $\mathbb{Z}$. Now, let $g = (1, 1)$. Then
$$N_{d_2}(m, g) = \{n \in \mathbb{Z} : d_2(m, n) < g\} = \{m\}.$$
This yields $\mathbb{Z}$ is a discrete G-metric space, whereas $G = \mathbb{Z} \times \mathbb{Z}$ is not a cyclic group (note, it is a finitely generated group which is generated by the set $\{(0, 1), (1, 0)\}$).

Definition 3.12. If (X, d_X) (resp. (Y, d_Y)) is a G_1- (resp. G_2-) metric space, a function $f : X \to Y$ is called *continuous* at $x_0 \in X$ if and only if for each $e_2 < g_2 \in G_2$ there is some $e_1 < g_1 \in G_1$ such that $d_Y(f(x_0), f(y)) < g_2$, whenever $d_X(x_0, y) < g_1$. f is called *continuous* on X, if it is continuous at every $x \in X$.

A simple translation of the above definition is:

Corollary 3.13. *A function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for each g_2-disk $N_{d_Y}(f(x_0), g_2)$ centered at $f(x_0)$, there is some g_1-disk $N_{d_X}(x_0, g_1)$ centered at x_0, such that $f(N_{d_X}(x_0, g_1)) \subseteq N_{d_Y}(f(x_0), g_2)$.*

Theorem 3.14. *If (X, d_X) and (Y, d_Y) are G_1- and G_2-metric spaces respectively, a function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for each open set V of Y containing $f(x_0)$, there exists an open set U of X containing x_0 such that $f(U) \subseteq V$.*

Proof. $(\Rightarrow)$: Suppose that f is continuous at x_0 and V is an open set in Y containing $f(x_0)$. By definition of open sets, there is $g_2 \in G_2$ such that $f(x_0) \in N_{d_Y}(f(x_0), g_2) \subseteq V$. By Corollary 3.13, there exists a g_1-disk $N_{d_X}(x_0, g_1)$ centered at x_0 such that $f(N_{d_X}(x_0, g_1)) \subseteq N_{d_Y}(f(x_0), g_2) \subseteq V$, where $g_1 \in G_1$. It now suffices to choose $U = N_{d_X}(x_0, g_1)$.

$(\Leftarrow)$: Consider $e_2 < g_2 \in G_2$ and $N_{d_Y}(f(x_0), g_2)$ as an open set in Y containing $f(x_0)$. By hypothesis, there exists an open set U in X containing x_0 such that $f(U) \subseteq N_{d_Y}(f(x_0), g_2)$. Also, we can take $e_1 < g_1 \in G_1$ such that $N_{d_X}(x_0, g_1) \subseteq U$. So $f(N_{d_X}(x_0, g_1)) \subseteq f(U) \subseteq N_{d_Y}(f(x_0), g_2)$, and we are done. □

The following lemma is the counterpart of Lemma 2.7 for a Dedekind-complete group G. The only difference is that there A and B were finite subsets of G but here these sets must be bounded.

Lemma 3.15. *Let G be a Dedekind-complete group and; A and B are bounded subsets of G such that $A \geq e$ (i.e., $a \geq e$, for all $a \in A$) and $B \geq e$. Then*

(i) *if $A \leq B$ (i.e., for each $a \in A$ there is $b \in B$ such that $a \leq b$) and $e \leq g \in G$, then $\sup(gA) = g \sup A \leq g \sup B = \sup(gB)$, where $gA = \{ga : a \in A\}$.*

(ii) *if $A \geq B$ (i.e., for each $a \in A$ there is $b \in B$ such that $b \leq a$) and $e \leq g \in G$, then $\inf(gB) = g \inf B \leq g \inf A = \inf(gA)$.*

(iii) *$\sup(AB) = \sup A \sup B$, and also $\inf(AB) = \inf A \inf B$, where $AB = \{ab : a \in A, b \in B\}$.*

In the remainder of this article, G is assumed to be a Dedekind-complete densely ordered group (i.e., a densely ordered group in which every bounded nonempty subset has a supremum and an infimum in G), (X, d) a G-metric space, and d is bounded. The *distance* of a point x to a set $A \, (\subseteq X)$ is defined by $d(x, A) = \inf\{d(x, a) : a \in A\}$, if $A \neq \varnothing$, and $d(x, \varnothing) = e$.

Theorem 3.16.

(i) *The mapping $f : X \to G$ defined by $f(x) = d(x, A)$ is continuous.*

(ii) *$x \in cl_X A$ if and only if $f(x) = d(x, A) = e$, in fact, $cl_X A = f^{-1}(e)$.*

Proof. (i). First, by Proposition 2.8, we have (G, d') is a G-metric space, where $d'(g_1, g_2) = |g_1 g_2^{-1}|$. Let $x_0 \in X, g_0 \in G$ and $N_{d'}(f(x_0), g_0)$ be an open set containing $f(x_0)$. Then

$$d(x_0, a) \leq d(x_0, x)d(x, a), \text{ and } d(x, a) \leq d(x, x_0)d(x_0, a). \tag{R_2}$$

Now, if we let $G_1 = \{d(x_0, a) : a \in A\}$ and $G_2 = \{d(x_0, x)d(x, a) : a \in A\}$, then G_1 and G_2 are two subsets of G with the same cardinality and $G_2 \geq G_1$. By Lemma 3.15 (ii), we have $\inf_{a\in A} G_1 \leq \inf_{a\in A} G_2$. In other words, taking infimum on both sides of each of the inequalities in (R_2) with respect to $a \in A$, we obtain

$$\inf_{a\in A} d(x_0, a) \leq d(x_0, x) \inf_{a\in A} d(x, a), \text{and } \inf_{a\in A} d(x, a) \leq d(x, x_0) \inf_{a\in A} d(x_0, a).$$

Thus, $f(x_0) \leq d(x, x_0)f(x)$ and $f(x) \leq d(x, x_0)f(x_0)$. Hence, $f(x_0)f^{-1}(x) \leq d(x, x_0)$ and also $f(x)f^{-1}(x_0) \leq d(x, x_0)$, i.e., $d(x, x_0)$ is a common upper bound for $f(x_0)f^{-1}(x)$ and $f^{-1}(x_0)f(x)$. Therefore,

$$d'(f(x_0), f(x)) = |f(x)f^{-1}(x_0)| = \sup\{f(x_0)f^{-1}(x), f^{-1}(x_0)f(x)\} \leq d(x, x_0).$$

Now, for the g_0-disk $N_d(x_0, g_0)$ we have $f(N_d(x_0, g_0)) \subseteq N_{d'}(f(x_0), g_0)$, and we are through.

(ii). *Necessity:* First, we note that by Corollary 3.4, $\mathrm{cl}A = \bar{A} = \{x \in X : N_d(x, g) \cap A \neq \varnothing, \text{for all } g > e\}$. If $d(x, A) = g > e$ then $d(x, a) \geq g > e$, for all $a \in A$. By assumption, since G is a densely ordered group, we can take $g_1 \in G$ such that $g > g_1 > e$. Now, we observe that $N_d(x, g_1) \cap A = \varnothing$. Hence, $x \notin \overline{A}$.

Sufficiency: Let $x \notin \overline{A}$. Then $N_d(x, g) \cap A = \varnothing$, for some $e < g \in G$. Hence, $d(x, a) \geq g$, for all $a \in A$. Therefore, $d(x, A) \geq g > e$. So $d(x, A) \neq e$, and we are done. □

Theorem 3.17. *Let G be a Dedekind-complete densely ordered group, (X, d) a G-metric space, d is bounded, $g \in G$, and let $\mathcal{F}(X)$ be the family of all nonempty closed subsets of X. For $A, B \in \mathcal{F}(X)$ define*

$$d_B(A) = \sup\{d(a, B) : a \in A\}, \textit{and } d'(A, B) = \sup\{d_A(B), d_B(A)\}.$$

Then the following statements hold.

(1) *d' is a G-metric on $\mathcal{F}(X)$. We call it the Hausdorff G-metric on $\mathcal{F}(X)$.*
(2) *$d'(A, B) \leq g$ if and only if $A \subseteq N_d(B, \bar{g})$ and $B \subseteq N_d(A, \bar{g})$, where $N_d(A, \bar{g}) = \{x \in X : d(x, A) \leq g\}$.*

Proof. (1). (i) and (iii) of Definition 2.4 are evident. Let $d'(A, B) = e$. Then $d_B(A) = e = d_A(B)$. So $d(a, B) = e$ for all $a \in A$. By Theorem 3.16 (ii), $a \in \mathrm{cl}B = B$, i,e., $A \subseteq B$. Similarly, $B \subseteq A$. This proves (ii) of Definition 2.4. For the proof of triangle inequality, let $A, B, C \in \mathcal{F}(X)$ and $a \in A, b \in B, c \in C$. We notice that $d(a, B) \leq d(a, b)$ and $d(b, C) \leq d_C(B)$. Thus,

$$d(a, B) \leq d(a, b) \leq d(a, c)d(c, b).$$

Taking infimum on both sides of the above inequality with respect to $c \in C$ plus Lemma 3.15 yield

$$d(a,B) \leq \inf_{c\in C}\{d(a,c)d(c,b)\} = \inf_{c\in C} d(a,c) \inf_{c\in C} d(c,b).$$

Therefore, $d(a,B) \leq d(a,C)d(b,C)$. Since $d(b,C) \leq d_C(B)$, we have $d(a,B) \leq d(a,C)d_C(B)$. Taking supremum on both sides of the latter inequality with respect to $a \in A$, we obtain

$$d_B(A) \leq d_C(A)d_C(B). \qquad (R_3)$$

On the other hand, taking infimum over $c \in C$ on both sides of the inequalities $d(b,A) \leq d(a,b) \leq d(a,c)d(c,b)$ we obtain $d(b,A) \leq d(a,C)d(b,C)$ (Lemma 3.15). Furthermore, $d(a,C) \leq d_C(A)$ gives $d(b,A) \leq d_C(A)d(b,C)$. Now, take supremum on both sides of the latter inequality respect to $b \in B$. Thus,

$$d_A(B) \leq d_C(A)d_C(B). \qquad (R_4)$$

Combining (R_3) and (R_4) we get

$$d'(A,B) \leq d_C(A)d_C(B) \leq d'(A,C)d'(C,B).$$

Hence, d' satisfies (iv) of Definition 2.4, and we are done.

(2). $(\Rightarrow)$: Let $d'(A,B) \leq g$. Then $d_B(A) \leq g$ and $d_A(B) \leq g$. Hence, $d(a,B) \leq g$, for all a in A. So $A \subseteq N_d(B,\bar{g})$. Similarly, $B \subseteq N_d(A,\bar{g})$.

$(\Leftarrow)$: Since $A \subseteq N_d(B,\bar{g})$, it gives $d(a,B) \leq g$, for all a in A, and therefore $d_B(A) = \sup_{a\in A} d(a,B) \leq g$. The assertion $d_A(B) \leq g$ is deduced similarly. So $d'(A,B) \leq g$, and we are through. □

Corollary 3.18. *Let G be a densely ordered group and further a closed subset of $\mathbb{R}$, $\mathcal{K}(X)$ the family of nonempty compact subsets of X and $A, B \in \mathcal{K}(X)$ such that X, d, g, d_A and d' be as defined in Theorem 3.17. Then $d'(A,B) < g$ if and only if $A \subseteq N_d(B,g)$ and $B \subseteq N_d(A,g)$, where $N_d(A,g) = \{x \in X : d(x,A) < g\}$.*

Proof. We first recall the fact that a nonempty subset of $\mathbb{R}$ has the least-upper-bound property (equivalently, the greatest-lower-bound property) if and only if it is closed in $\mathbb{R}$. So G has the least-upper-bound property and hence it is a Dedekind-complete densely ordered group. Moreover, by Proposition 3.9, X is Hausdorff and therefore every compact set in X is closed. Thus, the conditions of Theorem 3.17 are satisfied. The necessary condition is obvious. To prove the sufficiency, let us define

$$f_1, f_2 : X \to G \text{ with } f_1(x) = d(x,A) \text{ and } f_2(x) = d(x,B).$$

Now, since A and B are compact subsets of X and further; f_1 and f_2 are continuous functions on X (Theorem 3.16), $f_1(B)$ and $f_2(A)$ are compact sets in G (note, since G is closed, f_1 and f_2 are well defined). Therefore, $\sup f_1(B) \in f_1(B)$ and also $\sup f_2(A) \in f_2(A)$. So we have

$$d_A(B) = \sup f_1(B) = f_1(b_1) = d(b_1,A), \text{ for some } b_1 \in B,$$

and also

$$d_B(A) = \sup f_2(A) = f_2(a_2) = d(a_2,B), \text{ for some } a_2 \in A.$$

By assumption, we now get $d_A(B) < g$ and $d_B(A) < g$. Hence, $d'(A, B) < g$, and we are through. $\square$

Acknowledgements. *The authors are grateful to the referee for providing helpful comments and recommendations to improve the quality of the paper.*

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