On the formal foundations of cash management systems

Francisco Salas-Molina · Juan A. Rodríguez-Aguilar · David Pla-Santamaria · Ana García-Bernabeu

Abstract Cash management aims to find a balance between what is held in cash and what is allocated in other investments in exchange for a given return. Dealing with cash management systems with multiple accounts and different links between is a complex task. Current cash management models provide analytic solutions without exploring the underlying structure of accounts and its main properties. There is a need for a formal definition of cash management systems. In this work, we introduce a formal approach to manage cash with multiple accounts based on graph theory. Our approach allows a formal reasoning on the relation between accounts in cash management systems. A critical part of this formal reasoning is the characterization of desirable and non-desirable cash management policies. Novel theoretical results guide cash managers in the analysis of complex cash management systems.

Keywords: Finance; graph theory; formal reasoning; multiple accounts.

Mathematics Subject Classification: 90B10.

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1 Introduction

Companies and many other organizations commonly use a high number of accounts to manage cash. In addition to regular bank accounts, idle cash balances can be allocated in other short-term investments in exchange for a given return. Large corporations with a hierarchical structure require cash management systems designed to deal with complex relations between multiple assets, namely, either bank accounts or other investments. Furthermore, different assets imply different features in terms of expected returns, liquidity, holding and transaction costs. As a result, determining the best set of transactions between assets within a given period of time, which is called a policy, is by no means straightforward.

Most previous works in the cash management literature (Gregory, 1976; Srinivasan and Kim, 1986; da Costa Moraes et al., 2015) limit the analysis to the common two-assets framework with a single bank account and another short-term investment. Baccarin (2009) described a multidimensional model under continuous fluctuations of cash balances given by homogeneous diffusion processes. However, the assumption of a continuous time diffusion cash flow process complicates the extension of the analysis to more complex cash management systems. Recently, Salas-Molina (2017) also proposed a multiobjective cost-risk cash management model for systems with multiple accounts.

In most cash management models, the particular relationship between accounts is neglected avoiding a complete understanding of particular cash movements between accounts. A few cash management works attempted to analyze the structure of cash management systems were related to graph theory. However, the use of graphs was limited to basic representations of assets and money transfers without exploiting the mathematical background of graph theory. Golden et al. (1979) were the first to propose graphical models in cash management that were later used by de Avila Pacheco and Morabito (2011) and Righetto et al. (2016). In these works, the evolution over time of systems was represented by horizontally replicating nodes, hence complicating the visualization when multiple accounts and long planning horizons are considered.

In this work, we propose a new formal representation of cash management systems based on graph theory (Bondy and Murty, 1976; Chartrand and Oellermann, 1993; Valiente, 2013). We connect the most relevant graph theory tools to the main requirements of cash managers. We guide them in the analysis of complex structures of cash management systems. When the number of assets under analysis is large, when the links established between alternative assets follow a complex structure, cash management cannot be performed by intuition. Formal architectures enrich the analysis of cash management systems by providing:

1. Visualization power to help cash managers understand the underlying structure of cash management systems.
2. Computational power to elicit the best feasible policies that are coherent with the structure of cash management systems.

Our formal approach ensures both visual and computational capabilities. We rely on graph theory to guarantee the previous requirements visual and to provide the formal foundations of cash management decision-making. First, we introduce a formal definition of cash management systems based on graphs. Second, we highlight the main structural features that most common cash management systems present. This formal reasoning on cash management systems, allow us to derive novel theoretical results on the necessary conditions for rational or non-trivial policies. More precisely, we show that previous results on cash management literature within the common two-assets framework can be generalized to multiple assets.

Summarizing, this paper reveals the ability of formal architectures to provide further insights on cash management by means of two main contributions:

1. We introduce formal systems in cash management as a first step for further development of models and results.
2. We show that important features of policies derive from the formal definition of cash management systems.

In what follows, we describe the structure of this work. In Section 2, we connect graph theory and cash management by means of a number some useful definitions. In Section 3, we formally introduce formal cash management systems. In Section 4, we present our main theoretical results. Finally, in Section 5, we conclude providing future interesting lines of work.

2 Graph theory background

In this section, we introduce concepts on graph theory that we later connect to basic functional elements in cash management such as accounts and transactions. For an introduction to graph theory, see e.g. Bondy and Murty (1976); Chartrand and Oellermann (1993) and Bollobás (2013). We later describe how common structures in cash management can be described in terms of graphs.

Definition 1 (Graph) A graph $G = (\mathcal{M}, \mathcal{N})$ is a non-empty finite set $\mathcal{M}$ of m nodes and a finite set $\mathcal{N} \subseteq \mathcal{M} \times \mathcal{M}$ of n arcs or edges.

Let us map the elements of set $\mathcal{M} = \{1, 2, \ldots, m\}$ to the accounts (or assets) that a cash manager wants to manage including regular accounts and other investments. Consider that $\mathcal{N} = \{x_1, x_2, \ldots, x_n\}$ is a set of n possible control actions $x_j = (v, w)$, with $j = 1, 2, \ldots, n$, to transfer cash from account v to account w. Both sets $\mathcal{M}$ and $\mathcal{N}$ are enough to describe the structure of cash management systems by means of a graph. As an example, we represent in Figure 1 the common two-assets framework described in most cash management models Gormley and Meade (2007); Salas-Molina et al. (2018). In this case, the elements of set $\mathcal{M} = \{1, 2\}$ are accounts represented by circles, and the elements of set $\mathcal{N} = \{x_1, x_2\}$ are transactions between accounts represented by directed links.

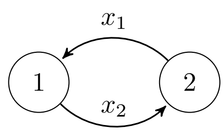

Fig. 1: The common two-assets framework.

In Figure 1, the transaction $x_2 = (1, 2)$ is a directed arc with respect to accounts 1 and 2 where 1 is the origin of funds and 2 is the destination of transferred funds. Note that we are dealing with a directed graph since the existence of transaction (v, w) does not necessarily imply the existence of transaction (w, v). This feature is quite common in cash management practice when a transaction is allowed in one direction but not in the opposite one. For instance, cash managers can transfer cash to their suppliers but they are not authorized to draw cash from their suppliers. When transferring cash is allowed in both directions, we can represent transactions as undirected arcs as shown in Figure 2.

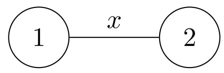

Fig. 2: An undirected system equivalent to the one in Figure 1.

Definition 2 (Undirected graph) A graph $G = (\mathcal{M}, \mathcal{N})$ is undirected when $x_i = (v, w) \in \mathcal{N}$ implies that $x_j = (w, v) \in \mathcal{N}$ for all $v, w \in \mathcal{M}$.

A common situation in many companies is that transactions are allowed between any pair of accounts in the cash management system. This case can be represente by means of a complete graph as shown in Figure 3.

Definition 3 (Complete graph) Un undirected graph $G = (\mathcal{M}, \mathcal{N})$ is complete if for all $v, w \in \mathcal{M}$ with $v \neq w$, with $(v, w) \in \mathcal{N}$.

Hierarchical structures are also quite common in cash management, specially in big corporations. We can represent this kind of cash management systems by means of a particular type of graph called trees. In order to introduce trees, we have to define some related concepts.

Definition 4 (Path) A path from node v_i to node v_j is a sequence of nodes $[v_i, v_{i+1}, \ldots, v_{j-1}, v_j]$ such that there is an arc $x_k = (v_{k-1}, v_k)$ for $k = i + 1, \ldots, j$.

Definition 5 (Cycle) A cycle is a closed path with no repeated nodes except for the first node which is the beginning and the end of the cycle.

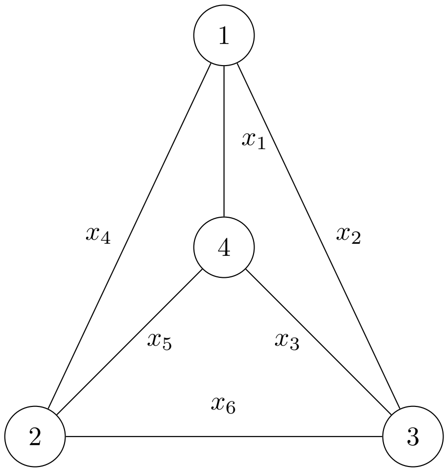

Fig. 3: A complete cash management system.

Definition 6 (Tree) A tree is a graph $G = (\mathcal{M}, \mathcal{N})$ in which for each pair of nodes $v, w \in \mathcal{M}$ there is a path between v and w with no cycles.

The graph in Figure 4 may represent a hierarchical cash management system for a group of companies in which account 1 aggregates funds from accounts 2 and 3. Account 3 aggregates accounts 4 and 5 for two related companies which, in turn, operate through accounts 6, 7 and 8, on the one hand, and 9 and 10, on the other hand.

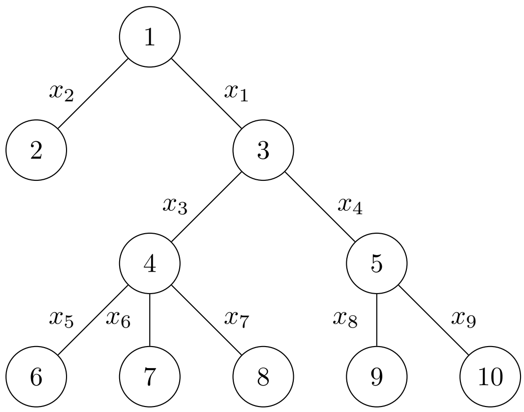

Fig. 4: A hierarchical cash management system.

A graph is a visual model that represents a real-world situation to facilitate its understanding. However, to fully characterize this graphical model, we need some additional algebraic tools such as incident matrices.

Definition 7 (Directed incidence matrix) Given a graph $(\mathcal{M},\mathcal{N})$ with m nodes and n arcs, its directed incidence matrix A is an $m \times n$ matrix in which element $a_{ij} = 1$ if arc $x_j = (v_k, v_i)$ ends in node v_i, $a_{ij} = -1$ if arc $x_j = (v_i, v_k)$ begins in node v_i, and $a_{ij} = 0$ otherwise.

To illustrate the previous definition, let us consider a cash management system with three accounts and four transactions as shown in Figure 5. The directed incidence matrix of this system is:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{1}$$

Directed matrix A of dimension 3×4 shows that transactions x_1 and x_3 add cash to account 1 while transactions x_2 and x_4 remove cash from account 1. An interesting characteristic of a directed matrix is that the sum of each column is necessarily zero since, by definition, a directed transaction (an arc), transfers cash from a single account to another account. This matrix is useful because it connect the visual model (the graph) with any possible mathematical model, hence allowing a quantitative analysis of cash management systems.

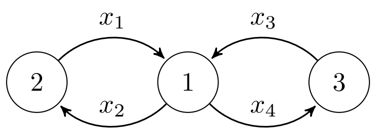

Fig. 5: A cash management systems with three accounts.

3 A formal definition of cash management systems

In this section, we introduce a formal definition of cash management systems based on graph theory as a sound visual and mathematical support. Let us begin by highlighting that cash management systems are not isolated. Regular bank accounts are used to receive payments from customers and to send payments to suppliers. We could use an additional node to represent each debtor or creditor of a company, or even more synthetically, a node representing all debtors and and another node representing all creditors. However, we can drastically simplify the visualization by considering net cash flows on accounts as the difference between inflows and outflows. Given a graph $G = (\mathcal{M},\mathcal{N})$, we propose to represent external net cash flows as an arc $f(v) = (w, v)$ with $w \in \mathcal{P}$ and $v \in \mathcal{M}$, where $\mathcal{P}$ is a virtual set summarizing all the debtors and creditors linked to a given account such that $\mathcal{P} \cap \mathcal{M} = \emptyset$. As a result, a cash management system is built through:

- A set $\mathcal{M}$ of accounts.

– A set $\mathcal{N} \subseteq \mathcal{M} \times \mathcal{M}$ of possible transactions between accounts.
– A virtual set $\mathcal{P}$ with debtors and creditors for accounts in $\mathcal{M}$.
– A set $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{M}$ of external net cash flows.

This approach leads us to define a special type of graph representing the relationship of a cash management system with its environment by means of a set of external net cash flows as shown in Figure 6.

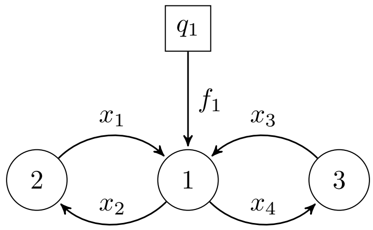

Fig. 6: A cash management systems with external net cash flows.

Definition 8 (Cash management system) A cash management system is a tuple $G = (\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{F})$ where $\mathcal{M}$ is a set of m accounts, $\mathcal{N}$ is a set of n transactions, $\mathcal{P}$ is a virtual set with debtors and creditors and $\mathcal{F}$ is a set of m possible external net cash flows from elements in $\mathcal{P}$ to each account in $\mathcal{M}$.

By convention, we assume that there is no cycle between the elements of $\mathcal{P}$ and $\mathcal{M}$, by restricting $f_i \in \mathcal{F}$ to be of the form $f_i = (w, v)$ with $v \in \mathcal{M}$ and $w \in \mathcal{P}$. Note also that an alternative approach would be considering arc $f(v) \in \mathcal{N}$ described by $f(v) = (v, v)$ with $v \in \mathcal{M}$ as external net cash flows without requiring an additional set $\mathcal{F}$. However, the existence of $\mathcal{F}$ presents the advantage of keeping $\mathcal{N}$ as a set of possible transactions to control balances, since external net cash flows are usually out of the control of cash managers.

We mentioned in Section 2 that the two-assets framework depicted in Figure 1 summarizes all regular accounts in account 1 and all other investments in account 2. We can reasonably assume that only regular accounts are affected by external cash flows, since the purpose of investment accounts is not accepting payments from debtors nor sending payments to creditors. Graphically, this common situation is equivalent to remove external arcs as in accounts 2 and 3 of Figure 6. Thus, although we define $\mathcal{F}$ as an m-dimensional set, $\mathcal{F}$ will frequently be an m_1-dimensional set according to:

– $\mathcal{M}_1 \subseteq \mathcal{M}$ with $m_1 = |\mathcal{M}_1|$ regular accounts.
– $\mathcal{M}_2 \subseteq \mathcal{M}$ with $m_2 = |\mathcal{M}_2|$ investments accounts.
– $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, with $|\mathcal{M}| = m_1 + m_2$.
– $f_i = 0 \;\; \forall \;\; v_i \in M_2$.

As a result, bipartite graphs become also useful in cash management.

Definition 9 (Bipartite cash management system) A bipartite cash management system is a tuple $G = (\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{F})$ where $\mathcal{M}$ can be divided in two disjoint subsets $\mathcal{M}_1$ and $\mathcal{M}_2$, with $m_1 = |\mathcal{M}_1|$ regular accounts and $m_2 = |\mathcal{M}_2|$ investments accounts, where for all $(u, v) \in \mathcal{N}$, $u \in \mathcal{M}_1$ and $v \in \mathcal{M}_2$, or $v \in \mathcal{M}_1$ and $u \in \mathcal{M}_2$, and where $\mathcal{F}$ is and m_1-dimensional set of external net cash flows for each account in set $\mathcal{M}_1$.

An example of a bipartite cash management system is shown in Figure 7, with three regular accounts ($\mathcal{M}_1 = \{1, 2, 3\}$) and two available investments accounts ($\mathcal{M}_2 = \{4, 5\}$). In this case, external net cash flows affect only accounts in $\mathcal{M}_1$.

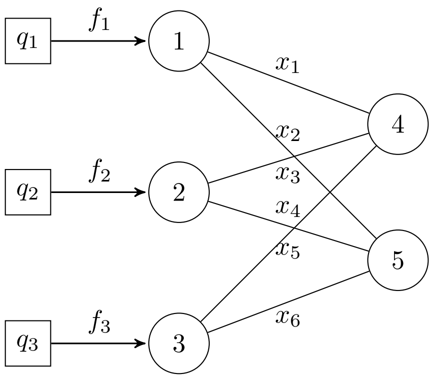

Fig. 7: Bipartite cash management system.

Once we have formally defined cash management systems by relying on graph theory, we are in a position to characterize them by means of a number of useful variables. The most important state variables are available cash balances indexed over time by $t : \{1, 2, \ldots, T\}$, where T is the planning horizon.

Definition 10 (Cash balance) Given an account $v \in \mathcal{M}$, cash balance $b_t(v) \in \mathbb{R}$ represents the amount of available cash in account v at time step t. Given a cash management system $G = (\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{F})$, vector $\boldsymbol{b}_t(G)$ of dimension $m \times 1$ contains cash balances $b(v)$ for each account v.

Cash managers can control cash balances by mapping each transaction between accounts to transferred amounts.

Definition 11 (Control action) Given a transaction $x = (v, w) \in \mathcal{N}$, a transfer or control action $u_t(v, w)$ at time t is a map $\mathcal{N} \times T \rightarrow \mathbb{R}_{\geq 0}$ between the transaction and the amount of cash transferred between accounts v and w. Given a cash management system $G = (\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{F})$, vector $\boldsymbol{u}_t(G)$ of dimension $n \times 1$ contains all control actions at time t.

A basic condition for feasibility of control actions is that the difference between outflows and inflows must not exceed the previous cash balance for each account:

$$\sum_{w \in M} u_t(v,w) - \sum_{w \in M} u_t(w,v) - f_t(v) \leq b_{t-1}(v) \quad \forall v \in M, \quad t = 1, 2, \dots T. \quad (2)$$

Equation (2) is equivalent to non-negativity of cash balances:

$$b_t(v) \geq 0 \quad \forall v \in M, \quad t = 1, 2, \dots T. \quad (3)$$

Indeed, from its initial state $\boldsymbol{b}_0$, we can represent the dynamics of cash management systems according to a linear law of motion. At the end of each time step, cash balances $\boldsymbol{b}_t$ are updated after control actions $\boldsymbol{u}_t$ and external cash flows $\boldsymbol{f}_t$ according to the following equation:

$$\boldsymbol{b}_{t+1} = \boldsymbol{b}_t + A\boldsymbol{u}_t + \boldsymbol{f}_t \quad (4)$$

where A is a directed incidence $m \times n$ matrix as in equation (1).

On the performance side, cash management systems are subject to returns on cash balances and costs associated to control decisions (transfers).

Definition 12 (Return on holdings) Given a cash balance vector $\boldsymbol{b}_t(G)$ at time t, the returns derived from holdings in a cash management system is a function $h(\boldsymbol{b}_t(G)) : \mathbb{R}^m \rightarrow \mathbb{R}$ that maps balances to the total amount obtained (or paid).

Returns on cash management systems are usually linear and computed as follows:

$$h(\boldsymbol{b}_t(G)) = \boldsymbol{r}' \cdot \boldsymbol{b}_t(G) \quad (5)$$

where $\boldsymbol{r}$ is an $m \times 1$ vector with constant returns $r_i \in \mathbb{R}$, with $i = 1, \dots, m$, and the prime symbol denotes transposition. However, more general return functions can be considered through piecewise linear return functions of the form:

$$h(\boldsymbol{b}_t(G)) = \boldsymbol{r}(\boldsymbol{b}_t)' \cdot \boldsymbol{b}_t(G) \quad (6)$$

where each element r_i of vector $\boldsymbol{r}(\boldsymbol{b}_t)$ depends on the actual balance $\boldsymbol{b}_t$ and is computed as follows:

$$r_i = \begin{cases} r_1 \text{ if } b_{it} < q_1, \\ r_2 \text{ if } q_1 \leq b_{it} < q_2, \\ \vdots \\ r_K \text{ if } q_{K-1} \leq b_{it} < q_K, \\ r_{K+1} \text{ if } b_{it} \geq q_K \end{cases} \quad (7)$$

for $K+1$ different intervals of balances b_{it} in vector $\boldsymbol{b}_t$ established by thresholds $q_k \in \mathbb{R}$ for each account i at time step t and with $k = 1, 2, \dots, K$. In the common situation of negative balances charged with penalty costs and linear returns for positive balances, we are dealing with a particular case of equation (7) with two intervals:

$$r_i = \begin{cases} r_1 \text{ if } b_{it} < 0, \\ r_2 \text{ if } b_{it} \geq 0. \end{cases} \tag{8}$$

Note that if $r_i \cdot b_{it} < 0$, the return obtained is not a benefit but a cost.

Definition 13 (Transfer cost) Given a vector $\boldsymbol{u_t}(G)$ with transfers between accounts in cash management system G at time t, the cost of controlling the system is a function $c(\boldsymbol{u}_t(G)) : \mathbb{R}^n \rightarrow \mathbb{R}$ that maps transfers to the total amount charged by banks.

A typical control cost function charges transfers with a fixed and a variable cost:

$$c(\boldsymbol{u}_t(G)) = \boldsymbol{\gamma_0}' \cdot \boldsymbol{z}_t(G) + \boldsymbol{\gamma_1}' \cdot \boldsymbol{u}_t(G) \tag{9}$$

where $\boldsymbol{\gamma_0}$ and $\boldsymbol{\gamma_1}$ are $n \times 1$ vectors with fixed costs $\gamma_{0,j} \in \mathbb{R}_{\geq 0}$ and variable costs $\gamma_{1,j} \in \mathbb{R}_{\geq 0}$, for $j = 1, \ldots, n$, and where $\boldsymbol{z}_(G)$ is a $n \times 1$ vector with binary elements set to $z_j = 1$ if element $u_j \neq 0$, and $z_j = 0$ otherwise.

Similarly to equations (6) and (7), piecewise linear cost functions could be used in practice to charge a different cost γ_{1j} in vector $\boldsymbol{\gamma}_1$ to transferred amounts through the j-th transaction. For simplicity, let us now consider that the aim of cash managers is to minimize the difference between transaction costs in equation (9) and returns in equation (5) within the planning horizon T. Formally, cash managers want to find a sequence of controls $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_T\}$ to minimize the following objective function:

$$y = \boldsymbol{\gamma}_0' \sum_{t=1}^{T} \boldsymbol{z}_t(G) + \boldsymbol{\gamma}_1' \sum_{t=1}^{T} \boldsymbol{u}_t(G) - \boldsymbol{r}' \sum_{t=1}^{T} \boldsymbol{b}_t(G). \tag{10}$$

Summarizing, a cash management system $G = (\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{F})$ is usually by two graphs. First, a directed graph $(\mathcal{M}, \mathcal{N})$ linking accounts in $\mathcal{M}$ trough transactions in $\mathcal{N}$. Second, a bipartite graph $(\mathcal{P}, \mathcal{M})$ linking the virtual set $\mathcal{P}$ with creditors and debtors and accounts to accounts in $\mathcal{M}$ through external net cash flows in $\mathcal{F}$. The underlying structure of a cash management system defined by G, its returns and costs associated to transfers determine the basic conditions for the existence of non-trivial policies as we next show.

4 Formal reasoning with cash management systems

Within the cash management problem for a single bank account, Constantinides and Richard (1978) pointed out the necessary conditions for rational or non-trivial policies. Next, we generalize these conditions for cash management systems with multiple accounts. To this end, let us first formally define the concept of policy under the context of cash management.

Definition 14 (Policy) Given a cash management system G, a policy $\pi_0 = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_T)$ is a sequence of control actions over planning horizon T.

We say the a policy is trivial when we use some naive method to set its elements. For instance, setting all control actions to zero would be a trivial policy. The notion of trivial policy is similar to that of naive forecasts within the context of fitting predictive models to existing data (see e.g. Makridakis et al. (2008)). In order to assess the fitness of a given model to a dataset, it is customary to compare the accuracy of the model to that obtained by using a naive method, for instance, by predicting always the mean of the data used to fit the model. Similarly, an example of a trivial policy in cash management is taking no control action.

Due to high penalty costs on negative cash balances, cash managers are usually interested only in policies that result in non-negative balances, hence restricting feasibility to non-negative cash balances.

Assumption 1 (Non-negativity) Given a cash management system G, a policy $\pi = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_T)$ ensures non-negativity of balances when $\boldsymbol{b}_t \geq \mathbf{0}$, for all $t = 1, 2, \ldots, T$.

The following theorem characterizes non-trivial policies for cash management systems with directed incidence matrix A as in Definition 7.

Theorem 1 *Given a cash management system G with directed incidence matrix A, in a linear cost scenario described by equation* (10)*, if the following condition holds:*

$$\boldsymbol{\gamma_1}' < \boldsymbol{r}' A \tag{11}$$

then, any policy π ensuring non-negativity is non-trivial.

Proof Let us consider a general policy $\pi = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_T)$ and a particular trivial policy $\pi_0 = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$, where $\mathbf{0}$ is an $n \times 1$ vector of zeros, equivalent to taking no control action. A policy π is non-trivial with respect to π_0 if the sum of transaction and holding costs derived from π are smaller than the holding costs derived from the trivial policy π_0:

$$\boldsymbol{\gamma}_0' \sum_{t=1}^{T} \boldsymbol{z}_t + \boldsymbol{\gamma}_1' \sum_{t=1}^{T} \boldsymbol{u}_t - \boldsymbol{r}' \sum_{t=1}^{T} \boldsymbol{b}_t(\pi) < -\boldsymbol{r}' \sum_{t=1}^{T} \boldsymbol{b}_t(\pi_0) \tag{12}$$

$$\boldsymbol{\gamma}_0' \sum_{t=1}^{T} \boldsymbol{z}_t + \boldsymbol{\gamma}_1' \sum_{t=1}^{T} \boldsymbol{u}_t < \boldsymbol{r}' \sum_{t=1}^{T} \boldsymbol{b}_t(\pi) - \boldsymbol{r}' \sum_{t=1}^{T} \boldsymbol{b}_t(\pi_0) \tag{13}$$

$$\boldsymbol{\gamma}_0' \sum_{t=1}^{T} \boldsymbol{z}_t + \boldsymbol{\gamma}_1' \sum_{t=1}^{T} \boldsymbol{u}_t < \boldsymbol{r}' \left(\boldsymbol{b}_0 + A \sum_{t=1}^{T} \boldsymbol{u}_t + \sum_{t=1}^{T} \boldsymbol{f}_t - \boldsymbol{b}_0 - \sum_{t=1}^{T} \boldsymbol{f}_t \right) \tag{14}$$

$$\boldsymbol{\gamma}_0' \sum_{t=1}^{T} \boldsymbol{z}_t + \boldsymbol{\gamma}_1' \sum_{t=1}^{T} \boldsymbol{u}_t < \boldsymbol{r}' A \sum_{t=1}^{T} \boldsymbol{u}_t. \tag{15}$$

Then, since $\boldsymbol{z}_t \neq 0$, when $\boldsymbol{u}_t \neq 0$, it is never optimal to transfer money through $\boldsymbol{u}_t$, unless $\boldsymbol{\gamma_1}' < \boldsymbol{r}'A$.

In other words, the cost of transferring one money unit through any transaction must be smaller than the increase $(r_j - r_l)$ in returns between the target account j and the source account l with $r_j \geq r_l$.

Next, we discuss an important result that stems from Theorem 1. Before that, we introduce an important concept regarding the structure of a cash management system.

Definition 15 (Loop) Given a cash management system with incidence matrix A, we say that there is a loop between accounts j and l iff there is a pair of transactions (x_i, x_k) such that $a_{ji} = 1$, $a_{li} = -1$, $a_{jk} = -1$ and $a_{lk} = 1$.

The definition above tells us that there is a transaction x_i from l to j, and another transaction x_k from j to l. An example of such a loop is depicted in Figure 8. Notice that a loop between accounts j and l indicates that transactions can eventually occur in both directions, namely from j to l and from l to j. In other words, loops may eventually involve bidirectional transactions. Nonetheless, the following theorem characterizes the conditions under which such bidirectional transactions cannot occur. Furthermore, it does indicate which transaction is actually preferred.

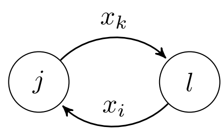

Fig. 8: An example of a loop between accounts.

Corollary 1 *Given a cash management system G with incidence matrix A that satisfies the necessary condition for non-triviality from Theorem 1, bidirectional transactions within loops cannot occur. Furthermore, the preferred transaction is either x_i when $\gamma_{1,i} < r_j - r_l$ or x_k when $\gamma_{1,k} < r_l - r_j$.*

Proof Given a cash management system defined by incidence matrix A, with elements a_{ij}, for any pair of transactions (x_i, x_k) bidirectionally connecting accounts (j, l) such that $a_{ji} = 1$, $a_{li} = -1$, $a_{jk} = -1$ and $a_{lk} = 1$, the condition in equation (11) is equivalent to the next double element-wise comparison:

$$\gamma_{1,i} < r_l - r_j \tag{16}$$

$$\gamma_{1,k} < r_j - r_l \tag{17}$$

which can only hold either when $r_j > r_l$ in (16), or when $r_j < r_l$ in (17), but never simultaneously in (16) and (17), provided that $\gamma_{1,i}, \gamma_{1,k} < |r_j - r_l|$, for any $\gamma_{1,i}, \gamma_{1,k} \geq 0$.

As an illustrative example, consider again the common two-assets framework described in Figure 1, with a regular account 1 and an investment account 2, where $r_2 > r_1$, Theorem 1 implies the following conditions:

$$\gamma_{1,1} < r_1 - r_2 \tag{18}$$

$$\gamma_{1,2} < r_2 - r_1 \tag{19}$$

where γ_{11} and γ_{12} are the elements of vector $\boldsymbol{\gamma}_1$, and r_1 and r_2 are the elements of vector $\boldsymbol{r}$. Since $r_2 > r_1$, only condition (19) holds, hence showing that the preferred transaction is x_2. Transaction x_1 will occur only when needed to guarantee non-negativity.

We can relax Assumption 1 by considering more general return functions. To this end, we need to introduce a new class of functions.

Definition 16 (Interval cumulative balance function) Given a policy π and a real valued interval $[q_k, q_{k+1}]$, an interval cumulative balance function $\delta(\pi, q_k, q_{k+1}, T)$ for planning horizon T is defined as follows:

$$\delta(\pi, q_k, q_{k+1}, T) = \sum_{t=1}^{T} \sum_{i=1}^{m} \{b_{it} \mid q_k \leq b_{it} < q_{k+1}\} \tag{20}$$

For instance, to account for the return function described by equations (6) and (8), we can denote $\delta_1(\pi) = \delta(\pi, -\infty, 0, T)$ and $\delta_2(\pi) = \delta(\pi, 0, \infty, T)$ as the cumulative balance functions and compute the returns by adding through the considered intervals:

$$\sum_{t=1}^{T} h(\boldsymbol{b}_t(G)) = \sum_{k=1}^{K+1} r_k \cdot \delta_k(\pi) = r_1 \cdot \delta_1(\pi) + r_2 \cdot \delta_2(\pi) \tag{21}$$

By relying on interval cumulative balance functions, we are in a position to extend Theorem 1 and Corollary 1 to relax the non-negativity assumption for balances.

Theorem 2 *Given a cash management system G in a linear cost scenario described by equation* (10) *with piecewise linear returns, if the following condition holds:*

$$\sum_{t=1}^{T} c(\boldsymbol{u}_t(G)) < \sum_{k=1}^{K+1} r_k(\delta_k(\pi) - \delta_k(\pi_0)) \tag{22}$$

then, any policy π is non-trivial.

Proof Let us consider again policy $\pi = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_T)$ and a particular trivial policy $\pi_0 = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$. A policy π is non-trivial with respect to π_0 if the sum of transaction and holding costs derived from π are smaller than the holding costs derived from the trivial policy π_0:

$$\sum_{t=1}^{T} c(\boldsymbol{u}_t(G)) - \sum_{k=1}^{K+1} r_k \cdot \delta_k(\pi) < -\sum_{k=1}^{K+1} r_k \cdot \delta_k(\pi_0) \tag{23}$$

$$\sum_{t=1}^{T} c(\boldsymbol{u}_t(G)) < \sum_{k=1}^{K+1} r_k(\delta_k(\pi) - \delta_k(\pi_0)) \tag{24}$$

Then, it is never optimal to transfer money through policy π unless the difference of cumulative returns with respect to trivial policy π_0 exceeds control costs.

Corollary 2 *Given a cash management system G with incidence matrix A that satisfies the necessary condition for non-triviality from Theorem 2, bidirectional transactions within loops cannot simultaneously occur.*

Proof The proof of this corolloary is analogous to the proof of Corollary 1. The only difference is that the preferred transaction may vary at each time step depending on balances and their respective piecewise returns.

A similar reasoning with more general linear cost and return functions would lead to more complex expressions of non-triviality.

5 Conclusions

In this paper, we introduce a formal definition of cash management systems with multiple accounts. We show that our formal approach is able to describe most of the situations faced by cash managers in practice such as complete systems, trees and bipartite systems. In addition to the graphical aid, we provide the necessary mathematical formulation including critical aspects for cash managers concerned with the optimization of cash balances by means of a sequence of control actions. From that, we conclude that our formal approach is able to encode cash management systems from a graphical and quantitative point of view.

The expressiveness of our formulation allows a formal reasoning on cash management. This formal reasoning includes the generalization of the common two-assets framework in Constantinides and Richard (1978) to account for particular transaction between multiple accounts. Transactions between accounts are no longer neglected, but considered as decision variables to elicit appropriate cash management policies. One of the main concerns of cash managers when dealing with multiple bank accounts is the characterization of alternative policies. The formal reasoning proposed in this paper allowed us to show the basic condition for the existence of non-trivial policies as a critical first step in cash management. We report theoretical results linking this non-triviality condition to the underlying structure of a cash management system.

An additional advantage of our approach is the possibility to extend this formal reasoning to derive further insights on cash management. The impact of misspecifications or uncertainty within cash management systems represents a promising future line of work. Since cash managers are usually interested in avoiding volatility in cash balances, we also expect that this framework can foster further research on risk analysis.

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