



Convergence Estimates of a Family of Approximation Operators of Exponential Type

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Abstract. The main object of this article is to consider a family of approximation operators of exponential type, which has presumably not been studied earlier due mainly to their seemingly complicated behavior. We estimate and establish a quantitative asymptotic formula in terms of the modulus of continuity with exponential growth, a Korovkin-type result for exponential functions and also a Voronovskaja-type asymptotic formula in the simultaneous approximation.

1. Introduction, Definitions and Preliminaries

Systematic investigations of approximation and some other basic properties of various linear and nonlinear operators are potentially useful in many different areas of researches in the mathematical, physical and engineering sciences. Some of the widely and extensively studied approximation operators include (for example) the Szász and Baskakov operators [3], the Post-Widder operators [7], the Srivastava-Gupta operators (see, for example, [8], [13] and [14]), the Szász-Mirakjan Beta-type operators [16], the Szász-Bézier operators [17], and so on (see, for example, [9]; see also [12] and [15] for other potentially useful developments leading to various approximation theorems).

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Over four decades ago, Ismail and May [11] proposed an approach to construct several exponential type operators. In fact, for the positive operators defined by

$$S_n(f, x) = \int_{-\infty}^{\infty} k_n(x, t) f(t) dt \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}),$$

the sequence of kernels $\{k_n(x, t)\}_{n \in \mathbb{N}}$ satisfy the following partial differential equation:

$$\frac{\partial}{\partial x} \{k_n(x, t)\} = \frac{n}{p(x)} (t - x) k_n(x, t),$$

where $p(x)$ denotes certain polynomials or functions of x . By taking different values of $p(x)$, Ismail and May [11] defined a number of known or new operators. In one of their examples, they considered $p(x) = 2x^{3/2}$ and, based upon this example, Ismail and May [11, (3.16)] deduced the following operators:

$$\begin{aligned} T_n(f, x) &= \int_0^{\infty} k_n(x, t) f(t) dt \\ &= e^{-n\sqrt{x}} \left(n \int_0^{\infty} e^{-nt/\sqrt{x}} t^{-1/2} I_1(2n\sqrt{t}) f(t) dt + f(0) \right), \end{aligned} \quad (1)$$

where $x \in (0, \infty)$, the kernel $k_n(x, t)$ is given by

$$k_n(x, t) = e^{-n\sqrt{x}} \left(n e^{-nt/\sqrt{x}} t^{-1/2} I_1(2n\sqrt{t}) + \delta(t) \right) \quad (2)$$

in terms of the Dirac delta function $\delta(t)$, and $I_\nu(z)$ is the modified Bessel function of the first kind defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{k! \Gamma(\nu + k + 1)}.$$

The Ismail-May operators $T_n(f, x)$ in (1) satisfy the following partial differential equation:

$$2x^{3/2} \frac{\partial}{\partial x} \left\{ e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \right\} = n(t - x) \left\{ e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \right\}. \quad (3)$$

Even though the operators $T_n(f, x)$ in (1) were presumably new at the time when Ismail and May [11] initiated their study, yet these operators do not appear to have been studied much ever since then. In order to preserve the constant function, the term $f(0)$ was incorporated in the above definition (1). These operators do preserve linear functions. Actually, by reproducing the linear functions, it is observed that better approximation can be achieved. Many operators as such preserve linear functions, but some well-known operators, which do not preserve linear functions, can be appropriately modified, so as to preserve linear functions. Some other general classes of linear positive operators, which preserve linear functions are studied recently in (for example) [5], [7] and [8].

Here, in our investigation, we present a systematic analysis of the Ismail-May approximation operators of exponential type, which was presumably not studied earlier due mainly to their seemingly complicated behavior. In particular, we estimate and establish a quantitative asymptotic formula in terms of the modulus of continuity with exponential growth, a Korovkin-type result for exponential functions and also a Voronovskaja-type asymptotic formula in the simultaneous approximation.

2. A Set of Auxiliary Results

We begin this section by presenting the following result.

Lemma 1. The moments $T_n(e_r, x)$ for $e_r(t) = t^r$ ($r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) satisfy the following recurrence relation:

$$T_n(e_{r+1}, x) = xT_n(e_r, x) + \frac{2x^{3/2}}{n} T'_n(e_r, x).$$

Furthermore, if the m th-order central moments $\mu_{n,m}(x)$ are given by

$$\mu_{n,m}(x) = T_n((t - x)^m, x),$$

then

$$\mu_{n,m+1}(x) = \frac{2x^{3/2}}{n} [\mu_{n,m}(x)]' + \frac{2mx^{3/2}}{n} \mu_{n,m-1}(x).$$

Proof. The proof of Lemma 1 follows immediately by making use of (3). \square

We now state the following consequence of Lemma 1.

Corollary 1. Let β and δ be any two positive real numbers. Also let $[a, b] \subset (0, \infty)$ be any bounded interval. Then, for any $m \in \mathbb{N}_0$, there exists a constant C , depending only on m , such that

$$\left\| \int_{|t-x| \geq \delta} k_n(x, t) e^{\beta t} dt \right\| \leq Cn^{-m}, \tag{4}$$

where $\|\cdot\|$ denotes the sup-norm over $[a, b]$.

Remark 1. (see [4]) Another approach to find the moments of the Ismail-May operators is by means of the moment generating function (m.g.f) which is given by

$$T_n(e^{At}, x) = e^{\frac{nAx}{(n-A\sqrt{x})}}. \tag{5}$$

Thus, upon expanding in powers of A , we have

$$\begin{aligned} T_n(e^{At}, x) &= 1 + xA + \left(\frac{2x^{3/2}}{n} + x^2\right) \frac{A^2}{2!} + \left(\frac{6x^2}{n^2} + \frac{6x^{5/2}}{n} + x^3\right) \frac{A^3}{3!} \\ &+ \left(\frac{24x^{5/2}}{n^3} + \frac{36x^3}{n^2} + \frac{12x^{7/2}}{n} + x^4\right) \frac{A^4}{4!} + \dots \end{aligned} \tag{6}$$

It may be observed here that the coefficients of $\frac{A^r}{r!}$ will provide the r th-order moment $T_n(e_r, x)$. Further, we have

$$T_n(e_r, x) = x^r + \frac{r(r-1)}{n} x^{(2r-1)/2} + \frac{r(r-1)^2(r-2)}{2n^2} x^{r-1} + O\left(\frac{1}{n^3}\right).$$

Remark 2. The central moments of the Ismail-May operators can be obtained as follows:

$$\begin{aligned} e^{\frac{nAx}{(n-A\sqrt{x})} - Ax} &= 1 + \frac{2x^{3/2}}{n} \frac{A^2}{2!} + \frac{6x^2}{n^2} \frac{A^3}{3!} \\ &+ \left(\frac{24x^{5/2}}{n^3} + \frac{12x^3}{n^2}\right) \frac{A^4}{4!} + \dots \end{aligned}$$

We may observe here the the coefficients of $\frac{A^r}{r!}$ will provide the r th-order central moment $\mu_{n,r}(x) = T_n((t - x)^r, x)$ of the Ismail-May operators.

Remark 3. In light of Remark 1, if we differentiate both sides partially with respect to A , we find that

$$\begin{aligned}
 T_n(e^{At}e_0, x) &= e^{\frac{nAx}{(n-A\sqrt{x})}} \\
 T_n(e^{At}e_1, x) &= e^{\frac{nAx}{(n-A\sqrt{x})}} \frac{n^2x}{(n-A\sqrt{x})^2} \\
 T_n(e^{At}e_2, x) &= e^{\frac{nAx}{(n-A\sqrt{x})}} \frac{n^2x^{3/2}(2n-2A\sqrt{x}+n^2x^{1/2})}{(n-A\sqrt{x})^4}.
 \end{aligned}$$

Thus, clearly, we have

$$\begin{aligned}
 T_n(e^{At}(e_2 - e_0x)^2, x) &= e^{\frac{nAx}{(n-A\sqrt{x})}} \frac{n^4}{(n-A\sqrt{x})^4} \mu_{n,2}(x) \\
 &\quad \cdot \left(1 + \frac{A\sqrt{x}}{n}(2Ax-1) - \frac{2A^3x^2}{n^2} + \frac{A^4x^{5/2}}{2n^3} \right).
 \end{aligned}$$

Suppose now that $n > 2A\sqrt{x}$ implies that

$$n - A\sqrt{x} > \frac{n}{2} \quad \text{or} \quad \frac{n}{n - A\sqrt{x}} < 2.$$

We then easily have

$$\begin{aligned}
 e^{\frac{nAx}{(n-A\sqrt{x})}} &< e^{2Ax}, \\
 \frac{A\sqrt{x}}{n}(2Ax-1) &< Ax - \frac{1}{2}, \\
 \frac{n^4}{(n-A\sqrt{x})^4} &< 16
 \end{aligned}$$

and

$$\frac{A^4x^{5/2}}{2n^3} < \frac{Ax}{16}.$$

Thus, by using the above bounds, we have

$$T_n(e^{At}(e_2 - e_0x)^2, x) \leq C(A, x)\mu_{n,2}(x),$$

where

$$C(A, x) = e^{2Ax}16\left(Ax + \frac{1}{2} + \frac{Ax}{16}\right) = e^{2Ax}(8 + 17Ax).$$

3. Convergence Estimates

The first-order modulus of continuity (see [18] and [9]) is defined as follows:

$$\omega_1(f, \delta, A) = \sup_{\substack{0 < x < \infty \\ (|h| \leq \delta)}} |f(x) - f(x+h)| e^{-Ax}.$$

We now recall the following known result.

Lemma 2. (see [18] and [9]) *For every positive number $h > 0$ and for $k \in \mathbb{N}$, the following inequality holds true:*

$$\omega_1(f, kh, A) \leq k \cdot e^{A(k-1)h} \cdot \omega_1(f, h, A).$$

Theorem 1 below provides a quantitative asymptotic formula in terms of the exponential modulus of continuity.

Theorem 1. For the operators $T_n : E \rightarrow C(0, \infty)$, where E is the space of functions f having exponential growth, let

$$f \in C^2(0, \infty) \cap E \quad \text{and} \quad f'' \in \text{Lip}(\beta, A) \quad (0 < \beta \leq 1).$$

Then, for $x \in (0, \infty)$ and $n > 2A\sqrt{x}$, it is asserted that

$$\begin{aligned} & \left| T_n(f, x) - f(x) - \frac{x^{3/2}}{n} f''(x) \right| \\ & \leq \left(e^{2Ax} + \frac{C(A, x)}{2} + \frac{\sqrt{C(2A, x)}}{2} \right) \cdot \frac{2x^{3/2}}{n} \\ & \quad \cdot \omega_1 \left(f'', \sqrt{\left(\frac{12x}{n^2} + \frac{6x^{3/2}}{n} \right)}, A \right), \end{aligned} \tag{7}$$

where

$$C(A, x) = e^{2Ax}(8 + 17Ax)$$

and the spaces $\text{Lip}(\beta, A)$ ($0 < \beta \leq 1$) consist of all functions f such that

$$\omega_1(f, \delta, A) \leq M\delta^\beta \quad (\forall \delta < 1).$$

Remark 4. The proof of Theorem 1 follows along the lines used earlier by [18] and [9]). It makes use of Lemma 2, Remark 2 and Remark 3. We, therefore, choose to omit the details involved.

We next turn to a Korovkin-type result for exponential functions, which was established in [2]. Subsequently, in another paper [10], the following quantitative version of such results was given as the following general theorem.

Theorem A. (see [10]) Let

$$\{L_n : \widehat{C}[0, \infty) \rightarrow \widehat{C}[0, \infty)\}_{n \in \mathbb{N}}$$

be a sequence of linear positive operators, where $\widehat{C}[0, \infty)$ denotes the class of all real-valued continuous functions f on $[0, \infty)$ having finite limit value as $x \rightarrow \infty$ and equipped with uniform norm. Suppose also that it satisfies the following equalities:

$$\|L_n(e^{-kt}) - e^{-kx}\|_\infty = \alpha_k(n) \quad (k = 0, 1, 2).$$

Then

$$\begin{aligned} \|L_n f - f\|_\infty & \leq \|f\|_\infty \alpha_0(n) + [2 + \alpha_0(n)] \\ & \quad \cdot \bar{\omega}(f, \sqrt{\alpha_0(n) + 2\alpha_1(n) + \alpha_2(n)}) \end{aligned} \tag{8}$$

for $f \in \widehat{C}[0, \infty)$. The modulus of continuity used in the above assertion is defined as follows:

$$\bar{\omega}(f, \delta) = \sup_{\substack{|e^{-x} - e^{-t}| \leq \delta \\ (x, t > 0)}} |f(t) - f(x)|.$$

For the Ismail-May operators defined by (1), Theorem A takes the form given by Theorem 2 below.

Theorem 2. For $f \in \widetilde{C}(0, \infty)$, and $n \in \mathbb{N}$, it is asserted that

$$\|T_n f - f\|_\infty \leq 2\bar{\omega}\left(f, \sqrt{2\alpha_1(n) + \alpha_2(n)}\right), \tag{9}$$

where convergence takes place if n is sufficiently large.

Proof. Since the operators T_n preserve only the constant function, we have $\alpha_0(n) = 0$. Also, by Remark 1, we find that

$$T_n(e^{-t}, x) - e^{-x} = e^{-\frac{nx}{n+\sqrt{x}}} - e^{-x}.$$

We observe here that

$$0 < \frac{n}{n + \sqrt{x}} < 1 \quad (n \geq 1; x \in (0, \infty)).$$

Thus, following [10, Lemma 3.1], we get

$$\frac{u - v}{\ln u - \ln v} < \frac{u + v}{2} \quad (0 < v < u)$$

and, for

$$u = e^{-\frac{nx}{n+\sqrt{x}}} > v = e^{-x} > 0,$$

we have

$$\begin{aligned} e^{-\frac{nx}{n+\sqrt{x}}} - e^{-x} &< \frac{e^{-\frac{nx}{n+\sqrt{x}}} + e^{-x}}{2} x \left(1 - \frac{n}{n + \sqrt{x}}\right) \\ &= \frac{1}{2} (xe^{-\frac{nx}{n+\sqrt{x}}} + xe^{-x}) \frac{\sqrt{x}}{n + \sqrt{x}}. \end{aligned}$$

Again, by using [10, Eq. (3.1)], we get

$$e^{-\frac{nx}{n+\sqrt{x}}} - e^{-x} < \frac{1}{2e} \left(\frac{n + \sqrt{x}}{n} + 1\right) \frac{\sqrt{x}}{n + \sqrt{x}},$$

which implies that $\alpha_1(n)$ tends to 0 as $n \rightarrow \infty$.

Similarly, the estimate $\alpha_2(n)$ has the upper bound given by

$$\begin{aligned} e^{-\frac{2nx}{n+2\sqrt{x}}} - e^{-2x} &< \frac{1}{2e} \left(\frac{n + 2\sqrt{x}}{2n} + \frac{1}{2}\right) \frac{2\sqrt{x}}{n + 2\sqrt{x}} \\ &= \frac{1}{2e} \left(\frac{n + 2\sqrt{x}}{n} + 1\right) \frac{\sqrt{x}}{n + 2\sqrt{x}}. \end{aligned}$$

This evidently completes the proof of Theorem 2. \square

4. Simultaneous Approximation

In this section, we establish a Voronovskaja-type asymptotic formula in simultaneous approximation. We first establish Lemma 3 below.

Lemma 3. There exist the functions $q_{i,j,r}(x)$, independent of n and t , such that

$$(2x^{3/2})^r \frac{\partial^r}{\partial x^r} \left\{ e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \right\} = \sum_{\substack{2+i+j \leq r \\ (i,j \geq 0)}} n^{i+j} (t-x)^j q_{i,j,r}(x) \left\{ e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \right\}.$$

Proof. In order to prove the result asserted by Lemma 3, it is sufficient to show that

$$\frac{\partial^r}{\partial x^r} \left\{ e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \right\} = \frac{1}{2^r} \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j} (t-x)^j q_{i,j,r}(x) x^{-3r/2} \left\{ e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \right\}.$$

We shall prove this last result by applying the principle of mathematical induction. Indeed, for $r = 1$, the result is true as indicated in (3) in this case when $i = 0, j = 1$ and $q_{i,j,r}(x) = 1$. Suppose now that the result is true for a give positive integer r . Then

$$\begin{aligned} & \frac{\partial^{r+1}}{\partial x^{r+1}} \left\{ e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \right\} \\ &= \frac{\partial}{\partial x} \left\{ \frac{1}{2^r} \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j} (t-x)^j q_{i,j,r}(x) x^{-3r/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \right\} \\ &= \frac{1}{2^r} \sum_{\substack{2i+j \leq r \\ (i \geq 0, j \geq 1)}} n^{i+j} (t-x)^{j-1} [-j q_{i,j,r}(x)] x^{-3r/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \\ &\quad + \frac{1}{2^r} \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j} (t-x)^j q'_{i,j,r}(x) x^{-3r/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \\ &\quad + \frac{1}{2^{r+1}} \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j} (t-x)^j [-3r q_{i,j,r}(x)] x^{(-3r-2)/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \\ &\quad + \frac{1}{2^{r+1}} \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j+1} (t-x)^j [-x^{-1/2} q_{i,j,r}(x)] x^{-3r/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \\ &\quad + \frac{1}{2^{r+1}} \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j+1} (t-x)^j [t x^{-3/2} q_{i,j,r}(x)] x^{-3r/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}}. \end{aligned}$$

We thus find that

$$\begin{aligned} & \frac{\partial^{r+1}}{\partial x^{r+1}} \left\{ e^{-n\sqrt{x}} \exp\left(-\frac{nt}{2x^2}\right) \right\} \\ &= \frac{1}{2^r} \sum_{\substack{2(i-1)+(j+1) \leq r \\ (i \geq 0, j \geq 1)}} n^{i+j} (t-x)^j [-(j+1) q_{i-1,j+1,r}(x)] x^{-3r/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \\ &\quad + \frac{1}{2^r} \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j} (t-x)^j q'_{i,j,r}(x) x^{-3r/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \\ &\quad + \frac{1}{2^{r+1}} \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j} (t-x)^j [-3rx^{1/2} q_{i,j,r}(x)] x^{-3(r+1)/2} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}} \\ &\quad + \frac{1}{2^{r+1}} \sum_{\substack{2i+(j-1) \leq r \\ (i,j \geq 0)}} n^{i+j} (t-x)^j [q_{i,j-1,r}(x)] x^{-3(r+1)} e^{-n\sqrt{x}} e^{-nt/\sqrt{x}}, \end{aligned}$$

which does have the required form where

$$q_{i,j,r+1}(x) = -x^{3/2} (j+1) q_{i-1,j+1,r}(x) + x^{3/2} q'_{i,j,r}(x) - 3rx^{1/2} q_{i,j,r}(x) + q_{i,j-1,r}(x)$$

$$(2i+j \leq r+1; i, j \geq 0)$$

with the convention that $q_{i,j,r}(x) = 0$ if any one of the constraints is violated. Thus the result holds true for $r + 1$. This completes the proof of Lemma 3. \square

Throughout the remainder of this section, $C_\gamma(0, \infty)$ ($\gamma > 0$) denotes the class of continuous functions on $(0, \infty)$ with

$$|f(t)| \leq Me^{t\gamma} \quad (M > 0).$$

Moreover, the norm $\|\cdot\|_\gamma$ on this class of functions is defined as follows:

$$\|f\|_\gamma = \sup_{x \in (0, \infty)} |f(t)|e^{-t\gamma}.$$

Theorem 3. Let $f \in C_\gamma(0, \infty)$. Also let $f^{(r+2)}(x)$ exists at a point $x \in (0, \infty)$. Then, for $r = 0, 1, 2$, it is asserted that

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \left[T_n^{(r)}(f, x) - f^{(r)}(x) \right] \\ &= x^{-3/2} \frac{f^{(r-1)}(x)}{(r-3)!} \left\{ \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) \right\} \\ &+ x^{-1/2} \frac{f^{(r)}(x)}{(r-2)!} \left[- (r-2) \left\{ \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) \right\} \right. \\ &+ \left. \left\{ \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) \right\} \right] \\ &+ x^{1/2} \frac{f^{(r+1)}(x)}{(r-1)!} \left[\frac{(r-1)(r-2)}{2} \left\{ \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) \right\} \right. \\ &- (r-1) \left\{ \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) \right\} \\ &+ \left. \left\{ \left(r + \frac{1}{2} \right) \left(r - \frac{1}{2} \right) \cdots \frac{3}{2} \right\} \right] \\ &+ x^{3/2} \frac{f^{(r+2)}(x)}{r!} \left[- \frac{r(r-1)(r-2)}{6} \left\{ \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) \right\} \right. \\ &+ \frac{r(r-1)}{2} \left\{ \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) \right\} \\ &- \left. r \left\{ \left(r + \frac{1}{2} \right) \left(r - \frac{1}{2} \right) \cdots \frac{3}{2} \right\} + \left\{ \left(r + \frac{3}{2} \right) \left(r + \frac{1}{2} \right) \cdots \frac{5}{2} \right\} \right], \end{aligned}$$

where the terms within the curly brackets end with the last terms as indicated in each bracket and, otherwise, its value is 1.

Proof. Making use of Taylor's theorem, we can write

$$f(t) = \sum_{v=0}^{r+2} \frac{f^{(v)}(x)}{v!} (t-x)^v + \psi(t, x)(t-x)^{r+2} \quad (0 < t < \infty), \quad (10)$$

where the function $\psi(t, x) \rightarrow 0$ as $t \rightarrow x$. From the equation (10), we obtain

$$\begin{aligned}
 T_n^{(r)}(f, x) &= \frac{d^r}{dw^r} \left\{ T_n(f(t), w) \right\} \Big|_{w=x} \\
 &= \sum_{v=0}^{r+2} \frac{f^{(v)}(x)}{v!} \frac{d^r}{dw^r} \left\{ T_n((t-x)^v, w) \right\} \Big|_{w=x} \\
 &\quad + \frac{d^r}{dw^r} \left\{ T_n(\psi(t, x)(t-x)^{r+2}, w) \right\} \Big|_{w=x} \\
 &= \sum_{v=0}^{r+2} \frac{f^{(v)}(x)}{v!} \sum_{j=0}^v \binom{v}{j} (-x)^{v-j} \frac{d^r}{dw^r} \left\{ T_n(t^j, w) \right\} \Big|_{w=x} \\
 &\quad + \frac{d^r}{dw^r} \left\{ T_n(\psi(t, x)(t-x)^{r+2}, w) \right\} \Big|_{w=x} \\
 &=: I_1 + I_2.
 \end{aligned} \tag{11}$$

Because of the fractional powers of x in $T_n(e_r, x)$, it does not seem to be possible to consider all terms. Following [6, Theorem 5], therefore, we estimate I_1 as per our assumption as follows:

$$\begin{aligned}
 I_1 &= \frac{f^{(r-1)}(x)}{(r-1)!} \frac{d^r}{dw^r} \left\{ T_n(t^{r-1}, w) \right\} \Big|_{w=x} \\
 &\quad + \frac{f^{(r)}(x)}{r!} \left[r(-x) \frac{d^r}{dw^r} \left\{ T_n(t^{r-1}, w) \right\} \Big|_{w=x} + \frac{d^r}{dw^r} \left\{ V_n(t^r, w) \right\} \Big|_{w=x} \right] \\
 &\quad + \frac{f^{(r+1)}(x)}{(r+1)!} \left[\frac{(r+1)r}{2} x^2 \frac{d^r}{dw^r} \left\{ T_n(t^{r-1}, w) \right\} \Big|_{w=x} + (r+1)(-x) \frac{d^r}{dw^r} \left\{ T_n(t^r, w) \right\} \Big|_{w=x} \right] \\
 &\quad + \frac{d^r}{dw^r} \left\{ T_n(t^{r+1}, w) \right\} \Big|_{w=x} \\
 &\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \left[\frac{(r+2)(r+1)r}{3!} (-x)^3 \frac{d^r}{dw^r} \left\{ T_n(t^{r-1}, w) \right\} \Big|_{w=x} \right] \\
 &\quad + \frac{(r+2)(r+1)}{2} x^2 \frac{d^r}{dw^r} \left\{ V_n(t^r, w) \right\} \Big|_{w=x} \\
 &\quad + (r+2)(-x) \frac{d^r}{dw^r} \left\{ T_n(t^{r+1}, w) \right\} \Big|_{w=x} + \frac{d^r}{dw^r} \left\{ T_n(t^{r+2}, w) \right\} \Big|_{w=x}.
 \end{aligned}$$

which, in view of Remark 1, yields

$$\begin{aligned}
 I_1 &= \frac{f^{(r-1)}(x)}{(r-1)!} \left[\frac{(r-1)(r-2)}{n} \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) x^{-3/2} \right] \\
 &\quad + \frac{f^{(r)}(x)}{r!} \left[r(-x) \frac{(r-1)(r-2)}{n} \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) x^{-3/2} \right] \\
 &\quad + r! + \frac{r(r-1)}{n} \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) x^{-1/2} \\
 &\quad + \frac{f^{(r+1)}(x)}{(r+1)!} \left[\frac{(r+1)r}{2} x^2 \frac{(r-1)(r-2)}{n} \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) x^{-3/2} \right] \\
 &\quad + (r+1)(-x) \cdot \left(r! + \frac{r(r-1)}{n} \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) x^{-1/2} \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ \left((r+1)! x + \frac{(r+1)r}{n} \left(r + \frac{1}{2} \right) \left(r - \frac{1}{2} \right) \cdots \frac{3}{2} x^{1/2} \right) \\
 &+ \frac{f^{(r+2)}(x)}{(r+2)!} \left[\frac{(r+2)(r+1)r}{3!} (-x)^3 \frac{(r-1)(r-2)}{n} \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) x^{-3/2} \right. \\
 &+ \frac{(r+2)(r+1)}{2} x^2 \cdot \left(r! + \frac{r(r-1)}{n} \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) x^{-1/2} \right) \\
 &+ (r+2)(-x) \left((r+1)! x + \frac{(r+1)r}{n} \left(r + \frac{1}{2} \right) \left(r - \frac{1}{2} \right) \cdots \frac{3}{2} x^{1/2} \right) \\
 &\left. + \left(\frac{(r+2)!}{2} x^2 + \frac{(r+2)(r+1)}{n} \left(r + \frac{3}{2} \right) \left(r + \frac{1}{2} \right) \cdots \frac{5}{2} x^{3/2} \right) \right] + O\left(\frac{1}{n^2}\right).
 \end{aligned}$$

We thus obtain

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \left\{ n \left[T_n^{(r)}(f, x) - f^{(r)}(x) \right] \right\} \\
 &= \frac{f^{(r-1)}(x)}{(r-1)!} \left[(r-1)(r-2) \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) x^{-3/2} \right] \\
 &\quad + \frac{f^{(r)}(x)}{r!} \left[-r(r-1)(r-2) \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) x^{-1/2} \right. \\
 &\quad \left. + r(r-1) \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) x^{-1/2} \right] \\
 &\quad + \frac{f^{(r+1)}(x)}{(r+1)!} \left[\frac{1}{2} (r+1)r(r-1)(r-2) \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) x^{1/2} \right. \\
 &\quad \left. + (r+1)(-x) \cdot \left(r(r-1) \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) x^{-1/2} \right) \right. \\
 &\quad \left. + \left((r+1)r \left(r + \frac{1}{2} \right) \left(r - \frac{1}{2} \right) \cdots \frac{3}{2} x^{1/2} \right) \right] \\
 &\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \left[-\frac{1}{3!} (r+2)(r+1)r(r-1)(r-2) \left(r - \frac{3}{2} \right) \left(r - \frac{5}{2} \right) \cdots \left(-\frac{1}{2} \right) x^{3/2} \right. \\
 &\quad + \frac{(r+2)(r+1)}{2} x^2 \cdot \left(r(r-1) \left(r - \frac{1}{2} \right) \left(r - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) x^{-1/2} \right) \\
 &\quad + (r+2)(-x) \left((r+1)r \left(r + \frac{1}{2} \right) \left(r - \frac{1}{2} \right) \cdots \frac{3}{2} x^{1/2} \right) \\
 &\quad \left. + \left((r+2)(r+1) \left(r + \frac{3}{2} \right) \left(r + \frac{1}{2} \right) \cdots \frac{5}{2} x^{3/2} \right) \right].
 \end{aligned}$$

In order to complete the proof of Theorem 3, it is sufficient to show that

$$\lim_{n \rightarrow \infty} \{nI_2\} = 0.$$

We begin by observing that

$$\begin{aligned}
 I_2 &= \frac{d^r}{dw^r} \left\{ T_n(\psi(t, x)(t-x)^{r+2}, w) \right\} \Big|_{w=x} \\
 &= \int_0^\infty k_n^{(r)}(w, t) \psi(t, x)(t-x)^{r+2} dt \Big|_{w=x}
 \end{aligned}$$

Also, in view of Lemma 3, we have

$$\begin{aligned}
 |I_2| &\leq \sum_{\substack{2+i \leq r \\ (i, j \geq 0)}} n^{i+j} \frac{|q_{i,j,r}(w)|}{2^r \cdot w^{3r/2}} \int_0^\infty |t-w|^j k_n(w, t) \psi(t, x) \cdot |t-x|^{r+2} dt \Big|_{w=x} \\
 &= \sum_{\substack{2+i \leq r \\ (i, j \geq 0)}} n^{i+j} \frac{|q_{i,j,r}(x)|}{2^r \cdot x^{3r/2}} \int_0^\infty |t-x|^j k_n(x, t) \psi(t, x) \cdot |t-x|^{r+2} dt.
 \end{aligned} \tag{12}$$

Now, since $\psi(t, x) \rightarrow 0$ as $t \rightarrow x$, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\psi(t, x)| < \epsilon \quad \text{whenever} \quad |t-x| < \delta.$$

Furthermore, we can find a constant $M > 0$ such that

$$|(t-x)^{r+2} \psi(t, x)| \leq M e^{\gamma t} \quad (|t-x| \geq \delta).$$

Hence, by using Lemma 3 once again, we get

$$\begin{aligned}
 |I_2| &\leq \sum_{\substack{2+i \leq r \\ (i, j \geq 0)}} n^{i+j} \frac{|q_{i,j,r}(x)|}{2^r \cdot x^{3r/2}} \int_0^\infty |t-x|^j k_n(x, t) \psi(t, x) \cdot |t-x|^{r+2} dt \\
 &\leq \sum_{\substack{2+i \leq r \\ (i, j \geq 0)}} n^{i+j} \frac{|q_{i,j,r}(x)|}{2^r \cdot x^{3r/2}} \left(\epsilon \int_{|t-x| < \delta} k_n(x, t) |t-x|^{j+r+2} dt \right. \\
 &\quad \left. + M \int_{|t-x| \geq \delta} k_n(x, t) |t-x|^j e^{\gamma t} dt \right) \\
 &=: I_{21} + I_{22}.
 \end{aligned}$$

Let us now take

$$K = \sup_{\substack{2+i \leq r \\ (i, j \geq 0)}} \frac{|q_{i,j,r}(x)|}{2^r \cdot x^{3r/2}}.$$

Thus, by applying the Schwarz inequality and Remark 2, we find that

$$\begin{aligned}
 I_{21} &= \epsilon K \sum_{\substack{2+i \leq r \\ (i, j \geq 0)}} n^{i+j} \left(\int_0^\infty k_n(x, t) dt \right)^{1/2} \left(\int_0^\infty k_n(x, t) (t-x)^{2j+2r+4} dt \right)^{1/2} \\
 &= \epsilon \sum_{\substack{2+i \leq r \\ (i, j \geq 0)}} n^{i+j} O\left(\frac{1}{n^{(j+r+2)/2}}\right) = \epsilon \cdot O\left(\frac{1}{n}\right).
 \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $nI_{21} \rightarrow 0$ as $n \rightarrow \infty$. Again, by using the Schwarz inequality, Corollary 1 (with $2\gamma = \beta$) and Remark 2, we obtain

$$\begin{aligned} I_{22} &\leq M_1 \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j} \left(\int_0^\infty k_n(x,t)(t-x)^{2j} dt \right)^{1/2} \cdot \left(\int_{|t-x| \geq \delta} k_n(x,t)e^{2\gamma t} dt \right)^{1/2} \\ &\leq M_1 \sum_{\substack{2i+j \leq r \\ (i,j \geq 0)}} n^{i+j} O\left(\frac{1}{n^{j/2}}\right) O\left(\frac{1}{n^{m/2}}\right) = O\left(\frac{1}{n^{(m-r)/2}}\right), \end{aligned}$$

which implies that $nI_{22} \rightarrow 0$ as $n \rightarrow \infty$ on choosing $m > r + 2$. Thus, from the above estimates of I_{21} and I_{22} , $nI_2 \rightarrow 0$ as $n \rightarrow \infty$. Clearly, the result asserted by Theorem 3 follows from the above estimates of I_1 and I_2 . \square

Corollary 2. Let $f \in C_\gamma(0, \infty)$. Also let $f''(x)$ exist at a point $x \in (0, \infty)$, then

$$\lim_{n \rightarrow \infty} \{n[T_n(f, x) - f(x)]\} = x^{3/2} f''(x).$$

Corollary 3. Let the function $f \in C_\gamma(0, \infty)$ admit the derivative of the third order at a fixed point $x \in (0, \infty)$. Then

$$\lim_{n \rightarrow \infty} \{n[T'_n(f, x) - f'(x)]\} = \frac{3x^{1/2}}{2} f''(x) + x^{3/2} f'''(x).$$

Remark 5. It may be remarked here that Theorem 3 can be extended for $r \geq 3$ if we also consider the coefficients for $f^{(r-i)}(x)$ ($i = 2, \dots, r$), which is because of the fact that, in Remark 1, the value of $T_n(e_r, x)$ has fractional powers of x .

5. Concluding Remarks and Observations

Our present investigation is motivated essentially by the fact that systematic investigations of approximation and some other basic properties of various linear and nonlinear operators are potentially useful in many different areas of researches in the mathematical, physical and engineering sciences. Here, in this article, we have considered a family of approximation operators of exponential type, which was presumably not studied earlier due mainly to their seemingly complicated behavior. In particular, we have estimated and established a quantitative asymptotic formula in terms of the modulus of continuity with exponential growth, a Korovkin-type result for exponential functions and also a Voronovskaja-type asymptotic formula in the simultaneous approximation. For motivating and encouraging further researches of such topics as those that we have considered here, we have chosen to include citations of earlier developments concerning other families for approximation operators.

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