Ergodic properties of composition operators on Banach spaces of analytic functions.

Enrique Jordá[*] and Alberto Rodríguez-Arenas [†,‡]

December 11, 2019

Abstract

The composition operators defined on little Bloch spaces, Bergman spaces, Hardy spaces or little weighted Bergman spaces of infinite type, when well defined, are shown to be mean ergodic if and only if they are power bounded if and only if the symbol has an interior fixed point. For these operators uniform mean ergodicity is equivalent to quasicompactness in the sense of Yosida and Kakutani.

Keywords: Composition operator, mean ergodic operator, uniformly mean ergodic operator, power bounded operator, Bloch space, Bergman space.
MSC Classification 2010: 47B33, 30H20, 30H30, 46E15, 47A35.

1 Introduction, notation and preliminaries

1.1 Introduction

We defer to Subsection 1.2 for notation, definitions and basic facts.

[*]Enrique Jordá: EPS Alcoy, Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, Plaza Ferrándiz y Carbonell s/n E-03801 Alcoy, Spain, ejorda@mat.upv.es

[†]Alberto Rodríguez-Arenas: Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, Camino de Vera, s/n E-46022 Valencia, Spain, alrodar3@posgrado.upv.es

[‡]The present research was partially supported by the project MTM2016-76647-P. The second author was also partially supported by the grant PAID-01-16 of the Universitat Politècnica de València.

The properties of the composition operator acting between various spaces of holomorphic functions have been thoroughly studied, especially relating the properties of C_φ to those of φ. See the books of Cowen and McCluer [14] and Shapiro [28]. We study power boundedness, mean ergodicity and uniform mean ergodicity for composition operators defined in these spaces. This line of research was first studied by Bonet and Domański [9] when they studied the operator acting on $H(U)$, with U a domain in a Stein manifold. Later Wolf [30] considered the composition operator on the general weighted Bergmann spaces of infinite order $H_v^\infty(\mathbb{D})$. Bonet and Ricker [12] characterized the power boundedness and mean ergodicity of the multiplication operator on the general weighted Bergman spaces of infinite type $H_v^\infty(\mathbb{D})$ and $H_v^0(\mathbb{D})$. More recently, Beltrán-Meneu et al [5, 6] studied the ergodic properties of the composition operator on $A(\mathbb{D})$ and on $H^\infty(\mathbb{D})$ as well as of weighted composition operators on $H(\mathbb{D})$. Arendt et al [1] extended the results of [5] to the convergence of the sequence of powers of operators besides the Cesáro means, including $A(\mathbb{D})$, $H^\infty(\mathbb{D})$ and also the Wiener algebra $W(\mathbb{D})$. The article [2] by the same authors was presented to us after preliminary submission and is quite related to ours, see Remark 3.13. Also during the process we have noticed the existence of [17], where the mean ergodicity and uniform mean ergodicity of composition operators on the Hardy spaces H^p, $1 \leq p < \infty$ is characterized, independently of both [2] and this paper.

Some of the spaces we focus on share some analogies with $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$. In this note we consider composition operators on $\mathcal{B}_p$, $\mathcal{B}_p^0$ (Bloch spaces of order $p \geq 1$), H_v, H_v^0 (weighted Bergman space of infinite order, for appropriate weights v), A^p (Bergman space of order $p \geq 1$) and H^p (Hardy space of order $p \geq 1$). We prove

Theorem 1. *Let v be a convenient weight and $p \geq 1$. Let X^0 stand for H_v^0, $\mathcal{B}_p^0$, A^p or H^p and let X stand for H_v or $\mathcal{B}_p$. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic. Then the following assertions hold:*

(i) $C_\varphi \in \mathcal{L}(X^0)$ is power bounded if and only if it is mean ergodic, and if and only if φ is elliptic.

(ii) $C_\varphi \in \mathcal{L}(X)$ is power bounded if and only if φ is elliptic.

(iii) if $C_\varphi \in \mathcal{L}(X^0)$ and φ has a Denjoy-Wolff point $z_0 \in \mathbb{D}$, then $(C_\varphi^n)_n$ converges weakly to C_{z_0} on X^0, and consequently C_φ is mean ergodic.

Considering the spaces $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$, summarizing the results of [5] and [1] we get the following characterization for uniformly mean ergodic composition operators. In $H^\infty(\mathbb{D})$ uniform mean ergodicity is equivalent to mean ergodicity for power bounded operators because it is a Grothendieck-Dunford-Pettis space and composition operators on it have norm 1.

Theorem A: *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be (uniformly continuous and) holomorphic. Then $C_\varphi : H^\infty(\mathbb{D}) \longrightarrow H^\infty(\mathbb{D})$ ($C_\varphi : A(\mathbb{D}) \longrightarrow A(\mathbb{D})$) is uniformly mean ergodic if and only if either φ is equivalent to a rational rotation or one of the following equivalent conditions is satisfied:*

(i) There exists $z_0 \in \mathbb{D}$ such that $\lim \varphi_n(z) = z_0$ uniformly on $\mathbb{D}$.

(ii) There exists $n_0 \in \mathbb{N}$ such that $\|C_\varphi^{n_0}\|_e < 1$.

(iii) There exists $n_0 \in \mathbb{N}$ such that $C_\varphi^{n_0}$ is a compact operator.

(iv) There exists $z_0 \in \mathbb{D}$ such that $\|C_\varphi^n - C_{z_0}\| \to 0$, where $C_{z_0}(f) = f(z_0)$.

Following Yosida and Kakutani [36] we call the operators satisfying condition (ii) of Theorem A *quasicompact*. In our work on Bloch spaces, weighted Bergmann spaces of infinite type, Bergman spaces and Hardy spaces (in these last two cases restricting to univalent symbols), we prove in Theorem 3.8 that uniform mean ergodicity of composition operators whose symbol is not equivalent to a rational rotation can be characterized by having an interior Denjoy–Wolff point together with condition (ii), which is equivalent to (iv) in Theorem A above as a consequence of [1, Theorem 3.4]. We provide examples showing that conditions (i) an (iii) are sufficient but not necessary. Briefly, the converse of the Yosida–Kakutani Mean Ergodic Theorem is also true in the spaces we consider, but here the condition of T^n being compact for some $n \in \mathbb{N}$ is only sufficient.

1.2 Notation and preliminaries

If X is a Banach space we denote the space of continuous and linear operators from X to itself by $\mathcal{L}(X)$. The space $\mathcal{L}(X)$ is endowed with the operator norm, unless explicitly mentioned. Recall the concepts of the *spectrum* $\sigma(T, X)$ (we write $\sigma(T)$ if the space on which T acts is clear) of an operator $T \in \mathcal{L}(X)$ on a Banach space X (the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible), the *point spectrum* $\sigma_p(T)$ (the set of $\lambda \in \mathbb{C}$

such that $T-\lambda I$ is not injective) and the *approximate spectrum* $\sigma_{ap}(T)$ (the set of $\lambda \in \mathbb{C}$ for which there is $(x_n)_n \subset X$, with $\|x_n\| = 1$, such that $\lim_n \|T(x_n) - \lambda x_n\| = 0$). It is well known that $\sigma_p(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T)$. The *spectral radius* of T is defined as $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$. Furthermore, if $\mathcal{K}(X) \subset \mathcal{L}(X)$ denotes the set of compact operators on X, then the *essential norm* $\|T\|_e := \inf\{\|T-K\| : K \in \mathcal{K}(X)\}$ defines indeed a norm on the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Denote by $r_e(T)$ the *essential spectral radius*, i.e. the spectral radius of the projection of the operator T to $\mathcal{L}(X)/\mathcal{K}(X)$. We write $r_e(T,X)$ if we need to stress the space X.

Given an operator $T \in \mathcal{L}(X)$ we say that it is *power bounded* if the set of its iterates is uniformly bounded, i.e. $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. The operator is called *mean ergodic* (resp. *uniformly mean ergodic*) if the sequence of its Cesàro means

$$T_{[n]} = \frac{1}{n}\sum_{m=1}^{n} T^m$$

converges in the strong operator topology (resp. in the operator norm topology). Yosida [34] characterized mean ergodicity of power bounded operators on Banach spaces as those operators $T \in \mathcal{L}(X)$ satisfying that $(T_{[n]}(x))$ is relatively weakly compact in X for every $x \in X$. The condition of power boundedness can be relaxed (see e.g. [36, Theorem 1], [27, Theorem 1.3],[35],[18]). As a consequence of the aforementioned *Mean Ergodic Theorem*, one deduces that a power bounded operator T such that $(T_{[n]})_n$ converges in the weak operator topology is mean ergodic.

Further, Yosida and Kakutani in [36] proved the *Yosida–Kakutani Mean Ergodic Theorem* which states that a power bounded quasicompact operator is already uniformly mean ergodic.

The necessity of the condition in the next theorem is due to Dunford [16, Theorem 3.16] and the sufficiency goes back to Lin [19]. The result connects spectral properties with the uniform mean ergodicity. In [18, Theorem 2.7] it is stated for power bounded operators.

Theorem 1.1 (Dunford–Lin) *An operator T on a Banach space X is uniformly mean ergodic if and only if both $(\|T^n\|/n)_n$ converges to 0 and either $1 \in \mathbb{C}\setminus\sigma(T)$ or 1 is a pole of order 1 of the resolvent $R_T : \mathbb{C}\setminus\sigma(T) \longrightarrow \mathcal{L}(X)$, $R_T(\lambda) := (T-\lambda I)^{-1}$. Consequently if 1 is an accumulation point of $\sigma(T)$, then T is not uniformly mean ergodic.*

The following result permits us to show that condition (c) in [1, Proposition 3.1] can be substituted by power boundedness. Notice that $r_e(T) < 1$ holds precisely when T is quasicompact. Indeed, if T is quasicompact, there is $n_0 \in \mathbb{N}$ such that $\|T^{n_0}\|_e < \rho < 1$. Then $\|T^{n_0 k}\|_e^{\frac{1}{n_0 k}} \leq \rho^{\frac{1}{n_0}} < 1$ for each $k \in \mathbb{N}$. Since the limit $r_e(T) = \lim_n \|T^n\|_e^{1/n}$ exists, we get $r_e(T) < 1$. The other implication is trivial.

Proposition 1.2 *The following conditions are equivalent for $T \in \mathcal{L}(X)$:*

(a) $(T^n)_n$ converges in $\mathcal{L}(X)$ to a finite rank projection P.

(b) T is power bounded, quasicompact and $\sigma_p(T) \cap \partial\mathbb{D} \subseteq \{1\}$.

(c) $r_e(T) < 1$, $\sigma_p(T) \cap \partial\mathbb{D} \subseteq \{1\}$ and if 1 is in the spectrum then it is a pole of order 1 of the resolvent R_T.

Proof. The equivalence between (a) and (c) is [1, Proposition 3.1] . If (a) is satisfied then T is power bounded and quasicompact. Hence the equivalence between (a) and (b) is due to [36, Theorem 3.4, Corollary (ii), (iii)]. □

Our interest falls on composition operators defined on Banach spaces of analytic functions on the disc. Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$ with the topology τ_c of uniform convergence on compact sets. Given a holomorphic function $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$, the *composition operator* $C_\varphi : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ is defined by $C_\varphi f = f \circ \varphi$. Let $X \hookrightarrow H(\mathbb{D})$ be a Banach space with continuous inclusion. A holomorphic self map $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is said to be *a symbol for X* if $C_\varphi(X) \subset X$. As a consequence of the Closed Graph Theorem this is equivalent to $C_\varphi \in \mathcal{L}(X)$. Since clearly $C_\varphi^n = C_{\varphi_n}$, where $\varphi_n := \varphi \circ \overset{(n)}{\cdots} \circ \varphi$, the study of the iterates of the operator shifts to the study of the iterates of the symbol φ. The behaviour of φ and its iterates has been deeply investigated and it has great importance in our study. The self-map φ is called *elliptic* if it has a fixed point p. In this case, by the Schwarz Lemma, $|\varphi'(p)| \leq 1$. Furthermore, if $|\varphi'(p)| = 1$, then φ is called an *elliptic automorphism*. The holomorphic automorphism of the disk $\phi_p(z) := (p - z)/(1 - \overline{p}z)$ interchanges the fixed point p of φ with 0 and $\Phi = \phi_p \circ \varphi \circ \phi_p$ defines a holomorphic function with fixed point 0 with the property that $C_\Phi = C_{\phi_p} C_\varphi C_{\phi_p}$ has the ergodic properties of C_φ. When φ is an elliptic automorphism, then $\Phi(z) = \lambda z$. In this case Φ is called *rational rotation* if there exists $n \in \mathbb{N}$ such that $\lambda^n = 1$ and it is called *irrational rotation* otherwise.

The results shown in this paper depend deeply on the well known Denjoy-Wolff Theorem (see [32, 31, 33, 15]). If $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic and not an elliptic automorphism, then there exists a unique point $p \in \overline{\mathbb{D}}$ (called the Denjoy-Wolff point of φ) such that the sequence $(\varphi_n)_n$ of the iterates converges to p uniformly on compact sets of $\mathbb{D}$. If the Denjoy-Wolff point of φ is in $\mathbb{D}$, it is clearly also a fixed point and as stated before, we may always consider it is 0. If it is on the boundary of $\mathbb{D}$, then, using a rotation, we may consider it is 1 for simplicity.

The next classical result is due to Koenigs and related to the Schroeder equation. We state a particular case of this theorem, which in fact characterizes completely the point spectrum from a functional analytic point of view.

Theorem 1.3 (Koenigs) *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic with a Denjoy-Wolff point $z_0 \in \mathbb{D}$. Then $\sigma_p(C_\varphi, H(\mathbb{D})) \cap \partial\mathbb{D} = \{1\}$.*

We consider composition operators on classical spaces of holomorphic functions on the unit disc. In particular, Bloch spaces of order $p \geq 1$, weighted Bergman spaces of infinite order, for appropriate weights, Bergman spaces of order $p \geq 1$ and Hardy spaces of order $p \geq 1$.

For $p > 0$, the space of Bloch functions or *Bloch space of order p* is

$$\mathcal{B}_p := \{f \in H(\mathbb{D}) \, : \, \|f\|_{\mathcal{B}_p} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^p |f'(z)| < \infty\}.$$

It is a Banach space when endowed with the norm $\|f\| := |f(0)| + \|f\|_{\mathcal{B}_p}$. The closed subspace

$$\mathcal{B}_p^0 := \{f \in H(\mathbb{D}) \, : \, \lim_{|z| \to 1} (1 - |z|^2)^p |f'(z)| = 0\}$$

is called the *little Bloch space of order p* and is also a Banach space with the norm $\| \cdot \|$. The spaces $\mathcal{B}_1$ and $\mathcal{B}_1^0$ are nothing but the classical Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$ respectively.

All these spaces are continuously included in $H(\mathbb{D})$. It holds $C_\varphi \in \mathcal{L}(\mathcal{B}_p)$ for every $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ analytic, with $\|C_\varphi f\|_{\mathcal{B}_p} \leq \|f\|_{\mathcal{B}_p}$ as a consequence of the Schwarz-Pick Lemma. On the other hand $C_\varphi^0 := C_\varphi|_{\mathcal{B}_p^0} \in \mathcal{L}(\mathcal{B}_p^0)$ if and only if $\varphi \in \mathcal{B}_p^0$.

We restrict our study to Bloch spaces of order $p \geq 1$ as each $\mathcal{B}_p$ with $0 < p < 1$ is a space of Lipschitz functions and hence included in $A(\mathbb{D})$, see [37].

A continuous function $v : \mathbb{D} \longrightarrow]0, \infty[$ is called a *weight*. Associated to a weight v the *weighted Bergman spaces of infinite type* are defined as follows

$$H_v := \{f \in H(\mathbb{D}) \ : \ \|f\|_v := \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty\},$$

$$H_v^0 := \{f \in H(\mathbb{D}) \ : \ \lim_{|z| \to 1} v(z)|f(z)| = 0\}.$$

Both of these are Banach spaces with the norm $\| \cdot \|_v$. A weight v is called *typical* if v is radial (i.e. $v(z) = v(|z|)$), decreasing (i.e. $v(z) \leq v(w)$ if $|z| \geq |w|$) and satisfies $\lim_{|z| \to 1} v(z) = 0$.

A composition operator C_φ is well-defined and continuous on each of H_v and H_v^0 whenever v is a typical weight satisfying the Lusky condition, that is, that

$$\inf_{n \in \mathbb{N}} \frac{\hat{v}(1 - 2^{-n})}{\hat{v}(1 - 2^{-(n-1)})} > 0,$$

where $\hat{v}(z) := \frac{1}{\|\delta_z\|_{H_v^*}}$ $(z \in \mathbb{D})$ is the associated weight, see [10, Theorem 2.3].

For $p \geq 1$, the *Bergman space of order* p is

$$A^p := \{f \in H(\mathbb{D}) \ : \ \|f\|_{A^p}^p := \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty\},$$

where dA is the normalized Lebesgue measure on $\mathbb{D}$. This is a Banach space with the norm $\| \cdot \|_{A^p}$.

For $p \geq 1$, the *Hardy space of order* p is

$$H^p := \{f \in H(\mathbb{D}) \ : \ \|f\|_{H^p}^p := \lim_{r \to 1^-} \int_{\mathbb{D}} |f(re^{i\theta})|^p d\theta < \infty\},$$

which is a Banach space with the norm $\| \cdot \|_{H^p}$.

For every $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ analytic the operator C_φ is in $\mathcal{L}(A^p)$ and $\mathcal{L}(H^p)$ (see [14]).

2 Ergodic theorems on general Banach spaces of analytic functions

We consider the following conditions on a Banach space X continuously embedded in $H(\mathbb{D})$.

(PB1) Automorphisms are symbols for X.

(PB2) If $\alpha \in \partial\mathbb{D}$ then there is $f \in X$ such that $\lim_{z\to\alpha} \mathrm{Re}\, f(z) = +\infty$.

(PB3) For each symbol ψ of X with $\psi(0) = 0$, the operator $C_\psi : X \to X$ is power bounded.

(ME) If $(f_n)_n$ is a bounded sequence in X which is pointwise open convergent to $f \in X$ then $(f_n)_n$ is weakly convergent to f.

(UME) $\overline{B(0, r_e(C_\psi))} \subseteq \sigma(C_\psi)$ for each symbol ψ with Denjoy-Wolff point $0 \in \mathbb{D}$.

Remark 2.1 Since $X \hookrightarrow H(\mathbb{D})$ and this space is Montel (even nuclear), the topology of pointwise convergence and the compact–open topology coincide on bounded sets. Hence condition (ME) above is in fact equivalent to requiring any bounded sequence (f_n) in X to be weakly convergent whenever it converges in the compact–open topology.

In this section we give results of power boundedness, (uniform) mean ergodicity and asymptotic behaviour of the powers of composition operators related to these conditions. Spaces satisfying these conditions are considered in Section 3.

Proposition 2.2 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ and containing the constants. Assume (PB1)–(PB3) hold for X. Let φ be a symbol for X. Then $C_\varphi : X \longrightarrow X$ is power bounded if and only if φ is elliptic.*

Proof. If φ is not elliptic, then it has a Denjoy-Wolff α in the boundary. Take f as in (PB2). We have that $(\delta_0(C_\varphi^n(f)))_n$ is not bounded since

$$\lim_{n\to\infty} \mathrm{Re}\, C_{\varphi_n}(f)(0) = +\infty, \tag{2.1}$$

and therefore $(C_\varphi^n(f))_n$ is not bounded.

Assume now φ to have an interior fixed point. (PB1) implies that there is $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ automorphism and (a symbol) $\psi : \mathbb{D} \longrightarrow \mathbb{D}$ with $\psi(0) = 0$ such that $\varphi = \phi^{-1} \circ \psi \circ \phi$. Hence power boundedness is a consequence of (PB1) and (PB3). □

Remark 2.3 In Proposition 2.2, the condition (PB2) can be relaxed to (PB2-) For each $\alpha \in \partial\mathbb{D}$ there exists $f \in X$ so that $\lim_{z\to\alpha} |f(z)| = +\infty$. We use the full assumption of (PB2) in Proposition 2.6.(i).

Corollary 2.4 *Let X be a Banach space satisfying (PB1) and (PB3). Then for any symbol φ of X that has an interior Denjoy-Wolff point and for which C_φ is quasicompact we have that C_φ is uniformly mean ergodic.*

Proof. The conditions (PB1) and (PB3) ensure that C_φ is power bounded. The conclusion now follows from the Yosida–Kakutani Mean Ergodic Theorem. □

Now we consider the mean ergodicity of C_φ giving a characterization in the case when the symbol is an elliptic automorphism.

Proposition 2.5 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ and containing the constants. Further assume (PB1) and (PB3) hold for X. Let φ be an elliptic automorphism with fixed point $z_0 \in \mathbb{D}$ which is a symbol for X. Then*

(i) if φ is equivalent to a rational rotation, then C_φ is uniformly mean ergodic with $(C_\varphi)_{[n]} \longrightarrow \frac{1}{k}(C_\varphi + \cdots + C_\varphi^k)$ for some $k \in \mathbb{N}$,

(ii) if φ is equivalent to an irrational rotation, then C_φ is not uniformly mean ergodic,

(iii) if φ is equivalent to an irrational rotation and X contains the polynomials as a dense subspace, then C_φ is mean ergodic, with $(C_\varphi)_{[n]}f \longrightarrow C_{z_0}f$, where $C_{z_0}(f) = f(z_0)$.

Proof. We may restrict ourselves to the case when $\varphi(z) = \lambda z$ is a rotation. If it is a rational rotation, i.e., there exists $k \in \mathbb{N}$ such that $\lambda^k = 1$, then C_φ is periodic with period k (take the smallest k). Then, (see [5, Theorem 2.2] and [30, Proposition 18])

$$\lim_{n\to\infty} \left\| (C_\varphi)_{[n]} - \frac{1}{k}\sum_{j=1}^{k} C_\varphi^j \right\| = 0.$$

In case $\varphi(z) = \lambda z$ and $\lambda^n \neq 1$ for every $n \in \mathbb{N}$, we proceed as in [5, Theorem 2.2 (ii)] to see that the operator is mean ergodic. Let $f_k(z) := z^k$, with $k \in \mathbb{N}$, $k \neq 0$, then

$$\|((C_\varphi)_{[n]} f_k)(z)\| = \left\| \frac{1}{n} \sum_{j=1}^{n} \lambda^{jk} z^k \right\| \leq \frac{2}{n|1-\lambda^k|} \|f_k\|,$$

and $\lim_n (C_\varphi)_{[n]} = C_0$ on the monomials. For $k = 0$, we have $(C_\varphi)_{[n]}(1) = I(1) = 1$. Since we are assuming (PB1) and (PB3), the sequence $(C_\varphi)_{[n]}$ is equicontinuous. By density, we deduce the mean ergodicity of C_φ.

Finally we see that $\alpha \in \sigma_{ap}(C_\varphi) \subset \sigma(C_\varphi)$ for all $\alpha \in \partial\mathbb{D}$. The hypothesis on λ yields that there exists an increasing sequence $(n_k)_k \subset \mathbb{N}$ such that $\lambda^{n_k} \to \alpha$ as $k \to \infty$. Let $g_n := f_n/\|f_n\|$. We have

$$\|C_\varphi g_{n_k} - \alpha g_{n_k}\| = \frac{1}{\|f_{n_k}\|} \|\lambda^{n_k} f_{n_k} - \alpha f_{n_k}\| = |\lambda^{n_k} - \alpha|,$$

which converges to 0 as $k \to \infty$. This shows that $\partial\mathbb{D} \subset \sigma_{ap}(C_\varphi)$. Then 1 is an accumulation point of the spectrum and by Theorem 1.1, C_φ is not uniformly mean ergodic. □

Proposition 2.6 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ and containing the constants. Assume (PB1) and (PB3) hold for X. Let φ be a symbol for X. Then we have:*

(i) If (PB2) holds for X and C_φ is mean ergodic, then φ is elliptic.

(ii) If φ is elliptic and X satisfies (ME) and contains the polynomials as a dense subspace, then C_φ is mean ergodic. In case φ has a Denjoy-Wolff point $z_0 \in \mathbb{D}$, then we even have that $(C_\varphi^n)_n$ converges to C_{z_0} in the weak operator topology.

Proof. If φ were not elliptic, it would have a boundary Denjoy-Wolff α. Take f as in (PB2). From (2.1) it also follows

$$\lim_{n\to\infty} \mathrm{Re}\,((C_\varphi)_{[n]} f)(0) = +\infty,$$

and hence $((C_\varphi)_{[n]}(f))_n$ is not a bounded sequence in X and C_φ is not mean ergodic. This shows (i).

Now assume that (ME) holds and that φ is elliptic. If φ is a rotation the mean ergodicity follows from Proposition 2.5. If z_0 is the Denjoy–Wolff point of φ then $C_\varphi^n(f)$ converges pointwise to $f(z_0)$ for every $f \in X$. Since condition (PB1) holds, we have that $C_\varphi^n(f)$ is a bounded sequence in X for each $f \in X$. Furthermore, (ME) ensures that $C_\varphi^n(f)$ converges weakly to $f(z_0)$ for every $f \in X$. Apply the Mean Ergodic Theorem [27, Theorem 1.3] to obtain *(ii)*. $\square$

Remark 2.7 Summarizing the results of Proposition 2.2, Proposition 2.5 and Proposition 2.6, we obtain that in a Banach space X continuously embedded in $H(\mathbb{D})$, containing the polynomials as a dense subspace and satisfying the conditions (PB1)–(PB3) and (ME) the operator C_φ is power bounded if and only if it is mean ergodic, and if and only if φ is elliptic.

Theorem 2.8 *Let X be a Banach space continuously embedded in $H(\mathbb{D})$ and containing the constants. Assume (PB1)–(PB3) and (UME) hold for X. Let φ be a symbol for X. Then C_φ is uniformly mean ergodic if and only if either*

(i) φ is similar to a rational rotation, or

(ii) $z_0 \in \mathbb{D}$ is the Denjoy–Wolff point of φ and C_φ is quasicompact, or, equivalently, $\lim_n \|C_\varphi^n - C_{z_0}\| = 0$.

Proof. If φ is not elliptic then neither (i) nor (ii) hold, and since X satisfies (PB2) by Proposition 2.6.(i) we have that C_φ is not mean ergodic, either. So we may assume φ is elliptic, and by (PB1) that $\varphi(0) = 0$. Now, since X satisfies (PB1)–(PB3), by Proposition 2.2 the operator C_φ is power bounded.

Notice that C_φ is uniformly mean ergodic whenever (i) φ is equivalent to a rational rotation or (ii) the operator C_φ is quasicompact. Indeed if φ is similar to a rational rotation, then C_φ is uniformly mean ergodic by Proposition 2.5.(i). On the other hand, if C_φ is quasicompact then C_φ is uniformly mean ergodic due to Yosida–Kakutani Mean Ergodic Theorem.

Finally assume C_φ is uniformly mean ergodic. It remains to show that either (i) or (ii) hold. If φ is an automorphism, then by Proposition 2.5.(i) and Proposition 2.5.(iii) the symbol φ is equivalent to a rational rotation. So we may assume that φ is an elliptic non-automorphism. Since C_φ is power bounded we have

$$\|C_\varphi^n\|_e \le \|C_\varphi^n\| \le M,$$

for some positive constant independent of n. Hence by the spectral radius formula for the Calkin algebra we get $r_e(C_\varphi) \le 1$. Now, if $r_e(C_\varphi) = 1$, then $\overline{B(0,1)} \subseteq \sigma(C_\varphi)$ by the assumption (UME) on X and 1 is an accumulation point of $\sigma(C_\varphi)$, contradicting that C_φ is uniformly mean ergodic by the Dunford–Lin Theorem 1.1. So we must have $r_e(C_\varphi) < 1$, that is, that C_φ is quasicompact.

The equivalence in (ii) follows from Proposition 1.2 and Theorem 1.3. □

3 Ergodic theorems on concrete spaces

In this section we apply the results of Section 2 to some classical spaces of holomorphic functions on the unit disk $\mathbb{D}$. We first study whether the spaces satisfy the properties defined in Section 2. These results are summarized below.

	H_v	H_v^0	$\mathcal{B}_p$	$\mathcal{B}_p^0$, $p > 1$	$\mathcal{B}_1^0$	A^p, $p > 1$	A^1	H^p, $p > 1$	H^1
(PB1)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(PB2)	$*$	$*$	✓	✓	?	✓	✓	✓	✓
(PB3)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(ME)	✗	✓	✗	✓	✓	✓	✗	✓	?
(UME)	✓	✓	✓	✓	✓	$**$	$**$	$**$	$**$

$*$: (PB2) holds if v is convenient (Definition 3.2).
$**$: (UME) holds if ψ is univalent.

Lemma 3.1 *The properties (PB1) and (PB3) hold for the following spaces:*

(1) H_v and H_v^0, for a typical weight v satisfying the Lusky condition,

(2) $\mathcal{B}_p$ and $\mathcal{B}_p^0$, for $p \ge 1$,

(3) A^p and H^p, for $p \ge 1$.

Proof. The fact that $\|C_\psi\| \le 1$ if $\psi(0) = 0$ follows for H_v and H_v^0 from the formulas for C_ψ for typical weights (see [10]). For the Bloch spaces it follows from the Schwarz-Pick lemma (see [23]). The case of A^p and H^p is a consequence of Littlewood subordination principle [20, 14]. □

We introduce a subset of typical weights in order to state our results for a wide class of these spaces.

Definition 3.2 *A typical weight v is said to be convenient if it satisfies the Lusky condition and there exists $f \in H_v^0$ such that $\lim_{z \to 1} \mathrm{Re}\,(f) = +\infty$.*

We remark that all the standard weights $v_p(z) = (1 - |z|)^p$ satisfy that $f(z) = \log(1 - z) \in H_{v_p}^0$, and also the Lusky condition, hence they are convenient. Also if v is a typical weight satisfying the Lusky condition with $v = O(v_p)$ as $|z| \to 1$ then v is convenient.

Lemma 3.3 *The property (PB2) holds for the following spaces:*

(1) H_v and H_v^0, for a convenient weight v,

(2) $\mathcal{B}_p$, for $p \geq 1$,

(3) $\mathcal{B}_p^0$, for $p > 1$,

(4) A^p and H^p, for $p \geq 1$.

Proof. The case of (1) is clear by the definition of convenient weight. For the rest of cases note that $f(z) = \log(1 - z)$ is in all of the spaces considered.

□

Lemma 3.4 *The property (ME) holds for the following spaces:*

(1) H_v^0, for a typical weight v satisfying the Lusky condition,

(2) $\mathcal{B}_p^0$, for $p \geq 1$,

(3) A^p and H^p, for $p > 1$.

Proof. (1) By means of $g \mapsto vg$, H_v^0 is isometric to a subspace of $C(\widehat{\mathbb{D}})$, where $\widehat{\mathbb{D}}$ is the Alexandroff compactification of $\mathbb{D}$. We denote by H this subspace, which is formed of functions vanishing at infinity. Therefore, if $(f_n)_n \subseteq H_v^0$ is pointwise convergent to f in $\mathbb{D}$ then $(vf_n)_n$ is a bounded sequence in $C(\widehat{\mathbb{D}})$ which is pointwise convergent in $\widehat{\mathbb{D}}$ to $vf \in H$. For every functional $u \in C(\widehat{\mathbb{D}})^*$ there is a finite Radon measure μ on $\widehat{\mathbb{D}}$ such that $u(g) = \int_{\widehat{\mathbb{D}}} g d\mu$ for every $g \in C(\widehat{\mathbb{D}})$. Lebesgue's dominated convergence theorem implies that $(vf_n)_n$ is weakly convergent to vf in $C(\widehat{\mathbb{D}})$, and hence $(f_n)_n$ is weakly convergent to f in H_v^0.

(2) Let us now consider the case of $\mathcal{B}_p^0$. Assume $(f_n)_n \subseteq \mathcal{B}_p^0$ is pointwise convergent to f in $\mathbb{D}$. Since $\mathcal{B}_p^0 \hookrightarrow (H(\mathbb{D}), \tau_c)$, we get that $(f_n)_n$ is relatively

compact in $H(\mathbb{D})$ and hence $(f_n)_n$ is actually convergent to f in $(H(\mathbb{D}), \tau_c)$. We have the isometric identification $\mathcal{B}_p^0 = H_{v_p}^0 \oplus_{l_1} \mathbb{C}$, $g \mapsto (g', g(0))$. Since $(f_n)_n$ is τ_c convergent to f and the diferentiation operator $g \mapsto g'$ is continuous on $(H(\mathbb{D}), \tau_c)$ we get that $(f_n')_n$ is a bounded sequence in $H_{v_p}^0$ which is pointwise convergent to f'. Thus we conclude by the previous case that $(f_n')_n$ is weakly convergent to f'. Since $(f_n(0))_n$ converges to $f(0)$ by hypothesis, the isometric identification gives us that $(f_n)_n$ is weakly convergent to f in $\mathcal{B}_p^0$.

(3) When $p > 1$ the spaces considered are reflexive and therefore (ME) holds since the pointwise topology in $\mathbb{D}$ is Hausdorff and bounded sets are relatively weakly compact in reflexive spaces. □

Lemma 3.5 *The property (UME) holds for the following spaces:*

(1) H_v and H_v^0, for a typical weight v satisfying the Lusky condition,

(2) $\mathcal{B}_p$ and $\mathcal{B}_p^0$, for $p \geq 1$,

(3) A^p and H^p, for $p \geq 1$, whenever ψ is univalent.

Proof. It is a consequence of the spectral radii formulas given in [4, 11, 13, 24], which, summarized, are

$$\sigma(C_\psi, X) = \overline{B(0, r_e(C_\psi))} \cup \{\psi'(0)^n \ : \ n \in \mathbb{N} \cup \{0\}\},$$

where X is any of the above spaces and ψ any symbol of X with Denjoy–Wolff point 0. □

Proposition 3.6 *Neither A^1, $\mathcal{B}_p$ for $p \geq 1$, nor H_v for any typical weight v satisfy the property (ME).*

Proof. For $v(z) := (1 - |z|)$ it follows from [29] that $(H_v^0)^* = A^1$ and $(A^1)^* = H_v$. Via the dual pairing defined in [29] we have that, for $w \in \mathbb{D}$, the evaluations δ_w can be identified with $K_w(z) = \frac{2}{(1-wz)^2}$, and K_w is in H_v^0. The topology of pointwise convergence is then a Hausdorff topology weaker than the weak* topology. Hence these topologies agree on bounded sets of A^1. From this we conclude that the bounded sequences (f_n) in A^1 which are pointwise convergent to some $f \in A^1$ are precisely those which are weak*–convergent. This set of sequences is strictly contained in that of (bounded) sequences which are weakly convergent because H_v^0 is not a Grothendieck space, since it is separable and not reflexive.

The case for H_v and $\mathcal{B}_p$ is similar, and both are analogous to that of A^1. We only give the proof for H_v. In [8] it is shown that the subspace X of H_v^* formed by functionals which are continuous on the unit ball B_v of H_v for the compact–open topology satisfies $X^* = H_v$. X contains the span of the evaluations $H := \{\delta_z : z \in \mathbb{D}\}$ as a separable subspace which is separating in H_v, i.e. H is dense in X. Hence the bounded sequences in H_v which are pointwise ($\sigma(H_v, H)$-) convergent are precisely those which are weak*–convergent, and X is not a Grothendieck space since it is separable and not reflexive. □

Theorem 3.7 *Let v be a convenient weight and $p \geq 1$. Let X^0 stand for H_v^0, $\mathcal{B}_p^0$, A^p or H^p and let X stand for H_v or $\mathcal{B}_p$. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic. Then the following assertions hold:*

(i) $C_\varphi \in \mathcal{L}(X^0)$ is power bounded if and only if it is mean ergodic, and if and only if φ is elliptic.

(ii) $C_\varphi \in \mathcal{L}(X)$ is power bounded if and only if φ is elliptic.

(iii) if $C_\varphi \in \mathcal{L}(X^0)$ and φ has a Denjoy-Wolff point $z_0 \in \mathbb{D}$, then $(C_\varphi^n)_n$ converges weakly to C_{z_0} on X^0, and consequently C_φ is mean ergodic.

Proof. We do not know whether $\mathcal{B}_0 = \mathcal{B}_1^0$ satisfies (PB2). Also, property (ME) does not hold for A^1 and we do not know if it holds for H^1. For the rest of spaces apply Remark 2.7, Proposition 2.1 and assertion (ii) of Proposition 2.6.

For $\mathcal{B}_0$ recall that $\mathcal{B}_0^{**} = \mathcal{B}$ and $(C_\varphi|_{\mathcal{B}_0})^{**} = C_\varphi|_{\mathcal{B}}$ with $\|C_\varphi|_{\mathcal{B}_0}\| = \|C_\varphi|_{\mathcal{B}}\|$. These equalities also hold for the iterates and the Cesàro means. Let us denote C_φ^0 when the operator is considered on $\mathcal{B}^0$. If C_φ^0 is power bounded, then also C_φ is and hence φ is elliptic by assertion (ii) with $X = \mathcal{B}$. The converse of assertion (i) follows from (PB3). If C_φ^0 is mean ergodic, then $(\|(C_\varphi^0)_{[n]}\|)_n = (\|(C_\varphi)_{[n]}\|)_n$ is a bounded sequence, and therefore by the proof of Proposition 2.6.(i), we get that φ is elliptic. Conversely, if φ has a Denjoy–Wolff point, then (iii) follows from Proposition 2.6.(ii).

For A^1 and H^1, since (PB1), (PB2) and (PB3) hold we can apply Proposition 2.2, Proposition 2.5 and Proposition 2.6 to get (i). We show (iii). The continuous inclusions $A^2 \hookrightarrow A^1$ and $H^2 \hookrightarrow H^1$ have dense range, therefore

$(C_\varphi^n)_n$ is an equicontinuous sequence which is convergent to C_{z_0} in the weak operator topology on a dense subspace. Thus $(C_\varphi^n)_n$ also converges in the weak operator topology to C_{z_0} in $\mathcal{L}(A^1)$ and $\mathcal{L}(H^1)$. $\square$

Theorem 3.8 *Let X^0, X and φ be as in Theorem 3.7. Further assume φ to be univalent if $X^0 = H^p$ or $X^0 = A^p$. Then the following assertions hold:*

(i) $C_\varphi \in \mathcal{L}(X^0)$ is uniformly mean ergodic if and only if either φ is equivalent to a rational rotation or φ has a Denjoy-Wolff point $z_0 \in \mathbb{D}$ and C_φ is quasicompact. In this last case we even have $\lim_n \|C_\varphi^n - C_{z_0}\| = 0$.

(ii) $C_\varphi \in \mathcal{L}(X)$ is uniformly mean ergodic if and only if it is mean ergodic if and only if either φ is equivalent to a rational rotation or φ has a Denjoy-Wolff point $z_0 \in \mathbb{D}$ and C_φ is quasicompact. In this last case we even have $\lim_n \|C_\varphi^n - C_{z_0}\| = 0$.

Proof. For (ii), Lotz proved in [21] that when X is a Grothendieck Banach space which satisfies the Dunford Pettis property (a GDP space) then mean ergodicity and uniform mean ergodicity are equivalent concepts. The spaces H_v and $\mathcal{B}_p$ are isomorphic to either l_∞ or H^∞ by [22, Theorem 1.1], and are therefore GDP spaces.

The rest of the statements follow directly from Lemmata 3.1, 3.3, 3.5 and Theorem 2.8. $\square$

Observation 3.9 Notice that for every power bounded operator T we have $r_e(T) \leq 1$ and $\frac{\|T^n\|}{n}$ converges to 0. So, by the Dunford–Lin Theorem 1.1, any power bounded operator T whose spectrum contains $\overline{B(0, r_e(T))}$ is uniformly mean ergodic if and only if $r_e(T) < 1$.

Corollary 3.10 *Let v be a convenient weight and $p \geq 1$. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic. The following assertions hold:*

(i) C_φ is mean ergodic on H_v if and only if C_φ is uniformly mean ergodic on H_v if and only if C_φ is uniformly mean ergodic on H_v^0.

(ii) If $\varphi \in \mathcal{B}_p^0$, then C_φ is mean ergodic on $\mathcal{B}_p$ if and only if C_φ is uniformly mean ergodic on $\mathcal{B}_p$ if and only if C_φ is uniformly mean ergodic on $\mathcal{B}_p^0$.

Example 3.11 Next we show examples of composition operators on H_{v_p} and $\mathcal{B}_p$ which are uniformly mean ergodic but such that none of its iterates is compact. Compare this with condition (iii) in Theorem A to see the difference with $H^\infty(\mathbb{D})$ and $A(\mathbb{D})$.

Let us consider $C_\varphi : H_{v_p} \longrightarrow H_{v_p}$ for $\varphi(z) = az + (1-a)z^2$ for $0 < a < 1$. From [10], since 0 is a fixed point, it follows $\|C_\varphi\| = 1$. From [25, Theorem 2.1] we get

$$\|C_{\varphi_n}\|_e = \lim_{z \to 1} \left(\frac{1 - |z|}{1 - |\varphi_n(z)|} \right)^p = \left(\frac{1}{\varphi'(1)^n} \right)^p = \frac{1}{(2-a)^{np}}.$$

Hence $0 < \|C_{\varphi_n}\|_e < 1$ for each $n \in \mathbb{N}$. This yields that C_φ is quasicompact but C_{φ_n} is not compact for any $n \in \mathbb{N}$.

Corollary 10 in [4] gives that $C_\varphi : \mathcal{B}_p \longrightarrow \mathcal{B}_p$ for $\varphi(z) = sz/(1 - sz)$ also satisfies $0 < \|C_{\varphi_n}\|_e < 1$ for each $n \in \mathbb{N}$.

We can also provide an extension of [1, Theorem 10, Theorem 12], where it is proved that if φ has an interior Denjoy-Wolff point then the sequence of iterates $(C_\varphi^n)_n$ is convergent in $\mathcal{L}(H^p)$ if and only if φ is not inner for $1 \leq p < \infty$. We see below that for the Hardy case we can drop the hypothesis of φ being univalent in Theorem 3.8 (i). Also this Theorem extends [17, Theorem 8].

Theorem 3.12 *Let $1 \leq p < \infty$. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic and not an elliptic automorphism. Then $C_\varphi \in \mathcal{L}(H^p)$ is uniformly mean ergodic if and only if φ has a Denjoy–Wolff z_0 in $\mathbb{D}$ and is not inner, and if and only if $\|C_\varphi^n - C_{z_0}\| \longrightarrow 0$.*

Proof. By Proposition 2.6 (i) we restrict ourselves to the case when φ has an interior Denjoy Wolff point, which we can as usual assume to be 0. If φ is not inner the sequence of iterates is norm convergent by [1, Theorem 12].

Let us assume that φ is inner. If φ is univalent, since $\varphi(0) = 0$, then φ is an elliptic automorphism ([14, Corollary 3.28]). If φ is not univalent, then $\sigma(C_\varphi) = \sigma_e(C_\varphi) = \overline{\mathbb{D}}$ (see [14, Theorem 7.43]), thus C_φ is not uniformly mean ergodic by Theorem 1.1. □

Remark 3.13 We have been informed that in [2] Arendt, Chalendar, Kumar and Srivastava have obtained independently some results presented by us and also others very related to ours. They prove there that whenever 0 is the

Denjoy-Wolff point of φ, the composition operators C_φ acting on H_{v_p}, on $H^0_{v_p}$, where $v_p(z) = (1 - |z|^2)^p$, on $\mathcal{B}_p$, for $p > 1$, and also on a weighted Bergman space A^p satisfy $r_e(C_\varphi) < 1$. Then from our Theorem 3.8 we can conclude (as they do in [2] even for weighted Bergman spaces A^p_β) that in these cases C_φ is mean ergodic if and only if C^n_φ converges in norm to C_0. We see below that the situation differs in H^0_v for a typical weight v, and also for Bloch spaces.

Proposition 3.14 *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic with $\varphi(0) = 0$. Assume φ is not a rotation and $\varphi([0, 1)) \subseteq [0, 1)$. Further assume the restriction $g(x) := \varphi|_{[0,1)}(x)$ admits a continuous extension to 1, with $g(1) = 1$, which is left differentiable at 1 and such that g' is left continuous at 1. The following assertions hold:*

(i) For $t > 0$ and $v(z) := (1 - \log(1 - |z|))^{-t}$, we have $r_e(C_\varphi, H^0_v) = 1$.

(ii) If, in addition, $\varphi \in \mathcal{B}_0$, then $r_e(C_\varphi, \mathcal{B}) = r_e(C_\varphi, \mathcal{B}_0) = 1$.

Proof. We show $\|C^n_\varphi\|_e = 1$ for each $n \in \mathbb{N}$, which forces $r_e(C_\varphi) = 1$. If φ satisfies the assumptions, so do its iterates φ_n. Hence, we only write the computation for $\|C_\varphi\|_e = 1$.

First we prove (i). Consider $C_\varphi : H^0_v \longrightarrow H^0_v$. By [26, Theorem 2.1], $\|C_\varphi\|_e = \limsup_{|z|\to 1^-} v(z)/\tilde{v}(\varphi(z))$, where $\tilde{v}$ is the so-called *associated weight*. However by [7, Corollary 1.6], we have $v = \tilde{v}$. Therefore, using L'Hôpital's rule

$$\begin{aligned} 1 \geq \|C_\varphi(z)\|_e &= \limsup_{|z|\to 1^-} \frac{v(z)}{v(\varphi(z))} \\ &\geq \lim_{x\to 1^-} \left(\frac{1 - \log(1-x)}{1 - \log(1 - g(x))} \right)^{-t} \\ &= \left(\lim_{x\to 1^-} \frac{1 - g(x)}{g'(x)(1-x)} \right)^{-t} = 1. \end{aligned}$$

In order to prove (ii), we use [25, Proposition 2.2],

$$1 \geq \|C_\varphi(z)\|_e = \limsup_{|\varphi(z)|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| \geq \lim_{x\to 1} \frac{1 - x^2}{1 - g(x)^2} g'(x) = 1.$$

□

Corollary 3.15 *Let $v(z) := (1-\log(1-|z|))^{-t}$ and let $\varphi(z) := az+(1-a)z^n$, $0 \le a < 1$. Then C_φ^n converges weakly to C_0 (hence is mean ergodic) on H_v^0 and on $\mathcal{B}_0$, but C_φ is not uniformly mean ergodic on these spaces. Therefore it is neither mean ergodic on H_v nor on $\mathcal{B}$.*

The Corollary above contradicts [30, Theorem 10], since φ has an attracting fixed point, but C_φ is not uniformly mean ergodic.

The next example shows that we can easily get mean ergodic operators on $\mathcal{B}_0$ whose sequence of iterates converges weakly but not pointwise.

Example 3.16 Let $e_n(z) := z^n$. Then

$$\|e_n\|_{\mathcal{B}} = \sup_{0<x<1} nx^{n-1}(1-x^2) = \left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}} \frac{2n}{n+1}.$$

This implies that, for $\varphi(z) := e_2(z) = z^2$, by Proposition 3.14, we have

$$\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = 1,$$

but $C_\varphi : \mathcal{B}_0 \longrightarrow \mathcal{B}_0$ is not an isometry since $1 = \|e_1\|_{\mathcal{B}} \neq \|C_\varphi(e_1)\|_{\mathcal{B}} = \|e_2\|_{\mathcal{B}} = \frac{4}{3\sqrt{3}}$. Then neither $C_\varphi : \mathcal{B} \longrightarrow \mathcal{B}$ is an isometry. This shows that [3, Theorem 1.1] is not correct. Neither are the results in Section 3 of [2] stemmed from this wrong assertion, particularly [2, Theorem 3.11, Proposition 3.12]. Here by Theorem 3.7, C_φ^n converges weakly to C_0. However the operator C_φ is not uniformly mean ergodic by Theorem 3.8, since $r_e(C_\varphi) = 1$ by Proposition 3.14. Furthermore C_φ^n does not converge pointwise, since $\lim_{n\to\infty} C_\varphi^n(e_1) = 0$ weakly but

$$\lim_{n\to\infty} \|C_\varphi^n(e_1)\|_{\mathcal{B}} = \lim_{n\to\infty} \|\varphi_n\|_{\mathcal{B}} = \lim_{n\to\infty} \|e_{2^n}\|_{\mathcal{B}} = \frac{2}{e}.$$

The following question arises naturally from the given examples.

Problem Is there any self map φ with Denjoy Wolff point 0 such that the operator C_φ on H_v^0, for some typical weight v, or on B_0 satisfies $\|C_\varphi\|_e = 1$ but $r_e(C_\varphi) < 1$?

Acknowledgements. The authors wish to thank J. Bonet for interesting discussions, his careful reading of the manuscript and his advice, which certainly improved the paper. We also would like to thank R. Aron for alerting us about the existence of [2]. Finally we are deeply indebted to the referee by the careful reading of the work, the observations and the suggestions, which considerably improved the final version. In particular he/she provides us the proof of Theorem 2.8 which is more transparent than our original one.

References

[1] W. Arendt, I. Chalendar, M. Kumar, S. Srivastava, Asymptotic behaviour of the powers of composition operators on Banach spaces of holomorphic functions. *Indiana Univ. Math. J.* **67** (2018), no. 4, 1571–1595.

[2] W. Arendt, I. Chalendar, M. Kumar, S. Srivastava, Powers of composition operators: asymptotic behaviour on Bergman, Dirichlet and Bloch spaces. *J. Aust. Math. Soc.* to appear.

[3] R. F. Allen, F. Colonna. On the isometric composition operators on the Bloch space in $\mathbb{C}^n$, *J. Math. Anal. Appl.* **355** (2009) 675–688.

[4] R. Aron, M. Lindström, Spectra of weighted composition operators on weighted Banach spaces of analytic functions, *Israel J. Math.* **141** (2004), 263–276.

[5] M.J. Beltrán Meneu, M.C. Gómez-Collado, E. Jordá, D. Jornet, Mean ergodic composition operators on Banach spaces of holomorphic functions, *J. Funct. Anal.* **270** (2016), no. 12, 4369–4385.

[6] M.J. Beltrán Meneu, M.C. Gómez-Collado, E. Jordá, D. Jornet, Mean ergodicity of weighted composition operators on spaces of holomorphic functions. *J. Math. Anal. Appl.* **444** (2016), no. 2, 1640–1651.

[7] K.D. Bierstedt, J. Bonet, J. Taskinen, Associated weights and spaces of holomorphic functions. *Studia Math.* **127** (1998), no. 2, 137–168.

[8] K. D. Bierstedt, W.H. Summers, Biduals of weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A* **54** (1993), no. 1, 70–79.

[9] J. Bonet, P. Domański, A note on mean ergodic composition operators on spaces of holomorphic functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **105** (2011), no. 2, 389-396.

[10] J. Bonet, P. Domański, M Lindström, M,J. Taskinen, Composition operators between weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A* **64** (1998), no. 1, 101–118.

[11] J. Bonet, P. Galindo, M. Lindström, Spectra and essential spectral radii of composition operators on weighted Banach spaces of analytic functions. *J. Math. Anal. Appl.* 340 (2008) 884–891.

[12] J. Bonet, W. Ricker, Mean ergodicity of multiplication operators in weighted spaces of holomorphic functions, *Arch. Math.* **92** (2009) 428437.

[13] M. D. Contreras Contreras, A. G. Hernandez-Diaz, Weighted composition operators in weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A* **69** (2000), no. 1, 41–60.

[14] C. Cowen, B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[15] A. Denjoy, Sur l'itération des fonctions analytiques, *C.R. Acad. Sci. Paris* **182** (1926) 255–257.

[16] N. Dunford, Spectral theory. I. Convergence to projections. *Trans. Amer. Math. Soc.* **54**, (1943). 185–217.

[17] S.A. Han, Z. H Zhou, Mean ergodicity of composition operators on Hardy space. *Proc. Indian Acad. Sci. Math. Sci.* **129** (2019), no. 4, Art. 45, 10 pp. 47B33 (47B38)

[18] U. Krengel, *Ergodic Theorems*, de Gruyter, Berlin, 1985.

[19] M. Lin, On the uniform ergodic theorem, *Proc. Amer. Math. Soc.* **43** (2) (1974) 337–340.

[20] J.E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.*, **23** (1925) 481–519.

[21] H.P. Lotz, Uniform convergence of operators on L^∞ and similar spaces, *Math. Z.*, **190** (1985), 207–220.

[22] W. Lusky, On the isomophism classes of weighted spaces of harmonic and holomorphic functions, *Studia Math.* **175** (2006), 19–45

[23] K. Madigan, A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347** (1995), no. 7, 2679-2687.

[24] B. McCluer, K. Saxe, Spectra of composition operators on Bloch and Bergman spaces. *Israel J. Math* **128** (2002), 325–354.

[25] A. Montes-Rodríguez, The essential norm of a composition operator on Bloch spaces, *Pac. J. Math.*, **188** (1999), No. 2, 338–351.

[26] A. Montes-Rodríguez, Weighted composition operators on weighted Banach spaces of analytic functions, *J. London Math. Soc* (2) **61** (2000), 872–884

[27] K. Petersen, *Ergodic theory.* Cambridge Studies in Advanced Mathematics, 2. Cambridge University Press, Cambridge, 1983.

[28] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, Berlin, 1993.

[29] A. L. Shields, D. L. Williams, Bonded projections, duality, and multipliers in spaces of analytic functions. *Trans. Amer. Math. Soc.* **162** (1971), 287–302.

[30] E. Wolf, Power bounded composition operators, *Comput. Methods Funct. Theory* **12** (1) (2012) 105–117.

[31] J. Wolff, Sur l'itération des fonctions bornées, *C.R. Acad. Sci. Paris* **182** (1926) 200–201.

[32] J. Wolff, Sur l'itération des fonctions holomorphes dans une région, et dont les valeurs appartiennent à cette région, *C.R. Acad. Sci. Paris* **182** (1926) 42–43.

[33] J. Wolff, Sur une généralisation d'un théorème de Schwarz, *C.R. Acad. Sci. Paris* **182** (1926) 918–920.

[34] K. Yosida, Mean ergodic theorem in Banach spaces. *Proc. Imp. Acad.* **14** (1938), no. 8, 292–294.

[35] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[36] K. Yosida, S. Kakutani, Operator-theoretical treatment of Markoff's process and mean ergodic theorem, *Ann. Math.*, **42** (1941), No. 1.

[37] K. H. Zhu, Bloch type spaces of analytic functions. *Rocky Mountain J. Math.* **23** (1993), no. 3, 1143–1177.