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Aggregation functions of topological structures

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Abstract

It is a natural question if from a family of sets with the same mathematical structure we can obtain a new set with the same structure. In particular, we can ask if from a family of metric spaces we can obtain a new metric in the Cartesian product of the spaces. This problem was solved by Borsík and Doboš who characterized functions that merge several metrics into a single one in the Cartesian product. The above question has been extended to other contexts like how to obtain a norm or an asymmetric norm from a family of norms or asymmetric norms. In addition, Pedraza, Rodríguez-López and Valero characterized the functions that aggregate metrics and quasi-metrics in the fuzzy context.

In this paper we have made a review of the existing literature about functions that aggregate metrics, quasi-metrics, norms and asymmetric norms on products and on sets in the classic sense, as well as functions that aggregate metrics and quasi-metrics on products and on sets in the fuzzy context. Moreover, we propose original results to characterize functions that aggregate norms and quasi-norms in the fuzzy context.

Keywords: Metric, quasi-metric, norm, asymmetric norm, fuzzy, aggregation

1 Introduction

Aggregation is a mathematical process whose aim is to combine a set of values into a single one which captures in some sense certain information contained in the original values. This process is usually made by means of a function that it is called an *aggregation function*. The first easy example of an aggregation function that comes to mind is the arithmetic mean but there is huge amount of aggregation functions [18]. Nevertheless, the emergence of the theory of aggregation functions as an independent and important mathematical research field is very recent. This has been caused by the appearance of aggregation functions in several mathematical areas where decision-making is important as probability [37], computer science [28], economics [16, 34], etc. This lead to the necessity of establish a theoretical basis for aggregation functions which started in the 1980s. Nowadays, a lot of work has been developed related with this kind of functions and some monographs have appeared tackling the basics of this theory [18, 5].
Following [18], given $\mathbb{I}$ a nonempty real interval, an aggregation function is a function $F : \mathbb{I}^n \to \mathbb{I}$ verifying that

- is isotone;
- satisfies the boundary conditions

$$\inf_{\boldsymbol{x}\in\mathbb{I}^n} F(\boldsymbol{x}) = \inf \mathbb{I} \qquad \text{and} \qquad \sup_{\boldsymbol{x}\in\mathbb{I}^n} F(\boldsymbol{x}) = \sup \mathbb{I}.$$

Notice that the requirements asked to a function in order to be an aggregation function are in concordance with the idea behind this kind of functions. First of all, if you consider a set of n values in the interval $\mathbb{I}$ then it is logic that if the aggregated value resumes in some sense the information of the original values then it must belong also to $\mathbb{I}$. This is the reason for considering that the image of F must be included in $\mathbb{I}$. Moreover, isotonicity is required for obtaining that if some of the values increases then the aggregated value must also increase. Finally, the boundary

conditions assure that if $\mathbb{I}$ contains its extreme values and the inputs of the function are all equal to one of the extremes then the output is also the extreme.
In this way, given $n \in \mathbb{N}$, the following functions $f, g, h : [0, +\infty)^n \to [0, +\infty)$ given by

1. Arithmetic mean
$$f(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$$

2. Geometric mean
$$g(x_1, \ldots, x_n) = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$$

3. Minimum
$$h(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$$

are examples of aggregation functions.
From the above definition we deduce that usually aggregation functions only aggregate a finite amount of values. Nevertheless, some extensions of these basic aggregation functions have been considered in the literature. In this way the so-called *extended aggregation functions* [18] are used to aggregate any fixed but arbitrary number of values. On its part, *infinitary aggregation functions* [18, Definition A.1.],[26] allow to aggregate infinitely but countably many inputs. Moreover, aggregation functions defined on bounded partially ordered sets rather than on the cartesian product of an interval have also appeared in the literature [10, 21].
As well as the aggregation of values, aggregation functions can be used for merging a family of mathematical structures of the same type into a single one. As it was pointed out in [30], finite unions and finite intersections of subsets of a nonempty set X fit into this aggregation procedure. In this way, let us identify a subset A of X with its characteristic function $\chi_A : X \to \{0, 1\}$. Consider the aggregation function $F : [0, 1]^n \to [0, 1]$ given by $F(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$. Given a nonempty family $\{A_1, \ldots, A_n\}$ of subsets of X, consider $\chi_{A_1} \triangle \ldots \triangle \chi_{A_n} : X \to [0, 1]^n$, the diagonal of the mappings $\{\chi_{A_i}\}_{i=1}^n$ [15] given by

$$(\chi_{A_1} \triangle \ldots \triangle \chi_{A_n})(x) = (\chi_{A_1}(x), \ldots, \chi_{A_n}(x))$$

for all $x \in X$. Then the composition $F \circ (\chi_{A_1} \triangle \ldots \triangle \chi_{A_n})$ is the characteristic function of the set $\cup_{i=1}^n A_i$. If F is taken as the minimum, then the characteristic function of the intersection $\cap_{i=1}^n A_i$ is obtained.
It is widely recognized that the first deep study about the aggregation of a certain topological structure is due to Borsík and Doboš who characterized the functions that merge several metrics into a single metric [7, 11]. We next explain in detail what this exactly means [11, Chapter 9].
Let I be a nonempty set of indices. Consider an indexed family $\{(\mathbb{X}_i, d_i)\}_{i \in I}$ of metric spaces. Define $\widetilde{\mathbf{d}} : \left(\prod_{i \in I} \mathbb{X}_i\right) \times \left(\prod_{i \in I} \mathbb{X}_i\right) \to [0, +\infty)^I$ as

$$\widetilde{\mathbf{d}}((x_i)_{i \in I}, (y_i)_{i \in I}) = (d_i(x_i, y_i))_{i \in I}.$$

Then the function $f : [0, +\infty)^I \to [0, +\infty)$ is said to be a *metric preserving function* (a metric aggregation function *on products* in our terminology (see Definition 2.2)) if for each indexed family $\{(\mathbb{X}_i, d_i)\}_{i \in I}$ of metric spaces then $f \circ \widetilde{\mathbf{d}}$ is a metric on $\prod_{i \in I} \mathbb{X}_i$. In 1981, Borsík and Doboš [7, 11] characterized completely these functions (see Theorem 2.1) which turned out to be aggregation

functions. Notice that the usual metric defined on the Cartesian product of metric spaces can be obtained by means of a metric preserving function (see Example 2.2).
The same problem for quasi-metrics (metrics which do not satisfy the symmetry axiom) was solved by Mayor and Valero [24] in 2010 (see Theorem 2.2). We should also mention that Pradera and Trillas [33] studied a similar but different problem for pseudometrics (metrics which do not satisfy a separation axiom). The problem is as follows. Let us consider a fixed nonempty set $\mathbb{X}$ and a family $\{d_i : i \in I\}$ of pseudometrics on $\mathbb{X}$. Define $\mathbf{d} : \mathbb{X} \times \mathbb{X} \to [0, +\infty)^I$ as

$$\mathbf{d}(x, y) = (d_i(x, y))_{i\in I}.$$

Then a function $f : [0, +\infty)^I \to [0, +\infty)$ is said to *aggregate pseudometrics* (a pseudometric aggregation function *on sets* in our terminology (see Definition 2.2)) if $f \circ \mathbf{d}$ is a pseudometric on $\mathbb{X}$. Pradera and Trillas charaterized this type of functions when I is finite although their proof is also valid for an arbitrary set of indices I. Furthermore, they showed that when I is finite these functions are equivalent to the pseudometric preserving functions in the sense of Borsík and Doboš. Nevertheless this is not true for metrics since, for example, the projection is an example of a function which aggregates metrics but it is not metric preserving. We also refer the reader to a recent work by Mayor and Valero [25] who have characterized the functions which aggregate metrics instead of pseudometrics. In all these cases, the functions are aggregation functions.
From the above, we have that there are two different but related approaches to the aggregation of mathematical structures: the one due to Borsík and Doboš and the other one due to Pradera and Trillas. Since the terminology used by these authors is very different for problems which are related we will use here the unified terminology used in [32]. In this way, we will speak about functions aggregating a certain mathematical structure *on products* in the first case, meanwhile for the second case we will use the terminology functions aggregating a certain mathematical structure *on sets* (see Definitions 2.2 and 2.9).
At this point, it is natural to wonder if we can find in the literature other results characterizing functions aggregating other mathematical structures. In 1991 [19], Herburt and Moszyńska analyzed the same problem solved by Borsík and Doboš but for norms (see Theorem 2.5). On its part, Martín, Mayor and Valero [22] solved the same problem for asymmetric norms. As for metrics, the functions are aggregation functions.
The aggregation of (asymmetric) norms on sets, that is, in the sense of Pradera and Trillas, has been recently studied [31].
On the other hand, the aggregation of some fuzzy structures has been an active area of research in the last years. For example, Saminger, Mesiar and Bodenhofer [36] have studied conditions under which an aggregation function preserves the $*$-transitiviy of a family of fuzzy binary relations, where $*$ is a t-norm. We explain in detail this question. Recall that a binary operation $*\colon [0, 1] \times [0, 1] \to [0, 1]$ is called a *triangular norm* or a *t-norm* if $([0, 1], *)$ is an Abelian monoid with unit 1, such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$, with $a, b, c, d \in [0, 1]$.
A *fuzzy binary relation* R on a nonempty set X is a function from $X \times X$ to $[0, 1]$, i. e. a fuzzy subset of $X \times X$. Moreover, R is said to be $*$-*transitive* if $R \circ_* R \le R$ where

$$(R \circ_* R)(x, z) = \bigvee_{y\in X} R(x, y) * R(y, z)$$

for every $x, z \in X$.

Suppose that R_i is an $*$-transitive fuzzy binary relation on a nonempty set X_i for every $i \in \{1, \ldots, n\}$. Define $\widetilde{\mathbf{R}} : (\prod_{i=1}^n X_i) \times (\prod_{i=1}^n X_i) \to [0,1]^n$ as

$$\widetilde{\mathbf{R}}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = (R_1(x_1, y_1), \ldots, R_n(x_n, y_n)).$$

We say that a function $F : [0,1]^n \to [0,1]$ preserves $*$-transitive fuzzy binary relations [36] if whenever R_i is an $*$-transitive fuzzy binary relation on a nonempty set X_i for every $i \in \{1, \ldots, n\}$ then $F \circ \widetilde{\mathbf{R}}$ is an $*$-transitive fuzzy binary relation on $\prod_{i=1}^n X_i$. Saminger, Mesiar and Bodenhofer [36] proved a characterization of this functions in terms of domination, a more general concept than the one of $*$-supmultiplicativity that will be key in our work (Definition 3.5). Observe that this problem follows the same model than the problem of the functions which aggregate metrics on products considered by Borsík and Doboš.

On the other hand, Drewniak and Dudziak [12, 13] studied also the aggregation of fuzzy binary relations but on sets. Concretely, they studied those functions $F : [0,1]^n \to [0,1]$ such that whenever $\{R_i : i = 1, \ldots, n\}$ is a family of $*$-transitive fuzzy binary relations on an arbitrary nonempty set X then $F \circ \mathbf{R}$ is an $*$-transitive fuzzy binary relation on X where $\mathbf{R} : X \times X \to [0,1]^n$ is given by

$$R(x, y) = (R_1(x, y), \ldots, R_n(x, y))$$

for every $x, y \in X$.

Other interesting references where aggregation of fuzzy structures has been studied are [6, 14, 20, 23].

We stress that Pedraza, Rodríguez-López and Valero have recently solved the question for the aggregation of fuzzy (quasi-)metrics on products and on sets [32], extending to the fuzzy context the results of Borsík and Dobŏs, and Mayor and Valero (see Section 3). Nevertheless, in the literature it has not been previously studied the problem of the aggregation of fuzzy norms. In this manner, the main objective of this work is to characterize those functions which aggregate fuzzy norms. This will be done in the last section of this work where we include the solution to this problem [29].

The other goal of this work has been to gather together the already known results about the aggregation of some mathematical structures like (quasi-)metrics, (asymmetric) norms and fuzzy (quasi-)metrics. We must emphasize that during our work we have detected a wrong example in [22] (see Section 2.3) as well as a mistake in the proof of [22, Theorem 12] that we correct (see Theorem 2.3).

2 Aggregation functions for metrics, quasi-metrics, norms and asymmetric norms

2.1 Aggregation functions for metrics and quasi-metrics on products

First of all we are going to remember the basic concepts of metrics and quasi-metrics.

Definition 2.1 ([22])**.** Let X be a nonempty set, and $d : X \times X \longrightarrow \mathbb{R}^+$ be a nonnegative real valued function. We say that d is a quasi-metric on X if $\forall x, y, z \in X$ we have:

(i) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$.

(ii) $d(x,z) \leq d(x,y) + d(y,z)$.

If the function d satisfies in addition that $\forall x, y \in X$

(iii) $d(x,y) = d(y,x)$

we say that d is a metric on X.

Given a quasi-metric d on a nonempty set X, we will call the pair (X,d) a quasi-metric space. We can define the nonnegative real function d^{-1} on $X \times X$ by:

$$d^{-1}(x,y) = d(y,x) \ \forall x, y \in X.$$

This function is a quasi-metric on X called the conjugate of d.
Note that every quasi-metric d on X induces a metric d^s on X as follows:

$$d^s(x,y) = d(x,y) \vee d(y,x) \ \forall x, y \in X,$$

where $\vee$ stands for the maximum operator.

Example 2.1. *Let us consider the function $l : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ defined by*

$$l(x,y) = max\{x - y, 0\} \ \forall x, y \in \mathbb{R}.$$

Then we have that l is a quasi-metric. Notice that for the quasi-metric l we have that

$$l^{-1}(x,y) = l(y,x) = \max\{y - x, 0\}$$

and

$$l^s(x,y) = \max\{l(x,y), l(y,x)\} = \max\{\max\{x - y, 0\}, \max\{y - x, 0\}\} = |x - y|.$$

Given a family of metric spaces $\{(\mathbb{X}_i, d_i) : i \in I\}$ our interest remains in obtain a new metric on $\prod_{i \in I} \mathbb{X}_i$. It is well known how to find a metric in the Cartesian product that induces the product topology. Following this idea, Borsík and Doboš give a characterization of functions $\phi : [0, +\infty)^I \longrightarrow [0, +\infty)$ that given a family of metric spaces $\{(\mathbb{X}_i, d_i) : i \in I\}$ hold that $\phi \circ \tilde{\mathbf{d}} : \left(\prod_{i \in I} \mathbb{X}_i\right) \times \left(\prod_{i \in I} \mathbb{X}_i\right) \longrightarrow [0, +\infty)$ is a metric on $\prod_{i \in I} \mathbb{X}_i$ where $\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \phi((d_i(\mathbf{x}_i, \mathbf{y}_i))_{i \in I})$. Mayor y Valero [24] give a characterization of aggregation functions but in the context of quasi-metrics.

Definition 2.2 ([11, 24])**.** Let $\phi : [0, +\infty)^I \to [0, +\infty)$ be a function. We will say that ϕ is a (quasi-)metric aggregation function on products if for every family of (quasi-)metric spaces $\{(\mathbb{X}_i, d_i) : i \in I\}$ the function $\phi \circ \tilde{\mathbf{d}} : \prod_{i \in I} \mathbb{X}_i \times \prod_{i \in I} \mathbb{X}_i \longrightarrow [0, \infty)$ is a (quasi-)metric on $\prod_{i \in I} \mathbb{X}_i$ where $\tilde{\mathbf{d}} : \prod_{i \in I} \mathbb{X}_i \times \prod_{i \in I} \mathbb{X}_i \longrightarrow [0, \infty)^I$ is defined as

$$\tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = (d_i(\mathbf{x}_i, \mathbf{y}_i))_{i \in I}$$

$\forall \mathbf{x}, \mathbf{y} \in \prod_{i \in I} \mathbb{X}_i$.

Example 2.2 ([29])**.** *Let $\{(\mathbb{X}_n, d_n) : n \in \mathbb{N}\}$ be a family of metric spaces and consider the function $\tilde{\mathbf{d}} : \prod_{n\in\mathbb{N}} \mathbb{X}_n \times \prod_{n\in\mathbb{N}} \mathbb{X}_n \longrightarrow [0, +\infty)^{\mathbb{N}}$ defined by $\tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = (d_n(\mathbf{x}_n, \mathbf{y}_n))_{n\in\mathbb{N}}$. If we define the function $\phi : [0, +\infty)^{\mathbb{N}} \longrightarrow [0, +\infty)$ such that*

$$\phi(\mathbf{x}) = \sum_{n\in\mathbf{N}} \frac{\min\{\mathbf{x}_n, 1\}}{2^n}$$

then we have that $\phi \circ \tilde{\mathbf{d}}$ is a metric on $\prod_{n\in\mathbb{N}} \mathbb{X}_n$.
Let us see that this fact is true. First we see that $\phi \circ \tilde{\mathbf{d}}$ is well defined. Consider $\mathbf{x}, \mathbf{y} \in \prod_{n\in\mathbb{N}} \mathbf{X}_n$, then we have that

$$\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \sum_{n\in\mathbb{N}} \frac{\min\{d_n(\mathbf{x}_n, \mathbf{y}_n), 1\}}{2^n} \leq \sum_{n\in\mathbb{N}} \frac{1}{2^n} < \infty.$$

The symmetry is obvious since d_n is a metric in $\mathbb{X}_n$ $\forall n \in \mathbb{N}$.
Suppose that there exist $\mathbf{x}, \mathbf{y} \in \prod_{n\in\mathbb{N}} \mathbb{X}_n$ such that $\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = 0$, then we have that

$$\sum_{n\in\mathbb{N}} \frac{\min\{d_n(\mathbf{x}_n, \mathbf{y}_n), 1\}}{2^n} = 0,$$

but $0 \leq \frac{\min\{d_n(\mathbf{x}_n,\mathbf{y}_n),1\}}{2^n}$ $\forall n \in \mathbb{N}$, and it takes the value 0 if and only if $\min\{d_n(\mathbf{x}_n, \mathbf{y}_n), 1\} = 0$ $\forall n \in \mathbb{N}$, i. e. if $d_n(\mathbf{x}_n, \mathbf{y}_n) = 0$ $\forall n \in \mathbb{N}$, and from the fact that d_n is a metric on $\mathbb{X}_n$ we have that this is possible if and only if $\mathbf{x}_n = \mathbf{y}_n$, so we have that

$$\sum_{n\in\mathbb{N}} \frac{\min\{d_n(\mathbf{x}_n, \mathbf{y}_n), 1\}}{2^n} = 0$$

if and only if $d_n(\mathbf{x}_n, \mathbf{y}_n) = 0$ $\forall n \in \mathbb{N}$, i. e. if and only if $\mathbf{x} = \mathbf{y}$.
Now we prove the triangle inequality. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \prod_{n\in\mathbb{N}} \mathbb{X}_n$. Then we have that $d_n(\mathbf{x}_n, \mathbf{y}_n) \leq d_n(\mathbf{x}_n, \mathbf{z}_n) + d_n(\mathbf{z}_n, \mathbf{y}_n)$ $\forall n \in \mathbb{N}$ and this implies that $\min\{d_n(\mathbf{x}_n, \mathbf{y}_n), 1\} \leq \min\{d_n(\mathbf{x}_n, \mathbf{z}_n), 1\} + \min\{d_n(\mathbf{z}_n, \mathbf{y}_n), 1\}$ $\forall n \in \mathbb{N}$. We can conclude
$\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \sum_{n\in\mathbb{N}} \frac{\min\{d_n(\mathbf{x}_n\mathbf{y}_n),1\}}{2^n} \leq \sum_{n\in\mathbb{N}} \frac{\min\{d_n(\mathbf{x}_n,\mathbf{z}_n),1\}}{2^n} + \sum_{n\in\mathbb{N}} \frac{\min\{d_n(\mathbf{z}_n,\mathbf{y}_n),1\}}{2^n} = \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{z}) + \phi \circ \tilde{\mathbf{d}}(\mathbf{z}, \mathbf{y})$.

A natural question is: When a function is a metric aggregation function? (or a quasi-metric aggregation function). Doboš in [11] gives a characterization of when a function preserves metrics. For this purpose he defined the concept of triangular triplet.
We will denote by $\mathbb{R}^I$ the set of all real-valued functions defined on a nonempty set of indices I. Analogously, we will denote by $[0, \infty)^I$ the set of nonnegative real-valued functions defined on I. The elements of the cartesian product will be denoted by boldface letters. We will write $\mathbf{x}_i$ instead of $\mathbf{x}(i)$ $\forall \mathbf{x} \in \mathbb{R}^I$ and we will denote by $\mathbf{0}$ the element of $\mathbb{R}^I$ given by $\mathbf{0}_i = 0$ $\forall i \in I$. In $\mathbb{R}^I$ we will consider the partial order $\preceq$ defined by $\mathbf{x} \preceq \mathbf{y}$ if and only if $\mathbf{x}_i \leq \mathbf{y}_i$ $\forall i \in I$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^I$.

Definition 2.3 ([11])**.** Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^I$. We say that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a triangular triplet if $\mathbf{a} \preceq \mathbf{b}+\mathbf{c}$, $\mathbf{b} \preceq \mathbf{a} + \mathbf{c}$ and $\mathbf{c} \preceq \mathbf{a} + \mathbf{b}$.
Analogously, if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (0, \infty)^I$. We say that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a positive triangular triplet if $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$, $\mathbf{b} \preceq \mathbf{a} + \mathbf{c}$ and $\mathbf{c} \preceq \mathbf{a} + \mathbf{b}$.

In Doboš [11] we can see two equivalences of being a triangular triplet. If $(a, b, c) \in \mathbb{R}^3$ is a triangular triplet it is equivalent to say that

1. $|a-b| \leq c \leq a+b$
2. $a+b+c \geq 2max\{a,b,c\}$.

Example 2.3. *We notice that* $\forall \mathbf{a}, \mathbf{b} \in [0,+\infty)^I$ *the triplet* $(\mathbf{a}, \mathbf{b}, \mathbf{a}+\mathbf{b})$ *is a triangular triplet.*

Example 2.4. *Let* $(\mathbb{X}, d)$ *be a metric (or quasi-metric) space. Then* $\forall x, y, z \in \mathbb{X}$ *we have by triangular inequality that* $(d(x,y), d(x,z), d(z,y))$ *is a triangular triplet (or asymmetric triangular triplet).*

It is obvious that every triangular triplet is an asymmetric triangular triplet, but we can find asymmetric triangular triplets that are not triangular triplets.

Example 2.5. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$ *be such that* $\mathbf{a} \preceq \mathbf{b}$*,* $\mathbf{a} \neq \mathbf{b}$*, then we have that* $(\mathbf{a}, \mathbf{b}, \mathbf{0})$ *is an asymmetric triangular triplet because*

$$\mathbf{a} \preceq \mathbf{b} + \mathbf{0} = \mathbf{b},$$

but it is not a triangular triplet, because

$$\mathbf{b} \npreceq \mathbf{a} + \mathbf{0} = \mathbf{a}.$$

Definition 2.4 ([11])**.** We will say that ϕ preserves triangular triplets, if given $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0,+\infty)^I$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a triangular triplet, we have that $(\phi(\mathbf{a}), \phi(\mathbf{b}), \phi(\mathbf{c}))$ is a triangular triplet. Analogously we can define when ϕ preserves positive triangular triplets.

Now we are ready to introduce theorem that gives a characterization of metric aggregation functions. First, we will show a proposition that we will need to prove the theorem.

Proposition 2.1 ([11])**.** *Let* $a, b, c \in \mathbb{R}^+$*. Then* (a,b,c) *is a triangular triplet if and only if there are different points* $x, y, z \in \mathbb{R}^2$*, such that*

$$a = e(x,y), \;\; b = e(x,z), \;\; c = e(z,y),$$

where e *denotes the Euclidean metric on* $\mathbb{R}^2$*.*

Proof. Suppose that (a,b,c) is a triangular triplet. Put

$$\begin{aligned} x &= \left(\frac{a}{2}, 0\right), \; y = \left(-\frac{a}{2}, 0\right) \\ z &= \left(\frac{c^2-b^2}{2a}, \frac{1}{2a} \cdot \sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}\right). \end{aligned}$$

Then $a = e(x,y)$, $b = e(x,z)$, $c = e(z,y)$.
On the other hand if $x, y, z \in \mathbb{R}^2$, x, y, z different points, then

$$(e(x,y), e(x,z), e(y,z)) \text{ is a triangular triplet.}$$

This is immediate from the triangular inequality. $\square$

Theorem 2.1 ([11])**.** *Let* $\phi : [0,\infty)^I \longrightarrow [0,+\infty)$*. Then the assertions below are equivalent:*

1. *ϕ is a metric aggregation function on products.*

2. *ϕ holds the following properties:*

   (i) $\phi^{-1}(0) = \{\mathbf{0}\}$,

   (ii) *ϕ preserves triangular triplets.*

Proof. Suppose that ϕ is a metric aggregation function on products. Let $\mathbf{x} \in [0, +\infty)^I$ such that $\phi(\mathbf{x}) = 0$ and consider the indexed family of metric spaces $\{(\mathbb{X}_i, d_i)\}_{i\in I}$, where $\mathbb{X}_i = \mathbb{R}$ and $d_i = e$ $\forall i \in I$, where e denotes the Euclidean metric on $\mathbb{R}$. Since ϕ is a metric aggregation function we have that $\phi \circ \tilde{\mathbf{d}}$ is a metric on $\mathbb{R}^I$ so

$$0 = \phi(\mathbf{x}) = \phi((\mathbf{x}_i)_{i\in I}) = \phi((e(\mathbf{x}_i, 0))_{i\in I}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{0}) \Leftrightarrow \mathbf{x} = \mathbf{0}.$$

We have proved that $\phi^{-1}(0) = \{\mathbf{0}\}$.
Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, +\infty)^I$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a triangular triplet. Then, $(\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i)$ is a triangular triplet. By proposition 2.1 there exist $x_i, y_i, z_i \in \mathbb{R}^2$ such that $\mathbf{a}_i = e(x_i, y_i)$, $\mathbf{b}_i = e(x_i, z_i)$ and $\mathbf{c}_i = e(z_i, y_i)$ $\forall i \in I$, where e denotes the Euclidean metric on $\mathbb{R}^2$. Consider the family of metric spaces $\{(\mathbb{Y}_i, q_i)\}_{i\in I}$ where $\mathbb{Y}_i = \mathbb{R}^2$ and $q_i = e$ $\forall i \in I$. Then, since ϕ is a metric aggregation function on products, then $\phi \circ \tilde{\mathbf{d}}$ is a metric on $(\mathbb{R}^2)^I$ so

$$\begin{aligned}
&\phi(\mathbf{a}) = \phi((\mathbf{a}_i)_{i\in I}) = \phi((e(x_i, y_i))_{i\in I}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) \leq \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{z}) + \phi \circ \tilde{\mathbf{d}}(\mathbf{z}, \mathbf{y}) \leq \\
&\ \leq \phi((e(x_i, z_i))_{i\in I}) + \phi((e(z_i, y_i))_{i\in I}) = \phi(\mathbf{b}) + \phi(\mathbf{c}), \\
&\phi(\mathbf{b}) = \phi((\mathbf{b}_i)_{i\in I}) = \phi((e(x_i, z_i))_{i\in I}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{z}) \leq \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) + \phi \circ \tilde{\mathbf{d}}(\mathbf{z}, \mathbf{y}) \leq \\
&\phi((e(x_i, y_i))_{i\in I}) + \phi((e(z_i, y_i))_{i\in I}) = \phi(\mathbf{a}) + \phi(\mathbf{c}), \\
&\phi(\mathbf{c}) = \phi((\mathbf{c}_i)_{i\in I}) = \phi((e(y_i, z_i))_{i\in I}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{y}, \mathbf{z}) \leq \phi \circ \tilde{\mathbf{d}}(\mathbf{y}, \mathbf{x}) + \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{z}) \leq \\
&\ \leq \phi((e(y_i, x_i))_{i\in I}) + \phi((e(x_i, z_i))_{i\in I}) = \phi(\mathbf{a}) + \phi(\mathbf{b}),
\end{aligned}$$

where $\mathbf{x} = (\mathbf{x}_i)_{i\in I}$, $\mathbf{y} = (\mathbf{y}_i)_{i\in I}$, $\mathbf{z} = (\mathbf{y}_i)_{i\in I}$. Then $(\phi(\mathbf{a}), \phi(\mathbf{b}), \phi(\mathbf{c}))$ is a triangular triplet.
Conversely, suppose that $\phi^{-1}(0) = \{\mathbf{0}\}$ and ϕ preserves triangular triplets. Let $\{(\mathbb{X}_i, d_i)\}_{i\in I}$ be an indexed family of metric spaces and $\mathbf{x}, \mathbf{y} \in \prod_{i\in I} \mathbb{X}_i$. Since d_i is a metric $\forall i \in I$ we have

$$\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \phi((d_i(\mathbf{x}_i, \mathbf{y}_i))_{i\in I}) = \phi((d_i(\mathbf{y}_i, \mathbf{x}_i))_{i\in I}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{y}, \mathbf{x})$$

Now, suppose that $\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = 0$. Since $\phi^{-1}(0) = \{\mathbf{0}\}$ we have that $\tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \Leftrightarrow d_i(\mathbf{x}_i, \mathbf{y}_i) = \mathbf{0}$ $\forall i \in I$. We conclude that $\mathbf{x}_i = \mathbf{y}_i$ $\forall i \in I$ i.e, $\mathbf{x} = \mathbf{y}$.
Now, let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \prod_{i\in I} \mathbb{X}_i$, we have that $(d_i(\mathbf{x}_i, \mathbf{y}_i), d_i(\mathbf{x}_i, \mathbf{z}_i, d(\mathbf{z}_i), \mathbf{y}_i))$ is a triangular triplet, and then, $(\tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}), \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{z}), \tilde{\mathbf{d}}(\mathbf{z}, \mathbf{y}))$ is a triangular triplet. Since ϕ preserves triangular triplets we have

$$\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) \leq \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{z}) + \phi \circ \tilde{\mathbf{d}}(\mathbf{z}, \mathbf{y}).$$

So we have that $\phi \circ \tilde{\mathbf{d}}$ is a metric on $\prod_{i\in I} \mathbb{X}_i$ i.e, ϕ is a metric aggregation function on products. □

From now on in this section, we will consider $\{(\mathbb{X}_i, d_i)\}_{i\in I}$ an indexed family of quasi-metric vector spaces.
We will show the conditions that a function ϕ needs to hold to be a quasi-metric aggregation function, but first, we will need some definitions.

Definition 2.5 ([24]). Let $\mathbf{x}, \mathbf{y} \in [0, \infty)^I$ we will say that ϕ is isotone if $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$.

Definition 2.6 ([24]). Let $\mathbf{x}, \mathbf{y} \in [0, \infty)^I$, we say that ϕ is subadditive if $\phi(\mathbf{x} + \mathbf{y}) \leq \phi(\mathbf{x}) + \phi(\mathbf{y})$. Analogously we will say that ϕ is positive subadditive if is subadditive $\forall \mathbf{x}, \mathbf{y} \in (0, +\infty)^I$.

The problem of characterizing functions that aggregate quasi-metrics on products was solved by Mayor and Valero [24]. We recall their characterization in the following results.

Lemma 2.1 ([24]). *For every $a, b, c \in \mathbb{R}^+$ such that $a \leq b + c$, there exists a quasi-metric $D : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ such that there exist $x, y, z \in \mathbb{R}^2$ with $D(x, y) = a$, $D(x, z) = b$ and $D(z, y) = c$.*

Theorem 2.2 ([24]). *Let $\phi : [0, +\infty)^I \longrightarrow [0, +\infty)$. Then the below statements are equivalent:*

1. *$\phi^{-1}(0) = \{\mathbf{0}\}$, ϕ is subadditive and isotone.*
2. *ϕ is a quasi-metric aggregation function on products.*
3. *$\phi^{-1}(0) = \{\mathbf{0}\}$ and ϕ preserves asymmetric triangular triplets.*

Proof. First, we will show that 1. implies 2. Let $\{(\mathbb{X}_i, d_i)\}_{i \in I}$ be an indexed family of quasi-metric spaces. It is clear that $\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{x}) = 0 \ \forall \mathbf{x} \in \prod_{i \in I} \mathbb{X}_i$. Assume that $\phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{y}, \mathbf{x}) = 0$. Since ϕ vanishes exactly at $\mathbf{0}$ we deduce that $\tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \tilde{\mathbf{d}}(\mathbf{y}, \mathbf{x}) = \mathbf{0}$. Consequently $d_i(\mathbf{x}_i, \mathbf{y}_i) = 0$ $\forall i \in I$. Since d_i is a quasi-metric, we conclude that $\mathbf{x}_i = \mathbf{y}_i \ \forall i \in I$, an thus $\mathbf{x} = \mathbf{y}$.
Now, we show the triangular inequality. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \prod_{i \in I} \mathbb{X}_i$. From the triangular inequality of $d_i \ \forall i \in I$ and from the isotonicity and subadditivity of ϕ we obtain that

$$\begin{aligned} \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \phi((d_i(\mathbf{x}_i, \mathbf{y}_i)_{i \in I}) \leq \phi((d_i(\mathbf{x}_i, \mathbf{z}_i))_{i \in I} + (d_i(\mathbf{z}_i, \mathbf{y_i}))_{i \in I}) \leq \\ \leq \phi((d_i(\mathbf{x}_i, \mathbf{z}_i))_{i \in I} + \phi((d_i(\mathbf{z}_i, \mathbf{y}_i))_{i \in I}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{z}) + \phi \circ \tilde{\mathbf{d}}(\mathbf{z}, \mathbf{y}), \end{aligned}$$

then $\phi \circ \tilde{\mathbf{d}}$ is quasi-metric on $\prod_{i \in I} \mathbb{X}_i$ so ϕ is a quasi-metric aggregation function on products and we have that 1. implies 2.
We prove now that 2. implies 3. Suppose that ϕ is a quasi-metric aggregation function on products and let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^I$ with $\mathbf{a}_i \leq \mathbf{b}_i + \mathbf{c}_i \ \forall i \in I$. By the previous lemma we have that there exist a quasi-metric D on $\mathbb{R}^2$ and $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i \in \mathbb{R}^2$ such that $D(\mathbf{x}_i, \mathbf{y}_i) = \mathbf{a}_i$, $D(\mathbf{x}_i, \mathbf{z}_i) = \mathbf{b}_i$ and $D(\mathbf{z}_i, \mathbf{y}_i) = \mathbf{c}_i \ \forall i \in I$. We consider the family of quasi-metric spaces $\{(\mathbb{X}_i, d_i)\}_{i \in I}$, where $\mathbb{X}_i = \mathbb{R}^2$ and $d_i = D \ \forall i \in I$. Since ϕ is a quasi-metric aggregation function on products we have that,

$$\phi(\mathbf{a}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{y}) \leq \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{z}) + \phi \circ \tilde{\mathbf{d}}(\mathbf{z}, \mathbf{y}) = \phi(\mathbf{b}) + \phi(\mathbf{c}),$$

where $\mathbf{x} = (\mathbf{x}_i)_{i \in I}, \mathbf{y} = (\mathbf{y}_i)_{i \in I}$ and $\mathbf{z} = (\mathbf{z}_i)_{i \in I}$. It remains to prove that $\phi^{-1}(0) = \{\mathbf{0}\}$. Let $\mathbf{x} \in [0, +\infty)^I$ such that $\phi(\mathbf{x}) = 0$, and let $\{(\mathbb{X}_i, d_i)\}_{i \in I}$ be the family of quasi-metric spaces where $\mathbb{X}_i = \mathbb{R}$ and $d_i = e \ \forall i \in I$, where e denotes the Euclidean metric on $\mathbb{R}$. Then we have that

$$\begin{aligned} 0 = \phi(\mathbf{x}) = \phi((\mathbf{x}_i)_{i \in I}) = \phi((e(\mathbf{x}_i, 0))_{i \in I}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{x}, \mathbf{0}), \\ 0 = \phi(\mathbf{x}) = \phi((\mathbf{x}_i)_{i \in I}) = \phi((e(0, \mathbf{x}_i))_{i \in I}) = \phi \circ \tilde{\mathbf{d}}(\mathbf{0}, \mathbf{x}). \end{aligned}$$

But, since ϕ is a quasi-metric aggregation function on products, $\phi \circ \tilde{\mathbf{d}}$ is a quasi-metric on $(\mathbb{R})^I$ and the previous equality holds if and only if $\mathbf{x} = \mathbf{0}$ i.e, $\phi^{-1}(0) = \mathbf{0}$. So we have that 2. implies 3. Finally we are going to show that 3. implies 1. Let $\mathbf{a}, \mathbf{b} \in [0, \infty)^I$, such that $\mathbf{a} \leq \mathbf{b}$, then we have that $\mathbf{a} \leq \mathbf{b} + \mathbf{0}$ and we deduce since ϕ preserves asymmetric triangular triplets

$$\phi(\mathbf{a}) \leq \phi(\mathbf{b}) + \phi(\mathbf{0}) = \phi(\mathbf{b}),$$

we can conclude that ϕ is isotone. Moreover for arbitrary $\mathbf{a}, \mathbf{b} \in [0, +\infty)^I$, from the fact that ϕ preserves asymmetric triangular triplets we obtain

$$\phi(\mathbf{a} + \mathbf{b}) = \phi(\mathbf{a} + \mathbf{b} + \mathbf{0}) \leq \phi(\mathbf{a}) + \phi(\mathbf{b} + \mathbf{0}) = \phi(\mathbf{a}) + \phi(\mathbf{b})$$

so ϕ is subadditive and the proof is complete. □

It is clear from the above results that quasi-metric aggregation functions on products are also metric aggregation functions on products. In general, the converse is not true [24].

2.2 Aggregation functions for metrics and quasi-metrics on sets

Another relevant question is when from a family of metrics (or quasi-metrics) defined on the same set, we can obtain a new metric (or quasi-metric) in that set. It is natural to ask if the equivalences given in the previous section holds for aggregation functions on sets, i. e. if $\{d_i : i \in I\}$ is a family of (quasi-)metrics defined on the same set $\mathbb{X}$, we want to know when a function $\phi : [0, +\infty)^I \longrightarrow [0, +\infty)$ holds that $\phi \circ \mathbf{d}(x, y) : \mathbb{X} \times \mathbb{X} \longrightarrow [0, +\infty)$ is a (quasi-)metric on $\mathbb{X}$, where $\phi \circ \mathbf{d}(x, y) = \phi((d_i(x, y))_{i \in I})$ $\forall x, y \in \mathbb{X}$. On one hand we have that Mayor y Valero studied this problem on [25] for metrics, and Miñana and Valero did the same thing for quasi-metrics on [27]. We start this section giving the definition of a function that aggregates metrics on sets.

Definition 2.7 ([25])**.** Let $\phi : [0, +\infty)^I \to [0, +\infty)$ be a function. We will say that ϕ is a (quasi-)metric aggregation function on sets if for every family of (quasi-)metrics $\{d_i : i \in I\}$ defined over the same nonempty set $\mathbb{X}$ the function $\phi \circ \mathbf{d} : \mathbb{X} \times \mathbb{X} \longrightarrow [0, \infty)$ is a (quasi-)metric on $\mathbb{X}$ where $\mathbf{d} : \mathbb{X} \times \mathbb{X} \longrightarrow [0, \infty)^I$ is defined as

$$\mathbf{d}(x, y) = (d_i(x, y))_{i \in I}$$

$\forall x, y \in \mathbb{X}$.

Example 2.6 ([27])**.** *Let $n \in \mathbb{N}$ and $\{\alpha_2, \ldots, \alpha_n\}$ be a fixed family of coefficients such that $\alpha_i \in (0, +\infty)$, $i \in \{2, \ldots, n\}$, then we have that the function $\phi : [0, +\infty)^n \longrightarrow [0, +\infty)$ given by*

$$\phi(\mathbf{a}) = \sum_{i=2}^{n} \alpha_i \mathbf{a}_i$$

$\forall \mathbf{a} \in [0, +\infty)^n$ is a quasi-metric aggregation function on sets. Let us show this.
Let $\mathbb{X}$ be a nonempty set and $\{d_i\}_{i=1}^n$ be a family of quasi-metrics on $\mathbb{X}$.
Let $x, y \in \mathbb{X}$, we have that if $\phi \circ \mathbf{d}(x, y) = \sum_{i=2}^n \alpha_i d_i(x, y) = \sum_{i=2}^n \alpha_i d_i(y, x) = \phi \circ \mathbf{d}(y, x) = 0$ then, since $\alpha_i > 0$, $i = 2, \ldots n$ and d_i is a quasi-metric on $\mathbb{X}$, $i = 1, \ldots, n$ we have that necessary $d_i(x, y) = d_i(y, x) = 0$ for $i = 2, \ldots, n$, i. e. $x = y$.

Let $x, y, z \in \mathbb{X}$. Then, we have that for $i = 1, \ldots, n$ $d_i(x, y) \leq d_i(x, z) + d_i(z, y)$ and then $\phi \circ \mathbf{d}(x, y) = \sum_{i=2}^{n} \alpha_i d_i(x, y) \leq \sum_{i=2}^{n}(d_i(x, z) + d_i(z, y)) = \sum_{i=2}^{n} \alpha_i d_i(x, z) + \sum_{i=2}^{n} \alpha_i d_i(z, y) = \phi \circ \mathbf{d}(x, z) + \phi \circ \mathbf{d}(z, y)$.
So we have that ϕ is a quasi-metric aggregation function.

The next theorem is based on Theorem 12 in [25]. That theorem shows a characterization of metric aggregation functions on sets for a finite family of metrics defined on the same set. The proof of the theorem in the article is not correct, so we propose in this paper an original version of the proof that we think is correct. In this case, we will consider an arbitrary family of metrics on a set $\mathbb{X}$ that is not necessary finite.

Theorem 2.3 ([25])**.** *Let $\phi : [0, +\infty)^I \longrightarrow [0, +\infty)$. The assertions below are equivalent:*

1. *ϕ is a metric aggregation function on sets.*
2. *$\phi(\mathbf{0}) = 0$ and if $\phi(\mathbf{a}) = 0$ then there exists $j \in I$ such that $\mathbf{a}_j = 0$ and ϕ preserves positive triangular triplets*

Proof. Let us prove first that 1. implies 2. Let $\mathbb{X}$ be a nonempty set and $\{d_i : i \in I\}$ be an indexed family of metrics on $\mathbb{X}$. Take $x \in \mathbb{X}$. The fact that ϕ is a metric aggregation function on sets provides that $\phi \circ \mathbf{d}$ is a metric on $\mathbb{X}$ so

$$0 = \phi \circ \mathbf{d}(x, x) = \phi((d_i(x, x))_{i\in I}) = \phi(\mathbf{0}).$$

To prove that if $\phi(\mathbf{a}) = 0$, $\mathbf{a} \in [0, +\infty)^I$ then there exists $j \in I$ such that $\mathbf{a}_j = 0$, assume for the purpose of contradiction that we have $\phi(\mathbf{a}) = 0$, but $\mathbf{a}_i > 0$ $\forall i \in I$. Consider a nonempty set $\mathbb{X}$ with at least two elements and $x, y \in \mathbb{X}$ with $x \neq y$. Define the metrics $d_i = \mathbf{a}_i d$ $\forall i \in I$, where d is the discrete metric on $\mathbb{X}$. Since ϕ is a metric aggregation function on sets, we have that $\phi \circ \mathbf{d}$ is a metric on $\mathbb{X}$, so

$$0 = \phi(\mathbf{a}) = \phi((\mathbf{a}_i)_{i\in I}) = \phi((d_i(x, y))_{i\in I}) = \phi \circ \tilde{\mathbf{d}}(x, y) > 0$$

which is a contradiction.
It remains to prove that ϕ preserves positive triangular triplets. Let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be a positive triangular triplet. Notice that $\phi(\mathbf{a})$, $\phi(\mathbf{b})$ and $\phi(\mathbf{c})$ are positive. In fact, if we suppose with lost of generality that $\phi(\mathbf{a}) = 0$ then there exists $j \in I$ such that $\mathbf{a}_j = 0$ which is not possible. Take $\mathbb{X} = \{x_1, x_2, x_3\}$ a set with three different elements and define the family of metrics on $\mathbb{X}$

$$d_i(x, y) = d_i(y, x) = \begin{cases} 0 & \text{if } x = y \\ \mathbf{a}_i & \text{if } x = x_1,\ y = x_2 \\ \mathbf{b}_i & \text{if } x = x_1,\ y = x_3 \\ \mathbf{c}_i & \text{if } x = x_3,\ y = x_2, \end{cases}$$

$\forall i \in I$. It is clear that d_i is a metric on $\mathbb{X}$ $\forall i \in I$. Let us show this. Obviously $d_i(x, y) = d_i(y, x)$ $\forall x, y \in \mathbb{X}$ and $d_i(x, y) = 0$ if and only if $x = y$ $\forall i \in I$. Moreover, the triangular inequality holds because $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a positive triangular triplet. Then, since ϕ is a metric aggregation function on sets we have that

$$\begin{aligned} \phi(\mathbf{a}) = \phi((\mathbf{a}_i)_{i\in I}) &= \phi((d_i(x_1, x_2))_{i\in I}) = \phi \circ \mathbf{d}(x_1, x_2) \leq \phi \circ \mathbf{d}(x_1, x_3) + \phi \circ \mathbf{d}(x_3, x_2) = \\ &= \phi((d_i(x_1, x_3))_{i\in I}) + \phi((d_i(x_3, x_2))_{i\in I}) = \phi((\mathbf{b}_i)_{i\in I}) + \phi((\mathbf{c}_i)_{i\in I}) = \phi(\mathbf{b}) + \phi(\mathbf{c}) \end{aligned}$$

Therefore $(\phi(\mathbf{a}), \phi(\mathbf{b}), \phi(\mathbf{c}))$ triangular triplet.

Now we will show that 2. implies 1.

Let $\mathbb{X}$ be a nonempty set and $\{d_i : i \in I\}$ be an indexed family of metrics on $\mathbb{X}$. Let us check that $\phi \circ \mathbf{d}$ is a metric on $\mathbb{X}$. Take $x \in \mathbb{X}$. Then, we have that $\phi \circ \mathbf{d}(x,x) = \phi((d_i(x,x))_{i\in I}) = \phi(\mathbf{0}) = 0$ and if $x, y \in \mathbb{X}$ we have that $\phi \circ \mathbf{d}(x,y) = \phi((d_i(x,y))_{i\in I}) = \phi((d_i(y,x))_{i\in I}) = \phi \circ \mathbf{d}(y,x)$ because d_i is a metric on $\mathbb{X}$ $\forall i \in I$. Let $x, y \in \mathbb{X}$ such that $\phi \circ \mathbf{d}(x,y) = 0$. Then, $\phi((d_i(x,y))_{i\in I}) = 0$. This implies that there exists $j \in I$ such that $d_j(x,y) = 0$, but this implies that $x = y$, so $\phi \circ \mathbf{d}(x,y) = 0$ if and only if $x = y$.

Finally, we will prove the triangle inequality. Let x, y, z be pairwise different elements in $\mathbb{X}$. And let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (0, +\infty)^I$ such that $\mathbf{a}_i = d_i(x,y)$, $\mathbf{b}_i = d_i(x,z)$ and $\mathbf{c}_i = d_i(z,y)$. Then, since every d_i is a metric on $\mathbb{X}$, by triangle inequality we have that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a positive triangular triplet, then by hypothesis we have that $(\phi(\mathbf{a}), \phi(\mathbf{b}), \phi(\mathbf{c}))$ is a positive triangular triplet, i. e.

$$\begin{aligned}\phi \circ \mathbf{d}(x,y) = \phi((d_i(x,y))_{i\in I}) = \phi((\mathbf{a}_i)_{i\in I}) = \phi(\mathbf{a}) \leq \phi(\mathbf{b}) + \phi(\mathbf{c}) = \\ = \phi((d_i(x,z))_{i\in I}) + \phi((d_i(z,y))_{i\in I}) = \phi \circ \mathbf{d}(x,z) + \phi \circ \mathbf{d}(z,y)\end{aligned}$$

and the proof is complete. □

In [27] the quasi-metric aggregation functions defined on $[0, +\infty)^n$ were characterized. We next show that this characterization is also valid when I is infinite and we simplify the proof.

Theorem 2.4 ([27])**.** *Let $\phi : [0, +\infty)^I \longrightarrow [0, +\infty)$. Then the assertions below are equivalent:*

1. *ϕ is a quasi-metric aggregation function on sets.*

2. *$\phi(\mathbf{0}) = 0$, if $\phi(\mathbf{a}) = \phi(\mathbf{b}) = 0$ there exists $j \in I$ such that $\mathbf{a}_j = \mathbf{b}_j = 0$ and ϕ preserves asymmetric triangular triplets.*

3. *$\phi(\mathbf{0}) = 0$, if $\phi(\mathbf{a}) = \phi(\mathbf{b}) = 0$ there exists $j \in I$ such that $\mathbf{a}_j = \mathbf{b}_j = 0$, ϕ is isotone and subadditive.*

Proof. First, we are going to prove that 1. implies 2. To prove that $\phi(\mathbf{0}) = 0$ we can reason as in Theorem 2.3.

On the other hand, suppose that we can find $\mathbf{a}, \mathbf{b} \in [0, +\infty)^I$ such that $\phi(\mathbf{a}) = \phi(\mathbf{b}) = 0$ but there does not exist $j \in I$ such that $\mathbf{a}_j = \mathbf{b}_j = 0$. Consider $\mathbb{X} = \{x, y\}$ a set with two different elements and for each $i \in I$ define $d_i : \mathbb{X} \times \mathbb{X} \to [0, +\infty)$ by

$$\begin{cases} d_i(x,y) = \mathbf{a}_i \\ d_i(y,x) = \mathbf{b}_i \\ d_i(x,x) = d_i(y,y) = 0 \end{cases}.$$

Then $\{d_i : i \in I\}$ is a family of quasi-metrics on $\mathbb{X}$. By assumption, $\phi \circ \mathbf{d}$ is a quasi-metric on $\mathbb{X}$. Nevertheless

$$\phi \circ \mathbf{d}(x,y) = \phi((d_i(x,y))_{i\in I}) = \phi((\mathbf{a}_i)_{i\in I}) = \phi(\mathbf{a}) = 0$$

$$\phi \circ \mathbf{d}(y,x) = \phi((d_i(y,x))_{i\in I}) = \phi((\mathbf{b}_i)_{i\in I}) = \phi(\mathbf{b}) = 0$$

but $x \neq y$ which is a contradiction. Therefore, there must exists $j \in J$ such that $\mathbf{a}_j = \mathbf{b}_j = 0$.

Now we prove that ϕ preserves asymmetric triangular triplets. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, +\infty)^I$ be such that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an asymmetric triangular triplet. Consider $\mathbb{X} = \{x_1, x_2, x_3\}$ a set with three different elements and consider the family $\{d_i : i \in I\}$ of quasi-metrics on $\mathbb{X}$ defined by

$$d_i(x,y) := \begin{cases} 0 & \text{if } x = y, \\ \mathbf{a}_i & \text{if } x = x_1,\ y = x_2, \\ \mathbf{b}_i & \text{if } x = x_1,\ y = x_3, \\ \mathbf{c}_i & \text{if } x = x_3,\ y = x_2,\ , \\ k_i + 1 & \text{if } x = x_2,\ y = x_1, \\ k_i + 1 & \text{if } x = x_3,\ y = x_1, \\ k_i + 1 & \text{if } x = x_2,\ y = x_3, \end{cases} \tag{1}$$

where $k_i := \max\{\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i\}$ $\forall i \in I$. It is easy to check that d_i is a quasi-metric $\forall i \in I$. Then, since ϕ is a quasi-metric aggregation function we have that

$$\begin{aligned} \phi(\mathbf{a}) = \phi((\mathbf{a})_{i\in I}) = \phi((d_i(x_1,x_2))_{i\in I}) = \phi \circ \mathbf{d}(x_1,x_2) \leq \phi \circ \mathbf{d}(x_1,x_3) + \phi \circ \mathbf{d}(x_3,x_2) = \\ = \phi((d_i(x_1,x_3))_{i\in I}) + \phi((d_i(x_3,x_2))_{i\in I}) = \phi((\mathbf{b}_i)_{i\in I}) + \phi((\mathbf{c}_i)_{i\in I}) = \phi(\mathbf{b}) + \phi(\mathbf{c}). \end{aligned}$$

Let us show that 2. implies 3. Suppose that ϕ preserves asymmetric triangular triplets, we need to prove that ϕ is isotone and subbaditive. Let $\mathbf{a}, \mathbf{b} \in [0, +\infty)^I$ be such that $\mathbf{a} \preceq \mathbf{b}$, then we have that $(\mathbf{a}, \mathbf{b}, \mathbf{0})$ is an asymmetric triangular triplet, so $(\phi(\mathbf{a}), \phi(\mathbf{b}), \phi(\mathbf{0}))$ is an asymmetric tringular triplet, i. e.

$$\phi(\mathbf{a}) \leq \phi(\mathbf{b}) + \phi(\mathbf{0}) = \phi(\mathbf{b})$$

and we can conclude that ϕ is isotone.
Now we are going to prove that ϕ is subadditive. Take $\mathbf{a}, \mathbf{b} \in [0, +\infty)^I$, then $(\mathbf{a}+\mathbf{b}, \mathbf{a}, \mathbf{b})$ is a triangular triplet, so it is an asymmetric triangular triplet. Since ϕ preserves asymmetric triangular triplets we have that $(\phi(\mathbf{a}+\mathbf{b}), \phi(\mathbf{a}), \phi(\mathbf{b}))$ is an asymmetric triangular triplet and then

$$\phi(\mathbf{a}+\mathbf{b}) \leq \phi(\mathbf{a}) + \phi(\mathbf{b}),$$

i. e. ϕ is subadditive.
We next prove that 3. implies 1. Let $\mathbb{X}$ be a nonempty set and $\{d_i : i \in I\}$ an indexed family of quasi-metrics on $\mathbb{X}$. Let us show that $\phi \circ \mathbf{d}$ is a quasi-metric on $\mathbb{X}$. Take $x \in \mathbb{X}$, then $\phi \circ \mathbf{d}(x,x) = \phi((d_i(x,x))_{i\in I}) = \phi(\mathbf{0}) = 0$ because d_i is a quasi-metric on $\mathbb{X}$ $\forall i \in I$ and $\phi(\mathbf{0}) = 0$ by hypothesis. Let $x, y \in \mathbb{X}$ such that $\phi \circ \mathbf{d}(x,y) = \phi \circ \mathbf{d}(y,x) = 0$. Then $\phi(\mathbf{d}(x,y)) = \phi(\mathbf{d}(y,x)) = 0$, so there exists $j \in I$ such that $(\mathbf{d}(x,y))_j = d_j(x,y) = (\mathbf{d}(y,x))_j = d_j(y,x) = 0$, since d_j is a quasi-metric on $\mathbb{X}$ then $x = y$. Let now $x, y, z \in \mathbb{X}$. Since d_i is a quasi-metric $\forall i \in I$ we deduce that $(d_i(x,y), d_i(x,z), d_i(z,y))$ is an asymmetric triangular triplet, i. e. $d_i(x,y) \leq d_i(x,z) + d_i(z,y)$ $\forall i \in I$. Then, since ϕ is subadditve and isotone we have that

$$\begin{aligned} \phi \circ \mathbf{d}(x,y) = \phi((d_i(x,y))_{i\in I}) \leq \phi((d_i(x,z))_{i\in I} + (d_i(z,y))_{i\in I}) \leq \phi((d_i(x,z))_{i\in I}) + \phi((d_i(z,y))_{i\in I}) = \\ = \phi \circ \mathbf{d}(x,z) + \phi \circ \mathbf{d}(z,y) \end{aligned}$$

and we have that the triangular inequality holds for $\phi \circ \mathbf{d}$ so the proof is complete. $\square$

We can see with the results of this section and the previous one that every quasi-metric aggregation function is also a metric aggregation function, but there exist metric aggregation functions that are not quasi-metric aggregation functions. That is because every quasi-metric aggregation function on products (sets) preserves asymmetric (positive asymmetric) triangular triplets, and it is obvious that every (positive) triangular triplet is also an (positive) asymmetric triangular triplet, then it is obvious that a function that preserves (positive) asymmetric triangular triplets, preserves (positive) triangular triplets, i. e. a quasi-metric aggregation function on products (sets) is a metric aggregation function on products (sets). Notice that an (a positive) asymmetric triangular triplet is not necessary a (positive) triangular triplet, i. e. metric aggregation functions on products (sets) are not necessary quasi-metric aggregation functions on products (sets).
Now we summarize the previous results of aggregation functions of metrics and quasi-metrics on products and sets.

Aggregation function of metrics on products	$\phi^{-1}(0) = \{\mathbf{0}\}$ and ϕ preserves triangular triplets	
Aggregation functions of quasi-metrics on products	$\phi^{-1}(0) = \{\mathbf{0}\}$ and ϕ preserves asymmetric triangular triplets	$\phi^{-1}(0) = \{\mathbf{0}\}$, ϕ is subadditive and isotone
Aggregation functions of metrics on sets	$\phi(\mathbf{0}) = 0$ and if $\phi(\mathbf{a}) = 0$ then there exists $j \in I$ such that $\mathbf{a}_j = 0$ and ϕ preserves positive triangular triplets	
Aggregation functions of quasi-metrics on sets	$\phi(\mathbf{0}) = 0$, if $\phi(\mathbf{a}) = \phi(\mathbf{b}) = 0$ there exists $j \in I$ such that $\mathbf{a}_j = \mathbf{b}_j = 0$ and ϕ preserves asymmetric triangular triplets	$\phi(\mathbf{0}) = 0$, if $\phi(\mathbf{a}) = \phi(\mathbf{b}) = 0$ there exists $j \in I$ such that $\mathbf{a}_j = \mathbf{b}_j = 0$, ϕ is isotone and subadditive

Table 1: Metrics and quasi-metrics

2.3 Aggregations functions for norms and asymmetric norms on products

In the previous section we have made a review of existing results of aggregation functions of metrics and quasi-metrics on products and on sets. It is natural to wonder if we can do the same thing to other mathematical structures. Herburt and Monszyńska [19] analyzed this problem in the context of norms and gave a characterization of functions that aggregate norms on products. Moreover, Martín, Mayor y Valero [22] did this for asymmetric norms on products. Like in the previous sections of metrics and quasi-metrics, we start giving basic definitions of norms and asymmetric norms.

Definition 2.8 ([9])**.** Let $(V, +, \cdot)$ be a vector space with neutral element 0_V. An asymmetric norm on V is a nonnegative real valued function $n : V \longrightarrow \mathbb{R}^+$ such that $\forall x, y \in V$:

(i) $n(x) = n(-x) = 0 \Leftrightarrow x = 0_V$.

(ii) $n(x + y) \leq n(x) + n(y)$.

(iii) $n(\lambda x) = \lambda n(x)\ \forall \lambda \in [0, \infty)$.

If the function n satisfies in addition that:

(iv) $n(x) = n(-x)\ \forall x \in V$,

then we call n a norm on V and the pair (V, n) is called a normed vector space.

When n is an asymmetric norm on a vector space V, we can define the conjugate of n as $n^{-1} : V \longrightarrow [0, +\infty)$ by

$$n^{-1}(x) = n(-x)\ \forall x \in V.$$

And (V, n^{-1}) is also an asymmetric normed vector space. Notice that an asymmetric norm on V induces a norm n^s on V defined by:

$$n^s(x) = n(x) \vee n^{-1}(x)\ \forall x \in V.$$

Obviously from an asymmetric norm n on a vector space V we can define a quasi-metric d_n on V as follows:

$$d_n(x, y) = n(y - x)\ \forall x, y \in V.$$

Example 2.7. *Define on the real line the function $n : \mathbb{R} \longrightarrow [0, \infty)$ as follows*

$$n(x) = \begin{cases} |x|, & x \leq 0 \\ 2x, & x > 0. \end{cases}$$

Then n is an asymmetric norm on $\mathbb{R}$. Notice that for the asymmetric norm n we have that if $x < 0$ then $n^{-1}(x) = n(-x) = n(|x|) = 2|x|$. If $x > 0$ we obtain $n^{-1}(x) = n(-x) = |x|$.
We can also compute $n^s(x)\ \forall x \in \mathbb{R}$. In this case

$$n^s(x) = n(x) \vee n^{-1}(x) = n(x) \vee n(-x) = 2|x|$$

where $\vee$ denotes the maximum operator.

It remains to give the definition of what a function that aggregates (asymmetric) norms on products is.

Definition 2.9 ([19, 22])**.** A function $\phi : [0, +\infty)^I \longrightarrow [0, +\infty)$ is said to be a(n) (asymmetric) norm aggregation function on products if whenever $\{(V_i, n_i) : i \in I\}$ is a family of (asymmetric) normed vector spaces then $f \circ \widetilde{\mathbf{n}}$ is a(n) (asymmetric) norm on $\prod_{i \in I} V_i$ where

$$f \circ \widetilde{\mathbf{n}}(\mathbf{v}) = f((n_i(\mathbf{v}_i))_{i \in I})$$

for all $\mathbf{v} \in \prod_{i \in I} V_i$.

The next result was proved by Herburt and Monszyńska [19] for characterizing norm aggregation functions on products $\phi : [0, +\infty)^I \longrightarrow [0, +\infty)$ when $I = |2|$. Nevertheless, it is also valid for an arbitrary I so we state it in this level of generality. Before show the result, we will introduce the next definition.

Definition 2.10 ([38])**.** We will say that ϕ is positive homogeneous if $\phi(\lambda \mathbf{x}) = \lambda \phi(\mathbf{x})\ \forall \mathbf{x} \in [0, \infty)^I$, $\forall \lambda \in (0, +\infty)$.

Theorem 2.5 ([19])**.** *Let $\phi : [0, +\infty)^I \longrightarrow [0, +\infty)$. Then the assertions below are equivalent:*

1. *ϕ is a norm aggregation function on products.*
2. *$((\mathbb{R}^2)^I, \|\cdot\|_\phi)$ is a normed space where $\| \mathbf{x} \|_\phi = \phi((\| \mathbf{x}_i \|_2)_{i \in I})$ and $\|\cdot\|_2$ is the Euclidean norm $\forall \mathbf{x} \in (\mathbb{R}^2)^I$.*
3. *$\phi^{-1}(0) = \{\mathbf{0}\}$, ϕ is positive homogeneous and it preserves triangular triplets.*
4. *$\phi^{-1}(0) = \{\mathbf{0}\}$, ϕ is positive homogeneous, isotone and subadditive.*

Proof. It is obvious that 1. implies 2.
First we are going to prove that 2. implies 3. i.e., we need to prove that $\phi^{-1}(0) = \mathbf{0}$, ϕ is positive homogeneous and it preserves triangular triplets.
Suppose that there exists $\mathbf{x} \in [0, +\infty)^I$ such that $\phi(\mathbf{x}) = 0$. Consider the family of normed vector spaces $\{(\mathbb{R}^2, n_i)\}_{i \in I}$ where n_i is the Euclidean norm on $\mathbb{R}^2\ \forall i \in I$. Consider the element on $(\mathbb{R}^2)^I$ $(\mathbf{x}, \mathbf{0}) = (\mathbf{x}_i, 0)_{i \in I}$. Then we have

$$\phi \circ \tilde{\mathbf{n}}((\mathbf{x}, \mathbf{0}) = \phi((n_i((\mathbf{x}_i, 0)))_{i \in I}) = \phi((\mathbf{x}_i)_{i \in I}) = \phi(\mathbf{x}) = 0.$$

Since $\phi \circ \tilde{\mathbf{n}}$ is a norm we deduce that $\mathbf{x} = \mathbf{0}$, hence we have $\phi^{-1}(0) = \mathbf{0}$.
Now we are going to show that ϕ is positive homogeneous. Let $\mathbf{x} \in [0, \infty)^I$ and $\lambda \in (0, \infty)$. Then, from the fact that ϕ is a norm aggregation function on products we deduce that

$$\begin{aligned}\phi(\lambda \mathbf{x}) &= \phi((\lambda \mathbf{x}_i)_{i \in I}) = \phi((n_i(\lambda(\mathbf{x}_i, 0)))_{i \in I}) = \phi \circ \tilde{\mathbf{n}}(\lambda(\mathbf{x}, \mathbf{0})) = \lambda(\phi \circ \tilde{\mathbf{n}})((\mathbf{x}, \mathbf{0})) = \lambda \phi((n_i((\mathbf{x}_i, 0)))_{i \in I}) = \\ &= \lambda \phi((\mathbf{x}_i)_{i \in I}) = \lambda \phi(\mathbf{x}).\end{aligned}$$

It remains to prove that ϕ preserves triangular triplets.
Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^I$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a triangular triplet in $[0, \infty)^I$. Then we have that $(\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i)$ is a triangular triplet $\forall i \in I$. Then by theorem 2.1, we have that there exist $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i \in \mathbb{R}^2$ such that

$$\mathbf{a}_i = e(\mathbf{x}_i, \mathbf{y}_i), \quad \mathbf{b}_i = e(\mathbf{x}_i, \mathbf{z}_i), \quad \mathbf{c}_i = e(\mathbf{z}_i, \mathbf{y}_i),$$

where e denotes the Euclidean metric on $\mathbb{R}^2$. Let $\|\cdot\|_2$ be the Euclidean norm in $\mathbb{R}^2$, we know that $\forall x, y \in \mathbb{R}^2$ $e(x, y) = \| x - y \|_2$. Since ϕ is a norm aggregation function we have

$$\begin{aligned}&\phi(\mathbf{a}) = \phi((\mathbf{a}_i)_{i \in I})) = \phi((e(\mathbf{x}_i, \mathbf{y}_i))_{i \in I}) = \phi((\| \mathbf{x}_i - \mathbf{y}_i \|_2)_{i \in I}) = \phi((\| \mathbf{x}_i - \mathbf{z}_i + \mathbf{z}_i - \mathbf{y}_i \|_2)_{i \in I}) \leq \\ &\leq \phi((\| \mathbf{x}_i - \mathbf{z}_i \|_2)_{i \in I}) + \phi((\| \mathbf{z}_i - \mathbf{y}_i \|_2)_{i \in I}) = \phi((\mathbf{b}_i)_{i \in I}) + \phi((\mathbf{c}_i)_{i \in i}) = \phi(\mathbf{b}) + \phi(\mathbf{c}).\end{aligned}$$

Analogously we have $\phi(\mathbf{b}) \leq \phi(\mathbf{a}) + \phi(\mathbf{c})$ and $\phi(\mathbf{c}) \leq \phi(\mathbf{a}) + \phi(\mathbf{b})$. We have proved that $(\phi(\mathbf{a}), \phi(\mathbf{b}), \phi(\mathbf{c}))$ is a triangular triplet, i.e. ϕ preserves triangular triplets.

To prove that 3. implies 4. we are going to show that if ϕ preserves triangular triplets then ϕ is isotone and positive homogeneous. Suppose that ϕ preserves triangular triplets. We know that $\forall \mathbf{a}, \mathbf{b} \in [0,\infty)^I$ $(\mathbf{a}, \mathbf{b}, \mathbf{a}+\mathbf{b})$ is a triangular triplet, then $(\phi(\mathbf{a}), \phi(\mathbf{b}), \phi(\mathbf{a}+\mathbf{b}))$ is a triangular triplet and we deduce

$$\phi(\mathbf{a}+\mathbf{b}) \leq \phi(\mathbf{a}) + \phi(\mathbf{b}),$$

so we have shown that ϕ is subadditive.
Now we are going to prove that ϕ is isotone. Let $\mathbf{a}, \mathbf{b} \in [0,\infty)^I$ such that $\mathbf{a} \preceq \mathbf{b}$, then $(\mathbf{a}, \frac{\mathbf{b}}{2}, \frac{\mathbf{b}}{2})$ is a triangular triplet. By assumption $(\phi(\mathbf{a}), \phi(\frac{\mathbf{b}}{2}), \phi(\frac{\mathbf{b}}{2}))$ is a triangular triplet so

$$\phi(\mathbf{a}) \leq \phi(\frac{\mathbf{b}}{2}) + \phi(\frac{\mathbf{b}}{2}).$$

Since ϕ is positive homogeneous, and we can conclude that

$$\phi(\mathbf{a}) \leq \frac{\phi(\mathbf{b})}{2} + \frac{\phi(\mathbf{b})}{2} = \phi(\mathbf{b}),$$

so ϕ is isotone.
It remains to prove that 4. implies 1. Suppose that ϕ is positive homogeneous, $\phi^{-1}(0) = \mathbf{0}$, ϕ is isotone and subadditive. Let $\{(V_i, n_i)\}_{i\in I}$ be an indexed family of normed vector spaces. Let us check that $\phi \circ \tilde{\mathbf{n}}$ is a norm on $\prod_{i\in I} V_i$. From the facts of ϕ being positive homogeneous and $\phi^{-1}(0) = \mathbf{0}$ we deduce that $\phi \circ \tilde{\mathbf{n}}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$ and $\phi \circ \tilde{\mathbf{n}}(\lambda \mathbf{x}) = |\lambda| \phi \circ \tilde{\mathbf{n}}(\mathbf{x})$ $\forall \mathbf{x} \in \prod_{i\in I} V_i$.
Let $\mathbf{u}, \mathbf{v} \in \prod_{i\in I} V_i$, we have that $n_i(\mathbf{u}_i + \mathbf{v}_i) \leq n_i(\mathbf{u}_i) + n_i(\mathbf{v}_i)$ because n_i is a norm $\forall i \in I$. Then, from the facts that ϕ is nondecreasing and subadditive we have

$$\begin{aligned}\phi \circ \tilde{\mathbf{n}}(\mathbf{u}+\mathbf{v}) &= \phi((n_i(\mathbf{u}_i + \mathbf{v}_i))_{i\in I}) \leq \phi((n_i(\mathbf{u}_i) + n_i(\mathbf{v}_i))_{i\in I}) \leq \phi((n_i(\mathbf{u}_i))_{i\in I}) + \phi((n_i(\mathbf{v}_i))_{i\in I}) = \\ &= \phi \circ \tilde{\mathbf{n}}(\mathbf{u}) + \phi \circ \tilde{\mathbf{n}}(\mathbf{v}).\end{aligned}$$

We have shown that the triangular inequality holds for $\phi \circ \tilde{\mathbf{n}}$ i.e, ϕ is a norm aggregation function on products. $\square$

For the case of metrics and quasi-metrics we have seen that every quasi-metric aggregation function is also a metric aggregation function, but conversely it is not true, so we can think that every asymmetric norm aggregation function is a norm aggregation function, this is obvious since every norm is also an asymmetric norm, but, is the converse true? The theorem 2.6 based on results in [22] give us a characterization of asymmetric norm aggregation functions on products. To prove it we will need the next lemma that is extracted from [22] too.

Lemma 2.2 ([22])**.** *Let $a, b \in \mathbb{R}$ such that $a \leq b$. Then there exists an asymmetric norm u on $\mathbb{R}^2$ in such a way that there exist $x, y \in \mathbb{R}^2$ with $u(x+y) = a$, $u(x) = b$ and $u(y) = 0$.*

Theorem 2.6 ([22])**.** *Let $\phi : [0,\infty)^I \longrightarrow [0,\infty)$. Then the assertions below are equivalent:*

1. *ϕ is an asymmetric norm aggregation function on products.*

2. *$\phi^{-1}(0) = \{\mathbf{0}\}$ and ϕ is positive homogeneous, subadditive and isotone.*

3. *$\phi^{-1}(0) = \{\mathbf{0}\}$, ϕ is positive homogeneous and ϕ preserves triangular triplets.*

Proof. Assume that ϕ is an asymmetric norm aggregation function, first we are going to prove that 1. implies 2.
Suppose that there exists $\mathbf{x} \in [0, +\infty)^I$ such that $\phi(\mathbf{x}) = 0$. Consider the indexed family of asymmetric normed spaces $\{(V_i, n_i)\}_{i\in I}$ where $V_i = \mathbb{R}$ and n_i is the Euclidean norm on $\mathbb{R}$ $\forall i \in I$. Since ϕ is an asymmetric norm aggregation function we have

$$0 = \phi(\mathbf{x}) = \phi((n_i(\mathbf{x}_i))_{i\in I}) = \phi \circ \tilde{\mathbf{n}}(\mathbf{x})$$
$$0 = \phi(\mathbf{x}) = \phi((n_i(-\mathbf{x}_i))_{i\in I}) = \phi \circ \tilde{\mathbf{n}}(-\mathbf{x}).$$

Whence we deduce that $\mathbf{x} = \mathbf{0}$ i.e, $\phi^{-1}(0) = \mathbf{0}$. Next we show that ϕ is subadditive.
Let $\mathbf{a}, \mathbf{b} \in [0, +\infty)^I$. Consider the same indexed family of asymmetric normed vector spaces. Since ϕ is an asymmetric norm aggregation function we have that

$$\begin{aligned}\phi(\mathbf{a} + \mathbf{b}) = \phi((n_i(\mathbf{a}_i + \mathbf{b}_i))_{i\in I}) = \phi \circ \tilde{\mathbf{n}}(\mathbf{a} + \mathbf{b}) \leq \phi \circ \tilde{\mathbf{n}}(\mathbf{a}) + \phi \circ \tilde{\mathbf{n}}(\mathbf{b}) = \\ = \phi((n_i(\mathbf{a}_i))_{i\in I}) + \phi((n_i(\mathbf{b}_i))_{i\in I}) = \phi(\mathbf{a}) + \phi(\mathbf{b}).\end{aligned}$$

It remains to prove that ϕ is isotone.
Let $\mathbf{a}, \mathbf{b} \in [0, +\infty)^I$ such that $\mathbf{a} \preceq \mathbf{b}$. By lemma 2.2 there exists an asymmetric norm u on $\mathbb{R}^2$ and there exist $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^2$ such that $u(\mathbf{x}_i + \mathbf{y}_i) = \mathbf{a}_i$, $u(\mathbf{x}_i) = \mathbf{b}_i$ and $u(\mathbf{y}_i) = 0$ $\forall i \in I$. Consider the indexed family of asymmetric normed vector spaces $\{(V_i, n_i)\}_{i\in I}$ where $V_i = \mathbb{R}^2$ and $n_i = u$ $\forall i \in I$, and consider $\mathbf{x} = (\mathbf{x}_i)_{i\in I}$, $\mathbf{y} = (\mathbf{y}_i)_{i\in I} \in (\mathbb{R}^2)^I$. Since ϕ is an asymmetric norm aggregation function and $\phi^{-1}(0) = \mathbf{0}$ we deduce

$$\begin{aligned}\phi(\mathbf{a}) = \phi((n_i(\mathbf{x}_i + \mathbf{y}_i))_{i\in I}) = \phi \circ \tilde{\mathbf{n}}(\mathbf{x} + \mathbf{y}) \leq \phi \circ \tilde{\mathbf{n}}(\mathbf{x}) + \phi \circ \tilde{\mathbf{n}}(\mathbf{y}) = \phi((n_i(\mathbf{x}_i))_{i\in I}) + \phi((n_i(\mathbf{y}_i))_{i\in I}) = \\ = \phi(\mathbf{b}) + \phi(\mathbf{0}) = \phi(\mathbf{b}).\end{aligned}$$

It remains to prove that ϕ is positive homogeneous. Let $\mathbf{x} \in [0, +\infty)^I$ and $\lambda \in [0, +\infty)$ and consider again the family $\{(V_i, n_i)\}_{i\in I}$ where $V_i = \mathbb{R}$ and n_i is the Euclidean norm on $\mathbb{R}$. Since ϕ is an asymmetric norm aggregation function on products we have

$$\phi(\lambda\mathbf{x}) = \phi((n_i(\lambda\mathbf{x}_i))_{i\in I}) = \phi \circ \tilde{\mathbf{n}}(\lambda\mathbf{x}) = \lambda\phi \circ \tilde{\mathbf{n}}(\mathbf{x}) = \lambda\phi((n_i(\mathbf{x}_i))_{i\in I} = \lambda\phi(\mathbf{x}).$$

Next we show that 2. implies 3.
Suppose that ϕ is positive homogeneous, isotone, subadditive and $\phi^{-1}(0) = \{\mathbf{0}\}$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, +\infty)^I$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a triangular triplet. Since ϕ is isotone we deduce that $\phi(\mathbf{a}) \leq \phi(\mathbf{b}+\mathbf{c})$, and from the fact of ϕ being subadditive we obtain that $\phi(\mathbf{b} + \mathbf{c}) \leq \phi(\mathbf{b}) + \phi(\mathbf{c})$. Analogously we obtain that $\phi(\mathbf{b}) \leq \phi(\mathbf{a} + \phi(\mathbf{c})$ and $\phi(\mathbf{c}) \leq \phi(\mathbf{a}) + \phi(\mathbf{b})$ i.e, $(\phi(\mathbf{a}), \phi(\mathbf{b}), \phi(\mathbf{c}))$ is a triangular triplet.
Finally, we prove that 3 implies 1. Suppose that $\phi^{-1}(0) = \mathbf{0}$, ϕ is positive homogenous and preserves asymmetric triangular triplets. Let $\{(V_i, n_i)\}_{i\in I}$ be an indexed family of asymmetric normed spaces, we want to show that $\phi \circ \tilde{\mathbf{n}}$ is an asymmetric norm on $\prod_{i\in I} V_i$. Let $\mathbf{x} \in \prod_{i\in I} V_i$ be such that $\phi \circ \tilde{\mathbf{n}}(\mathbf{x}) = \phi \circ \tilde{\mathbf{n}}(-\mathbf{x}) = 0$, since $\phi^{-1}(0) = \{\mathbf{0}\}$ we have that $\tilde{\mathbf{n}}(-\mathbf{x}) = \tilde{\mathbf{n}}(\mathbf{x}) = \mathbf{0}$ i.e, $n_i(-\mathbf{x}_i) = n_i(\mathbf{x}_i) = 0$ $\forall i \in I$ i.e, $\mathbf{x} = \mathbf{0}$.
Now we are going to show that the triangle inequality holds for $\phi \circ \tilde{\mathbf{n}}$. Let $\mathbf{x}, \mathbf{y} \in \prod_{i\in I} V_i$, it is obvious that $(\tilde{\mathbf{n}}(\mathbf{x}+\mathbf{y}), \tilde{\mathbf{n}}(\mathbf{x}), \tilde{\mathbf{n}}(\mathbf{y}))$ is an asymmetric triangular triplet because n_i is an asymmetric

norm $\forall i \in I$. Then, from the fact that ϕ preserves asymmetric triangular triplets we obtain that $(\phi \circ \tilde{\mathbf{n}}(\mathbf{x}+\mathbf{y}), \phi \circ \tilde{\mathbf{n}}(\mathbf{x}), \phi \circ \tilde{\mathbf{n}}(\mathbf{y}))$ is a triangular triplet i.e,

$$\phi \circ \tilde{\mathbf{n}}(\mathbf{x}+\mathbf{y}) \leq \phi \circ \tilde{\mathbf{n}}(\mathbf{x}) + \phi \circ \tilde{\mathbf{n}}(\mathbf{y}). \tag{2}$$

Finally, we will show that if $\lambda \in [0, +\infty)$ $\phi \circ \tilde{\mathbf{n}}(\lambda \mathbf{x}) = \lambda(\phi \circ \tilde{\mathbf{n}})(\mathbf{x})$ $\forall \mathbf{x} \in \prod_{i\in I} V_i$. Let $\mathbf{x} \in \prod_{i\in I} V_i$ and $\lambda \in [0, +\infty)$, since n_i is an asymmetric norm $\forall i \in I$ and ϕ is positive homogeneous we have

$$\phi \circ \tilde{\mathbf{n}}(\lambda \mathbf{x}) = \phi((n_i(\lambda \mathbf{x}_i))_{i\in I}) = \phi(\lambda(n_i(\mathbf{x}_i))_{i\in I}) = \lambda\phi((n_i(\mathbf{x}_i))_{i\in I} = \lambda\phi \circ \tilde{\mathbf{n}}(\mathbf{x}).$$

□

Notice that condition 2 in theorem 2.6 is the same that condition 4 in 2.5, so we have that every norm aggregation function is an asymmetric norm aggregation function, i. e. the two concepts are equivalent. In [22] they gave an example of a function that is a norm aggregation function on products that is not an asymmetric norm aggregation function on products, but this is not correct.

2.4 Aggregation functions for norms and asymmetric norms on sets

In the case of metrics and quasi-metrics we have shown results of aggregation on products and sets, so it makes sense to try to characterize norms and asymmetric norms on sets too. Rodríguez-López and Pedraza [31] present results of aggregation of norms and asymmetric norms on sets.
First we show the definition of a(n) (asymmetric) norm aggregation function on sets.

Definition 2.11 ([31])**.** A function $\phi : [0, +\infty)^I \to [0, +\infty)$ is said to be a(n) (*asymmetric*) *norm aggregation function on sets* if whenever $\{(V, n_i) : i \in I\}$ is a family of (asymmetric) normed vector spaces then $\phi \circ \boldsymbol{n}$ is a(n) (asymmetric) norm on V where

$$\phi \circ \boldsymbol{n}(v) = f((n_i(v))_{i\in I})$$

for all $v \in V$.

The next result gives us a characterization of norm aggregation functions on sets

Theorem 2.7 ([31])**.** *Let $\phi : [0, +\infty)^I \to [0, +\infty)$ be a function and let φ be the restriction of ϕ to $(0, +\infty)^I \cup \{\mathbf{0}\}$. The following statements are equivalent:*

1. *ϕ is a norm aggregation function on sets;*
2. *for every family of norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$, $(\mathbb{R}^2, \phi \circ \boldsymbol{n})$ is a normed space;*
3. *$\varphi^{-1}(0) = \mathbf{0}$ and φ is an isotone, positive homogeneous and subadditive function;*
4. *$\varphi^{-1}(0) = \mathbf{0}$, φ is positive homogeneous and it preserves asymmetric triangular triplets;*
5. *$\varphi^{-1}(0) = \mathbf{0}$, φ is positive homogeneous and it preserves triangular triplets.*

In the case of norms and asymmetric norms on products we have concluded by theorems 2.5 and 2.6 that it is equivalent to be an asymmetric norm aggregation function on products and a norm aggregation function on products. So it is natural to think that this equivalence will be true when we work on sets. In the next theorem extracted from [31] we can see the characterization of asymmetric norm aggregation functions on sets.

Theorem 2.8 ([31]). *Let $\phi : [0,+\infty)^I \to [0,+\infty)$ be a function. The following statements are equivalent:*

1. *ϕ is an asymmetric norm aggregation function on sets;*
2. *for every family of asymmetric norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$, $(\mathbb{R}^2, \phi \circ \boldsymbol{n})$ is an asymmetric normed space;*
3. *$\phi(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in \phi^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; ϕ is an isotone, positive homogeneous and subadditive function;*
4. *$\phi(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in \phi^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; f is positive homogeneous and it preserves asymmetric triangular triplets;*
5. *$\phi(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in \phi^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; ϕ is positive homogeneous and it preserves triangular triplets.*

Notice that characterization for the aggregation of norms on sets is not the same that the one for aggregation asymmetric norms on sets. In fact, an example on [31] show us a function that is a norm aggregation function on sets but it is not an asymmetric norm aggregation function on sets. The next table summarizes the characterizations given for functions aggregating metrics, quasi-metrics, norms and asymmetric norms on sets and products. Remember that φ is the restriction of the function $\phi : [0,+\infty)^I \longrightarrow [0,+\infty)$ to $(0,+\infty)^I \cap \{\mathbf{0}\}$.

Aggregation function of norms on products	$((\mathbb{R}^2)^I, \|\cdot\|_\phi)$ is a normed space where $\| \mathbf{x} \|_\phi = \phi((\| \mathbf{x}_i \|_2)_{i\in I})$ and $\|\cdot\|_2$ is the Euclidean norm $\forall \mathbf{x} \in (\mathbb{R}^2)^I$	$\phi^{-1}(0) = \{\mathbf{0}\}$, ϕ is positive homogeneous and it preserves triangular triplets	$\phi^{-1}(0) = \{\mathbf{0}\}$, ϕ is positive homogeneous, isotone and subadditive	**Aggregation functions of asymmetric norms on products**
Aggregation functions of norms on sets	for every family of norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$, $(\mathbb{R}^2, \phi \circ \boldsymbol{n})$ is a normed space	$\varphi^{-1}(0) = \mathbf{0}$ and φ is an isotone, positive homogeneous and subadditive function	$\varphi^{-1}(0) = \mathbf{0}$, φ is positive homogeneous and it preserves asymmetric triangular triplets	$\varphi^{-1}(0) = \mathbf{0}$, φ is positive homogeneous and it preserves triangular triplets
Aggregation functions of asymmetric norms on sets	for every family of asymmetric norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$, $(\mathbb{R}^2, \phi \circ \boldsymbol{n})$ is an asymmetric normed space	$\phi(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in \phi^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; ϕ is an isotone, positive homogeneous and subadditive function	$\phi(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in \phi^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; f is positive homogeneous and it preserves asymmetric triangular triplets	$\phi(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in \phi^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; ϕ is positive homogeneous and it preserves triangular triplets

Table 2: Norms and asymmetric norms

3 Aggregation functions of fuzzy metrics and fuzzy quasi-metrics

In the previous section we have shown characterizations of aggregation functions for metrics, quasi-metrics, norms and asymmetric norms on products and sets, but recently several authors have been working in the fuzzy context. From now on in this paper we will work with fuzzy metrics and fuzzy norms. The problem of characterizing fuzzy (quasi-)metric aggregation function was solved by Pedraza, Rodríguez-López and Valero [32].

3.1 t-norms, fuzzy metrics and fuzzy quasi-metrics

We start introducing the concept of fuzzy set. Let X be a nonempty set, we say that M is a fuzzy set in X if M is a function $M : X \longrightarrow [0,1]$.
Our next step is to define a fuzzy metric (or a fuzzy quasi-metric). To do this, first we need to introduce the concept of triangular norm.

Definition 3.1. A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a triangular norm or a t-norm if $([0,1], *)$ is an Abelian monoid with unit 1, such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

If $*$ is also continuous we will say that it is a continuous t-norm.

Now we can see some examples of t-norms.

Example 3.1 ([32]). *1. Minimum t-norm:* $x \wedge y := min\{x, y\}$

2. Product t-norm: $x *_P y := x \cdot y$

3. Lukasiewicz t-norm: $x *_L y := max\{x + y - 1, 0\}$

4. Drastic t-norm: $x *_D y := \begin{cases} min\{x, y\} & \textit{if } x = 1 \textit{ or } y = 1 \\ 0 & \textit{otherwise} \end{cases}$

Definition 3.2. For a given t-norm $*$ we can define:

1. its residuation o residuation implication as the function $\overset{*}{\rightarrow}: [0, 1] \times [0, 1] \to [0, 1]$, given by $x \overset{*}{\rightarrow} y = sup\{z \in [0, 1] : x * z \leq y\}$, $\forall x, y \in [0, 1]$.
2. its biresiduation as the function $\overset{*}{\leftrightarrow}: [0, 1] \times [0, 1] \longrightarrow [0, 1]$ given by $x \overset{*}{\leftrightarrow} y = min\{x \overset{*}{\rightarrow} y, y \overset{*}{\rightarrow} x\}$.

Example 3.2 ([32]). *1. For the t-norm $\wedge$ we have* $x \overset{\wedge}{\rightarrow} y = \begin{cases} 1 & \textit{if } x \leq y \\ y & \textit{if } x > y. \end{cases}$ *and* $x \overset{\wedge}{\leftrightarrow} y = \begin{cases} 1 & \textit{if } x = y \\ x \wedge y & \textit{if } x \neq y \end{cases}.$

2. For the t-norm $*_L$ we have $x \overset{*_L}{\rightarrow} y = min\{1 - x + y, 1\}$ *and* $x \overset{*_L}{\leftrightarrow} y = 1 - |x - y|$.

Now we can define a fuzzy quasi-metric and a fuzzy metric.

Definition 3.3 ([3]). A fuzzy quasi-metric on a nonempty set X is a pair $(M, *)$ such that $*$ is a t-norm and M is a fuzzy set on $X \times X \times [0, +\infty)$ such that for every $x, y, z \in X$ and $t, s > 0$ it verifies

(FQM1) $M(x, y, 0) = 0$;

(FQM2) $M(x, y, t) = M(y, x, t) = 1$ $\forall t > 0$ if and only if $x = y$;

(FQM3) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(FQM4) $M(x, y, \cdot) : [0, +\infty) \longrightarrow [0, 1]$ is left-continuous;

If in addition M satisfies

(FM) $M(x, y, t) = M(y, x, t)$ $\forall x, y \in X$;

Then we will say that $(M, *)$ is a fuzzy metric on X.

Given a (quasi-)metric space (X, d), we have that the (quasi-)metric d induces a fuzzy (quasi-)metric on X as we can see in this example:

Example 3.3 ([32]). *Let (X,d) be a (quasi-)metric space and let M_d be the fuzzy set defined on $X \times X \times [0,+\infty)$ defined by*

$$M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}, \tag{3}$$

then we have that for every t-norm $*$, $(M_d, *)$ is a fuzzy (quasi-)metric on X called the standard fuzzy (quasi-)metric induced by d.

Example 3.4 ([32]). *Let $*$ be a t-norm. Define $M^{\overset{*}{\rightarrow}}, M^{\overset{*}{\leftrightarrow}} : [0,1] \times [0,1] \times [0,+\infty) \longrightarrow [0,1]$ by*

$$M^{\overset{*}{\rightarrow}}(x,y,t) = \begin{cases} x \overset{*}{\rightarrow} y & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases},$$
$$M^{\overset{*}{\leftrightarrow}}(x,y,t) = \begin{cases} x \overset{*}{\leftrightarrow} y & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases},$$

$\forall x, y \in [0,1]$, and $\forall t > 0$. Then $(M^{\overset{*}{\rightarrow}}, *)$ is a quasi-metric on $[0,1]$ and $(M^{\overset{*}{\leftrightarrow}}, *)$ is a fuzzy metric on $[0,1]$.

To show the characterization of when a function $F : [0,1]^I \longrightarrow [0,1]$ preserves fuzzy metrics or fuzzy quasi-metrics first we will need some definitions and properties that F can hold.
We will consider in $[0,1]^I$ the same partial order than in the previous section. Moreover, if $*$ is a t-norm, we can define an operation $*^I$ on $[0,1]^I$ given by $(\mathbf{a} *^I \mathbf{b})_i = \mathbf{a}_i * \mathbf{b}_i$ $\forall \mathbf{a}, \mathbf{b} \in [0,1]^I$ and $\forall i \in I$.

Definition 3.4 ([32]). Let $F : [0,1]^I \longrightarrow [0,1]$. We will say that F is an aggregation function if:

1. F is isotone, i.e. if $\mathbf{a} \preceq \mathbf{b}$ then $F(\mathbf{a}) \leq F(\mathbf{b})$;
2. $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$.

The concept of a subadditive function has been important in the previous section, so it is natural to look for a new concept that can help us to find characterizations for fuzzy metrics and fuzzy quasi-metrics. In the next definition we will gave this concept called $* -$ *supmultiplicativity.*

Definition 3.5 ([32]). Let $F : [0,1]^I \longrightarrow [0,1]$ be an aggregation function and $*$ be a t-norm which is also an aggregation function. Then F is $*-$supmultiplicative whenever

$$F(\mathbf{a}) * F(\mathbf{b}) \leq F(\mathbf{a} *^I \mathbf{b})$$

for every $\mathbf{a}, \mathbf{b} \in [0,1]^I$.
If F is $*$-supmultiplicative for every t-norm $*$ then it will be called supmultiplicative.
Moreover, isotone functions that are $*$-supmultiplicative will be called $* -$ *closed.*

Let see some examples.

Example 3.5 ([32]). *1. Given $i \in I$, the projection function $P_i : [0,1]^I \longrightarrow [0,1]$ given by $P_i(\mathbf{x}) = \mathbf{x}_i$ is $* -$ closed for every t-norm $*$.*

2. *The function* $\mathbf{Inf} : [0,1]^I \longrightarrow [0,1]$ *given by* $\mathbf{Inf}(\mathbf{x}) = \inf_{i \in I} \mathbf{x}_i$ *is* $*-$*closed for every t-norm* $*$.

   Nevertheless, the function $\mathbf{Sup} : [0,1]^I \longrightarrow [0,1]$ *given by* $\mathbf{Sup}(\mathbf{x}) = \sup_{i \in I} \mathbf{x}_i$ *is not* $*-$ *supmultiplicative for any t-norm* $*$ *whenever* $|I| > 1$*. Indeed, let* $j, k \in I$ *with* $j \neq k$ *and let* $\mathbf{x}, \mathbf{y} \in [0,1]^I$ *such that* $\mathbf{x}_i = 0 \ \forall i \in I,\ i \neq j$ *and* $\mathbf{x}_j = 1$ *meanwhile* $\mathbf{y}_i = 0 \ \forall i \in I,\ i \neq k$ *and* $\mathbf{y}_k = 1$*. Then* $\mathbf{Sup}(\mathbf{x}) * \mathbf{Sup}(\mathbf{y}) = 1 \not\leq \mathbf{Sup}(\mathbf{x} *^I \mathbf{y}) = 0$.

Another important concept that we have studied to obtain the results for aggregation functions on metrics and quasi-metrics is the one of triangular and asymmetric triangular triplets. Now, we are going to redefine this concepts in the $t-norm$ sense.

Definition 3.6 ([32])**.** Let $*$ be a t-norm and I a set of indices.

1. A triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in ([0,1]^I)^3$ is said to be $*$-triangular if

$$\mathbf{a} *^I \mathbf{b} \preceq \mathbf{c}, \quad \mathbf{a} *^I \mathbf{c} \preceq \mathbf{b}, \quad \text{and} \quad \mathbf{b} *^I \mathbf{c} \preceq \mathbf{a},$$

   i. e.

$$\mathbf{a}_i * \mathbf{b}_i \leq \mathbf{c}_i \quad \mathbf{a}_i * \mathbf{c}_i \leq \mathbf{b}_i \quad \text{and} \quad \mathbf{b}_i * \mathbf{c}_i \leq \mathbf{c}_i$$

   $\forall i \in I$.

2. A triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in ([0,1]^I)^3$ is said to be asymmetric $*$-triangular if $\mathbf{a} *^I \mathbf{b} \preceq \mathbf{c}$.

If $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a (an asymmetric) $*$-triangular triplet for every t-norm, then we will say that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a (an asymmetric) triangular triplet.

Example 3.6 ([32])**.** *Given a t-norm* $*$*, the triplet* $(\mathbf{a}, \mathbf{b}, \mathbf{a} *^I \mathbf{b})$ *is* $*$*-triangular for every* $\mathbf{a}, \mathbf{b} \in [0,1]^I$*. Furthermore,* $(\mathbf{a}, \mathbf{a}, \mathbf{1})$ *is* $*$*-triangular for every* $\mathbf{a} \in [0,1]^I$.

Example 3.7 ([32])**.** *Given a t-norm* $*$*. The triplet* $(\mathbf{a}, \mathbf{b}, \mathbf{1})$ *is always an asymmetric* $*$*-triangular triplet.*
Furthermore, if $(X, M, *)$ *is a fuzzy metric space (or a fuzzy quasi-metric space). Given* $x, y, z \in X$ *and* $s, t > 0$ *we have that* $(M(x,y,t), M(y,z,s), M(x,z,t+s))$ *is an asymmetric* $*$*-triangular triplet.*

Remark 1. Note that like in the case of triangular and asymmetric triangular triplets on $[0, +\infty)^I$ we have that every $*$-triangular triplet is an asymmetric $*$-triangular triplet.

Now we can define when a function $F : [0,1]^I \longrightarrow [0,1]$ preserves $*$- triangular (asymmetric $*$-triangular) triplets.

Definition 3.7 ([32])**.** Let $*$ be a t-norm. A function $F : [0,1]^I \longrightarrow [0,1]$ preserves $*$-triangular (asymmetric $*$-triangular) triplets if given a $*$-triangular (asymmetric $*$-triangular) triplet we have that $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is a $*$-triangular (an asymmetric $*$-triangular) triplet, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0,1]^I$.
Analogously we can define when F preserves (asymmetric) triangular triplets.

Proposition 3.1 ([32])**.** *Let* $F : [0,1]^I \longrightarrow [0,1]$ *be a function and* $*$ *be a t-norm. Each of the following statements implies its successor:*

1. *F preserves asymmetric triangular triplets;*

2. *F preserves $*$-triangular triplets;*

3. *F dominates $*$, i. e. F is $*$-supmultiplicative;*

If F is isotone, then all the above statements are equivalent.

Proof. 1. implies 2. This is obvious. Take $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0,1]^I$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a $*$-triangular triplet, then it is an asymmetric $*$-triangular triplet. Since F preserves asymmetric $*$-triangular triplet we have that $F(\mathbf{a}) * F(\mathbf{b}) \leq F(\mathbf{c})$. Analogously, reordering the elements of $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ we obtain $F(\mathbf{a}) * F(\mathbf{c}) \leq F(\mathbf{b})$ and $F(\mathbf{b}) * F(\mathbf{c}) \leq F(\mathbf{a})$, i. e. $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is a $*$-triangular triplet and then F preserves $*$-triangular triplets.
Now we prove that 2. implies 3. Let $\mathbf{a}, \mathbf{b} \in [0,1]^I$. Then, we know that $(\mathbf{a}, \mathbf{b}, \mathbf{a} *^I \mathbf{b})$ is a triangular triplet. Hence $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{a} *^I \mathbf{b}))$ is a triangular triplet, i. e. $F(\mathbf{a}) * F(\mathbf{b}) \leq F(\mathbf{a} * \mathbf{b})$. Then F is $*$-supmultiplicative.
Suppose that F is isotone. Let us prove 3. implies 1. Let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be an asymmetric $*$-triangular triplet, i. e. $\mathbf{a} *^I \mathbf{b} \preceq \mathbf{c}$. By assumption, $F(\mathbf{a}) * F(\mathbf{b}) \leq F(\mathbf{a} *^I \mathbf{b})$. Since F is isotone then $F(\mathbf{a}) * F(\mathbf{b}) \leq F(\mathbf{a} *^I \mathbf{b}) \leq F(\mathbf{c})$ so $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is an asymmetric $*$- triangular triplet. □

Corollary 3.1 ([32])**.** *Let $F : [0,1]^I \longrightarrow [0,1]$ such that $F^{-1}(1) \neq \emptyset$ and let $*$ be a t-norm. Then F preserves asymmetric $*$-triangular triplets if and only if it is $*$-closed.*

Proof. From the previous proposition 3.1 we only need to show that if F preserves asymmetric $*$-triangular triplets then it is isotone. Let $\mathbf{x} \in [0,1]^I$ such that $F(\mathbf{x}) = 1$. Given $\mathbf{a}, \mathbf{b} \in [0,1]^I$ verifying $\mathbf{a} \preceq \mathbf{b}$ then $(\mathbf{a}, \mathbf{x}, \mathbf{b})$ is an asymmetric $*$-triangular triplet so $(F(\mathbf{a}), F(\mathbf{x}), F(\mathbf{b}))$ also is. Hence $F(\mathbf{a}) * F(\mathbf{x}) = F(\mathbf{a}) \leq F(\mathbf{b})$ and, therefore F is isotone. □

3.2 Fuzzy metric and quasi-metric aggregation functions on products and sets

Now we are going to focus in the same problem that in the previous sections, i.e. we want to know when a function is an aggregation function for fuzzy metrics or fuzzy quasi-metrics. Let us start giving the definition of a fuzzy metric (quasi-metric) aggregation function on products and a fuzzy metric (quasi-metric) aggregation function on sets. All the results of this section are extracted from [32].

Definition 3.8 ([32])**.** A function $F : [0,1]^I \to [0,1]$ is said to be:

- a *fuzzy (quasi-)metric aggregation function on products* if whenever $*$ is a t-norm and $\{(X_i, M_i, *) : i \in I\}$ is a family of fuzzy (quasi-)metric spaces then $(F \circ \widetilde{\mathbf{M}}, *)$ is a fuzzy (quasi-)metric on $\prod_{i \in I} X_i$ where $\widetilde{\mathbf{M}} : (\prod_{i \in I} X_i)^2 \times [0, +\infty) \to [0,1]^I$ is given by

  $$(\widetilde{\mathbf{M}}(\mathbf{x}, \mathbf{y}, t))_i = M_i(\mathbf{x}_i, \mathbf{y}_i, t)$$

  for every $\mathbf{x}, \mathbf{y} \in \prod_{i \in I} X_i$ and $t \geq 0$.

  If F only satisfies the above condition for a fixed t-norm $*$ then it is said to be an $*$-fuzzy (quasi-)metric aggregation function on products.

- a *fuzzy (quasi-)metric aggregation function on sets* if whenever $*$ is a t-norm and $\{(M_i, *) : i \in I\}$ is a family of fuzzy (quasi-)metrics on the same set X then $(F \circ \mathbf{M}, *)$ is a fuzzy (quasi-)metric on X where $\mathbf{M} : X^2 \times [0, +\infty) \to [0, 1]^I$ is given by

$$(\mathbf{M}(x, y, t))_i = M_i(x, y, t)$$

for every $x, y \in X$ and $t \geq 0$.

If F only satisfies the above condition for a fixed t-norm $*$ then it is said to be an $*$-fuzzy (quasi-)metric aggregation function on sets.

Remark 2. It is clear that if $F : [0, 1]^I \longrightarrow [0, 1]$ is a fuzzy (quasi-)metric aggregation function on products then it is a fuzzy (quasi-)metric aggregation function on sets. Moreover if $|I| = 1$ the two concepts coincide, but in general if $|I| > 1$ the two concepts are different.

Example 3.8 ([32])**.**

1. *If $P_i : [0, 1]^I \longrightarrow [0, 1]$ denotes the i-th projection, then P_i is a fuzzy (quasi-)metric aggregation function on sets but not on products. It is immediate to check that $F \circ \tilde{\mathbf{M}}$ does not verify (FQM2) for every family of fuzzy metrics.*

2. *If $*$ is a t-norm and $F_* : [0, 1]^n \longrightarrow [0, 1]$ is given by $F_*(a_1, \ldots, a_n) = a_1 * \cdots * a_n$ then F_* is a fuzzy (quasi-)metric aggregation function on products.*

Notice that a requirement for the characterization of a (quasi-)metric aggregation function on products ϕ is $\phi^{-1}(0) = \mathbf{0}$, and in sets we need weaker properties. These conditions must be adapted to the fuzzy context by asking some properties of the so-called core of a function F as we next recall

Definition 3.9 ([32])**.** The core of a function $F : [0, 1]^I \longrightarrow [0, 1]$ is the set $F^{-1}(1)$. If $F^{-1}(1) \neq \emptyset$ we will say that

1. F has a trivial core if $F^{-1}(1) = \{\mathbf{1}\}$.

2. The core of F is countably included in a unitary face if for a given $\{\mathbf{x}_n : n \in \mathbb{N}\} \subset F^{-1}(1)$ there exists $i \in I$ such that $(\mathbf{x}_n)_i = 1 \ \forall n \in \mathbb{N}$.

With the next definition we are ready to give a characterization of fuzzy metric and fuzzy quasi-metric aggregation functions.

Definition 3.10 ([18, 26])**.** Let I be a set of indices. A function $F : [0, 1]^I \to [0, 1]$ is said to be *sequentially left-continuous* if F is sequentially continuous when $[0, 1]^I$ is endowed with the product topology of the upper limit topology and $[0, 1]$ carries the usual topology.

Remark 3. Notice that if F is isotone then, for proving that F is sequentially left-continuous it is enough to consider nondecreasing sequences on $[0, 1]^I$. In fact, let $\{\boldsymbol{t}_n\}_{n \in \mathbb{N}}$ be a sequence in $[0, 1]^I$ converging to $\boldsymbol{t}$ with respect to the product topology of the upper limit topology. For all $n \in \mathbb{N}$, define $\boldsymbol{s}_n \in [0, 1]^I$ as $(\boldsymbol{s}_n)_i = \inf_{k \geq n} (\boldsymbol{t}_k)_i$ for all $i \in I$. It is obvious that $\{\boldsymbol{s}_n\}_{n \in \mathbb{N}}$ is a nondecreasing sequence and $\boldsymbol{s}_n \preceq \boldsymbol{t}_n$ for all $n \in \mathbb{N}$. Moreover, $\bigvee_{n \in \mathbb{N}} \boldsymbol{t}_n = \bigvee_{n \in \mathbb{N}} \boldsymbol{s}_n = \boldsymbol{t}$. Suppose that $\{F(\boldsymbol{s}_n)\}_{n \in \mathbb{N}}$ converges to $F(\boldsymbol{t}) = F(\bigvee_{n \in \mathbb{N}} \boldsymbol{t}_n)$, that is, $\bigvee_{n \in \mathbb{N}} F(\boldsymbol{s}_n) = F(\bigvee_{n \in \mathbb{N}} \boldsymbol{t}_n)$. Then

$$F(\bigvee_{n \in \mathbb{N}} \boldsymbol{t}_n) = \bigvee_{n \in \mathbb{N}} F(\boldsymbol{s}_n) \leq \bigvee_{n \in \mathbb{N}} F(\boldsymbol{t}_n) \leq F(\bigvee_{n \in \mathbb{N}} \boldsymbol{t}_n)$$

since F is isotone. Therefore $\bigvee_{n \in \mathbb{N}} F(\boldsymbol{t}_n) = F(\bigvee_{n \in \mathbb{N}} \boldsymbol{t}_n) = F(\boldsymbol{t})$.

Now, we are ready to give results about when a function is a fuzzy (quasi-)metric aggregation function on products or a fuzzy (quasi-)metric aggregation function on sets.

Theorem 3.1 ([32])**.** *Let $F : [0,1]^I \longrightarrow [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

1. *F is a ($*$-)fuzzy quasi-metric aggregation function on products;*

2. *F is a ($*$-)fuzzy metric aggregation function on products;*

3. *F is an aggregation function, ($*$-)supmultiplicative, sequentially left-continuous and F has a trivial core;*

4. *$F(\mathbf{0}) = 0$, F is sequentially left-continuous with a trivial core and F preserves asymmetric ($*$-)triangular triplets.*

Proof. The fact of 1. implies 2. is obvious.
Let us show that 2. implies 3. We first check that F is an aggregation function. We begin proving that F is isotone. Let $\mathbf{a}, \mathbf{b} \in [0,1]^I$ such that $\mathbf{a} \preceq \mathbf{b}$. Let us consider a set with two different elements $X = \{c, d\}$ and for every $i \in I$, let us define a fuzzy metric $(N_i, *)$ on X as

$$N_i(x,y,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = y \text{ and } t > 0, \\ 0 & \text{if } x \neq y \text{ and } 0 < t \leq 1, \\ \mathbf{a}_i & \text{if } x \neq y \text{ and } 1 < t \leq 2 \\ \mathbf{b}_i & \text{if } x \neq y \text{ and } 2 < t. \end{cases}$$

Then $\{(X, N_i, *) : i \in I\}$ is a family of fuzzy metric spaces so $(F \circ \tilde{\mathbf{N}}, *)$ is a fuzzy metric on X^I. Consequently if we fix $\mathbf{x}, \mathbf{y} \in X^I$ then $F \circ \tilde{\mathbf{N}}(\mathbf{x}, \mathbf{y}, \cdot)$ is an increasing function. In particular, if we choose $\mathbf{c}, \mathbf{d} \in X^I$ such that $\mathbf{c}_i = c$ and $\mathbf{d}_i = d$ $\forall i \in I$, we have that

$$\begin{aligned} &F \circ \tilde{\mathbf{N}}(\mathbf{c}, \mathbf{d}, 2) \leq F \circ \tilde{\mathbf{N}}(\mathbf{c}, \mathbf{d}, 3) \\ &F((N_i(\mathbf{c}_i, \mathbf{d}_i, 2))_{i \in I}) \leq F((N_i(\mathbf{c}_i, \mathbf{d}_i, 3))_{i \in I}) \\ &F(\mathbf{a}) \leq F(\mathbf{b}). \end{aligned}$$

Therefore F is isotone.
We check now that $F(\mathbf{0}) = 0$ and F has a trivial core, i. e. $F^{-1}(1) = \{\mathbf{1}\}$. Let $(X, M, *)$ be an arbitrary fuzzy metric space, $x \in X$ and $t > 0$. Considering the family of fuzzy metric spaces $\{(X_i, M_i, *) : i \in I\}$ where $(X_i, M_i, *) = (X, M, *)$ $\forall i \in I$, we have that $(F, \tilde{\mathbf{M}}, *)$ is a fuzzy metric on X^I so given $\mathbf{x} \in X^I$ with $\mathbf{x}_i = x$ $\forall i \in I$ and $t > 0$ then $1 = F \circ \tilde{\mathbf{M}}(\mathbf{x}, \mathbf{x}, t) = F((M(x,x,t))_{i \in I}) = F(\mathbf{1})$. Furthermore, $0 = F \circ \tilde{\mathbf{M}}(\mathbf{x}, \mathbf{x}, 0) = F(\mathbf{0})$. Suppose that we can find $\mathbf{a} \in [0,1]^I$ such that $F(\mathbf{a}) = 1$ but $\mathbf{a} \neq 1$. Consider the family of fuzzy metric spaces $\{(X_i, M_i, *) : i \in I\}$ where $X_i = [0,1]$ and $(M_i, *) = (M^*, *)$ $\forall i \in I$ where the fuzzy metric M^* is defined by:

$$M^*(x,y,t) = \begin{cases} 1 & \text{if } x = y \text{ and } t > 0 \\ x * y & \text{if } x \neq y \text{ and } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Then $(F \circ \tilde{\mathbf{M}}^*, *)$ is a fuzzy metric on $[0,1]^I$, but given $t > 0$ we have that $F \circ \tilde{\mathbf{M}}^*(\mathbf{1}, \mathbf{a}, t) = F((M^*(1, \mathbf{a}_i, t))_{i \in I}) = F(\mathbf{a}) = 1$. Since $(F \circ \tilde{\mathbf{M}}^*, *)$ is a fuzzy metric on $[0,1]^I$ we obtain that $\mathbf{a} = \mathbf{1}$. This contradicts our assumption, so F is an aggregation function with a trivial core.
We next prove that sequentially F is left-continuous. Let us consider a sequence $(\mathbf{t}_n)_n$ in $[0,1]^I$ such that $\mathbf{t}_n \preceq \mathbf{t}_{n+1}$ $\forall n \in \mathbb{N}$. We also denote by $\mathbf{s}$ the supremum of this sequence. Then $(\mathbf{t}_{n,i})_{n \in \mathbb{N}} := ((\mathbf{t}_n)_i)_{n \in \mathbb{N}}$ converges to $\mathbf{s}_i$ $\forall i \in I$. Let $X = \{a, b\}$ be a set with two different elements. Then, given $i \in I$ define $M_i : X \times X \times [0, +\infty) \longrightarrow [0,1]$ as

$$M_i(x,y,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = y,\ t > 0, \\ 0 & \text{if } x \neq y,\ 0 < t \leq \frac{1}{2}, \\ \mathbf{t}_{n,i} & \text{if } x \neq y,\ 1 - \frac{1}{n+1} < t \leq 1 - \frac{1}{n+2}, \\ \mathbf{s}_i & \text{if } x \neq y,\ t \geq 1. \end{cases}$$

It is easy to check that $(X, M_i, *)$ is a fuzzy metric space $\forall i \in I$. Then $(F \circ \tilde{\mathbf{M}}, *)$ is a fuzzy metric on X^I associated with the family of fuzzy metric spaces $\{(X, M_i, *) : i \in I\}$. Consequently $F \circ \tilde{\mathbf{M}}(\mathbf{a}, \mathbf{b}, \cdot)$ is left-continuous where $\mathbf{a}_i = a$ and $\mathbf{b}_i = b$ $\forall i \in I$. Hence $(F \circ \tilde{\mathbf{M}}\left(\mathbf{a}, \mathbf{b}, 1 - \frac{1}{n+2}\right)_{n \in \mathbb{N}}$ converges to $F \circ \tilde{\mathbf{N}}(\mathbf{a}, \mathbf{b}, 1)$, but

$$F \circ \tilde{\mathbf{M}}\left(\mathbf{a}, \mathbf{b}, 1 - \frac{1}{n+2}\right) = F\left(\left(M_i\left(\mathbf{a}_i, \mathbf{b}_i, 1 - \frac{1}{n+2}\right)\right)_{i \in I}\right) = F((\mathbf{t}_{n,i})_{i \in I}) = F(\mathbf{t}_n)$$

for every $n \in \mathbb{N}$ and

$$F \circ \tilde{\mathbf{M}}(\mathbf{a}, \mathbf{b}, 1) = F((M_i(a, b, 1))_{i \in I}) = F((\mathbf{s}_i)_{i \in I}) = F(\mathbf{s}).$$

Consequently F is sequentially left-continuous.
Let us check that F is $*$-supmultiplicative. Let $\mathbf{a}, \mathbf{b} \in [0,1]^I$. Define $I_1 = \{i \in I : \mathbf{a}_i \neq 1, \mathbf{b}_i \neq 1\}$, $I_2 = \{i \in I : \mathbf{a}_i \neq 1, \mathbf{b}_i = 1\}$, $I_3 = \{i \in I : \mathbf{a}_i = 1, \mathbf{b}_i \neq 1\}$ and $I_4 = \{i \in I : \mathbf{a}_i = \mathbf{b}_i = 1\}$. Let us consider a fixed set $X = \{x_1, x_2, x_3\}$ with three different elements and a fuzzy metric $(M_i, *)$ on X given by

1. If $i \in I_1$ then

$$M_i(x,y,t) = M_i(y,x,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = y,\ t > 0, \\ \mathbf{a}_i & \text{if } x = x_1,\ y = x_2,\ t > 0, \\ \mathbf{b}_i & \text{if } x = x_2,\ y = x_3,\ t > 0, \\ \mathbf{a}_i * \mathbf{b}_i & \text{if } x = x_1,\ y = x_3,\ t > 0. \end{cases}$$

2. if $i \in I_2$ then

$$M_i(x,y,t) = M_i(y,x,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = y,\ t > 0, \\ \mathbf{a}_i & \text{if } x = x_1,\ y = x_2,\ t > 0, \\ \mathbf{a}_i * \mathbf{a}_i & \text{if } x = x_2,\ y = x_3,\ 0 < t \leq 1 \\ 1 & \text{if } x = x_2,\ y = x_3,\ t > 1 \\ \mathbf{a}_i & \text{if } x = x_1,\ y = x_3, t > 0. \end{cases}$$

3. if $i \in I_3$ then

$$M_i(x,y,t) = M_i(y,x,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = y,\ t > 0, \\ \mathbf{b}_i * \mathbf{b}_i & \text{if } x = x_1,\ y = x_2,\ 0 < t \leq 1, \\ 1 & \text{if } x = x_1,\ y = x_2,\ t > 1, \\ \mathbf{b_i} & \text{if } x = x_2,\ y = x_3,\ t > 0, \\ \mathbf{b}_i & \text{if } x = x_1,\ y = x_3\ , t > 0. \end{cases}$$

4. if $i \in I_4$ then

$$M_i(x,y,t) = M_i(y,x,t) = \begin{cases} 0 & \text{if } t = 0, \\ 0 & \text{if } x \neq y,\ 0 < t \leq 1, \\ 1 & \text{if } x \neq y, t > 1, \\ 1 & \text{if } x = y,\ t > 0. \end{cases}$$

Then $\{(X, M_i, *) : i \in I\}$ is a family of fuzzy metric spaces. Hence $F \circ \tilde{\mathbf{M}}$ is a fuzzy metric on X^I. Let us define $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in X^I$ such that $(\mathbf{x}_i)_j = x_i\ \forall j \in I$ and $i = 1,2,3$. the we have that

$$\begin{aligned} &F \circ \tilde{\mathbf{M}}(\mathbf{x}_1, \mathbf{x}_2, 2) * F \circ \tilde{\mathbf{M}}(\mathbf{x}_2, \mathbf{x}_3, 2) \leq F \circ \tilde{\mathbf{M}}(\mathbf{x}_1, \mathbf{x}_3, 4) \\ &F((M_i(x_1, x_2, 2))_{i \in I}) * F((M_i(x_2, x_3, 2))_{i \in I}) \leq F((M_i(x_1, x_3, 4))_{i \in I}) \\ &F(\mathbf{a}) * F(\mathbf{b}) \leq F(\mathbf{a} *^I \mathbf{b}) \end{aligned}$$

and, thus, F is $*$-supmultiplicative.

It is easy to prove that 3. implies 4. Let $*$ be a t-norm. Since F is an aggregation function we have that F is isotone and $F(\mathbf{0}) = 0$. By corollary 3.1 we have that F preserves asymmetric $*$-triangular triplets.

It remains to prove that 4. implies 1. Let $\{(X_i, M_i, *) : i \in I\}$ be a family of fuzzy quasi-metric spaces. Let us check that $(F \circ \tilde{\mathbf{M}}, *)$ is a quasi-metric on $\prod_{i \in I} X_i$. Since F is an aggregation function then $F \circ \tilde{\mathbf{M}}(\mathbf{x}, \mathbf{y}, 0) = F((M_i(\mathbf{x}_i, \mathbf{y}_i, 0))_{i \in I}) = F(\mathbf{0}) = 0$ for every $\mathbf{x}, \mathbf{y} \in \prod_{i \in I} X_i$, so (FQM1) holds. Moreover, if $t > 0$ then $F \circ \tilde{\mathbf{M}}(\mathbf{x}, \mathbf{x}, t) = F((M_i(\mathbf{x}_i, \mathbf{x}_i, t))_{i \in I}) = F(\mathbf{1}) = 1\ \forall x \in \prod_{i \in I} X_i$. On the other hand, given $\mathbf{x}, \mathbf{y} \in \prod_{i \in I} X_i$ if $F \circ \tilde{\mathbf{M}}(\mathbf{x}, \mathbf{y}, t) = F \circ \tilde{\mathbf{M}}(\mathbf{y}, \mathbf{x}, t) = 1\ \forall t > 0$ then $F((M_i(\mathbf{x}_i, \mathbf{y}_i, t))_{i \in I}) = F((M_i(\mathbf{y}_i, \mathbf{x}_i, t))_{i \in I}) = 1\ \forall t > 0$. Since F has a trivial core we have that

for every $i \in I$, $M_i(\mathbf{x}_i, \mathbf{y}_i, t) = M_i(\mathbf{y}_i, \mathbf{x}_i, t) = 1$ $\forall t > 0$ so $\mathbf{x}_i = \mathbf{y}_i = 1$ $\forall i \in I$, i. e. $\mathbf{x} = \mathbf{y}$. Thus (FQM2) holds.
Moreover, given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \prod_{i\in I} X_i$ and $t, s > 0$ we have that $M_i(\mathbf{x}_i, \mathbf{y}_i, t) * M_i(\mathbf{y}_i, \mathbf{z}_i, s) \leq M_i(\mathbf{x}_i, \mathbf{z}_i, t+s)$ is an asymmetric $*$-triangular triplet. Then by assumption,
$(F((M_i(\mathbf{x}_i, \mathbf{y}_i, t))_{i\in I}), F((M_i(\mathbf{y}_i, \mathbf{z_i}, \mathbf{s}))_{i\in I}), F((M_i(\mathbf{x}_i, \mathbf{z}_i, t+s))_{i\in I}))$ is an asymmetric $*$-triangular triplet so

$$F((M_i(\mathbf{x}_i, \mathbf{y}_i, t))_{i\in I}) * F((M_i(\mathbf{y}_i, \mathbf{z}_i, s))_{i\in I}) \leq F((M_i(\mathbf{x}_i, \mathbf{z}_i, t+s))_{i\in I})$$

and, thus, (FQM3) holds for $F \circ \tilde{\mathbf{M}}$.
Finally, (FMQ4) is clear since F is sequentially left-continuous and $M_i(x, y, \cdot)$ is left-continuous and isotone for every $x, y \in X$ and every $i \in I$. □

Example 3.9 ([32])**.** *1. Given a t-norm $*$ and $n \in \mathbb{N}$, the function $F_* : [0,1]^n \longrightarrow [0,1]$ given by $F_*(a_1, \ldots, a_n) = a_1 * \cdots * a_n$ is a sequentially left-continuous $(*-)$supmultiplicative aggregation function with a trivial core. So it is a $*$-fuzzy (quasi-)metric aggregation function on products.*

2. The function $F : [0,1] \longrightarrow [0,1]$ given by $F(x) = \sqrt{x}$ is a continuous $*_p$-supmultiplicative aggregation function with a trivial core. Consequently, it is a $*_P$-fuzzy (quasi-)metric aggregation function.

Notice that in the case of fuzzy metric and fuzzy quasi-metric spaces, the concept of a fuzzy metric aggregation function and fuzzy quasi-metric aggregation function on products coincide, in contrast with the case of metric aggregation functions and quasi-metric aggregation functions on products. It makes us thing that probably a fuzzy quasi-metric aggregation function and fuzzy aggregation function on sets will be equivalent. Pedraza, Rodríguez-López and Valero [32] gave a characterization of fuzzy (quasi-)metric aggregation functions on sets.

Theorem 3.2 ([32])**.** *Let $F : [0,1]^I \longrightarrow [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

(1) F is a ($*$-)fuzzy quasi-metric aggregation function on sets;

(2) F is a ($*$-)fuzzy metric aggregation function on sets;

(3) F is an aggregation function, ($*$-)supmultiplicative, sequentially left-continuous and the core of F is countably included in a unitary face;

(4) $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, F is sequentially left-continuous, the core of F is countably included in a unitary face and F preserves asymmetric ($*$-)triangular triplets.

Proof. 1. implies 2. This is clear. 2. implies 3. Everything follows with a simple adaptation of the proof of theorem 3.1 except for proving that the core of F is countably included in a unitary face. Assume for the sake of contradiction that we can find a sequence $\{\mathbf{x}_n : n \in \mathbb{N}\} \subset F^{-1}(1)$ such that for any $i \in I$ we could find $n_i \in \mathbb{N}$ such that $(\mathbf{x}_{n_i})_i \neq 1$. Let us consider a set $X = \{a, b\}$ with two different elements and for each $i \in I$ define $M_i : X \times X \times [0, +\infty) \longrightarrow [0,1]$ as

$$M_i(x,y,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = y,\ t > 0, \\ (\mathbf{x}_1)_i & \text{if } x \neq y,\ t > 1, \\ (\mathbf{x}_1)_i * \cdots * (\mathbf{x}_{n+1})_i & \text{if } x \neq y,\ \frac{1}{n+1} < t \leq \frac{1}{n}. \end{cases}$$

Notice that $(X, M_i, *)$ is a fuzzy metric space $\forall i \in I$. We only show that (FQM2) holds since the other conditions follow trivially. If $M_i(x,y,t) = 1\ \forall t > 0$ then $x = y$. Otherwise, we can suppose that $x = a$ and $y = b$ so $M_i(a,b,t) = 1\ \forall t > 0$. Nevertheless, this is impossible because $M_i(a, b, \frac{1}{n_i}) = (\mathbf{x}_1)_i * \cdots * (\mathbf{x}_{n_i})_i \neq 1$.
By assumption, $(F \circ \mathbf{M}, *)$ is a fuzzy metric on X. However, given $t > 0$ and since F is $(*-)$supmultiplicative we have that

$$F \circ \mathbf{M}(a,b,t) = F((M_i(a,b,t))_{i \in I}) = F(\mathbf{x}_1 *^I \cdots *^I \mathbf{x}_{n+1}) \geq F(\mathbf{x}_1) * \ldots F(\mathbf{x}_{n+1}) = 1 * \cdots * 1 = 1$$

for some $n \in \mathbb{N} \cup \{0\}$. Since $a \neq b$ this contradicts (FQM2).
3. implies 4. is a direct consequence of corollary 3.1.
The proof of 4. implies 1. is similar to 4. implies 1. of theorem 3.1 and the only difference is with proving the condition (FQM2). Let $\{(X, M_i, *) : i \in I\}$ be a family of fuzzy quasi-metric spaces. Let $x, y \in X$ such that $F \circ \mathbf{M}(x,y,t) = F \circ \mathbf{M}(y,x,t) = 1\ \forall t > 0$. By corollary 3.1 F is $*$-supmultiplicative. hence we have that $1 = 1 * 1 = F(\mathbf{M}(x,y,t)) * F(\mathbf{M}(y,x,t)) \leq F(\mathbf{M}(x,y,t) *^I \mathbf{M}(y,x,t))\ \forall t > 0$, so $F((M_i(x,y,t))_{i \in I} *^I (M_i(y,x,t))_{i \in I}) = 1$. Define $\mathbf{x}_n = (M_i(x,y,t))_{i \in I} *^I (M_i(y,x,t))_{i \in I}$. Then $\{\mathbf{x}_n : n \in \mathbb{N}\} \subset F^{-1}(1)$. By assumption, there exist $i \in I$ such that $(\mathbf{x}_n)_i = 1\ \forall n \in \mathbb{N}$, i. e. $M_i(x, y, \frac{1}{n}) = M_i(y, x, \frac{1}{n}) = 1\ \forall n \in \mathbb{N}$. Since $M_i(x,y,\cdot)$ and $M_i(y,x,\cdot)$ are increasing we deduce that $M_i(x,y,t) = M_i(y,x,t) = 1\ \forall t > 0$. Consequently, $x = y$ since $(M_i, *)$ is a fuzzy quasi-metric. $\square$

Now we summarize the characterizations of fuzzy metric and fuzzy quasi-metric aggregation functions on sets and products.

Aggregation functions of fuzzy metrics on products	F is an aggregation function, ($*$-)supmultiplicative, sequentially left-continuous and F has a trivial core	$F(\mathbf{0}) = 0$, F is sequentially left-continuous with a trivial core and F preserves asymmetric ($*$-)triangular triplets	**Aggregation functions of fuzzy quasi-metrics on products**
Aggregation functions of fuzzy metrics on sets	F is an aggregation function, ($*$-)supmultiplicative, sequentially left-continuous and the core of F is countably included in a unitary face	$F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, F is sequentially left-continuous, the core of F is countably included in a unitary face and F preserves asymmetric ($*$-)triangular triplets	**Aggregation functions of fuzzy quasi-metrics on sets**

Table 3: Fuzzy metrics and fuzzy quasi-metrics

4 Aggregation functions of fuzzy norms and fuzzy asymmetric norms

In the classic context we have shown different results of aggregation functions of metrics, quasi-metrics, norms and asymmetric norms on products and on sets. Golet [17] gave a definition of a fuzzy norm. It makes sense to look for characterization of functions that aggregate norms and asymmetric norms on products and on sets in the fuzzy context. In this section we are going to show results that characterize aggregation functions of fuzzy norms and fuzzy quasi-norms on products and on sets. All the results of this section are original and are included in [29].

4.1 Fuzzy norms and fuzzy asymmetric norms

Our next step is to characterize aggregation function of norms and quasi-norms in the fuzzy context, but for this purpose, first we need to introduce some definitions and results.

Definition 4.1 ([17, 1, 2])**.** A *weak fuzzy quasi-norm* on a real vector space V is a pair $(N, *)$ such that $*$ is a continuous t-norm and N is a fuzzy set on $V \times [0, +\infty)$ such that for every $x, y \in V$ and $t, s > 0$ it verifies

(FQN1) $N(x, 0) = 0$;

(FQN2) $N(x, t) = N(-x, t) = 1$ for all $t > 0$ if and only if $x = 0_V$;

(FQN3) $N(\lambda x, t) = N\left(x, \frac{t}{\lambda}\right)$ for all $\lambda > 0$;

(FQN4) $N(x, t) * N(y, s) \leq N(x + y, t + s)$;

(FQN5) $N(x, \cdot) : [0, \infty) \to [0, 1]$ is left-continuous.

If N also satisfies

(FQN6) $\lim_{t\to+\infty} N(x, t) = 1$

then $(N, *)$ is called a *fuzzy quasi-norm.*
A (weak) fuzzy (quasi-)norm on a real vector space V is a (weak) fuzzy (quasi-)norm $(N, *)$ on V such that

(FQN3') $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right)$ for all $\lambda \in \mathbb{R}\backslash\{0\}$.

A *(weak) fuzzy (quasi-)normed* space is a triple $(V, N, *)$ such that V is a real vector space and $(N, *)$ is a (weak) fuzzy (quasi-)norm on V.

Remark 4. Notice that if $(N, *)$ is a weak fuzzy norm on a real vector space V, where $*$ is a t-norm, then $N(x, \cdot)$ is isotone. Let $t, s > 0$ such that $t < s$ and let $x \in V$, then we have that there exists $r > 0$ such that $s = t + r$, then, since $(N, *)$ is a weak fuzzy quasi-norm we have that

$$N(x, t) = N(x, t) * 1 = N(x, t) * N(0, r) \leq N(x + 0, t + r) = N(x, s).$$

Remark 5. Notice that the definition of fuzzy norm that we have considered is that of Goleţ. It differs slightly from that defined in [8] since Cheng and Mordeson considered a real or complex vector space V, they allow that the parameter t takes also negative values by considering that $N(x, t) = 0$ for every $t < 0$ and they only make the definition for the minimum t-norm. The above definition is similar to that in [2, 1].
There are also other notions of a fuzzy norm which modify the previous conditions as the elimination of (FQN5) [4].

Example 4.1 (cf. [35, Example 1],[1, Example 1])**.** *Let (V, q) be a quasi-normed space. Let $k, m, n \in \mathbb{R}^+$ be fixed. Define $N : V \times [0, +\infty) \to [0, 1]$ as*

$$N(x, t) = \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{kt^n}{kt^n + mq(x)} & \text{if } t > 0. \end{cases}$$

Then $(V, N, *)$ is a fuzzy quasi-normed space for every continuous t-norm $*$. If $k = n = m = 1$ then $(N, *)$ is called the standard fuzzy quasi-norm and it will be denoted by $(N_q, *)$.

Example 4.2. *Let $a \in [0, 1[$ and consider $N_a : \mathbb{R} \times [0, +\infty)$ defined as*

$$N_a(x, t) = \begin{cases} 1 & \text{if } t > |x| \\ a & \text{if } 0 < t \leq |x| \\ 0 & \text{if } t = 0. \end{cases}$$

It is straightforward to check that $(\mathbb{R}, N_a, *)$ is a fuzzy normed space for every continuous t-norm $*$.

Let us show that $(N_a, *)$ is a fuzzy norm for every continuous t-norm $*$.

It is obvious that $N_a(x, 0) = 0\ \forall x \in \mathbb{R}$, so (FQN1) holds.

It is obvious that $N_a(x, t) = N_a(-x, t)\ \forall t > 0$, then we only have to prove that $N(x, t) = 1\ \forall t > 0$ if and only if $x = 0$. Let $x \in \mathbb{R}$ such that $N_a(x, t) = 1\ \forall t > 0$, i. e. $t > |x|\ \forall t > 0$, but this is possible only if $x = 0$. Conversely if $x = 0$ it is obvious that $N(x, t) = 1$ and (FQN2) holds.

Now we prove (FQN3'). Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $x \in \mathbb{R}$. Then we have that

$$N_a(\lambda x, t) = \begin{cases} 1 & \text{if } t > |\lambda x| = |\lambda||x|, \\ a & \text{if } 0 < t \leq |\lambda x| = |\lambda||x|, \\ 0 & \text{if } t = 0 \end{cases} = \begin{cases} 1 & \text{if } \frac{t}{|\lambda|} > |x|, \\ a & \text{if } 0 < \frac{t}{|\lambda|} \leq |x|, \\ 0 & \text{if } t = 0 \end{cases} = N_a\left(x, \frac{t}{\lambda}\right).$$

and $(FQN3')$ holds.

Consider $x, y \in \mathbb{R}$ and $t, s > 0$, we will consider different cases:

1. *$N(x, t) = N(y, s) = a$, then we have that $N(x + y, t + s) = 1$ or $N(x + y, t + s) = a$ and we deduce*

$$N(x, t) * N(y, s) = a * a \leq a * 1 = N(x + y, t + s) * 1 = N(x + y, t + s),$$

or

$$N(x, t) * N(y, s) = a * a \leq 1 * 1 = N(x + y, t + s) * 1 = N(x + y, t + s).$$

2. *If $N(x, t) = a$, $N(y, s) = 1$ we have again that $N(x+y, t+s) = a$ or $N(x+y, t+s) = 1$ and we obtain*

$$N(x, t) * N(y, s) = a * 1 = N(x + y, t + s) * 1 = N(x + y, t + s)$$

or

$$N(x, t) * N(y, s) = a * 1 \leq 1 * 1 = N(x + y, t + s) * 1 = N(x + y, t + s).$$

3. *$N(x, t) = N(y, s) = 1$, then we have that $t + s > |x| + |y| \geq |x + y|$, i. e. $N(x + y, t + s) = 1$ and then $N(x, t) * N(y, s) = 1 * 1 = 1 = N(x + y, t + s)$.*

Let $(t_n)_{n \in N}$ be a sequence in $[0, +\infty)$ such that $t_n \leq t_{n+1}\ \forall n \in \mathbb{N}$ and let s be the supremum of this sequence. Let $x \in \mathbb{R}$, $x \neq 0$ then we have that if $t_n \leq |x|$, then $N(x, t_n) = a\ \forall n \in \mathbb{N}$, but, since s is the supremum of the sequence, we have that $s \leq |x|$, i. e. $N(x, s) = a$ and $N(x, \cdot)$ is left-continuous. If $x = 0$, then we have that $N(x, t_n) = 1 = N(x, s)\ \forall n \in \mathbb{N}$ and $N(x, \cdot)$ is left continuous. If $t_n > |x|\ \forall n \in \mathbb{N}$ or there exists $n_0 \in \mathbb{N}$ such that $t_n > |x|\ \forall n \geq n_0$ we have that $N(x, t_n) = 1\ \forall n \geq n_0$, but, since s is the supremum of the sequence, we have that $N(x, s) = 1$. So we can conclude that $\lim_{n\to\infty} N(x, t_n) = 1 = N(x, s)$.

It remains to prove that $\lim_{t\to\infty} N(x, t) = 1$, but this is obvious.

Remark 6. Let $(V, N, *)$ be a weak fuzzy quasi-normed space. We define the open ball $B_N(x, r, t)$ with center x, radius r, $0 < r < 1$, and $t > 0$ as

$$B_n(x, r, t) = \{y \in V : N(y - x, t) > 1 - r\}.$$

The family of open balls is a base for a topology $\tau(N)$ on V. Notice that this topology is the weak topology on V making the functionals $N(x,\cdot) : [0,+\infty) \to [0,1]$ lower semicontinuous, for every $x \in V$.

4.2 Aggregation functions of fuzzy norms and fuzzy quasi-norms on products and sets

Finally, we will study when a function preserves weak fuzzy norms and weak fuzzy quasi-norms. We start giving the definitions of a weak fuzzy (quasi-)norm on products and a weak fuzzy (quasi-)norm aggregation function on sets.

Definition 4.2 ([29])**.** A function $F : [0,1]^I \to [0,1]$ is said to be:

- a *weak fuzzy (quasi-)norm aggregation function on products* if whenever $*$ is a t-norm and $\{(V_i, N_i, *) : i \in I\}$ is a family of weak fuzzy (quasi-)normed spaces then $(F \circ \widetilde{\boldsymbol{N}}, *)$ is a weak fuzzy (quasi-)norm on $\prod_{i\in I} V_i$ where $\widetilde{\boldsymbol{N}} : \prod_{i\in I} V_i \times [0,+\infty) \to [0,1]^I$ is given by

  $$(\widetilde{\boldsymbol{N}}(\boldsymbol{x}, t))_i = N_i(\mathbf{x}_i, t)$$

  for every $\boldsymbol{x} \in \prod_{i\in I} V_i$ and $t \geq 0$.

  If F only satisfies the above condition for a fixed t-norm $*$ then it is said to be an $*$-weak fuzzy (quasi-)norm aggregation function on products.

- a *weak fuzzy (quasi-)norm aggregation function on sets* if whenever $*$ is a t-norm and $\{(N_i, *) : i \in I\}$ is a family of weak fuzzy (quasi-)norms on the same real vector space V then $(F \circ \boldsymbol{N}, *)$ is a weak fuzzy (quasi-)norm on V where $\boldsymbol{N} : V \times [0,+\infty) \to [0,1]^I$ is given by

  $$(\boldsymbol{N}(x,t))_i = N_i(x,t)$$

  for every $x \in V$ and $t \geq 0$.

  If F only satisfies the above condition for a fixed t-norm $*$ then it is said to be an $*$-weak fuzzy (quasi-)norm aggregation function on sets.

Theorem 4.1 ([29])**.** *Let $F : [0,1]^I \longrightarrow [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

1. *F is a $(*$-$)$weak fuzzy quasi-norm aggregation function on products;*
2. *F is a $(*$-$)$weak fuzzy norm aggregation function on products;*
3. *$F(\mathbf{0}) = 0$, F is isotone, $(*$-$)$supmultiplicative, sequentially left-continuous and F has trivial core;*
4. *$F(\mathbf{0}) = 0$, F is sequentially left-continuous with trivial core and F preserves asymmetric $(*$-$)$triangular triplets.*

Proof. 1. implies 2. is obvious.
Let us show that 2. implies 3. First, let $\{(V_i, N_i, *)\}_{i\in I}$ be a family of weak fuzzy normed spaces. Then, we have that $F(\mathbf{0}) = F((N_i(x_i, 0))_{i\in I}) = F \circ \tilde{\mathbf{N}}(\mathbf{x}, 0) = 0\ \forall \mathbf{x} \in \prod_{i\in I} V_i$, so we have that $F(\mathbf{0}) = 0$.
Let us prove that F has trivial core. If $\{(\mathbb{R}, N_i, *) : i \in I\}$ is a family of weak fuzzy normed spaces we have that $(F \circ \tilde{\mathbf{N}}, *)$ is a weak fuzzy norm on $\mathbb{R}^I$. Let $t > 0$, then we have that $F(\mathbf{1}) = F((N_i(0, t))_{i\in I}) = F \circ \tilde{\mathbf{N}}(\mathbf{0}, t) = 1$. Suppose that we can find $\mathbf{a} \in [0, 1]^I$ such that $F(\mathbf{a}) = 1$ but $\mathbf{a} \neq \mathbf{1}$. Let $J = \{i \in I : \mathbf{a}_i \neq 1\}$, which is nonempty. Consider the family of weak fuzzy normed spaces $\{(\mathbb{R}, N_i, *) : i \in I\}$ where

- if $i \in J$ then $(N_i, *) = (N_{\mathbf{a}_i}, *)$ is the fuzzy norm of Example 4.2;
- if $i \notin J$ then $(N_i, *) = (N, *)$ is an arbitrary fixed weak fuzzy norm on $\mathbb{R}$.

By assumption $(F \circ \tilde{\mathbf{N}}, *)$ is a weak fuzzy norm on $\mathbb{R}^I$. Let $\mathbf{x} \in \mathbb{R}^I$ such that $\mathbf{x}_i = 1$ if $i \in J$ and $\mathbf{x}_i = 0$ otherwise. Then given $t > 0$ we have that

$$F \circ \tilde{\mathbf{N}}(\mathbf{x}, t) = F((N_i(\mathbf{x}_i, t))_{i\in I}) = \begin{cases} F(\mathbf{1}) = 1 & \text{if } t > 1 \\ F(\mathbf{a}) = 1 & \text{if } 0 < t \leq 1. \end{cases}.$$

which contradicts (FQN2). Hence F has trivial core.
Now we prove that F is isotone. Let $\mathbf{a}, \mathbf{b} \in [0, 1]^I$ be such that $\mathbf{a} \preceq \mathbf{b}$ and consider a t-norm $*$ and the family of weak fuzzy norms $\{(N_i, *) : i \in I\}$ defined on $\mathbb{R}$ given by

$$N_i(x, t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{|x|}{4}, \\ \mathbf{a}_i & \text{if } \frac{|x|}{4} < t \leq \frac{|x|}{2}, \\ \mathbf{b}_i & \text{if } \frac{|x|}{2} < t \leq |x|, \\ 1 & \text{if } t > |x|. \end{cases}$$

It is easy to check that $(N_i, *)$ is a weak fuzzy norm on $\mathbb{R}\ \forall i \in I$. Since F is a weak fuzzy norm aggregation function on products we have that $(F \circ \tilde{\mathbf{N}}, *)$ is a weak fuzzy norm on $\mathbb{R}$. Consequently

$$\begin{aligned} F(\mathbf{a}) &= F((\mathbf{a}_i)_{i\in I}) = F((N_i(6, 2))_{i\in I}) = F \circ \tilde{\mathbf{N}}(\mathbf{6}, 2) = F \circ \tilde{\mathbf{N}}(\mathbf{6}, 2) * 1 = \\ &= F \circ \tilde{\mathbf{N}}(\mathbf{6}, 2) * F \circ \tilde{\mathbf{N}}(\mathbf{0}, 2) \leq F \circ \tilde{\mathbf{N}}(\mathbf{6} + \mathbf{0}, 2 + 2) = F((N_i(\mathbf{6}, 4))_{i\in I}) = \\ &= F((\mathbf{b}_i)_{i\in I}) = F(\mathbf{b}) \end{aligned}$$

i. e. F is isotone.
We next show that F is $*$-supmultiplicative. Let $\mathbf{a}, \mathbf{b} \in [0, 1]^I$. Define $L_1 = \{(x, y) \in \mathbb{R}^2 : x \neq 0, y = 0\}$, $L_2 = \{(x, y) \in \mathbb{R}^2 : x = 0, y \neq 0\}$ and $L_3 = \mathbb{R}^2 \backslash (L_1 \cup L_2 \cup \{(0, 0)\})$. For each $i \in I$ we will consider a function $N_i : \mathbb{R}^2 \times [0, +\infty) \to [0, 1]$ as follows:

$$N_i(\mathbf{x}, t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \|\mathbf{x}\|, \\ \mathbf{a}_i & \text{if } \mathbf{x} \in L_1 \text{ and } t > \|\mathbf{x}\|, \\ \mathbf{b}_i & \text{if } \mathbf{x} \in L_2 \text{ and } t > \|\mathbf{x}\|, \\ \mathbf{a}_i * \mathbf{b}_i & \text{if } \mathbf{x} \in L_3 \text{ and } t > \|\mathbf{x}\|, \\ 1 & \text{if } \mathbf{x} = \mathbf{0} \text{ and } t > 0, \end{cases}$$

where $\|\cdot\|$ is the Euclidean norm. Then $(N_i, *)$ is a weak fuzzy norm on $\mathbb{R}^2$ for all $i \in I$. We only check that $(N_i, *)$ satisfies (FQN4). Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and $t, s > 0$. If $\mathbf{x} + \mathbf{y} = \mathbf{0}$ the inequality is trivially true. If $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ we also obtain the inequality since $N_i(\mathbf{z}, \cdot)$ is isotone for every $\mathbf{z} \in \mathbb{R}^2$. So let us suppose that $\mathbf{x} + \mathbf{y} \neq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$. Let $j \in \{1, 2, 3\}$ such that $\mathbf{x} + \mathbf{y} \in L_j$. If $\mathbf{x} \in L_j$ or $\mathbf{y} \in L_j$ then it is clear that $N_i(\mathbf{x}, t) * N_i(\mathbf{y}, s) \leq N_i(\mathbf{x} + \mathbf{y}, t + s)$. If $\mathbf{x} \notin L_j$ and $\mathbf{y} \notin L_j$ we distinguish three cases:

- if $j = 3$ then $\mathbf{x} \in L_1$ and $\mathbf{y} \in L_2$ or viceversa. If $t \leq \|\mathbf{x}\|$ or $s \leq \|\mathbf{y}\|$ then $N_i(\mathbf{x}, t) = 0$ or $N_i(\mathbf{y}, s) = 0$ so the inequality holds. Otherwise $t > \|\mathbf{x}\|$ and $s > \|\mathbf{y}\|$ which implies that $t + s > \|\mathbf{x}\| + \|\mathbf{y}\| \geq \|\mathbf{x} + \mathbf{y}\|$. Hence $N_i(\mathbf{x}, t) * N_i(\mathbf{y}, s) = \mathbf{a}_i * \mathbf{b}_i \leq N_i(\mathbf{x} + \mathbf{y}, t + s) = \mathbf{a}_i * \mathbf{b}_i$.

- if $j = 2$ then at least one of $\mathbf{x}, \mathbf{y}$ belongs to L_3. Without loss of generality we can suppose that $\mathbf{x} \in L_3$. As above, if $t \leq \|\mathbf{x}\|$ or $s \leq \|\mathbf{y}\|$ then $N_i(\mathbf{x}, t) = 0$ or $N_i(\mathbf{y}, s) = 0$ so the inequality holds. Otherwise $t > \|\mathbf{x}\|$ and $s > \|\mathbf{y}\|$ which implies that $t + s > \|\mathbf{x}\| + \|\mathbf{y}\| \geq \|\mathbf{x} + \mathbf{y}\|$. Hence $N_i(\mathbf{x}, t) * N_i(\mathbf{y}, s) = \mathbf{a}_i * \mathbf{b}_i * N_i(\mathbf{y}, s) \leq \mathbf{b}_i = N_i(\mathbf{x} + \mathbf{y}, t + s)$.

- if $j = 1$ we can reason as in the previous case.

Since $(F \circ \tilde{\mathbf{N}}, *)$ is a weak fuzzy norm on $(\mathbb{R}^2)^I$ then it verifies (FQN4). Defining $(\mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{1}) \in (\mathbb{R}^2)^I$ such that $(\mathbf{1}, \mathbf{0})_i = (1, 0)$, $(\mathbf{0}, \mathbf{1})_i = (0, 1)$ and $(\mathbf{1}, \mathbf{1})_i = (1, 1)$ for all $i \in I$ we have

$$\begin{aligned} F \circ \tilde{\mathbf{N}}((\mathbf{1}, \mathbf{0}), 2) * F \circ \tilde{\mathbf{N}}((\mathbf{0}, \mathbf{1}), 2) &\leq F \circ \tilde{\mathbf{N}}((\mathbf{1}, \mathbf{1}), 4) \\ F((\mathbf{a}_i)_{i \in I}) * F((\mathbf{b}_i)_{i \in I}) &\leq F((\mathbf{a}_i * \mathbf{b}_i)_{i \in I}) \end{aligned}$$

so F is $*$-supmultiplicative.

Finally, we prove that F is sequentially left-continuous. By Remark 3, let us consider a nondecreasing sequence $\{\mathbf{s}_n\}_{n \in \mathbb{N}}$ in $[0, 1]^I$ converging to $\mathbf{s} \in [0, 1]^I$ in the product topology of the lower limit topology.

For each $i \in I$ define $N_i : \mathbb{R} \times [0, +\infty) \to [0, 1]$ as

$$N_i(x, t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = 0,\ t > 0, \\ 0 & \text{if } 0 < t \leq \frac{|x|}{2}, \\ (\mathbf{s}_n)_i & \text{if } |x|\left(1 - \frac{1}{n+1}\right) < t \leq |x|\left(1 - \frac{1}{n+2}\right),\ n \in \mathbb{N} \\ \mathbf{s}_i & \text{if } 0 < |x| \leq t. \end{cases}$$

Then $(\mathbb{R}, N_i, *)$ is a weak fuzzy normed space for all $i \in I$. We only check (FQN4). Let $x, y \in \mathbb{R}$ and $t, s > 0$. If $N_i(x, t) = 0$ or $N_i(y, s) = 0$ then the conclusion is obvious so we suppose that $N_i(x, t) \neq 0$ and $N_i(y, s) \neq 0$. We may also assume that $x \neq 0$, $y \neq 0$ and $x + y \neq 0$ (otherwise, the conclusion follows trivially). If $|x| \leq t$ and $|y| \leq s$ then $|x + y| \leq |x| + |y| \leq t + s$ so $N_i(x + y, t + s) = \mathbf{s}_i \geq N_i(x, t) * N_i(y, s) = \mathbf{s}_i * \mathbf{s}_i$.

Now suppose that $|x| > t$ and $|y| \leq s$. Since $N_i(x, t) \neq 0$ and $x \neq 0$ we also have that $0 < \frac{|x|}{2} < t$. Then there exists $n_x \in \mathbb{N}$ such that

$$|x|\left(1 - \frac{1}{n_x + 1}\right) < t \leq |x|\left(1 - \frac{1}{n_x + 2}\right).$$

Therefore,

$$\begin{aligned} t+s &> |x|\left(1-\frac{1}{n_x+1}\right)+|y| > |x|\left(1-\frac{1}{n_x+1}\right)+|y|\left(1-\frac{1}{n_x+1}\right) \\ &= (|x|+|y|)\left(1-\frac{1}{n_x+1}\right) \geq |x+y|\left(1-\frac{1}{n_x+1}\right). \end{aligned}$$

From this and since $\{(\mathbf{s}_n)_i\}_{n\in\mathbb{N}}$ is nondecreasing then $N_i(x+y,t+s) \geq (\mathbf{s}_{n_x})_i = N_i(x,t) \geq N_i(x,t) * N_i(y,s)$.
If $|x| \leq t$ and $|y| > s$ we can reason as above.
Finally, let us suppose that $|x| > t$ and $|y| > s$. Then we can find $n_x, n_y \in \mathbb{N}$ such that

$$|x|\left(1-\frac{1}{n_x+1}\right) < t \leq |x|\left(1-\frac{1}{n_x+2}\right) \text{ and } |y|\left(1-\frac{1}{n_y+1}\right) < s \leq |y|\left(1-\frac{1}{n_y+2}\right).$$

Then

$$\begin{aligned} |x+y|\left(1-\frac{1}{(n_x\wedge n_y)+1}\right) &\leq |x|+|y|-\frac{|x|}{(n_x\wedge n_y)+1}-\frac{|y|}{(n_x\wedge n_y)+1} \\ &\leq |x|+|y|-\frac{|x|}{n_x+1}-\frac{|y|}{n_y+1} < t+s. \end{aligned}$$

Hence

$$N_i(x+y,t+s) \geq (\mathbf{s}_{n_x\wedge n_y})_i \geq (\mathbf{s}_{n_x})_i * (\mathbf{s}_{n_y})_i = N_i(x,t) * N_i(y,s).$$

Consequently, $(N_i, *)$ satisfies (FQN4).
By assumption $(F \circ \tilde{\mathbf{N}}, *)$ is a weak fuzzy norm on $\mathbb{R}^I$ associated with the family of fuzzy normed spaces $\{(\mathbb{R}, N_i, *) : i \in I\}$. Then $F \circ \tilde{\mathbf{N}}(\mathbf{1}, \cdot)$ is left-continuous so $\{F \circ \tilde{\mathbf{N}}(\mathbf{1}, 1-\frac{1}{n+2})\}_{n\in\mathbb{N}}$ converges to $F \circ \tilde{\mathbf{N}}(\mathbf{1}, 1)$. We observe that

$$F \circ \tilde{\mathbf{N}}\left(\mathbf{1}, 1-\frac{1}{n+2}\right) = F\left(\left(N_i\left(1, 1-\frac{1}{n+2}\right)\right)_{i\in I}\right) = F(((\mathbf{s}_n)_i)_{i\in I}) = F(\mathbf{s}_n)$$

for every $n \in \mathbb{N}$ and

$$F \circ \tilde{\mathbf{N}}(\mathbf{1}, 1) = F((N_i(1,1))_{i\in I}) = F((\mathbf{s}_i)_{i\in I}) = F(\mathbf{s}).$$

Consequently F is sequentially left-continuous.
3. implies 4. is the same that 3. implies 4. in theorem 3.1.
Finally we will show that 4. implies 1. Suppose that $F(\mathbf{0}) = 0$, F is sequentially left-continuous with a trivial core and F preserves asymmetric triangular triplets. Let $\{(V_i, N_i, *)\}_{i\in I}$ be a family of weak fuzzy quasi-normed spaces. Let us check that $F \circ \tilde{\mathbf{N}}$ is a weak fuzzy quasi-norm on $\prod_{i\in I} V_i$. Take $\mathbf{x} \in \prod_{i\in I} V_i$, then we have that $F \circ \tilde{\mathbf{N}}(\mathbf{x}, 0) = F((N_i(\mathbf{x}_i, 0))_{i\in I}) = F(\mathbf{0}) = 0$ so we have that (FQN1) holds. Consider $\mathbf{x} \in \prod_{i\in I} V_i$ such that $F \circ \tilde{\mathbf{N}}(\mathbf{x}, t) = F \circ \tilde{\mathbf{N}}(-\mathbf{x}, t) = 1$ $\forall t > 0$, then we have that $F((N_i(\mathbf{x}, t))_{i\in I}) = 1$ and $F((N_i(-\mathbf{x}, t))_{i\in I}) = 1$, but F has a trivial core, i. e. $N_i(\mathbf{x}_i, t) = N_i(-\mathbf{x}_i, t) = 1$ $\forall i \in I$. Since $(N_i, *)$ is a weak fuzzy quasi-norm on V_i for every $i \in I$ we can conclude that $\mathbf{x}_i = 0_{V_i}$ $\forall i \in I$, so $\mathbf{x} = \mathbf{0}$ and we have (FQN2). Now take $\lambda > 0$, $t > 0$ and $\mathbf{x} \in \prod_{i\in I} V_i$. From the fact that $(N_i, *)$ is a weak fuzzy quasi-norm $\forall i \in I$ we deduce that

$$F \circ \tilde{\mathbf{N}}(\lambda\mathbf{x}, t) = F((N_i(\lambda\mathbf{x}_i, t))_{i\in I}) = F\left(\left(N_i\left(\mathbf{x}_i, \frac{t}{\lambda}\right)\right)_{i\in I}\right) = F \circ \tilde{\mathbf{N}}\left(\mathbf{x}, \frac{t}{\lambda}\right)$$

, i. e. (FQN3) holds.
Consider $\mathbf{x}, \mathbf{y} \in \prod_{i\in I} V_i$ and $t, s > 0$ we want to prove that $F \circ \tilde{\mathbf{N}}(\mathbf{x}, t) * F \circ \tilde{\mathbf{N}}(\mathbf{y}, s) \leq F \circ \tilde{\mathbf{N}}(\mathbf{x}+\mathbf{y}, t+s)$. From the fact that $(N_i, *)$ is a weak fuzzy norm for all $i \in I$ we have that $(N_i(\mathbf{x}_i, t), N_i(\mathbf{y}_i, s), N_i(\mathbf{x}_i+\mathbf{y}_i, t+s))$ is an asymmetric $*$-triangular triplet, i. e. $(\tilde{\mathbf{N}}(\mathbf{x}, t), \tilde{\mathbf{N}}(\mathbf{y}, s), \tilde{\mathbf{N}}(\mathbf{x}+\mathbf{y}, t+s))$ is an asymmetric $*$-triangular triplet. Since F preserves asymmetric $*$-triangular triplets we have that

$$F \circ \tilde{\mathbf{N}}(\mathbf{x}, t) *^I F \circ \tilde{\mathbf{N}}(\mathbf{y}, s) \leq F \circ \tilde{\mathbf{N}}(\mathbf{x}+\mathbf{y}, t+s)$$

and we have (FQN4).
(FQN5) is clear since F is sequentially left-continuous and $N_i(x, \cdot)$ is left-continuous and isotone $\forall i \in I$. □

Notice that by theorem 4.1 and by theorem 3.1 we can conlude that in the fuzzy context the concept of a fuzzy metric, fuzzy quasi-metric, weak fuzzy norm and weak fuzzy quasi-norm aggregation function coincide, in contrast with the classical context.
Finally we will show the result that gives us a characterization of weak fuzzy quasi-norm aggregation functions on sets.

Theorem 4.2 ([29])**.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

(1) F is a ($*$-)weak fuzzy quasi-norm aggregation function on sets;

(2) F is a ($*$-)weak fuzzy norm aggregation function on sets;

(3) $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, the core of F is countably included in a unitary face, F is isotone, ($*$-)supmultiplicative and sequentially left-continuous ;

(4) $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, the core of F is countably included in a unitary face, F is sequentially left-continuous and F preserves asymmetric ($*$-)triangular triplets.

Proof. (1) $\Rightarrow$ (2) This is trivial.
(2) $\Rightarrow$ (3) We first prove that $F(\mathbf{0}) = 0$. Let $(V, N, *)$ be an arbitrary weak fuzzy normed space, $v \in V$ and $t > 0$. Considering the family of weak fuzzy normed spaces $\{(V, N_i, *) : i \in I\}$ where $N_i = N$ for all $i \in I$, we have that $(F \circ \boldsymbol{N}, *)$ is a weak fuzzy norm on V so $0 = F \circ \boldsymbol{N}(v, 0) = F((N(v, 0))_{i\in I}) = F(\mathbf{0})$.
On the other hand, $F(\mathbf{1}) = F((N(0_V, t))_{i\in I}) = F \circ \boldsymbol{N}(0_V, t) = 1$ by (FQN2).
For proving that F is $*$-supmultiplicative we can proceed as in the proof of this fact in the implication (2) $\Rightarrow$ (3) of Theorem 4.1.
Now, we check that the core of F is countably included in a unitary face.
Now suppose, contrary to our claim, that there exists a sequence $\{\boldsymbol{a}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$ such that for any $i \in I$ we could find $n_i \in \mathbb{N}$ verifying $(\boldsymbol{a}_{n_i})_i \neq 1$. Let us consider the vector space $\mathbb{R}$ and for

each $i \in I$ define $N_i : \mathbb{R} \times [0, +\infty) \to [0, 1]$ as

$$N_i(x,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = 0,\ t > 0, \\ (\boldsymbol{a}_1)_i & \text{if } x \neq 0,\ t > |x|, \\ (\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n+1})_i & \text{if } x \neq 0,\ \frac{|x|}{n+1} < t \leq \frac{|x|}{n},\ n \in \mathbb{N}. \end{cases}$$

Notice that $(\mathbb{R}, N_i, *)$ is a weak fuzzy normed space for all $i \in I$. Let us check it. It is obvious that (FQN1) is satisfied. On the other hand, let $x \in \mathbb{R}$ such that $N_i(x,t) = N_i(-x,t) = 1$ for all $t > 0$. By assumption, we can find $n_i \in \mathbb{N}$ such that $(\boldsymbol{a}_{n_i}) \neq 1$. Hence, if $x \neq 0$ then $N_i(x, \frac{|x|}{n_i}) = (\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n_i+1})_i \leq (\boldsymbol{a}_1)_i \wedge \ldots \wedge (\boldsymbol{a}_{n_i+1})_i < 1$ which is a contradiction. Therefore $x = 0$.
Furthermore, let $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}\backslash\{0\}$. If $x = 0$ it is clear that $N_i(\lambda 0, t) = 1 = N_i\left(0, \frac{t}{|\lambda|}\right)$. If $x \neq 0$ the equality $N_i(\lambda x, t) = N_i\left(x, \frac{t}{|\lambda|}\right)$ follows from the equivalences of the inequalities $t > |\lambda x|$ and $\frac{|\lambda x|}{n+1} < t \leq \frac{|\lambda x|}{n}$ with $\frac{t}{|\lambda|} > x$ and $\frac{|x|}{n+1} < \frac{t}{|\lambda|} \leq \frac{|x|}{n}$ respectively, so (FQN3') is proved.
We next check (FQN4). Let $x, y \in \mathbb{R}$ and $t, s > 0$. If $x+y = 0$ it is obvious that $N_i(x,t) * N_i(y,s) \leq N_i(x+y, t+s) = 1$. Moreover, if $x+y \neq 0$ and $x = 0$ or $y = 0$ the inequality is also clear since if, for example $y = 0$ then $N_i(x,t) * N_i(y,s) = N_i(x,t) = (\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n+1})_i$ for some $n \in \mathbb{N}$ and since $t < s+t$ the factors which appears in multiplication by the t-norm $*$ in the value of $N_i(x, t+s)$ are less or equal than the factors in $N_i(x,t)$, so $N_i(x,t) \leq N_i(x, t+s)$. Finally suppose that $x+y \neq 0$ and $x \neq 0$, $y \neq 0$. If $t+s > |x+y|$ the inequality is clear since $N_i(x+y, t+s) = (\boldsymbol{a}_1)_i$. Otherwise, $t+s \leq |x+y| \leq |x| + |y|$. Then $t \leq |x|$ or $s \leq |y|$. We distinguish some cases:

- $t \leq |x|$ and $s > |y|$. Then there exists $n_x \in \mathbb{N}$ such that $\frac{|x|}{n_x+1} < t \leq \frac{|x|}{n_x}$. Then

$$t + s > \frac{|x|}{n_x+1} + |y| > \frac{|x|+|y|}{n_x+1} \geq \frac{|x+y|}{n_x+1}$$

  This means that the number of factors which appear in $N_i(x+y, t+s)$ are less than or equal to $n_x + 1$ which is the number of factors which appear in $N_i(x,t)$. Consequently $N_i(x,t) * N_i(y,s) \leq N_i(x,t) \leq N_i(x+y, t+s)$.

- $t > |x|$ and $s \leq |y|$. In this case we can reason as above.

- $t \leq |x|$ and $s \leq |y|$. Let $n_x, n_y \in \mathbb{N}$ such that

$$\frac{|x|}{n_x+1} < t \leq \frac{|x|}{n_x} \text{ and } \frac{|y|}{n_y+1} < s \leq \frac{|y|}{n_y}.$$

  Then

$$t + s > \frac{|x|}{n_x+1} + \frac{|y|}{n_y+1} \geq \frac{|x|+|y|}{\max\{n_x, n_y\}+1} \geq \frac{|x+y|}{\max\{n_x, n_y\}+1}.$$

  This means that the number of factors which appear in $N_i(x+y, t+s)$ are less than or equal to $\max\{n_x, n_y\} + 1$. Reasoning as above we obtain the desired inequality.

It remains to prove that $N_i(x, \cdot)$ is left-continuous. If $x = 0$ it is obvious. Suppose now that $x \neq 0$ and let $t > 0$. By construction, if $\{t_n\}_{n\in\mathbb{N}}$ is a sequence in $(0, +\infty)$ whose upper limit is t, we can

find $n_0 \in \mathbb{N}$ such that $N(x, t_n)$ is constant for every $n \geq n_0$, so the conclusion follows. We conclude that $(N_i, *)$ is a weak fuzzy norm on $\mathbb{R}$ for all $i \in I$.
Notice that if $t > 1$ then $N_i(1, t) = (\boldsymbol{a}_1)_i$ so

$$F \circ \boldsymbol{N}(1, t) = F((N_i(1, t))_{i\in I}) = F(((\boldsymbol{a}_1)_i)_{i\in I}) = F(\boldsymbol{a}_1) = 1.$$

If $0 < t \leq 1$ then we can find $n \in \mathbb{N}$ such that $N_i(1, t) = (\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n+1})_i$ for all $i \in I$. Since F is $*$-supmultiplicative then

$$\begin{aligned} F \circ \boldsymbol{N}(1, t) = F((N_i(1, t))_{i\in I}) &= F(((\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n+1})_i)_{i\in I}) = F(\boldsymbol{a}_1 * \ldots * \boldsymbol{a}_{n+1}) \\ &\geq F(\boldsymbol{a}_1) * \ldots * F(\boldsymbol{a}_n) = 1. \end{aligned}$$

Therefore, $F \circ \boldsymbol{N}(1, t) = 1$ for all $t > 0$ which contradicts that $(F \circ \boldsymbol{N}, *)$ is a weak fuzzy norm. Consequently the core of F is countably included in a unitary face.
The proof that F is isotone and the proof that F is sequentially left-continuous are similar to the same proofs in the implication (2)$\Rightarrow$ (3) of Theorem 4.1.
(3) $\Rightarrow$ (4) This is similar to the implication (3) $\Rightarrow$ (4) of Theorem 4.1.
(4) $\Rightarrow$ (1) Let $\{(V, N_i, *) : i \in I\}$ be a family of weak fuzzy quasi-normed spaces. Let us check that $(F \circ \boldsymbol{N}, *)$ is a weak fuzzy quasi-norm on V.
Given $x \in V$ then $F \circ \boldsymbol{N}(x, 0) = F((N_i(x, 0))_{i\in I}) = F(\boldsymbol{0}) = 0$ so (FQN1) holds.
Now, suppose that there exists $x \in V$ such that $F \circ \boldsymbol{N}(x, t) = F \circ \boldsymbol{N}(-x, t) = 1$ for all $t > 0$. Since $(N_i(x, t), N_i(-x, t), N_i(x_i, t) * N_i(-x, t))$ is an asymmetric $*$-triangular triplet for all $i \in I$, by assumption $(F((N_i(x, t))_{i\in I}), F((N_i(-x, t))_{i\in I}), F((N_i(x, t) * N_i(-x, t))_{i\in I}))$ is also an asymmetric $*$-triangular triplet so

$$1 = 1 * 1 = F((N_i(x, t))_{i\in I}) * F((N_i(-x, t))_{i\in I}) \leq F((N_i(x, t) * N_i(-x, t))_{i\in I}).$$

Hence $F((N_i(x, t) * N_i(-x, t))_{i\in I}) = 1$ for all $t > 0$. Let us define $\boldsymbol{a}_n = (N_i(x, \frac{1}{n}) * N_i(-x, \frac{1}{n}))_{i\in I}$. Then $\{\boldsymbol{a}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$ and since the core of F is countably included in a unitary face we can find $j \in I$ such that $(\boldsymbol{a}_n)_j = 1$ for all $n \in \mathbb{N}$, that is, $N_j(x, \frac{1}{n}) * N_j(-x, \frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Consequently, $N_j(x, \frac{1}{n}) = N_j(-x, \frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Moreover, since $N_i(x, \cdot)$ and $N_i(-x, \cdot)$ are isotone we immediately obtain that $N_j(x, t) = N_j(-x, t) = 1$ for all $t > 0$. Since $(N_i, *)$ is a weak fuzzy quasi-metric on V then $x = 0_V$. Therefore, $F \circ \boldsymbol{N}$ satisfies (FQN2).
It is clear that $F \circ \boldsymbol{N}$ verifies (FQN3) since given $\lambda, t > 0$ and $x \in V$ we have that

$$F \circ \boldsymbol{N}(\lambda x, t) = F((N_i(\lambda x, t))_{i\in I}) = F\left(\left(N_i\left(x, \frac{t}{\lambda}\right)\right)_{i\in I}\right) = F \circ \boldsymbol{N}\left(x, \frac{t}{\lambda}\right).$$

In order to prove (FQN4), let $x, y \in V$ and $t, s > 0$. Since $(N_i, *)$ is a weak fuzzy quasi-norm for all $i \in I$ then it is obvious that $((N_i(x, t))_{i\in I}, (N_i(y, s))_{i\in I}, (N_i(x + y, t + s))_{i\in I})$ is an asymmetric $*$-triangular triplet. Then, by assumption, $(F((N_i(x, t))_{i\in I}), F((N_i(y, s))_{i\in I}), F((N_i(x+y, t+s))_{i\in I}))$ is an asymmetric $*$-triangular triplet so

$$F((N_i(x, t))_{i\in I}) * F((N_i(y, s))_{i\in I}) \leq F((N_i(x + y, t + s))_{i\in I})$$

so $F \circ \boldsymbol{N}$ verifies (FQN4).
To this end, (FQN5) follows from the fact that F is sequentially left-continuous and $N_i(x, \cdot)$ is left-continuous and isotone for every $x \in V$ and every $i \in I$. $\square$

Like in the case of products, we can see that for a function F is equivalent to be a weak fuzzy norm (or quasi-norm) aggregation function on sets than to be a fuzzy metric (or quasi-metric) aggregation function on sets.
Finally we summarize results about aggregation functions of (weak) fuzzy norms and (weak) fuzzy quasi-norms.

Aggregation functions of fuzzy norms on products	$F(\mathbf{0}) = 0$, F is isotone, ($*$-)supmultiplicative, sequentially left-continuous and F has trivial core	$F(\mathbf{0}) = 0$, F is sequentially left-continuous with trivial core and F preserves asymmetric ($*$-)triangular triplets	**Aggregation functions of fuzzy quasi-norms on products**
Aggregation functions of fuzzy norms on sets	$F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, the core of F is countably included in a unitary face, F is isotone, ($*$-)supmultiplicative and sequentially left-continuous	$F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, the core of F is countably included in a unitary face, F is sequentially left-continuous and F preserves asymmetric ($*$-)triangular triplets	**Aggregation functions of fuzzy quasi-norms on sets**

Table 4: Fuzzy metrics and fuzzy quasi-metrics

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