Weighted composition operators on Korenblum type spaces of analytic functions

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Abstract

We investigate the continuity, compactness and invertibility of weighted composition operators $W_{\psi,\varphi} : f \to \psi(f \circ \varphi)$ when they act on the classical Korenblum space $A^{-\infty}$ and other related spaces of analytic functions on the open unit disc. Some results about the spectrum of these operators are presented, as well. Rather complete results are obtained in the case of the Korenblum space.

1 Introduction, notation and preliminaries

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic selfmap on the unit disc $\mathbb{D}$ of the complex plane, and let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic map. The aim of this article is to investigate the continuity, compactness, invertibility and the spectrum of weighted composition operators $W_{\psi,\varphi} : f \to \psi(f \circ \varphi)$ when they act on certain weighted Fréchet or (LB)-spaces of analytic functions on $\mathbb{D}$ of the complex plane. The operator $W_{\psi,\varphi}$ can be written as the composition $W_{\psi,\varphi} = M_\psi \circ C_\varphi$, where $M_\psi f := \psi f$ is the multiplication operator and $C_\varphi f := f \circ \varphi$ is the composition operator. In this paper the operator $W_{\psi,\varphi}$ acts on spaces which appear as intersections or unions of the growth Banach spaces of analytic functions defined below. Weighted composition operators have been investigated by many authors. We refer the books by Cowen and McCluer [15] and Shapiro [28]. Further references will be given later.

The space $H(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ is endowed with the Fréchet topology of uniform convergence on compact sets. When we write a space, we mean a Hausdorff locally convex space. We refer the reader to [25] for results and terminology about functional analysis, and in particular about Fréchet and (LB)-spaces.

For each $\alpha > 0$, the growth Banach spaces of analytic functions are defined as

$$H_\alpha^\infty := \{f \in H(\mathbb{D})\colon \|f\|_\alpha := \sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f(z)| < \infty\}$$

and

$$H_\alpha^0 := \{f \in H(\mathbb{D})\colon \lim_{|z|\to 1^-} (1-|z|)^\alpha |f(z)| = 0\}\,.$$

These spaces are sometimes defined using the weight $(1-|z|^2)$ instead of $(1-|z|)$. Since $1-|z| \le 1-|z|^2 \le 2(1-|z|)$, the spaces coincide and the norms are equivalent. Both H_α^∞ and H_α^0 are Banach spaces when endowed with the norm $\|\cdot\|_\alpha$. The space H_α^0, which is a closed subspace of H_α^∞, coincides with the closure of the polynomials on H_α^∞; see e.g. [9]. The space H^∞ of bounded analytic functions on $\mathbb{D}$ is contained in H_α^0 for each $\alpha > 0$. These Banach spaces, as well as their intersections and unions, play a relevant and important role in connection with the interpolation and sampling of analytic functions; see [19, Section 4.3].

For each pair $0 < \beta_1 < \beta_2$ we have $H_{\beta_1}^\infty \subset H_{\beta_2}^0$, with continuous inclusion. Moreover, for each $\alpha > 0$, H_α^∞ is canonically isomorphic to the bidual Banach space $(H_\alpha^0)''$ of H_α^0, [9, 29].

The spaces of analytic functions we consider are defined in the following way.

$$A_+^{-\alpha} := \bigcap_{\beta>\alpha} H_\beta^\infty = \{f \in H(\mathbb{D})\colon \sup_{z\in\mathbb{D}} (1-|z|)^\beta |f(z)| < \infty \ \forall \beta > \alpha\},$$

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in which case also

$$A_+^{-\alpha} = \bigcap_{\beta>\alpha} H_\beta^0$$

for each $\alpha \geq 0$. And

$$A_-^{-\alpha} := \bigcup_{\beta<\alpha} H_\beta^\infty = \{f \in H(\mathbb{D})\colon \sup_{z\in\mathbb{D}}(1-|z|)^\beta |f(z)| < \infty \text{ for some } \beta < \alpha\}$$

in which case also

$$A_-^{-\alpha} = \bigcup_{\beta<\alpha} H_\beta^0,$$

for each $0 < \alpha \leq \infty$.

The space $A_+^{-\alpha}$ is a Fréchet space, that is a metrizable and complete locally convex space, when endowed with the locally convex topology generated by the increasing sequence of norms $||f||_k := \sup_{z\in\mathbb{D}}(1-|z|)^{\alpha+\frac{1}{k}}|f(z)|$, for $f \in A_+^{-\alpha}$ and each $k \in \mathbb{N}$. We note, for $0 < \beta_1 < \beta_2$, that the natural inclusion $H_{\beta_1}^\infty \subset H_{\beta_2}^0$ is actually a compact operator. Therefore $A_+^{-\alpha}$ is a Fréchet Schwartz space, [20, §21.1 Example 1(b)]. In particular, bounded subsets of $A_+^{-\alpha}$ are relatively compact, [25, Remark 24.24]. Moreover, for every $\beta > \alpha > 0$ we have $H_\alpha^\infty \subset A_+^{-\alpha} \subset H_\beta^0$ with continuous inclusions.

Each space $A_-^{-\alpha}$ is endowed with the finest locally convex topology such that all the natural inclusion maps $H_\beta^\infty \subset A_-^{-\alpha}$, for $\beta < \alpha$, are continuous. In particular, the space $A_-^{-\alpha}$ is the complete (DFS)-space

$$A_-^{-\alpha} := \operatorname*{ind}_k H^\infty_{\alpha-\frac{1}{k}} = \operatorname*{ind}_k H^0_{\alpha-\frac{1}{k}},$$

which is necessarily a Schwartz space, [25, Proposition 25.20]. The inductive limit is taken over all $k \in \mathbb{N}$ such that $(\alpha - \frac{1}{k}) > 0$. Furthermore, the (LB)-space $A_-^{-\alpha}$ satisfies that every bounded set $B \subset A_-^{-\alpha}$ is contained and bounded in the Banach space H_β^∞ for some $0 < \beta < \alpha$. (LB)-spaces satisfying this property are called *regular*.

The classical Korenblum space $A_-^{-\infty}$, [22], denoted $A^{-\infty}$, is defined by

$$A^{-\infty} := \bigcup_{0<\alpha<\infty} H_\alpha^\infty = \bigcup_{n\in\mathbb{N}} H_n^\infty$$

and is endowed with the finest locally convex topology such that all the natural inclusion maps $H_n^\infty \subset A^{-\infty}$ are continuous, that is, $A^{-\infty} = \operatorname{ind}_n H_n^\infty$. It is well-known that $A^{-\infty}$ is an algebra with continuous pointwise multiplication. For the reader's convenience, we give some details for proving that: since, clearly, the multiplication is separately continuous, we may apply [23, pp. 158] because $A^{-\infty}$ is a barreled DF-space. Also, it is closed under derivation and integration. Further information can be found in [19, Section 4.3]. We mention that $A_-^{-\alpha} \subset H_\alpha^0 \subset H_\alpha^\infty \subset A_+^{-\alpha}$, for all $\alpha > 0$, with continuous inclusions.

2 Continuous, compact and invertible weighted composition operators

An operator $T : X \to X$ on a space X is called *compact* (resp. *bounded*) if there exists a neighbourhood U of 0 such that $T(U)$ is a relatively compact (resp. bounded) subset of X. Every bounded operator is continuous. If the bounded subsets of X are relatively compact, as it happens if X is one of the spaces $A_+^{-\alpha}$ or $A_-^{-\alpha}$, then bounded and compact operators $T : X \to X$ coincide. The following two lemmas follow from the definitions involved and Grothendieck's factorization theorem [25, Theorem 24.33].

Lemma 2.1 (Lemma 25 of [4]) *Let $E := \mathrm{proj}_m E_m$ and $F := \mathrm{proj}_n F_n$ be Fréchet spaces which are projective limits of Banach spaces. Assume that E is dense in E_m and that $E_{m+1} \subset E_m$ with a continuous inclusion for each $m \in \mathbb{N}$ (resp. $F_{n+1} \subset F_n$ with a continuous inclusion for each $n \in \mathbb{N}$). Let $T : E \to F$ be a linear operator.*

(i) T is continuous if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that T has a unique continuous linear extension $T_{m,n} : E_m \to F_n$.

(ii) Assume T is continuous. Then T is bounded if and only if there exists $m_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, the operator T has a unique continuous linear extension $T_{m_0,n} : E_{m_0} \to F_n$.

Lemma 2.2 (Lemma 4.1 of [5]) *Let $X = \mathrm{ind}_n X_n$ and $Y = \mathrm{ind}_m Y_m$ be two (LB)-spaces which are increasing unions of Banach spaces $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{m=1}^{\infty} Y_m$. Let $T\colon X \to Y$ be a linear map.*

(i) T is continuous if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(X_n) \subset Y_m$ and the restriction $T_{n,m} : X_n \to Y_m$ is continuous.

(ii) Assume that Y is a regular (LB)-space. Then T is bounded if and only if there exists $m \in \mathbb{N}$ such that $T(X_n) \subset Y_m$ and $T\colon X_n \to Y_m$ is continuous for all $n \geq m$.

2.1 Continuity

The continuity and compactness of weighted composition operators between spaces of type H_α^∞ was described in [16] even in a more general context; see also [11] and [26]. Combining Lemmas 2.1 and 2.2 with Propositions 3.1 and 3.2 of [16], we obtain the following result.

Proposition 2.3 *Let $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$.*

(1) Let $\alpha \geq 0$. The operator $W_{\psi,\varphi} : A_+^{-\alpha} \to A_+^{-\alpha}$ is continuous if and only if for each $\varepsilon > 0$ there is $\delta \in]0, \varepsilon]$ such that

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)|(1-|z|)^{\alpha+\varepsilon}}{(1-|\varphi(z)|)^{\alpha+\delta}} < \infty.$$

If this is the case, then $\psi \in A_+^{-\alpha}$.

(2) Let $0 < \alpha \leq \infty$. The operator $W_{\psi,\varphi} : A_-^{-\alpha} \to A_-^{-\alpha}$ is continuous if and only if for each $0 < \beta < \alpha$ there is $\gamma \in [\beta, \alpha[$ such that

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)|(1-|z|)^{\gamma}}{(1-|\varphi(z)|)^{\beta}} < \infty.$$

If this is the case, then $\psi \in A_-^{-\alpha}$.

Corollary 2.4 *Let $\alpha \geq 0$, $\psi \in H(\mathbb{D})$. $M_\psi \in \mathcal{L}(A_+^{-\alpha})$ if and only if $\psi \in A_+^{-0}$.*

Proof. Setting $\varphi(z) = z$ in Proposition 2.3 (1), we get that $M_\psi \in \mathcal{L}(A_+^{-\alpha})$ if and only if for each $\varepsilon > 0$ there is $\delta \in]0, \varepsilon]$ such that $\sup_{z\in\mathbb{D}} |\psi(z)|(1-|z|)^{\varepsilon-\delta} < \infty$. This happens exactly when $\psi \in A_+^{-0}$. □

Corollary 2.5 *Let $\alpha \geq 0$, $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$. If $\psi \in A_+^{-0}$, then $W_{\psi,\varphi} \in \mathcal{L}(A_+^{-\alpha})$.*

Proof. By [11, Theorem 2.3], the operators $C_\varphi : H_\beta^\infty \to H_\beta^\infty$ and $C_\varphi : H_\beta^0 \to H_\beta^0$ are continuous for each $\beta > 0$. Therefore, Lemma 2.1 implies that C_φ is continuous on $A_+^{-\alpha}$. Since $W_{\psi,\varphi} = M_\psi \circ C_\varphi$, the conclusion follows from Corollary 2.4. □

Example 2.6 In this example it is shown that the converse of Corollary 2.5 does not hold. Take $\alpha > 0$ and $\varphi(z) = z/2, z \in \mathbb{D}$. For each $\psi \in A_+^{-\alpha} \setminus A_+^{-0}$ the operator $W_{\psi,\varphi}$ is continuous on $A_+^{-\alpha}$. Indeed, we apply Proposition 2.3 (1), given $\varepsilon > 0$ take $\delta = \varepsilon$. Then

$$\sup_{z\in\mathbb{D}} |\psi(z)| \frac{(1-|z|)^{\alpha+\varepsilon}}{(1-|z|/2)^{\alpha+\varepsilon}} \leq 2^{\alpha+\varepsilon} \sup_{z\in\mathbb{D}} |\psi(z)|(1-|z|)^{\alpha+\varepsilon} < \infty.$$

Corollary 2.7 *Let $0 < \alpha < \infty$, $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$. The operator $M_\psi \in \mathcal{L}(A_-^{-\alpha})$ if and only if $\psi \in A_+^{-0}$.*

Proof. The proof is similar to that of Corollary 2.4, using Proposition 2.3 (2). $\square$

Corollary 2.8 *Let $0 < \alpha < \infty$, $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$. If $\psi \in A_+^{-0}$, then $W_{\psi,\varphi} \in \mathcal{L}(A_-^{-\alpha})$.*

Proof. Proceed as in the proof of Corollary 2.5. $\square$

The converse of Corollary 2.8 also fails, as can be seen is not true taking $\varphi(z) = z/2$, similarly as in Example 2.6.

Proposition 2.9 *Let $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then $W_{\psi,\varphi} \in \mathcal{L}(A^{-\infty})$ if and only if $\psi \in A^{-\infty}$.*

Proof. Assume $W_{\psi,\varphi}$ is continuous on $A^{-\infty}$. The constant function $\mathbf{1} \in A^{-\infty}$ so, $W_{\psi,\varphi}(\mathbf{1})(z) = \psi(z)$ for all $z \in \mathbb{D}$. Then $\psi \in A^{-\infty}$.

Conversely, assume now $\psi \in A^{-\infty}$. Since $A^{-\infty}$ is an algebra under pointwise multiplication, the operator M_ψ is continuous on $A^{-\infty}$. On the other hand, [11, Theorem 2.3] and Lemma 2.2 imply that the operator C_φ is also continuous on $A^{-\infty}$. Consequently $W_{\psi,\varphi} = M_\psi \circ C_\varphi$ is continuous, too. $\square$

2.2 Compactness

Proceeding similarly as we did for the continuity, using Lemmas 2.1, 2.2 and Propositions 3.1, 3.2 of [16] we obtain the following results.

Proposition 2.10 *Let $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$.*

(1) Let $\alpha \geq 0$. The operator $W_{\psi,\varphi} : A_+^{-\alpha} \to A_+^{-\alpha}$ is compact if and only if it is continuous and there exists $\eta > \alpha$ such that for each $\tau \in]\alpha, \eta]$ we have

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)|(1-|z|)^\tau}{(1-|\varphi(z)|)^\eta} < \infty.$$

(2) Let $0 < \alpha \leq \infty$. The operator $W_{\psi,\varphi} : A_-^{-\alpha} \to A_-^{-\alpha}$ is compact if and only if it is continuous and there exists $\beta < \alpha$ such that for every $\gamma \in [\beta, \alpha[$ we have

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)|(1-|z|)^\beta}{(1-|\varphi(z)|)^\gamma} < \infty.$$

Corollary 2.11 *Let $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$.*

(1) Let $\alpha \geq 0$. If $W_{\psi,\varphi} : A_+^{-\alpha} \to A_+^{-\alpha}$ is compact, then there is $\eta > \alpha$ such that $W_{\psi,\varphi} : H_\eta^0 \to H_\eta^0$ is compact.

(2) Let $0 < \alpha \leq \infty$. If $W_{\psi,\varphi} : A_-^{-\alpha} \to A_-^{-\alpha}$ is compact, then there is $\gamma < \alpha$ such that $W_{\psi,\varphi} : H_\gamma^0 \to H_\gamma^0$ is compact.

Proof. This is a direct consequence of Proposition 2.10 and [16, Corollary 4.5]. $\square$

Corollary 2.12 *Assume that there exists an r, $0 < r < 1$, such that $|\varphi(z)| \leq r$ for all $z \in \mathbb{D}$. If $W_{\psi,\varphi} : A_+^{-\alpha} \to A_+^{-\alpha}, \alpha \geq 0$, (resp. $W_{\psi,\varphi} : A_-^{-\alpha} \to A_-^{-\alpha}, \alpha > 0$) is continuous, then $W_{\psi,\varphi}$ is compact.*

Corollary 2.13 *Assume that M_ψ is continuous on $A_+^{-\alpha}$ or on $A_-^{-\alpha}$. If M_ψ is compact, then $\psi \equiv 0$.*

2.3 Invertibility

The characterization of invertible weighted composition operators on $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$, is a consequence of the following results due to Bourdon [13, Theorem 2.2 and Corollary 2.3].

Theorem 2.14 (Theorem 2.2 of [13]) *Suppose that X is a set of functions analytic on $\mathbb{D}$ such that*

(i) $W_{\psi,\varphi}$ maps X to X,

(ii) X contains a nonzero constant function,

(iii) X contains a function of the form $z \mapsto z + c$ for some constant c,

(iv) there is a dense subset S of the unit circle such that for each point in S there is a function in X that does not extend analytically to a neighbourhood of that point.

If $W_{\psi,\varphi} : X \to X$ is invertible, then φ is an automorphism of $\mathbb{D}$.

Theorem 2.15 (Corollary 2.3 of [13]) *If X, ψ and φ satisfy the hypotheses of Theorem 2.14 and X is automorphism invariant, i.e., $f \circ \varphi \in X$ for all automorphism φ of $\mathbb{D}$, then $W_{\psi,\varphi}$ is invertible on X if and only if φ is an automorphism of $\mathbb{D}$ and ψ as well as $1/\psi$ are multipliers of X, that is, the corresponding multiplication operators map X into X.*

Remark 2.16 $W_{\psi,\varphi}$ satisfies hypothesis *(i)* of Theorem 2.14 whenever it is continuous, and the spaces $A_+^{-\alpha}$, $A_-^{-\alpha}$ and $A^{-\infty}$ verify hypothesis *(ii)*. Moreover, they are linear spaces which contain the constants and the polynomials, thus *(iii)* is equally satisfied. The hypothesis *(iv)* is also satisfied. In fact, for an $a \in \partial\mathbb{D}$ and $p > 0$ the function $g_{p,a}(z) := 1/(a-z)^p \in A_+^{-\alpha}, A_-^{-\alpha}, A^{-\infty}$ and does not extend analytically to any neighbourhood of a. In case $p = 0$, one can take $\log(a-z)$ instead of $g_{p,a}$ because $\log(a-z) \in A_+^{-\alpha}, A_-^{-\alpha}, A^{-\infty}$.

Proposition 2.17 *Let $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$.*

(1) Let $\alpha \geq 0$. The operator $W_{\psi,\varphi}$ is invertible on $A_+^{-\alpha}$ if and only if φ is an automorphism of $\mathbb{D}$ and $\psi, 1/\psi \in A_+^{-0}$.

(2) Let $\alpha > 0$. The operator $W_{\psi,\varphi}$ is invertible on $A_-^{-\alpha}$ if and only if φ is an automorphism of $\mathbb{D}$ and $\psi, 1/\psi \in A_+^{-0}$.

(3) The operator $W_{\psi,\varphi}$ is invertible on $A^{-\infty}$ if and only if φ is an automorphism of $\mathbb{D}$ and $\psi, 1/\psi \in A^{-\infty}$.

Proof. We only prove part (1); the other proofs are analogous. By Remark 2.16, hypotheses *(i)*, *(ii)*, *(iii)* and *(iv)* of Theorem 2.14 are satisfied. We apply Theorem 2.15 to get that $W_{\psi,\varphi}$ is invertible if and only if φ is an automorphism of $\mathbb{D}$ and ψ and $1/\psi$ are multipliers of $A_+^{-\alpha}$ or, equivalently, that M_ψ and $M_{1/\psi}$ are continuous on $A_+^{-\alpha}$. The conclusion follows from Corollary 2.4. $\square$

Example 2.18 The condition $\psi, 1/\psi \in A_+^{-0}$ need not imply that $\psi, 1/\psi \in H^\infty$. To see this, consider the function $\psi(z) := \log(z+1) - 5, z \in \mathbb{D}$, which is not bounded on $\mathbb{D}$. Moreover, $\psi \in A_+^{-0}, A^{-\infty}$. We show that $1/\psi \in H^\infty$. Indeed, if we call $z = a + bi$, with $a, b \in [0, 1[$,

$$|\log(a+bi+1) - 5| = \left|\log\sqrt{(a+1)^2+b^2} - 5 + i\mathrm{Arg}(a+bi)\right| \geq \left|\log\sqrt{(a+1)^2+b^2} - 5\right|.$$

If $\log\sqrt{(a+1)^2+b^2} < 0$, then $\left|\log\sqrt{(a+1)^2+b^2} - 5\right| > 5$.

On the other hand, if $\log\sqrt{(a+1)^2+b^2} \geq 0$, as $|a+1| \leq |a| + 1 < 2$, then $(a+1)^2 + b^2 < 5$. Thus, $0 \leq \log\sqrt{(a+1)^2+b^2} < \log\sqrt{5} < 2$. Hence, $\left|\log\sqrt{(a+1)^2+b^2} - 5\right| > 3$.

Therefore $\frac{1}{|\log(z+1)-5|} \leq \frac{1}{3}$ for all $z \in \mathbb{D}$. Then $1/\psi \in H^\infty$ and $\psi \in A_+^{-0} \setminus H^\infty$.

3 Some results about the spectrum

3.1 Preliminaries and point spectrum

Let $T : X \to X$ be a continuous operator on a space X. We write $T \in \mathcal{L}(X)$. The *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ is a continuous linear operator, that is $T - \lambda I : X \to X$ is bijective and has a continuous inverse. Here I stands for the identity operator on X. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_p(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space X, then we write $\sigma(T, X)$, $\sigma_p(T, X)$ and $\rho(T, X)$. Unlike for Banach spaces X, it may happen that $\rho(T) = \emptyset$ or that $\rho(T)$ is not open.

The following two abstract lemmas will be useful in the rest of the section.

Lemma 3.1 (Lemma 2.1 of [2]) *Let $X = \cap_{n\in\mathbb{N}} X_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $((X_n, ||\cdot||_n))_{n\in\mathbb{N}}$ satisfying $X_{n+1} \subset X_n$ with $||x||_n \leq ||x||_{n+1}$ for each $n \in \mathbb{N}$ and $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy that for each $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of T_n to X (resp. of T_n to X_{n+1}) coincides with T (resp. with T_{n+1}). Then, $\sigma(T, X) \subset \cup_{n\in\mathbb{N}} \sigma(T_n, X_n)$.*

Lemma 3.2 (Lemma 5.2 of [3]) *Let $E = ind_n(E_n, ||\cdot||_n)$ be a Hausdorff inductive limit of Banach spaces. Let $T \in \mathcal{L}(E)$ satisfy that for each $n \in \mathbb{N}$ the restriction T_n of T to E_n maps E_n into itself and $T_n \in \mathcal{L}(E_n)$. Then, the following properties are satisfied.*

(i) $\sigma_p(T, E) = \cup_{n\in\mathbb{N}} \sigma_p(T_n, E_n)$.

(ii) $\sigma(T, E) \subset \cap_{m\in\mathbb{N}} (\cup_{n=m}^{\infty} \sigma(T_n, E_n))$.

We will also need a general version of two lemmas due to Kamowitz [21, Lemmas 2.3 and 2.4]. We only mention the changes needed in the proofs.

Lemma 3.3 *Let E be a space of holomorphic functions containing the polynomials such that the inclusion $E \subset H(\mathbb{D})$ is continuous. Consider $\varphi, \psi \in H(\mathbb{D})$, with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$. Then, $\psi(0) \in \sigma(W_{\psi,\varphi})$ and $\psi(0)\varphi'(0)^n \in \sigma(W_{\psi,\varphi})$ for all $n \in \mathbb{N}$.*

Proof. The proof is the same as [21, Lemma 2.3]. □

Lemma 3.4 *Let E be a space of holomorphic functions containing the polynomials, and such that the inclusion $E \subset H(\mathbb{D})$ is continuous. Consider $\varphi, \psi \in H(\mathbb{D})$, $\psi \not\equiv 0$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$ and φ is not a constant function. If λ is an eigenvalue of $W_{\psi,\varphi}$, then $\lambda \in \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty}$.*

Proof. Suppose λ is an eigenvalue of $W_{\psi,\varphi}$ with $f \in E$ as corresponding eigenvector. If $\lambda = 0$, $\psi(z)f(\varphi(z)) = 0$ for all $z \in \mathbb{D}$. Define $G := \{z \in \mathbb{D} : \psi(z) \neq 0\}$, which is non-empty and open (because $\psi \not\equiv 0$). Then, $f \circ \varphi \equiv 0$ on G. Now, φ cannot be constant on G because if it were, it would be necessarily constant on $\mathbb{D}$ by the Identity Principle, which contradicts the hypothesis. So, φ is not constant on G, which means $\varphi(G)$ is open and non empty. If $f \equiv 0$ on $\varphi(G)$, $f \equiv 0$, which is again a contradiction. Hence, $\lambda \neq 0$. The rest of the proof follows as in the proof of [21, Lemma 2.4]. □

The *essential norm* $||T||_e$ of an operator T on a Banach space X is the distance of the operator to the set of compact operators on X. More details about the essential norm and the essential spectral radius $r_e(T, X)$ can be seen in the book [1].

By [16, Theorem 4.2] or [26, Theorem 2.1] the continuous weighted composition operators $W_{\psi,\varphi} : H_p^\infty \to H_p^\infty$ and $W_{\psi,\varphi} : H_p^0 \to H_p^0$ satisfy that their essential norm is given by

$$||W_{\psi,\varphi}||_e = \lim_{r\to 1} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{(1-|z|)^p}{(1-|\varphi(z)|)^p} .$$

And, its essential spectral radius is

$$r_e(W_{\psi,\varphi}, H_p^\infty) = r_e(W_{\psi,\varphi}, H_p^0) = \lim_n ||W_{\psi,\varphi}^n||_e^{1/n} ,$$

where

$$||W_{\psi,\varphi}^n||_e = \lim_{r\to 1} \sup_{|\varphi(z)|>r} |\psi(z)| \cdots |\psi(\varphi_{n-1}(z))| \frac{(1-|z|)^p}{(1-|\varphi_n(z)|)^p} .$$

Here φ_n is the composition of φ with itself n-times.

Theorem 3.5 (Theorem 7 of [7]) *Let $p > 0$ and suppose φ, not an automorphism, has fixed point $a \in \mathbb{D}$ and $W_{\psi,\varphi} : H_p^\infty \to H_p^\infty$ is continuous. Then*

$$\sigma(W_{\psi,\varphi}, H_{v_p}^\infty) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_e(W_{\psi,\varphi}, H_p^\infty)\} \cup \{\psi(a)\varphi'(a)^n\}_{n=0}^\infty .$$

Remark 3.6 It is not hard to prove that the essential norm can be obtained for the weights v_p in the following way:

$$||W_{\psi,\varphi}||_e = \lim_{|z|\to 1^-} |\psi(z)| \frac{(1-|z|)^p}{(1-|\varphi(z)|)^p} .$$

Lemma 3.7 *Let $\varphi, \psi \in H(\mathbb{D})$, $\varphi(0) = 0$, φ non constant. Then,*

$$r_e(W_{\psi,\varphi}, H_\beta^\infty) \leq r_e(W_{\psi,\varphi}, H_\alpha^\infty)$$

whenever $0 < \alpha \leq \beta < \infty$.

Proof. By Schwarz's Lemma, $|\varphi(z)| \leq |z|$ for all $z \in \mathbb{D}$. Then, since $0 < \alpha \leq \beta < \infty$,

$$\left(\frac{1-|z|}{1-|\varphi(z)|}\right)^\beta \leq \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^\alpha$$

for every $z \in \mathbb{D}$. Thus, $||W_{\psi,\varphi}||_{e,\beta} \leq ||W_{\psi,\varphi}||_{e,\alpha}$. Since φ_n has the same properties as φ for each $n \in \mathbb{N}$, we also get $||W_{\psi,\varphi}^n||_{e,\beta} \leq ||W_{\psi,\varphi}^n||_{e,\alpha}$ for all $n \in \mathbb{N}$. □

For the essential spectral radius of composition operators we have the following result.

Proposition 3.8 *Let $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. Then*

(i) ([6, Theorem 5.1]) $r_e(C_\varphi, H_\alpha^\infty) < 1$ for each $\alpha > 0$.

(ii) $r_e(C_\varphi, H_\alpha^\infty) = r_e(C_\varphi, H_1^\infty)^\alpha$ for each $\alpha > 0$.

(iii) $\lim_{\beta\to\alpha} r_e(C_\varphi, H_\beta^\infty) = r_e(C_\varphi, H_\alpha^\infty)$ for each $\alpha > 0$.

(iv) $\lim_{\beta\to\infty} r_e(C_\varphi, H_\beta^\infty) = 0$.

Proof. Part (i) is a consequence of [6, Theorem 5.1].

(ii) We apply Remark 3.6 to get

$$r_e(C_\varphi, H_\alpha^\infty) = \lim_n \left(\lim_{|z|\to 1^-} \left(\frac{1-|z|}{1-|\varphi_n(z)|}\right)^\alpha \right)^{\frac{1}{n}} =$$

$$= \lim_n \left(\lim_{|z|\to 1^-} \left(\frac{1-|z|}{1-|\varphi_n(z)|}\right)^{\frac{1}{n}} \right)^\alpha = r_e(C_\varphi, H_1^\infty)^\alpha.$$

(iii) is a direct consequence of (ii), and (iv) follows from (i) and (ii). □

We have the following consequence of Lemmas 3.3 and 3.4 for composition operators on $A_+^{-\alpha}$ or $A_-^{-\alpha}$.

Proposition 3.9 *Let $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. Then*

(i) for $\alpha \geq 0$,

$$\{\varphi'(0)^n\}_{n=0}^\infty \setminus \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \subset \sigma_p(C_\varphi, A_+^{-\alpha}) \subset \{\varphi'(0)^n\}_{n=0}^\infty ,$$

(ii) and for $0 < \alpha < \infty$,

$$\{\varphi'(0)^n\}_{n=0}^{\infty} \setminus \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \subset \sigma_p(C_\varphi, A_-^{-\alpha}) \subset \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

Proof. (i) By Lemma 3.4 we have $\sigma_p(C_\varphi, A_+^{-\alpha}) \subset \{\varphi'(0)^n\}_{n=0}^{\infty}$. Now fix $n \in \mathbb{N}$ such that $|\varphi'(0)^n| > r_e(C_\varphi, H_\alpha^\infty)$. By Lemma 3.3 we have $\varphi'(0)^n \in \sigma(C_\varphi, H_\alpha^0)$. We apply [1, Theorem 7.44] to get $\varphi'(0)^n \in \sigma_p(C_\varphi, H_\alpha^0)$. That is, there exists $f \in H_\alpha^0$ such that $C_\varphi f = \varphi'(0)^n f$. Since $H_\alpha^0 \subset A_+^{-\alpha}$ we get that $\varphi'(0)^n \in \sigma_p(C_\varphi, A_+^{-\alpha})$.

(ii) Lemma 3.4 implies $\sigma_p(C_\varphi, A_-^{-\alpha}) \subset \{\varphi'(0)^n\}_{n=0}^{\infty}$. Fix $n \in \mathbb{N}$ such that $|\varphi'(0)^n| > r_e(C_\varphi, H_\alpha^\infty)$. We apply Proposition 3.8 (iii) to find $\beta < \alpha$ such that $|\varphi'(0)^n| > r_e(C_\varphi, H_\beta^\infty)$. By Lemma 3.3 for $E = H_\beta^0$ we have $\varphi'(0)^n \in \sigma(C_\varphi, H_\beta^0)$. From [1, Theorem 7.44] we conclude $\varphi'(0)^n \in \sigma_p(C_\varphi, H_\beta^0)$. Hence there is $f \in H_\beta^0 \subset A_-^{-\alpha}$ such that $C_\varphi f = \varphi'(0)^n f$. Thus $\varphi'(0)^n \in \sigma_p(C_\varphi, A_-^{-\alpha})$. □

Observation 3.10 Notice that if $\varphi'(0) \in \sigma_p(C_\varphi, A_-^{-\alpha})$ then $\varphi'(0) \notin \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)$. In such case, Königs eigenfunction σ belongs to $A_-^{-\alpha}$, so there is $\beta < \alpha$ such that $\sigma \in H_\beta^0$. Then, by a result of [12], $|\varphi'(0)| > r_e(C_\varphi, H_\beta^\infty) \geq r_e(C_\varphi, H_\alpha^\infty)$.

The case of the Korenblum space will be treated below in Theorem 3.13.

3.2 About the spectra of weighted composition operators

Proposition 3.11 *Let* $\psi, \varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$.

(i) If $\alpha \geq 0$ *and* $W_{\psi,\varphi}$ *is continuous on* $A_+^{-\alpha}$*, then*

$$\{0\} \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \subset \sigma(W_{\psi,\varphi}, A_+^{-\alpha}) \subset \overline{B}(0, \lim_{\beta\to\alpha^+} r_e(W_{\psi,\varphi}, H_\beta^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty},$$

(ii) If $0 < \alpha < \infty$ *and* $W_{\psi,\varphi}$ *is continuous on* $A_-^{-\alpha}$*, then*

$$\{0\} \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \subset \sigma(W_{\psi,\varphi}, A_-^{-\alpha}) \subset \overline{B}(0, \lim_{\beta\to\alpha^-} r_e(W_{\psi,\varphi}, H_\beta^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty}.$$

Proof. (i) Under the present assumptions φ is not an automorphism. This implies that the operator $W_{\psi,\varphi} : A_+^{-\alpha} \to A_+^{-\alpha}$ is not an isomorphism by Proposition 2.17. Therefore $0 \in \sigma(W_{\psi,\varphi}, A_+^{-\alpha})$. Moreover, Lemma 3.3 yields $\{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \subset \sigma(W_{\psi,\varphi}, A_+^{-\alpha})$, which completes the proof of the left inclusion.

To prove the other inclusion, we first apply Lemma 3.1 and Theorem 3.5 to obtain

$$\sigma(W_{\psi,\varphi}, A_+^{-\alpha}) \subset \bigcup_{\beta>\alpha} \left(\overline{B}(0, r_e(W_{\psi,\varphi}, H_\beta^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \right).$$

If $\lambda \in \cup_{\beta>\alpha}\overline{B}(0, r_e(W_{\psi,\varphi}, H_\beta^\infty))$, there exists $\beta > \alpha$ such that $|\lambda| \leq \sup_{\beta>\alpha} r_e(W_{\psi,\varphi}, H_\beta^\infty)$. By Lemma 3.7 $\sup_{\beta>\alpha} r_e(W_{\psi,\varphi}, H_\beta^\infty) = \lim_{\beta\to\alpha^+} r_e(W_{\psi,\varphi}, H_\beta^\infty)$, from where the inclusion follows.

(ii) The left inclusion follows similarly as in part (i).

For the other inclusion, we first apply Lemma 3.2 and Theorem 3.5 to get

$$\sigma(W_{\psi,\varphi}, A_-^{-\alpha}) \subset \bigcap_{\beta<\alpha} \left(\bigcup_{\gamma\geq\beta} \left(\overline{B}(0, r_e(W_{\psi,\varphi}, H_\gamma^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} \right) \right).$$

By Lemma 3.7 $\cup_{\gamma\geq\beta}\overline{B}(0, r_e(W_{\psi,\varphi}, H_\gamma^\infty)) = \overline{B}(0, r_e(W_{\psi,\varphi}, H_\beta^\infty))$. Therefore

$$\begin{aligned}\sigma(W_{\psi,\varphi}, A_-^{-\alpha}) &\subset \bigcap_{\beta<\alpha} \overline{B}(0, r_e(W_{\psi,\varphi}, H_\beta^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} = \\ &= \overline{B}(0, \inf_{\beta<\alpha} r_e(W_{\psi,\varphi}, H_\beta^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty} = \\ &= \overline{B}(0, \lim_{\beta\to\alpha^-} r_e(W_{\psi,\varphi}, H_\beta^\infty)) \cup \{\psi(0)\varphi'(0)^n\}_{n=0}^{\infty}.\end{aligned}$$

□

Corollary 3.12 *Let $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$.*

(i) If $\alpha \geq 0$, then

$$\{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} \subset \sigma(C_\varphi, A_+^{-\alpha}) \subset \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^{\infty},$$

(ii) If $0 < \alpha < \infty$, then

$$\{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} \subset \sigma(C_\varphi, A_-^{-\alpha}) \subset \overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Proof. The operator C_φ is continuous since it is continuous in each H_β^∞ by [11, Theorem 2.3]. The proof follows from Proposition 3.11 and Proposition 3.8 (iii). $\square$

Theorem 3.13 *Let $\varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Then,*

$$\sigma(C_\varphi, A^{-\infty}) = \{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty},$$

and

$$\sigma_p(C_\varphi, A^{-\infty}) = \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Proof. The operator $C_\varphi : A^{-\infty} \to A^{-\infty}$ is continuous since it is continuous in each H_n^∞ by [11].

By Lemma 3.3, $\varphi'(0)^n \in \sigma(C_\varphi, A^{-\infty})$ for each $n \in \mathbb{N}$. The assumptions imply that φ is not an automorphism. Therefore, Proposition 2.17 (3) yields that $C_\varphi : A^{-\infty} \to A^{-\infty}$ is not a surjective isomorphism, thus $0 \in \sigma(C_\varphi, A^{-\infty})$.

Now, denote $T_k = C_\varphi : H_k^\infty \to H_k^\infty$. By Theorem 3.5,

$$\sigma(C_\varphi, H_k^\infty) = \sigma(T_k, H_k^\infty) = \overline{B(0, r_e(C_\varphi, H_k^\infty))} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

By Lemma 3.7 $r_e(C_\varphi, H_{k+1}^\infty) \leq r_e(C_\varphi, H_k^\infty)$ for each $k \in \mathbb{N}$. Therefore

$$\bigcup_{j \geq k} \sigma(T_j) = \overline{B(0, r_e(T_k))} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

Now Lemma 3.2 yields

$$\sigma(C_\varphi, A^{-\infty}) \subset \bigcap_k \bigcup_{j \geq k} \sigma(T_j) = \left(\bigcap_k \overline{B(0, r_e(T_k))} \right) \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

Since $r_e(T_k)$ tends to 0 when k goes to infinity by Lemma 3.8 (iv), we conclude

$$\sigma(C_\varphi, A^{-\infty}) \subset \{0\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty} .$$

Now we prove the statement about the point spectrum. Lemma 3.4 implies

$$\sigma_p(C_\varphi, A^{-\infty}) \subset \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Since $\lim_{n\to\infty} r_e(C_\varphi, H_n^\infty) = 0$ by Lemma 3.8(iv), there is $m \in \mathbb{N}$ such that $|\varphi'(0)| > r_e(C_\varphi, H_m^\infty)$. We apply a result of Bourdon [12] to get that Königs's eigenfunction σ is in H_m^0 and $\varphi'(0)$ is an eigenvalue of C_φ in $A^{-\infty}$. Since Korenblum space $A^{-\infty}$ is an algebra and C_φ is an algebra homomorphism, we easily conclude $\varphi'(0)^n$ is also an eigenvalue for all $n \in \mathbb{N}$. $\square$

3.3 The spectra of multiplication operators

Given $\psi \in H(\mathbb{D})$, the multiplication operator M_ψ is a weighted composition operator for the selfmap $\varphi(z) = z$. In this section we study the spectrum of multiplication operators.

Proposition 3.14 *Let E be a space which is continuously included in $H(\mathbb{D})$, containing the polynomials, such that for each $\xi \in H^\infty$ the multiplication operator $M_\xi : E \to E$ is continuous. Let $\psi \in H(\mathbb{D})$ be not constant. If $M_\psi : E \to E$ is continuous, then $\sigma_p(M_\psi, E) = \emptyset$ and $\psi(\mathbb{D}) \subset \sigma(M_\psi, E) \subset \overline{\psi(\mathbb{D})}$.*

Proof. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of $M_\psi : E \to E$. There would exist $f \in E$, $f \not\equiv 0$ such that $\psi(z)f(z) = \psi(0)f(z)$ for each $z \in \mathbb{D}$. But as $f \not\equiv 0$, the set of points where f does not vanish is an open set. And, by the Identity Principle, ψ would be constant, which contradicts the hypothesis. Accordingly $\sigma_p(M_\psi, E) = \emptyset$.

If $\lambda \in \psi(\mathbb{D})$, there is $z_0 \in \mathbb{D}$ with $\psi(z_0) = \lambda$. Thus, all functions in $(M_\psi - \lambda I)(E)$ vanish at z_0 and $M_\psi - \lambda I$ is not surjective. Therefore $\psi(\mathbb{D}) \subset \sigma(M_\psi, E)$.

Finally, if $\lambda \notin \overline{\psi(\mathbb{D})}$, there exists $\varepsilon > 0$ such that $|\psi(z) - \lambda| \geq \varepsilon$ for all $z \in \mathbb{D}$. Thus the function $\xi := 1/(\psi(z) - \lambda) \in H^\infty$ and $M_\xi : E \to E$ is continuous by assumption. This implies $M_\psi - \lambda I$ is a bijective operator. Indeed, it is injective since the point spectrum of M_ψ is empty as we have already proved. Moreover, it is surjective because for each $g \in E$, the function $f := M_\xi g \in E$ verifies $(M_\psi - \lambda I)f = g$. We have shown that $\lambda \notin \sigma(M_\psi, E)$. □

Corollary 3.15 *If M_ψ is continuous on any of the spaces $A_+^{-\alpha}, \alpha \geq 0$, or $A_-^{-\alpha}, 0 < \alpha \leq \infty$ for some non-constant function $\psi \in H(\mathbb{D})$, then the point spectrum of M_ψ is empty and the spectrum contains $\psi(\mathbb{D})$ and it is contained in $\overline{\psi(\mathbb{D})}$.*

Proof. This is a direct consequence of Proposition 3.14, since by Corollaries 2.4 and 2.7 and Proposition 2.9 its assumptions are satisfied. □

Unlike in Banach spaces, the spectrum of M_ψ is not necessarily a closed set. The following example shows that the spectrum may not coincide with $\overline{\psi(\mathbb{D})}$.

Example 3.16 The analytic function $\psi(z) := \frac{1}{1-z}, z \in \mathbb{D}$, belongs to $A^{-\infty}$, so M_ψ is continuous on $A^{-\infty}$ by Theorem 2.9. Observe $\frac{1}{2} = \psi(-1) \in \overline{\psi(\mathbb{D})}$. But $\frac{1}{2} \in \rho(M_\psi, A^{-\infty})$. In fact, it is easy to check that $\frac{1}{\psi - \frac{1}{2}} \in A^{-\infty}$ and $M_{\frac{1}{\psi-\frac{1}{2}}}$ is the inverse of $M_\psi - \frac{1}{2}I$.

3.4 The spectra of composition operators whose symbol is a rotation

If φ is an automorphism of the disc such that $\varphi(0) = 0$, then it is a rotation, that is, there is $c \in \mathbb{C}$ with $|c| = 1$ such that $\varphi(z) = cz$ for all $z \in \mathbb{D}$. In this section we present a few results about the spectrum of composition operators on Korenblum type spaces when the selfmap is a rotation. Our first result is well-known.

Lemma 3.17 *Let $\varphi \in H(\mathbb{D})$, $\varphi(z) = cz$ for all $z \in \mathbb{D}$, with $|c| = 1$. Then*

(i) $\sigma_p(C_\varphi, H_\alpha^\infty) = \{c^n\}_{n=0}^\infty$.

(ii) If c is a root of unity, then $\sigma(C_\varphi, H_\alpha^\infty) = \sigma_p(C_\varphi, H_\alpha^\infty) = \{c^n\}_{n=0}^\infty$.

(iii) If c is not a root of unity, then $\sigma(C_\varphi, H_\alpha^\infty) = \partial\mathbb{D}$.

Proof. (i) Lemma 3.4 implies $\sigma_p(C_\varphi, H_\alpha^\infty) \subset \{c^n\}_{n=0}^\infty$. For each $m = 0, 1, 2, \ldots$, the function $f_m(z) := z^m$ belongs to H_α^∞. Moreover, for each m, $f_m(\varphi(z)) = c^m f_m(z)$ for all $z \in \mathbb{D}$ and $f_m(\varphi(z)) = c^m z^m$. So, every c^m is eigenvalue with eigenvector f_m.

(ii) If c is a root of unity, there exists $m \in \mathbb{N}$ with $\varphi^m(z) = z$ for every $z \in \mathbb{D}$. That is, $C_\varphi^m = I$. By Spectral Mapping Theorem [1, Theorem 6.31], $(\sigma(C_\varphi))^m = \sigma(C_\varphi^m) = \sigma(I) = \{1\}$. Then $\sigma(C_\varphi) \subset \{\lambda : \lambda^m = 1\}$. This implies $\{\lambda : \lambda^n = 1\} = \{c^n\}_{n=0}^\infty$.

(iii) Suppose that c is not a root of unity. If $|\lambda| > 1 = ||C_\varphi|| = ||C_\varphi^{-1}||$, then $\lambda \in \rho(C_\varphi)$ and $\lambda \in \rho(C_\varphi^{-1})$ by [1, Theorem 6.31]. It is easy to check that $-\lambda C_\varphi^{-1}(\lambda I - C_\varphi^{-1})^{-1}$ is the inverse of $\frac{1}{\lambda}I - C_\varphi$, which implies $1/\lambda \in \rho(C_\varphi)$. This shows that $\{c^n\}_{n=0}^\infty \subset \sigma(C_\varphi, H_\alpha^\infty) \subset \partial\mathbb{D}$. Since c is not a root of unity, Kronecker's Theorem [27, Theorem 2.2.4] implies that $\{c^n\}_{n=0}^\infty$ is dense in $\partial\mathbb{D}$. Since the spectrum of an operator on a Banach space is compact, this completes the proof of part (iii). □

Corollary 3.18 *Let $\varphi \in H(\mathbb{D})$, $\varphi(z) = cz$ for all $z \in \mathbb{D}$, with $|c| = 1$. Let E be any of the spaces $A_+^{-\alpha}, \alpha \geq 0$, or $A_-^{-\alpha}, 0 < \alpha \leq \infty$. Then* *(i)* $\sigma_p(C_\varphi, E) = \{c^n\}_{n=0}^\infty$.

(ii) If c *is a root of unity, then* $\sigma(C_\varphi, E) = \sigma_p(C_\varphi, E) = \{c^n\}_{n=0}^\infty$.

(iii) If c *is not a root of unity, then* $\{c^n\}_{n=0}^\infty \subset \sigma(C_\varphi, E) \subset \partial\mathbb{D}$.

Proof. The point spectrum is obtained with the same argument as in Lemma 3.17. And, for the spectrum, both cases follow from Lemmas 3.2 and 3.17. $\square$

In the case of the Korenblum space we can characterize exactly which points of the unit circle belong to the spectrum of C_φ when $\varphi(z) = cz$ for all $z \in \mathbb{D}$ and $c \in \partial\mathbb{D}$ is not a root of unity. This is a similar result as Theorem 1 of [10]. We first need the following known characterization of the functions in the Korenblum space in terms of their Taylor expansion. Its proof follows from an application of Cauchy inequalities.

Lemma 3.19 *A function* $f(z) = \sum_{n=0}^\infty a_n z^n$ *on the unit disc* $\mathbb{D}$ *belongs to* $A^{-\infty}$ *if and only if there is* $k \in \mathbb{N}$ *such that* $\sup_n n^{-k}|a_n| < \infty$.

Theorem 3.20 *Let* $c \in \mathbb{C}$ *be an element of the unit circle* $|c| = 1$ *which is not a root of unity. Let* $C_\varphi : A^{-\infty} \to A^{-\infty}$ *be the composition operator with selfmap* $\varphi(z) = cz, z \in \mathbb{D}$. *A complex number* $\lambda \neq 1, |\lambda| = 1$, *belongs to the resolvent* $\rho(C_\varphi, A^{-\infty})$ *if and only if there are* $s \geq 1$ *and* $\varepsilon > 0$ *such that* $|c^n - \lambda| \geq \varepsilon n^{-s}$ *for each* $n \in \mathbb{N}$.

Proof. First assume that there are $s \geq 1$ and $\varepsilon > 0$ such that $|c^n - \lambda| \geq \varepsilon n^{-s}$ for each $n \in \mathbb{N}$. In particular $C_\varphi - \lambda I$ is injective by Corollary 3.18 (i). We prove that it is also surjective. Given $g(z) = \sum_{n=0}^\infty a_n z^n \in A^{-\infty}$, we define $f(z) := \sum_{n=0}^\infty \frac{a_n}{c^n - \lambda} z^n$, $z \in \mathbb{D}$. It is easy to see that $(C_\varphi - \lambda I)f = g$. To conclude the proof of this implication it is enough to show that $f \in A^{-\infty}$. Since $g \in A^{-\infty}$, we apply Lemma 3.19 to find $k \in \mathbb{N}$ and $M > 0$ such that $n^{-k}|a_n| \leq M$ for each $n = 0, 1, 2, \ldots$. Hence, for each $n = 0, 1, 2, \ldots$, we get

$$\frac{|a_n|}{|c^n - \lambda|} \leq \frac{M n^k n^s}{\varepsilon} = \frac{M}{\varepsilon} n^{k+s},$$

which implies that f is analytic and belongs to $A^{-\infty}$ by Lemma 3.19.

Now suppose that $\lambda \neq 1, |\lambda| = 1$, belongs to $\rho(C_\varphi, A^{-\infty})$. Then the inverse operator $(C_\varphi - \lambda I)^{-1} : A^{-\infty} \to A^{-\infty}$ exists, it is continuous and it necessarily has the form

$$(C_\varphi - \lambda I)^{-1}(\sum_{n=0}^\infty a_n z^n) = \sum_{n=0}^\infty \frac{a_n}{c^n - \lambda} z^n.$$

The continuity of this inverse implies that for $m = 1$ there are $k > 1$ and $M > 0$ such that for each $\sum_{n=0}^\infty a_n z^n \in A^{-\infty}$ we have

$$\sup_{z \in \mathbb{D}} (1 - |z|)^k \left| \sum_{n=0}^\infty \frac{a_n}{c^n - \lambda} z^n \right| \leq M \sup_{z \in \mathbb{D}} (1 - |z|) \left| \sum_{n=0}^\infty a_n z^n \right|.$$

Evaluating this inequality for each monomial z^n, $n = 0, 1, 2...$ we get, for each $n = 0, 1, 2, ...$,

$$\sup_{z \in \mathbb{D}} (1 - |z|)^k \frac{|z^n|}{|c^n - \lambda|} \leq M \sup_{z \in \mathbb{D}} (1 - |z|)|z^n|.$$

Therefore, evaluating the maximum of $r^n(1 - r)^k$, we get for each $n = 0, 1, 2, ...$

$$\frac{k^k n^n}{(n + k)^{n+k}} \leq M|c^n - \lambda| \frac{n^n}{(n + 1)^{n+1}}.$$

This implies, for each $n = 0, 1, 2, ...$,

$$|c^n - \lambda| \geq \frac{1}{M} \left(\frac{n + 1}{n + k} \right)^{n+1} \frac{1}{(n + k)^{k-1}},$$

which yields the desired inequality. □

A real number $x \in \mathbb{R}$ is called *Diophantine* if $\exists \delta \geq 1$ and $d(x) > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{d(x)}{q^{1+\delta}}$$

for all rational numbers p/q.

In the next proposition, a characterization of the complex number 1 belonging to the resolvent set in relation with Diophantine numbers is stated. In this result, $A_0^{-\infty}$ denotes the space of all functions $f \in A^{-\infty}$ such that $f(0) = 0$. This proposition should be compared with Theorem 2 of [10].

Proposition 3.21 *Let $\varphi(z) = cz, z \in \mathbb{D}$, where $|c| = 1$ and c is not a root of unity. Let $C_\varphi : A_0^{-\infty} \to A_0^{-\infty}$ be the composition operator with symbol φ. Then $1 \in \rho(C_\varphi, A_0^{-\infty})$ if and only if $c = e^{i2\pi x}$, where x is a Diophantine number.*

Proof. Notice $1 \notin \sigma_p(C_\varphi)$ because if it were true, it would exist $f(z) = \sum_{n=1}^\infty a_n z^n \in A_0^{-\infty}$, $f \not\equiv 0$ such that

$$\sum_{n=1}^\infty a_n z^n \in A_0^{-\infty} = \sum_{n=1}^\infty a_n c^n z^n, \quad z \in \mathbb{D}.$$

However, as $f \not\equiv 0$, there exists $k \in \mathbb{N}$ with $a_k \neq 0$, what implies $c^k = 1$, which is a contradiction.

Now, suppose $c = e^{2\pi i x}$ with x Diophantine. Then, by [14, pp. 43] there exist $s \geq 1$ and $\varepsilon > 0$ such that $|c^n - 1| \geq \varepsilon n^{-s}$ for each $n \in \mathbb{N}$. Since $1 \notin \sigma_p(C_\varphi)$, to see $1 \notin \sigma(C_\varphi)$ it only remains to show $C_\varphi - I$ is surjective.

Given $g(z) = \sum_{n=1}^\infty a_n z^n$, $z \in \mathbb{D}$, define $f(z) = \sum_{n=1}^\infty \frac{a_n}{c^n - 1} z^n$, $z \in \mathbb{D}$. Clearly, $g = (C_\varphi - I)f$ and $f(0) = 0$. We want to check Lemma 3.19 for the function f. Since $g \in A^{-\infty}$, for some $k \in \mathbb{N}$, $M > 0$, $|a_n| \leq M n^k$ for all $n \in \mathbb{N}$. And, by hypothesis, $\frac{1}{|c^n - 1|} \leq \frac{n^s}{\varepsilon}$. Thus,

$$\frac{|a_n|}{|c^n - 1|} \leq \frac{M}{\varepsilon} n^{k+s} .$$

Therefore, $f \in A^{-\infty}$.

The converse follows as in the proof of Theorem 3.20, taking into account [14, pp. 43]. □

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References

[1] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory.* Graduate Studies in Mathematics, Amer. Math. Soc., **50** (2002).

[2] A. A. Albanese, J. Bonet, W. J. Ricker, *The Cesàro operator in the Fréchet spaces ℓ^{p+} and L^{p-}*. Glasgow Math. J., **59** (2017), 273–287.

[3] A. A. Albanese, J. Bonet, W. J. Ricker, *The Cesàro operator on Korenblum type spaces of analytic functions.* Collect. Math., **69**(2) (2018), 263–281.

[4] A. A. Albanese, J. Bonet, W. J. Ricker, *Operators on the Fréchet sequence spaces $ces(p+), 1 = p \leq \infty$.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113 (2019), no. 2, 1533–1556.

[5] A. A. Albanese, J. Bonet, W. J. Ricker, *Linear operators on the (LB)-sequence spaces $ces(p-), 1 = p \leq \infty$.* Descriptive topology and functional analysis. II, 43–67, Springer Proc. Math. Stat., 286, Springer, Cham, 2019.

[6] W. Arendt, I. Chalendar, M. Kumar, S. Srivastava, *Powers of composition operators: asymptotic behaviour on Bergman, Dirichlet and Bloch spaces.* Journal Austral. Math. Soc. 1-32. doi:10.1017/S1446788719000235.

[7] R. Aron, M. Lindström, *Spectra of weighted composition operators on weighted Banach spaces of analytic funcions.* Israel J. of Math., **141** (2004), 263–276.

[8] K.D. Bierstedt, J. Bonet, J. Taskinen, *Associated weights and spaces of holomorphic functions.* Studia Math., **127** (1998), 137-168.

[9] K.D. Bierstedt, W. H. Summers, *Biduals of weighted Banach spaces of analytic functions.* J. Austral. Math. Soc., Ser. A, **54** (1993), no. 1, 70-79.

[10] Bonet, J. *A note about the spectrum of composition operators induced by a rotation.* RACSAM **114**, 63 (2020) doi:10.1007/s13398-020-00788-5

[11] J. Bonet, P. Domański, M. Lindström, J. Taskinen, *Composition operators between weighted Banach spaces of analytic functions.* J. Austral. Math. Soc., Ser. A, **64** (1998), no. 1, 101–118.

[12] P. S. Bourdon, *Essential angular derivatives and maximum growth of Königs eigenfunctions.* J. Func. Anal., **160** (1998), 561–580.

[13] P. S. Bourdon, *Invertible weighted composition operators.* Proc. Amer. Math. Soc., **142**(1) (2014), 289–299.

[14] L. Carleson, T. Gamelin, *Complex dynamics.* Springer-Verlag, (1991).

[15] C. Cowen, B. MacCluer, *Composition operators on spaces of analytic functions.* CRC Press, Boca Raton, FL, (1995).

[16] M. Contreras, A.G. Hernández-Díaz, *Weighted composition operators in weighted Banach spacs of analytic functions.* J. Austral. Math. Soc., Ser. A, **69** (2000), 41–60.

[17] T. Eklund, P. Galindo, M. Lindström, *Königs eigenfunction for composition operators on Bloch and H^∞ spaces.* J. Math. Anal. Appl. **445** (2017), 1300–1309.

[18] A. Grothendieck, *Topological vector spaces. Notes on Mathematics and its Applications.* Gordon and Breach Science Publishers, New York–London–Paris (1973).

[19] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman Spaces.* Grad. Texts in Math. **199**, Springer-Verlag, New York, (2000).

[20] H. Jarchow, *Locally convex spaces.* Teubner, Stuttgart, (1981).

[21] H. Kamowitz, *Compact operators of the form uC_φ.* Pacific Journal of Math. **80**(1) (1979).

[22] B. Korenblum, *An extension of the Nevanlinna theory.* Acta Math. **135** (1975), 187–219.

[23] G. Köthe, *Topological vector spaces II.* Springer-Verlag New York Inc. (1979)

[24] W. Lusky, *On the isomorphism classes of weighted spaces of harmonic and holomophic functions.* Studia Math. **75** (2006), 19–45.

[25] R. Meise and D. Vogt, *Introduction to functional analysis.* Oxford Grad. Texts in Math. **2**, New York, (1997).

[26] A. Montes–Rodríguez, *Weighted composition operators on weighted Banach spaces of analytic functions.* J. London Math. Soc. **61**(2) no. 3 (2000), 872–884.

[27] H. Queffélec, M. Queffélec, *Diophantine Approximation and Dirichlet series.* Hindustain Book Agency, New Delhi (2013).

[28] J.H. Shapiro, *Composition operators and classical function theory.* New York, Springer, (1993).

[29] A.L. Shields, D.L. Williams, *Bounded projections, duality and multipliers in spaces of analytic functions.* Trans. Amer. Math. Soc. **162** (1971), 287–302.

[30] F. H Vasilescu, *Analytic functional calculus and spectral decompositions.* Translated from the Romanian. Mathematics and its Applications (East European Series), 1. D. Reidel Publishing Co., Dordrecht, (1982).

[31] K. Zhu, *Operator Theory on Function Spaces*, Math. Surveys and Monographs, Amer. Math. Soc. **138** (2007).

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