The final publication is available at

https://doi.org/10.1016/j.ijheatmasstransfer.2020.119677

Additional Information

The time fractional approach for the modeling of thermal therapies: Temperature analysis in laser heating

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Abstract

In this study we assessed the use of the fractional derivative formulation of the heat transmission equation (FDHTE) as an alternative to the classical or parabolic heat transfer equation (PHTE) in the mathematical modeling of some thermal therapy processes. We obtained the FDHTE analytical solutions in two cases: a general case of heat transfer in a finite bar with different and constant temperature in its extremes (without heat source), and the heating by a laser source of a semi-infnite medium which includes a heat source and it was considered in thermal therapies to destroy or alter biological tissue. Both solutions were obtained analytically and compared with the PHTE results. We also compared the FDHTE solution with the results of the hyperbolic heat transfer equation (HHTE), which is another alternative to the PHTE, but is only used for problems in which intense heat is applied to materials for very short times. The results show that the FDHTE can be used as an alternative to the PHTE in thermal therapy processes in which the PHTE theoretical models underestimate or overestimate the temperatures achieved in tissue. Unlike the HHTE, the FDHTE is not restricted to special problems only. We have thus laid the groundwork for the analytical resolution of the problems considered by the FDHTE.

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1 Introduction

The parabolic heat transfer equation (PHTE) is commonly used as the governing equation in a heating theoretical model

$$\rho c \frac{\partial T}{\partial t}(\mathbf{x}, t) = \nabla \cdot (k \nabla T(\mathbf{x}, t)) + S(\mathbf{x}, t) \tag{1.1}$$

where $T(\mathbf{x}, t)$ is temperature at point $\mathbf{x}$ at time t, k is the thermal conductivity, ρ is the density, c is the specific heat and $S(\mathbf{x}, t)$ is the heat source in the material.

Since the beginning of the 20th century, it has been known that Fourier's law of heat transfer in solids, on which the PHTE is based, leads to two physically unacceptable conclusions: an infinite speed of heat conduction and the existence of infinite value heat flows. However, as the results provided by the PHTE are in good agreement with the experimental results in most ordinary engineering applications, only theoretical physicists needed to improve the formulation of the heat equation. Due to the advances in heating technology (such as the increasing use of laser pulses in material processing since the 1960s) new physical situations arose in which great amounts of heat are applied to materials for very short times. Under these conditions, more significant differences were found between the PHTE and the experimental results. Some studies proposed the hyperbolic heat transfer equation (HHTE) as a mathematical formulation that correctly solves these particular heating situations ([9], [22], [27]). The HHTE assumes a delay in heating characterized by the thermal relaxation time (τ)

$$\rho c \left(\frac{\partial T}{\partial t}(\mathbf{x}, t) + \tau \frac{\partial^2 T}{\partial t^2}(\mathbf{x}, t) \right) = \nabla \cdot (k \nabla T(\mathbf{x}, t)) + S(\mathbf{x}, t) + \tau \frac{\partial S}{\partial t}(\mathbf{x}, t) \tag{1.2}$$

However, from a theoretical point of view, it has been shown that the HHTE violates the second law of thermodynamics. To overcome this limitation, the relativistic heat conduction equation [2] was formulated, which uses the same formulation as the HHTE but with a different physical interpretation of the parameter τ. The dual phase lag HHTE (DHHTE) approach was also used to overcome the problems of the single phase lag HHTE (eq. (1.2)). The DHHTE provides better agreement with the experimental results in some applications (see e.g. [36]). In the DHHTE formulation appears a second relaxation time which affects the heat flux. A Lorentz-covariant heat conduction model has recently been proposed [23] as a model which obeys Lorentz covariance and does not violate the second law of thermodynamics. The Lorentz-covariant heat conduction model has the same terms of application as the HHTE and the relativistic heat conduction model: processes in which intense heat is applied for very short times.

An alternative was proposed by Gurtin and Pipkin [12] which is valid for other cases than briefly applied intense heat. These authors found that the law of heat conduction can be given by a general non-local dependence on time, obtaining a heat conduction equation with memory

$$\rho c \frac{\partial T}{\partial t}(\mathbf{x}, t) = \int_0^t K(t - s) \nabla \cdot (k \nabla T(\mathbf{x}, s)) ds. \tag{1.3}$$

Fourier's law is obtained by choosing the kernel function $K(t)$ as the Dirac delta function [12].

With a memory proper power law kernel i.e. $K(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, the flux can be interpreted in terms of fractional integrals and derivatives. This leads to the time fractional heat equation

$$\rho c \frac{\partial^\beta T}{\partial t^\beta}(\mathbf{x}, t) = \nabla \cdot (k \nabla T(\mathbf{x}, t)) + S(\mathbf{x}, t) \tag{1.4}$$

Thermal therapies are commonly used to destroy tumors or other types of altered tissue such as arrhythmias, osteoid osteoma, corneal curvature, etc. [10, 26, 30]. Different energy sources can be used (radiofrequency, laser, microwave, ultrasound) to control heating in order to alter or destroy biological tissues. Sometimes this thermal therapies are mixed with other alternative techniques such as the addition of nanoparticles [17]. The theoretical modeling of thermal therapies is commonly used to study the biophysical characteristics of these processes and to assess or improve the suitability of the techniques. As in the theoretical modeling of the heating of other materials, the PHTE is used as the governing equation of biological tissues. Specifically, the governing equation for the heating of biological tissues is called the Bioheat Equation, which was proposed by Pennes [29]. The Bioheat Equation presents the same formulation than Equation (1.1), but the source $S(\mathbf{x}, t)$ also includes metabolic heat generation and heat losses due to blood perfusion. Despite the broad use of the Bioheat Equation (i.e. PHTE) in the theoretical modeling of thermal therapies, in some cases the theoretical and experimental results revealed differences in terms of temperature and thermal damage predictions (see for exam-ple [5, 24]). These authors suggest several reasons for these discrepancies, one is that the PHTE simply does not provide an accurate solution. These differences have been found in different types of problems, and not only in those which involve the application of high fluxes or short times. For example, in [24] heating times are around several minutes. This means that the usual alternatives for the PHTE (the HHTE, the relativistic and Lorentz-covariant heat conduction models) are not suitable and an appropriate formulation has still to be found.

The interest of this study was thus to propose an alternative to the PHTE that better fits with the experimental results of thermal therapies and which could be of significance for all times and values of heat fluxes.

Our hypothesis was that the time fractional derivative formulation of the heat transfer equation (FDHTE) can assume a non-ideal behavior of heat conduction which differs to the PHTE, and can provide more accurate theoretical results that better agree with experimental data. The FDHTE equations have the form (1.4) where $\beta > 0$ plays the role of a tuner that is able to better capture the experimental data.

The objective was thus to assess the effect of the FDHTE on the theoretical modeling of thermal therapies. With this aim, firstly, we assessed a general heat conduction case to observe how the different PHTE and FDHTE formulations affect the solution and also analyze the necessary tools to obtain an analytic solution by means of Laplace transform methods (section 2.1). In this first case no heat

source was considered. Secondly, we studied the laser irradiation of a semi-infinite medium, with was related with the use of laser in thermal therapies (section 2.2). In this second case a heat source from the laser irradiation was considered.

2 Mathematical modeling with a time fractional approach

2.1 Heat conduction in a finite bar without heat source

The first case involved heat conduction in a finite bar of length $L = 1$, with null initial temperature. For $t > 0$ we applied a constant temperature T_1 in $x = 0$, while the extreme $x = 1$ always remained at null temperature.

We considered the initial and boundary value problem:

$$\begin{cases} \partial_x^2 T(x,t) = \dfrac{1}{\alpha}\partial_t^\beta T(x,t), & t > 0, \quad 0 < x < 1, \quad 0 < \beta \le 2; \\ T(0,t) = T_1, & t > 0; \\ T(1,t) = 0, & t > 0; \\ T(x,0) = 0, & 0 < x < 1. \end{cases} \tag{2.1}$$

Here, we denote by ∂_t^β the time fractional derivative of order β in the sense of Caputo. We recall that given a differentiable function $f : \mathbb{R}_+ \to \mathbb{R}$ we define

$$\partial_t^\beta f(t) := (g_{1-\beta} * f')(t) := \int_0^t g_{1-\beta}(t - \tau)f'(\tau)d\tau, \quad t > 0, \tag{2.2}$$

whenever $0 < \beta \le 1$ and, for f twice differentiable,

$$\partial_t^\beta f(t) := (g_{2-\beta} * f')(t) := \int_0^t g_{2-\beta}(t - \tau)f''(\tau)d\tau, \quad t > 0,$$

in case $1 < \beta \le 2$. In the above formulation, g_γ denotes the standard kernel

$$g_\gamma(t) := \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \quad \gamma > 0, \quad t > 0,$$

where Γ denotes the Gamma function. In our formulation, each local integer derivative $f'(t)$ (respectively, $f''(t)$) at each time position τ ($0 < \tau < s$) in the time interval $(0, t)$ contributes with weight $g_{1-\beta}(t)$ (respectively, $g_{2-\beta}(t)$) to the Caputo fractional derivative of $f(t)$ during the time interval $(0, t)$. Hence, the Caputo derivative is a nonlocal quantity, pertaining to a time interval, versus the conventional derivative of $f(t)$, $f'(t)$, which is defined for the particular time location t. Within this framework, the effect of the initial condition at the initial time location 0 is still accounted for any time t ($0 \le t \le T$) during a whole simulation period $(0, T)$ by means of the fractional time derivative that appears in the above governing energy equation (2.1) in fractional time. It also follows from equation (2.2) that this memory effect is modulated by the value of the fractional power β. As shown by Podlubny [31], as

$\beta \to 1$, the Caputo fractional time derivative of $f(t)$, as given by equation (2.2), converges to the local time derivative $f'(t)$ at t.

From above it follows that the fractional governing equation (2.1) is nonlocal. Accordingly, it can account for the influence of the initial conditions on the flow process more effectively than the corresponding local-scale integer-order conventional governing equations. In fractional differentiation with respect to time, the parameter β can be physically interpreted as an existence of memory effects which correspond to intrinsic dissipation in our system. For more information related to the subject of fractional calculus, we refer the reader to e.g. [3]. Some methods to solve time fractional models have been proposed by several authors, see for instance the references [14], [15] and [16].

Note that the case $\beta = 1$ is the PHTE formulation of the problem, whereas the FDHTE formulation corresponds to $\beta \neq 1$.

Remark 2.1. *Using the change of variable $\overline{x} = x$, $\overline{t} = \alpha t$ and*

$$\overline{T}(\overline{x}, \overline{t}) = \frac{1}{T_1} T(\overline{x}, \frac{\overline{t}}{\alpha^\beta}),$$

we solve the dimensionless problem. We observe that this is possible only because the change of variable $\varphi(\tau) := \alpha \tau$ is linear and hence the Caputo fractional derivative obey to the composite rule: $D_\tau^\beta (f \circ \varphi)(\tau) = \alpha^\beta D_\tau^\beta f(\varphi(\tau))$. In general, the Caputo fractional derivative fails to have these property.

Consequently, we have to solve the initial value problem

$$\begin{cases} \partial_{\overline{x}}^2 \overline{T}(\overline{x}, \overline{t}) &= \partial_{\overline{t}}^\beta \overline{T}(\overline{x}, \overline{t}), \overline{t} > 0, & 0 < \overline{x} < 1, & 0 < \beta \leq 2; \\ \overline{T}(\overline{x}, 0) &= 0, & 0 < \overline{x} < 1; \\ \partial_t \overline{T}(\overline{x}, 0) &= 0, & 0 < \overline{x} < 1, & \text{(only if } 1 < \beta \leq 2), \end{cases} \tag{2.3}$$

with boundary conditions $\overline{T}(0, \overline{t}) = 1$ and $\overline{T}(1, \overline{t}) = 0$. One of the main features of this formulation, is that the temperature at $\overline{t} = 0$ is equal to zero. This is known as an ill-posed problem, because the solution does not depend on the initial data. From a mathematical point of view, several methods fail to solve this problem, since they deal only with well-posed problems. For instance the semigroup method [28], based on operator theory, produces the trivial solution. The reason of this drawback is that mathematically is customary to assume non-zero initial conditions in most of the theoretical studies. Therefore, we have to adopt in this paper a direct method, using the Laplace transform as main tool.

Recall that the Laplace transform of a function $f : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\widehat{f}(s) := \mathcal{L}(f)(s) := \int_0^\infty e^{-st} f(t) dt,$$

for all $s \in \mathbb{C}$ such that $Re(s)$ is sufficiently large, say, $Re(s) > \omega$ $(\omega \in \mathbb{R})$.

For example, for $Re(s) > \omega$ the identities

$$\widehat{\partial_t^\beta f}(s) = s^\beta \widehat{f}(s) - s^{\beta-1} f(0), \quad 0 < \beta \leq 1,$$

and

$$\widehat{\partial_t^\beta f}(s) = s^\beta \widehat{f}(s) - s^{\beta-1} f(0) - s^{\beta-2} f'(0), \quad 1 < \beta \le 2,$$

are valid [31, Chapter 4, p.138, formula (4.1)].

We first analyze $\beta = 1$. In such case, we claim that the unique solution of the PHTE problem (2.1) is given by the explicit formula

$$T(x,t) = T_1 \sum_{k=0}^\infty \left(\mathrm{Erfc}\left(\frac{x+2k}{2\sqrt{\alpha t}} \right) - \mathrm{Erfc}\left(\frac{2-x+2k}{2\sqrt{\alpha t}} \right) \right), \qquad (2.4)$$

where $Erfc$ denotes the complementary error function, defined by

$$Erfc(z) := \frac{2}{\sqrt\pi} \int_z^\infty e^{-t^2} dt, \quad z \in \mathbb{R}.$$

Indeed, taking Laplace transform with respect to $\bar t$ in equation (2.3) and using the initial condition $\overline{T}(\overline{x},0) = 0$ we get

$$\partial_{\overline{x}}^2 \widehat{\overline{T}}(\overline{x},s) - s\widehat{\overline{T}}(\overline{x},s) = 0 \qquad (2.5)$$

where we denote $\widehat{\overline{T}}(\overline{x},s) := \mathcal{L}[\overline{T}(\overline{x},\cdot)](s)$. Equation (2.5) is a second order differential equation with constant coefficients whose characteristic equation is $\lambda^2 - s = 0$, and so its solution is of the form

$$\widehat{\overline{T}}(\overline{x},s) = A(s) \, e^{\sqrt{s}\overline{x}} + B(s) \, e^{-\sqrt{s}\overline{x}}. \qquad (2.6)$$

From the boundary conditions $\overline{T}(0,\bar t) = 1$ and $\overline{T}(1,\bar t) = 0$ we get $\widehat{\overline{T}}(0,s) = \frac{1}{s}$ and $\widehat{\overline{T}}(1,s) = 0$. Thus, we deduce that $A(s)$ and $B(s)$ are given by

$$A(s) = \frac{e^{-2\sqrt{s}}}{(e^{-2\sqrt{s}} - 1)s} \qquad B(s) = \frac{-1}{(e^{-2\sqrt{s}} - 1)s}.$$

Therefore, the solution of equation (2.5) is

$$\widehat{\overline{T}}(\overline{x},s) = \frac{e^{-2\sqrt{s}(1+\overline{x})} - e^{-\sqrt{s}\overline{x}}}{(e^{-2\sqrt{s}} - 1)s}.$$

Since $Re(s) > \omega$, we can recast the fraction $\frac{1}{1-e^{-2\sqrt{s}}}$ as a geometric series of common ratio $e^{-2\sqrt{s}} < 1$, and the above equation can be rewritten as:

$$\widehat{\overline{T}}(\overline{x},s) = \frac{1}{s} \sum_{k=0}^\infty [e^{-\sqrt{s}\overline{x}-2k\sqrt{s}} - e^{-2\sqrt{s}+\overline{x}\sqrt{s}-2k\sqrt{s}}].$$

In order to obtain the inverse Laplace transform of the above expression, we divide it into two parts: $\widehat{\overline{T}}(x,s) = C(\overline{x},s) + D(\overline{x},s)$, being

$$C(\overline{x},s) := \sum_{k=0}^\infty \frac{e^{-\sqrt{s}[2-\overline{x}+2k]}}{s} \qquad \text{and} \qquad D(\overline{x},s) := \sum_{k=0}^\infty \frac{e^{-\sqrt{s}[\overline{x}+2k]}}{s}.$$

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From [13, Formula 102, p.1116] we know that the Laplace transform of the function $\psi(\tau) = Erfc(\frac{a}{\sqrt{\tau}})$ is given by

$$\widehat{\psi}(s) = \frac{e^{-2a\sqrt{s}}}{s}, \quad Re(a) > 0.$$

Therefore, the inverse Laplace transform of C and D are:

$$\mathcal{L}^{-1}(C)(\overline{x}, \overline{t}) = \sum_{k=0}^{\infty} Erfc\left(\frac{2 - \overline{x} + 2k}{2\sqrt{\overline{t}}}\right), \quad \mathcal{L}^{-1}(D)(\overline{x}, \overline{t}) = \sum_{k=0}^{\infty} Erfc\left(\frac{\overline{x} + 2k}{2\sqrt{\overline{t}}}\right).$$

Since $\overline{T}(\overline{x}, \overline{t}) = \mathcal{L}^{-1}(D)(\overline{x}, \overline{t}) - \mathcal{L}^{-1}(C)(\overline{x}, \overline{t})$, returning to dimension variables, we obtain the solution of the problem given by the formula (2.4) and proves the claim.

In what follows, we need to recall the Wright function, that we denote by $W_{\lambda,\mu}$. It was introduced and investigated by E. Maitland Wright in a series of notes starting from 1933 in the framework of the theory of partitions; see [34]. This entire function is defined by the series representation, convergent in the whole complex plane:

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C}.$$

For $0 < \nu < 1$ and $\mu \geq 0$ the function $\psi_{\nu,\mu}$ in two variables defined by

$$\psi_{\nu,\mu}(t, a) := t^{\mu-1} W_{-\nu,\mu}(-at^{-\nu}), \quad t > 0, \quad a \in \mathbb{C},$$

is called scaled Wright function, and was introduced in [1, Section 3] in connection with subordination principles. Among others, this function satisfy the following property [1, Theorem 3]:

$$\int_0^{\infty} e^{-st}\psi_{\nu,1}(t, a)dt = \frac{e^{-as^{\nu}}}{s}, \quad s > 0, \quad a > 0, \quad 0 < \nu < 1. \tag{2.7}$$

In particular, we have the following precise description of the function $\psi_{\nu,1}$

$$\psi_{\nu,1}(t, a) = \sum_{j=0}^{\infty} \frac{(-1)^j a^j t^{-\nu j}}{j!\Gamma(1 - \nu j)}, \quad a, t > 0, \quad 0 < \nu < 1.$$

Since $\Gamma(1) = 1$, we obtain using the Euler's reflection formula

$$\psi_{\nu,1}(t, a) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j a^j t^{-\nu j}}{j!} \frac{\sin(\pi\nu j)\Gamma(\nu j)}{\pi}, \quad a, t > 0, \quad 0 < \nu < 1.$$

Then, by using Legendre duplication formula for the Gamma function, we arrive at the following representation

$$\psi_{\nu,1}(t, a) = 1 + \frac{2}{\sqrt{\pi}} \sum_{j=1}^{\infty} \frac{(-1)^j a^j t^{-\nu j}}{j!} \frac{\sin(j\pi\nu)}{2^{2\nu j}} \frac{\Gamma(2\nu j)}{\Gamma(\nu j + \frac{1}{2})}, \quad a, t > 0, \quad 0 < \nu < 1. \tag{2.8}$$

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In case $\nu = 1/2$ we have

$$Erfc(\frac{a}{2\sqrt{t}}) = \psi_{1/2,1}(t,a) \tag{2.9}$$

Although the formula (2.8) is a proper mathematical generalization of (2.9), we observe that the series (2.8) is very unstable for computer simulations. This task will be treated in what follows.

There is another representation in terms of known functions. This can be derived observing that by [1, (v) of Theorem 3] we have

$$\psi_{\nu,1}(t,a) = \int_0^t f_{a,\nu}(s)ds, \quad t \geq 0,$$

where $f_{a,\nu}(s)$ is the Lévy density function, which according to [35, p.263, Formula (17) with $\theta = \pi$] is given by

$$f_{a,\nu}(s) = \frac{1}{\pi}\int_0^\infty e^{-sr-ar^\nu\cos(\pi\nu)}\sin(ar^\nu\sin(\pi\nu))dr$$

Therefore

$$\begin{aligned}
\psi_{\nu,1}(t,a) &= \frac{1}{\pi}\int_0^\infty e^{-ar^\nu\cos(\pi\nu)}\sin(ar^\nu\sin(\pi\nu))[\frac{1-e^{-tr}}{r}]dr \\
&= 1 - \frac{1}{\pi}\int_0^\infty e^{-tr-ar^\nu\cos(\pi\nu)}\frac{\sin(ar^\nu\sin(\pi\nu))}{r}dr \\
&= \frac{1}{\pi\nu}\int_0^\infty e^{-a\cos(\pi\nu)s}\frac{\sin(a\sin(\pi\nu)s)}{s}ds \\
&\quad - \frac{1}{\pi\nu}\int_0^\infty \frac{\sin(ar\sin(\pi\nu))}{r}e^{-tr^{1/\nu}-ar\cos(\pi\nu)}dr \\
&= 1 - \frac{1}{\pi\nu}\int_0^\infty \frac{\sin(ar\sin(\pi\nu))}{r}e^{-tr^{1/\nu}-ar\cos(\pi\nu)}dr,
\end{aligned}$$

where we used the identity

$$\frac{1}{\pi\nu}\int_0^\infty e^{-a\cos(\pi\nu)s}\frac{\sin(a\sin(\pi\nu)s)}{s}ds = 1, \tag{2.10}$$

which is valid only for $0 < \nu \leq \frac{1}{2}$ since we should have $\cos(\pi\nu) > 0$ in order to guarantee the convergence of the integral in (2.10).

We observe that, for our purposes, our interest is the case $\nu > 1/2$ and therefore the above representation for $\psi_{\nu,1}$, although valid, is not enough.

An equivalent representation, taking $\theta = 0$ in [35, p.263, Formula (17)] and then valid for $0 < \nu < 1$ is

$$\psi_{\nu,1}(t,a) = \frac{1}{2} + \frac{1}{\pi\nu}\int_0^\infty [e^{-a\cos(\pi\nu/2)r}]\frac{\sin(tr^{1/\nu} - ar\sin(\pi\nu/2))}{r}dr. \tag{2.11}$$

It happens that, when we tuning the parameter μ, that represents the fractional derivative in the explicit solution of the models, the above representation turns out

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to be very stable for computational simulations. This very important fact, will be made clear for the reader in the remaining of this paper.

Using (2.11), we claim that the unique solution of the PDHTE problem (2.1) in case $0 < \beta < 2$ is given by the following formula

$$T(x,t) = T_1 \sum_{k=0}^{\infty} [\psi_{\beta/2,1}(\alpha^\beta t, x + 2k) - \psi_{\beta/2,1}(\alpha^\beta t, 2 - x + 2k)]. \qquad (2.12)$$

Indeed, taking Laplace transform in equation (2.3) and using the initial conditions $\overline{T}(\overline{x}, 0) = 0$ and $\partial_t \overline{T}(\overline{x}, 0) = 0$ (the last one only in case $1 < \beta < 2$) we get

$$\partial_{\overline{x}}^2 \widehat{\overline{T}}(x, s) - s^\beta \widehat{\overline{T}}(\overline{x}, s) = 0.$$

Since the characteristic equation is $\lambda^2 - s^\beta = 0$, using the given boundary conditions, we deduce that the solution is

$$\widehat{\overline{T}}(\overline{x}, s) = \frac{e^{-2s^{\beta/2}(1+\overline{x})} - e^{-s^{\beta/2}\overline{x}}}{(e^{-2s^{\beta/2}} - 1)s}.$$

Rewriting the fraction $\frac{1}{1 - e^{-2s^{\beta/2}}}$ as a geometric series, we obtain

$$\widehat{\overline{T}}(\overline{x}, s) = \sum_{k=0}^{\infty} \frac{e^{-s^{\beta/2}[\overline{x}+2k]} - e^{-s^{\beta/2}[2-\overline{x}+2k]}}{s}.$$

Using the formula (2.7) we conclude that

$$\overline{T}(\overline{x}, \overline{t}) = \sum_{k=0}^{\infty} [\psi_{\beta/2,1}(\overline{t}, \overline{x} + 2k) - \psi_{\beta/2,1}(\overline{t}, 2 - \overline{x} + 2k)].$$

Going back to the dimensional variables, we obtain the representation of the solution given in (2.12).

2.2 Laser irradiation

The second case in study dealt with a type of thermal therapy that uses a laser to irradiate biological tissue. Radiofrequency, microwaves and laser are among the techniques based on raising the temperature above $50°$C to alter or destroy biological tissues. Laser can better focus heating on the target zone, although it is more expensive than radiofrequency or microwaves. However, the accuracy required for example in thermokeratoplasty make laser the most suitable heat source. As using intense heat for short times also makes the problem interesting from the HHTE viewpoint, in this case we compared the FDHTE solution with those obtained by PHTE and HHTE.

In laser irradiation, a laser beam is directly applied to the biological tissue, as can be seen in Figure 1. The most interesting result was the temperature of the
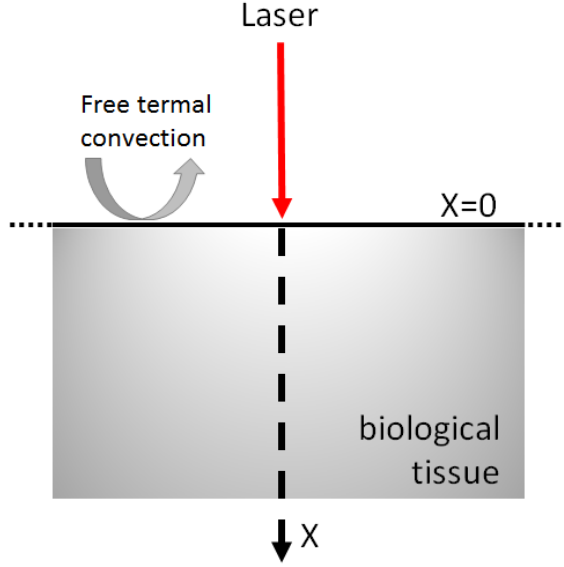
Figure 1: Schematic representation of the model geometry considered in the study of the tissue laser irradiation.

biological tissue in the laser beam axis, so that a 1-dimensional analysis can be conducted.

The main difference with the heat conduction case of section 2.1 is that the source $S(x, t)$ which appears in the governing equation is not zero. We assumed the same expression for all the formulations considered (PHTE, HHTE and FDHTE) obtained from Beer-Lambert's law

$$S(x,t) = (1 - R)bE_0 e^{-bx}\left[H(t) - H(t - \Delta t)\right] , \qquad (2.13)$$

where R is Fresnel surface reflectance, b is the absorption coefficient, E_0 is the incident irradiance on the tissue surface. The laser beam has a pulse duration Δt, and $H(t)$ represents the Heaviside function used to model it.

Expression (2.13) should be included in the governing equations (1.1), (1.2) and (1.4) in order to solve the thermal coupled problem.

2.2.1   PHTE and HHTE solutions

This problem was solved in [33] in which the solution of the problem for the PHTE and HHTE formulations was obtained. The solution was also based on the Laplace transform in equation (2.4). The most important feature of the formulation is the Dirac's distribution in the HHTE governing equation as the temporal derivative of the heat source $S(x, t)$ which includes Heaviside functions.

2.2.2   FDHTE solution

Substituting in the equation (1.4) the heat source (2.13) and after to a dimensionless procedure, one arrives at the governing equation of the problem from the

viewpoint of the FDHTE

$$
\begin{cases}
\partial_t^\beta V_F(x,t) = \alpha \partial_x^2 V_F(x,t) + S(x,t) & t > 0, \quad x > 0, \quad 0 < \beta \le 2; \\
V_F(x,0) = 0, & x > 0; \\
\lim\limits_{x \to \infty} V_F(x,t) = 0, & t > 0; \\
\partial_x V_F(0,t) = B(V_F(0,t) + C), & t > 0.
\end{cases}
\tag{2.14}
$$

where $V_F(x,t) = T(x,t) - T_0$ and $C = T_0 - T_a$, i.e. the initial temperature of the tissue is not zero, but we made a change of variable for simplicity of the theoretical treatment. As it is shown in Figure 1 the domain is a semi-infinite fragment of homogeneous isotropic biological tissue. The laser incidence is produced at $x = 0$ and we consider the heating process produced along the x-axis. At the surface $x = 0$ the tissue is affected by free convection of the surrounding air (last condition in (2.14)).

We finally arrive to the main theoretical result of this paper: We claim that the unique solution of the problem (2.14) in case $0 < \beta < 2$ has the form

$$
T(x,t) = T_0 + F_1(x,t) - \sum_{j=2}^{4} F_j(x,t) + F_5(x,t),
\tag{2.15}
$$

where F_i are defined below. Indeed, following the same steps as in [33, Section 3.2] we arrive at

$$
V_F(x,t) = \mathcal{L}^{-1}[f_1(x,\cdot)](t) - \sum_{j=2}^{4} \mathcal{L}^{-1}[f_j(x,\cdot)](t) + \mathcal{L}^{-1}[f_5(x,\cdot)](t)
$$

where $f_1(x,s) = \dfrac{Me^{-bx}}{s(s - \alpha b^2)}$, $f_2(x,s) = f_1(x,s)e^{-\Delta t s}$, $f_3(x,s) = \dfrac{BCe^{-x\frac{s^{\beta/2}}{\sqrt{\alpha}}}}{s(B + \frac{s^{\beta/2}}{\sqrt{\alpha}})}$,

$f_4(x,s) = \dfrac{M(b+B)e^{-x\frac{s^{\beta/2}}{\sqrt{\alpha}}}}{s(B + \frac{s^{\beta/2}}{\sqrt{\alpha}})(s - \alpha b^2)}$ and $f_5(x,s) = f_4(x,s)e^{-\Delta t s}$. Additionally, we ob-

serve that we can rewrite $f_4(x,s) = \dfrac{M(b+B)}{BC(s - \alpha b^2)} f_3(x,s)$. Consequently, we obtain

$$
V_F(x,t) = F_1(x,t) - \sum_{j=2}^{4} F_j(x,t) + F_5(x,t),
$$

where, following [33], we have

$$
F_1(x,t) = Me^{-bx}\left(\frac{e^{ab^2 t} - 1}{ab^2}\right) \qquad F_2(x,t) = H(t - \Delta t)F_1(x, t - \Delta t);
$$

$$
F_4(x,t) = \frac{M(b+B)}{BC} \int_0^t e^{ab^2(t-\tau)} F_3(x,\tau)d\tau,
$$

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and

$$F_5(x, t) = H(t - \Delta t)F_4(x, t - \Delta t).$$

Moreover

$$F_3(x, t) = \sqrt{\alpha}BC \int_0^t (t - \tau)^{\beta/2 - 1} E_{\beta/2, \beta/2}(-\sqrt{\alpha}B(t - \tau)^{\beta/2})\psi_{\beta/2, 1}(\tau, \frac{x}{\sqrt{\alpha}})d\tau,$$

where

$$E_{\beta/2, \beta/2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\frac{\beta}{2}n + \frac{\beta}{2})}.$$

is a Mittag-Leffler function, which proves the claim.

Remark 2.2. *We emphasize that the use in $F_3(x, t)$ of the representation for $\psi_{\beta/2, 1}$ given by the formula (2.11) will be essential in the computational simulations obtained later in this paper.*

3   Results and discussion

We first compared the PHTE and FDHTE solutions for the heat conduction problem in a finite bar (see Section 2.1) to assess the main differences between both formulations. Then, we assess the results of the FDHTE solution obtained for laser irradiation of biological tissue (section 2.2) and we compared them with the results obtained in [33] for an analogous problem from the viewpoint of the PHTE. Since the specific characteristics of this problem implied short times and high heat flux, a comparison with the HHTE solution obtained in [33] was also showed.

3.1   Heat conduction with null initial temperature

Aluminum was used to represent the solutions obtained in Section 2.1, i.e. $\alpha = 85 \times 10^{-9}$ m^2/s and 25°C as the initial bar temperature. Figure 2 shows the temperature distribution throughout the bar $x \in [0, 1]$ m at $t = 100$ s for different β values ($\beta = 0.9$, 0.95, 1, 1.05 and 1.1). The case $\beta = 1$ corresponds to the PHTE solution. As shown in Figure 2 the FDHTE solutions provide different temperature profiles to the PHTE, since obey other heat transmission behaviors. When $\beta < 1$, the heat transmission by means of the FDHTE ($\beta = 0.95$ and $\beta = 0.9$) is slower than the PHTE, and in the case $\beta > 1$, the heat transmission in the FDHTE ($\beta = 1.05$, $\beta = 1.1$) is faster than in the PHTE. As the speed at which the temperature arrives at each point of the bar changes according to the value of β., we noted that the FD-HTE formulation is an alternative to use in thermal problems in which the PHTE results overestimate or underestimate the experimental temperatures profiles.
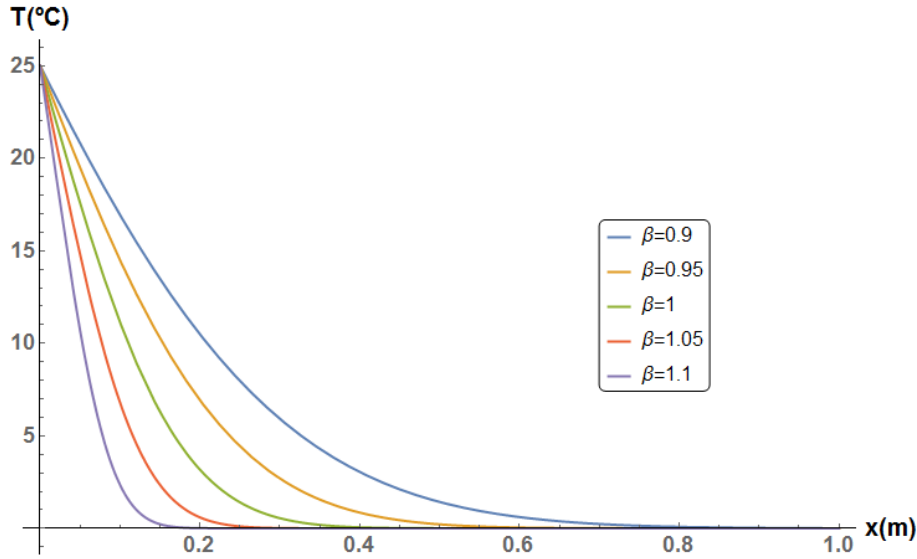
Figure 2: Temperature distribution along the finite bar for PHTE ($\beta = 1$ and FDHTE ($\beta = 0.95$, $\beta = 0.9$, $\beta = 1.05$ and $\beta = 1.1$) at time $t = 100$ s. The solutions corresponds to the case of the heat conduction in a finite bar.

Figure 3 shows the temperature evolution from 0 up to 1000 s at point $x = 0.4$ m for PHTE ($\beta = 1$) and FDHTE ($\beta = 0.95$, $\beta = 0.9$, $\beta = 1.05$ and $\beta = 1.1$). Although the time considered in Figure 3 is quite high, differences were found between FDHTE and PHTE solutions. In [4] the HHTE solution of the problem studied in section 2.1 was obtained. The HHTE solution is represented at the same point and time interval as used in Figure 3 and the temperature evolution exactly coincides with those obtained for the PHTE. To find any differences between the PHTE and HHTE time intervals in the order of 10^{-9} s using $\tau = 10^{-13}$ s were used (i.e. the thermal relaxation time of aluminum), which shows the suitability of the FHDE as an alternative to the PHTE, as it is not only valid for short times.

3.2 Laser irradiation

We were interested in comparing PHTE and FDHTE solutions in the problem related with a thermal therapy, which corresponds to the case of laser irradiation. We used for the PHTE solution the obtained in [33], in which authors solved an analogous problem, but applied for a specific treatment. The characteristics of the laser irradiation (short pulse for heating and great amount of heat applied by the laser source) make that the consideration of the HHTE as an alternative to the PHTE has sense, and this fact allows us to assess what is the role of the FDHTE.

To compare the FDHTE with the PHTE and HHTE solutions, we reproduced Figure 2 in [33] and adding the FDHTE solution, with the same laser pulse length (Δt = 200 ns), incident radiance $E_0 = 5 \times 10^8$, initial temperature 35°C and material characteristics cited in [33].
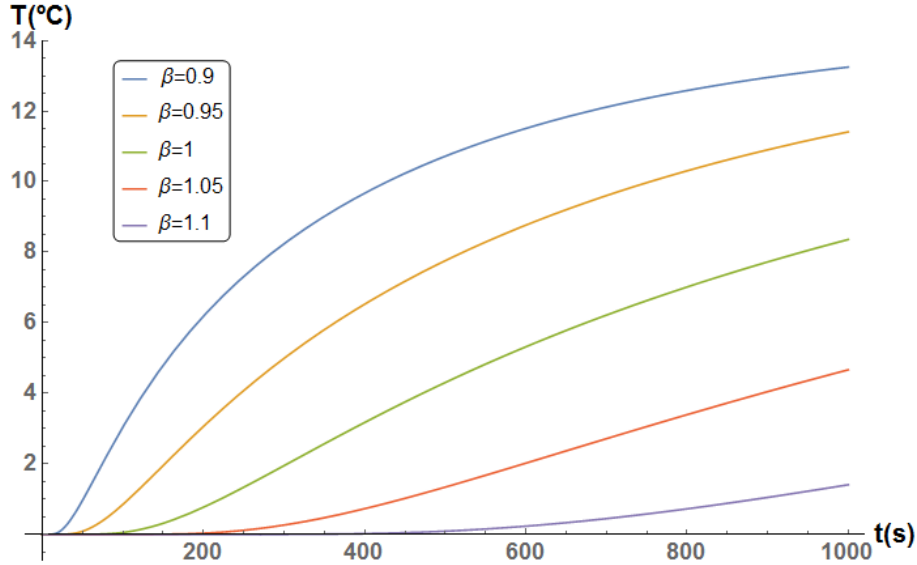
13

Figure 3: Temperature evolution from 0 to 1000 s at point $x = 0.4$ m for PHTE ($\beta = 1$) and FDHTE ($\beta = 0.95$, $\beta = 0.9$, $\beta = 1.05$ and $\beta = 1.1$). The solutions corresponds to the case of the heat conduction in a finite bar.

Figure 4 shows the temperature evolution on the surface $x = 0$ during the heating and cooling phases, i.e at time intervals $[0, 200]$ ns and $[200 \times 10^{-9}, 1]$ s, respectively. This figure is similar to Figure 2 of [33], with the addition of the FDHTE solution with $\beta = 1.1, 1.2$ and 1.5 and only the extreme values of τ for the HHTE solution ($\tau = 0.1$ and 10). From Figure 4 two main results can be extracted: 1) there is no significant difference between the models during the heating phase; and 2) there are differences in the way the heat is transmitted during cooling between the PHTE, HHTE and FDHTE. In the heating phase the heat source term has a predominant effect over the other terms in the heat equations, so that the differences between the PHTE, HHTE and FDHTE regarding the temporal derivatives formulation are negligible. When the heat source is switched off the differences between the models are in the form of temperature distribution and evolution. As the PHTE assumes an infinite heat conduction speed, there is an *ideal* heat transmission in the tissue and cooling is faster. The HHTE follows the same heat transmission behavior as the PHTE, but considering a *delay* in the heat flux. For this reason, when $\tau = 0.1$ s the temperature drop is slower at the beginning of the cooling period (up to 0.5 s), after which both solutions are similar. In the case of $\tau = 10$ there is no convergence at Figure 4, but that the temperature differences between HHTE and PHTE decrease with time. With the FDHTE, higher temperatures are obtained for $\beta > 1$ than those obtained by PHTE, but with different behavior to HTTE. In the case of the FDHTE the differences are observed gradually with cooling, and not only at the beginning. The temperature profiles provided by the HHTE show an abrupt drop (see Figure 4 in [33]), however, while those of the FDHTE solutions gradually decrease.

We used only $\beta \in [1, 2]$ for the FDHTE, since the aim was to compare it with
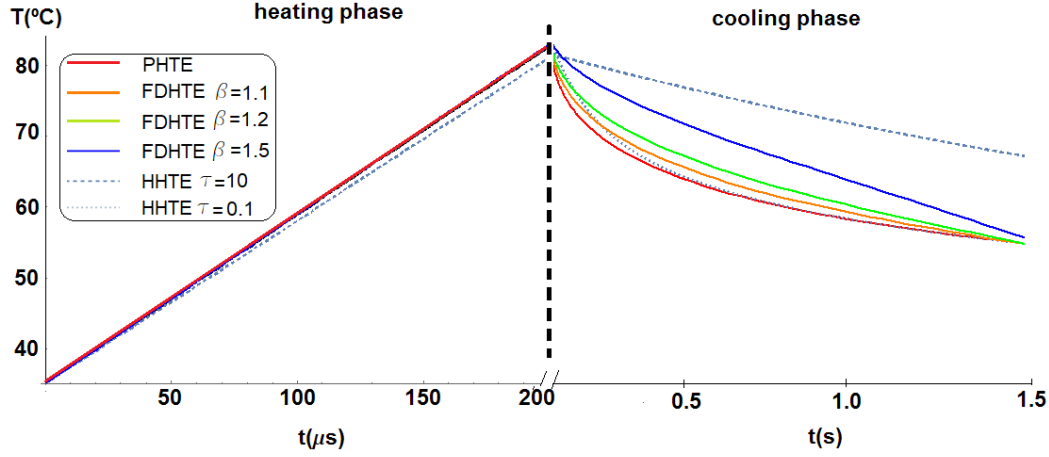
14

Figure 4: Temperature evolution at surface $x = 0$ during the heating and cooling phases for PHTE, HHTE ($\tau = 0.1$, $\tau = 10$) and FDHTE ($\beta = 1.1$, $\beta = 1.2$ and $\beta = 1.5$). The solutions corresponds to the case of laser irradiation.

both the PHTE ($\beta = 1$) and HHTE models, which use second partial derivative of temperature with respect to time.

In Figure 5 we have reproduced some curves of Figure 3 of [33] to assess differences between models in a point different to the surface. Specifically, Figure 5 shows the evolution of temperature at point $x = 10^{-5}$ m in the temporal interval $[0, 0.3]$ s for PHTE, HHTE ($\tau = 0.1$, $\tau = 10$) and FDHTE ($\beta = 1.2$). With this curves we can remark the differences in behaviour of the three heat conduction models. It is observed that PHTE assumes an higher speed of heat conduction than the FDHTE, and that HHTE differs from the PHTE in a delay of the heat transmission, which is clearly showed as an abrupt drop in the curve corresponding to $\tau = 10$.
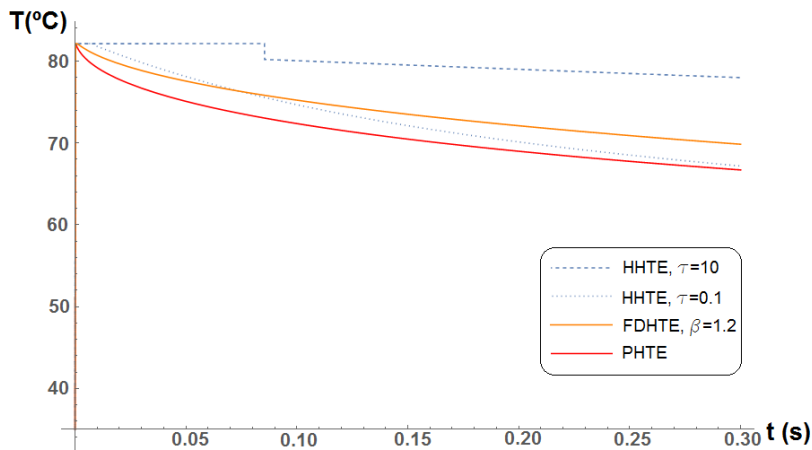

Figure 5: Temperature evolution at point $x = 10^{-5}$ m in the temporal interval $[0, 0.3]$ s for PHTE, HHTE ($\tau = 0.1$, $\tau = 10$) and FDHTE ($\beta = 1.2$). The solutions corresponds to the case of laser irradiation.

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Figures 4 and 5 show that the FDHTE solutions differ from the PHTE during the cooling phase. These differences in temperature are greater if the β value increases. In figure 4 these differences in the cooling phase are in the range of $2 - 10°$ C, which means between $3 - 17\%$. As the PHTE solution obtained in [33] were not compared with experimental data, we cannot state whether the FDHTE solution was closer to the experimental data than the PHTE. However, this was not the aim of this study. Our effort was focused in exploring a new approach for the heat transfer modeling of the thermal therapies which could be applied in cases in which the PHTE overestimates or underestimates the temperature values.

3.3   Theoretical considerations

The fractional partial differential equation has been analyzed by a number of authors in the context of biomedical engineering [6, 7, 8]. In the present study the time fractional derivative considered was the Caputo fractional derivative of order $\beta \in (0, 2]$, which was more suitable. One of our contributions is that, in contrast with earlier works, the FDHTE equations (1.4) are solved analytically, obtaining an explicit formula valid for the full range $0 < \beta \leq 2$ which is stable in computer simulations. In this line of research, as far as we know, only the recent paper [11] has dealt with the case of skin tissue. Our proposal makes it unnecessary to use numerical treatments of the fractional model, as used in most of the references on the subject, e.g. [18, 19, 20, 21, 32], and provide an exact solution of the problem.

One limitation of using of analytical solutions is the time needed to compute the results. Numerical computations are quicker than analytical. Although with numerical methods we cannot obtain the exact analytical solution, the errors can be negligible when they are well-formulated. So, it is a feature to consider for future works. Moreover, in order to compare experimental and theoretical results, theoretical models have to include realistic characteristics of the process. In the case of thermal therapies this implies to consider non linear problems (some characteristics of tissue depend on temperature or also the vaporization phenomenon) and complex geometries. These features make that problems have to be solved mainly using numerical methods. The analytical solutions are restricted to study some theoretical considerations, such us the exposed in this work. We point out that the novelty of the present study was not related with a better approximation to experimental results. Our main achievement was to study an alternative approach to the PHTE in thermal therapies, and to have an analytical solution for a problem of laser irradiation from the FDHTE viewpoint.

4   Concluding remarks

In this work we propose the FDHTE as an alternative to the PHTE in some thermal therapies problems in which the PHTE overestimates or underestimates the experimental temperatures. The advantage of the FDHTE over the HHTE and other alternatives to the PHTE (relativistic equation or Lorentz covariance equation)

is that the FDHTE is not restricted to problems in which intense heat is applied for short times. PHTE accuracy can be improved by adjusting the β parameter in the FDHTE.

The heat source term plays an important role in the formulation for thermal therapies. As seen by the results of the laser irradiation, the characteristics of the heat source make the use of PHTE or alternatively FDHTE have sense. It is not related with the fact of the kind of source like in HHTE, it is a more complex balance between application times of the source (heating and cooling phases), characteristics of the material, type of source (laser, RF, microwave, ultrasounds). So, if we have solved a specific problem using the PHTE and we observe that computational temperature distributions and evolution curves are always above or under the experimental ones, we can try to solve the problem under the viewpoint of the FDHTE because it can provide an accurate computational solution by adjusting the β parameter.

On the other hand, we have solved analytically the considered problems using the FDHTE and established a theoretical scheme for the time fractional approach of a broad type of problems.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgements

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