A New Quadrature Method for Singular Integrals of Boundary Integral Equations in Electromagnetism

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***Abstract*— In this paper we present a new method to compute singular integrals and nearly singular integrals in the context of boundary integral equations for electromagnetism. In particular the method is well suited for integral equations of the second kind. The method consist of splitting the integral in two parts, one regular, which is computed via adaptive integration, and another singular and local (with very small support of the integrand) which is computed using asymptotic expansion. This method can be applied to any second kind integral equation arising in CEM, like MFIE, Charge-Current integral equation, Non Resonant Charge Current Integral Equation (NRCCIE),..**

I. Introduction

Integral equation methods are very popular for solving electromagnetic scattering problems. The use of surface integral equations is particularly interesting for analyzing homogeneous and perfect electric conducting objects. One of the key aspects of the boundary integral equation method is the calculation of the singular integrals. [1–3].

The method that we propose here can be applied to boundary integral equations of the second kind (see [4, 5]). Those kind of integral equations are particularly interesting because of the absence of low frequency breakdown and the absence of high density mesh breakdown.

The discretization method used is the Nyström method ([3]).

II. Mathematical Formulation

Let's consider the Laplace single layer acting on a closed smooth surface (on-surface evaluation):

$$S_0[\rho](\boldsymbol{x}) = \int_{\partial D} \frac{1}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|}\rho(\boldsymbol{y})dS_{\boldsymbol{y}} \tag{1}$$

Consider an Ewald-type decomposition determined by a small parameter σ:

$$\begin{aligned} S_0[\rho](\boldsymbol{x}) &= \int_{\partial D} \frac{1}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|}\rho(\boldsymbol{y})dS_{\boldsymbol{y}} = \\ &= \int_{\partial D} \frac{\operatorname{erf}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\sqrt{2}\sigma}\right)}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|}\rho(\boldsymbol{y}) + \frac{\operatorname{erfc}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\sqrt{2}\sigma}\right)}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|}\rho(\boldsymbol{y})dS_{\boldsymbol{y}} \end{aligned} \tag{2}$$

Where σ is a parameter constant. The first integral in the decomposition is 'smooth'. For small values of the parameter σ we need adaptive quadrature. No more than 4-5 levels of refinement for each triangle for a suitable election of the constant σ.

The second integral in the decomposition is singular and local (it has exponential decay for $r \to +\infty$), and will be evaluated asymptotically with a high order local asymptotic expansion of the Jacobian in the change of variables.

Writing the second integral as:

$$\int_{\partial D} \frac{\operatorname{erfc}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\sqrt{2}\sigma}\right)}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|}\rho(\boldsymbol{y})dS_{\boldsymbol{y}} = \int_{\partial D} H(r)\rho(\boldsymbol{y})dS_{\boldsymbol{y}} \tag{3}$$

where $r = \|\boldsymbol{x}-\boldsymbol{y}\|$ and:

$$H(r) = \frac{\operatorname{erfc}\left(\frac{r}{\sqrt{2}\sigma}\right)}{4\pi r} \tag{4}$$

The function H is singular and decays exponentially for large r. Now we consider a chart $\boldsymbol{r}(u,v)$ centered at a fixed target point $\boldsymbol{x} \in \partial D$, that is $\boldsymbol{x} = \boldsymbol{r}(0,0)$, and orthonormal at this point, that is $\|\boldsymbol{r}_u(0,0)\| = 1, \|\boldsymbol{r}_v(0,0)\| = 1,$
$< \boldsymbol{r}_u(0,0), \boldsymbol{r}_v(0,0) >= 0$

Using this chart we can write r as:

$$r^2 = u^2(1+f(u,v)) + v^2(1+g(u,v)) \tag{5}$$

where $f \to 0$ and $g \to 0$ when $u, v \to 0$.

We can write 3 as:

$$\int_{\partial D} H(r(u,v))\rho(u,v)dS_{u,v} \tag{6}$$

In order to find an asymptotic expansion of 3 we do formally the following change of variables $(x,y) \to (u,v)$:

$$x = u\sqrt{1+f(u,v)}; y = v\sqrt{1+g(u,v)} \tag{7}$$

Notice that equations 7 define a map from a neighborhood of $(0,0)$ onto a neighborhood of $(0,0)$ due to Morse Lemma (see page 237 of [6]). With respect to the new variables (x,y), the function r^2 can be written as:

$$r^2 = u^2(1+f(u,v)) + v^2(1+g(u,v)) = x^2+y^2$$

and therefore:

$$\begin{aligned} &\int_{\partial D} H(r(u,v))\rho(u,v)dS_{u,v} = \\ &= \int_{\partial D} H(\sqrt{x^2+y^2})\rho(x,y)dS(x,y)J(x,y)dxdy \end{aligned} \tag{8}$$

Where $J(x, y)$ is the Jacobian determinant of the change of variables $(x, y) \to (u, v)$. Notice also that equations 7 are an explicit expression of the inverse of this map. Now we can Taylor expand the function $\rho(x, y)dS(x, y)J(x, y)$ around $(0, 0)$ and obtain a good approximation of the integral for small values of σ by computing analytically integrals of the form:

$$\begin{aligned} \alpha_{mn} &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H(\sqrt{x^2 + y^2}) x^m y^n dx dy = \\ &= \int_0^{+\infty} H(r) r^{m+n+1} dr \int_0^{2\pi} \cos^m(\phi) \sin^n(\phi) d\phi \end{aligned} \tag{9}$$

we obtain:

$$\int_0^{+\infty} H(r) r^{m+n+1} dr = \frac{(\sqrt{2}\sigma)^{m+n+1}}{4\pi} \frac{\Gamma(\frac{m+n}{2} + 1)}{\sqrt{\pi}(m + n + 1)} \tag{10}$$

we get $\alpha_{01} = \alpha_{10} = \alpha_{11} = \alpha_{30} = \alpha_{21} = \alpha_{12} = \alpha_{03} = = \alpha_{50} = \alpha_{41} = \alpha_{32} = \alpha_{23} = \alpha_{14} = \alpha_{05} = \alpha_{31} = \alpha_{13} = 0$ and:

$$\begin{aligned} \alpha_{00} =& \frac{\sigma}{\sqrt{2\pi}}; \alpha_{20} = \alpha_{02} = \frac{\sigma^3}{3\sqrt{2\pi}} \\ \alpha_{40} =& \alpha_{22} = \alpha_{04} = \frac{3\sigma^5}{5\sqrt{2\pi}} \end{aligned} \tag{11}$$

The error term is $O(\sigma^7)$ for the single layer with coefficients up to order 4 in $m + n$

The Taylor expansion of $J(x, y)$ can be obtained by computing the partial derivatives (to any desired order) of the map $(u, v) \to (u, v)$.

We can easily reduce the non-static case into the static case doing the following trick:

$$\begin{aligned} S_k[\rho](\boldsymbol{x}) &= \int_{\partial D} \frac{e^{ik\|\boldsymbol{x}-\boldsymbol{y}\|}}{4\pi\|\boldsymbol{x} - \boldsymbol{y}\|} \rho(\boldsymbol{y}) dS_{\boldsymbol{y}} = \\ &= \int_{\partial D} \frac{\cos(k\|\boldsymbol{x} - \boldsymbol{y}\|)}{4\pi\|\boldsymbol{x} - \boldsymbol{y}\|} \rho(\boldsymbol{y}) + \frac{i \sin(k\|\boldsymbol{x} - \boldsymbol{y}\|)}{4\pi\|\boldsymbol{x} - \boldsymbol{y}\|} \rho(\boldsymbol{y}) dS_{\boldsymbol{y}} \end{aligned} \tag{12}$$

The second integral is a smooth function, and the first integral can be written as the one in the static case with a target dependent smooth source given by $\overline{\rho}(\boldsymbol{y}) = \rho(\boldsymbol{y}) \cos(k\|\boldsymbol{x} - \boldsymbol{y}\|)$.

Provided the integral equation is of the second kind, the singular part is integrable ($O(1/r)$) and similar tricks can be used. Notice also that with a minor modification, the source term can be a tangent vector field.

III. Numerical Example

Next we apply this method for the Non Resonant Charge Current Integral Equation [7]. The geometry used is globally smooth and is obtained using a method described in [8]. The far field is computed using the FMM [9]. The total number of degrees of freedom is 166800, The number of triangles in the geometry is 11120 and the number of polynomials on each triangle is 15. The estimated accuracy is 4 digits.

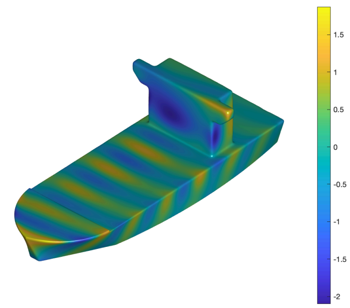

Fig. 1. Induced charge on a Cargo Ship produced by an incoming plane wave

Acknowledgement

This work was supported by the Generalitat Valenciana under the projects AICO/2019/018.

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