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Mean ergodic composition operators on spaces of holomorphic functions on a Banach space

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A B S T R A C T

We study mean ergodic composition operators on infinite dimensional spaces of holomorphic functions of different types when defined on the unit ball of a Banach or a Hilbert space: that of all holomorphic functions, that of holomorphic functions of bounded type and that of bounded holomorphic functions. Several examples in the different settings are given.

1. Introduction

If X and Y are Banach spaces and $U \subseteq X$ is open, then a function $f : U \to Y$ is *holomorphic* if it is Fréchet differentiable at every point of U. If the open unit ball B_X of X satisfies $B_X \subseteq U$ and $\varphi : B_X \to B_X$ is a holomorphic self-map on B_X, the associated composition operator is defined by $C_\varphi(f) := f \circ \varphi$. The function φ is called *symbol* of the composition operator. When $Y = \mathbb{C}$ and $U = B_X$, the space of holomorphic functions $f : B_X \to \mathbb{C}$ is simply denoted by $H(B_X)$. Our aim is to study the power boundedness and (uniform) mean ergodicity of the composition operator $C_\varphi : H(B_X) \to H(B_X)$ in terms of the properties of the symbol φ when $H(B_X)$ is equipped with its natural topology, the compact-open topology, and also when $H(B_X)$ is replaced by the space of holomorphic functions of bounded type $H_b(B_X)$ or that of bounded holomorphic functions $H^\infty(B_X)$. We study also the case when X is a Hilbert space for each of the settings considered above.

Several authors have studied different properties of composition operators on spaces of holomorphic functions on the unit ball of a Banach space. See, for instance, [1,14,15,17] and the references therein. However, it seems that there is no previous literature about the dynamics of such operators. The present

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work can be considered a sequel of [22] by the same authors, where we study some dynamical properties (especially mean ergodicity) of composition operators in spaces of homogeneous polynomials. As in [22], the motivation and inspiration of our investigation comes from several previous works, as [7], where the authors characterise those composition operators $C_\varphi : H(U) \to H(U)$ which are power bounded, where $H(U)$ is the space of holomorphic functions on a connected domain of holomorphy U of $\mathbb{C}^d$. It was proved in [7] that C_φ is power bounded if and only if it is (uniformly) mean ergodic, and this happens if and only if the symbol φ has stable orbits. On the other hand, if the domain is the unit disc, it was characterised in [3] when C_φ is mean ergodic or uniformly mean ergodic on the disc algebra or on the space of bounded holomorphic functions in terms of the asymptotic behaviour of the symbol. Power boundedness and (uniform) mean ergodicity of weighted composition operators on the space of holomorphic functions on the unit disc was analysed in [4] in terms of the symbol and the multiplier. In [23] power boundedness and mean ergodicity for (weighted) composition operators on function spaces defined by local properties was studied in a very general framework which extends previous work. In particular, it permits to characterise (uniform) mean ergodicity for composition operators on a large class of function spaces which are Fréchet-Montel spaces when equipped with the compact-open topology. Here, the results of [23] do not apply since $H(B_X)$, $H_b(B_X)$ or $H^\infty(B_X)$ are not Fréchet-Montel spaces. Other recent contributions to this topic can be found in [24], where mean ergodicity of composition operators on the space of bounded holomorphic functions on the n-dimensional Euclidean ball is studied, and in [21], where the authors consider composition operators on weighted spaces of holomorphic functions on the disc.

The paper is organised as follows. In Section 2 we give some basic definitions and fix the notation used throughout the paper. Moreover, we recall a specific result for Hilbert spaces which is useful along the text. In Section 3 we analyse some properties of stable and B_X-stable orbits. In Section 4 we study the mean ergodicity of the composition operator in the space of holomorphic functions on the unit ball of a Banach space. In Section 5 we consider the same problem for holomorphic functions of bounded type, while in Section 6 we consider the space of bounded holomorphic functions. In each section we treat the Hilbert-space case also.

2. Preliminaries

All along this paper E will always denote a locally convex Hausdorff space. The set of continuous seminorms on E is denoted by Γ_E and $L(E)$ is the space of continuous linear maps $T : E \to E$. We denote $T^0 = \mathrm{id}$ (the identity), $T^1 = T$ and, for $n \in \mathbb{N}$, we write $T^n = T^{n-1} \circ T$ (that is, the n-th composition of T with itself). With this notation, the n-the Cesàro mean of the sequence $(T^k)_{k=0}^\infty$ is defined as

$$T_{[n]} := \frac{1}{n} \sum_{k=0}^{n-1} T^k \,.$$

The operator T is said to be *power bounded* if $\{T^n : n \geq 0\}$ is equicontinuous. It is called *mean ergodic* if there is $L \in L(E)$ such that $(T_{[n]}x)_n$ is convergent (in E) to Lx for every $x \in E$. It is *uniformly mean ergodic* if $(T_{[n]})_n$ converges uniformly on the bounded subsets of E (we will refer to the topology so defined as *the topology of bounded convergence of* $L(E)$). Finally, we say that T is *topologizable* if for each $q \in \Gamma_E$ there exist a sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers and $p \in \Gamma_E$ such that

$$q(a_n T^n x) \leq p(x), \tag{1}$$

for all $x \in E$ and all $n \in \mathbb{N}$ (see [5,34]).

Also X will always denote a Banach space and H a Hilbert space. We write E', X' and H' for the corresponding dual spaces. The set B_X is the open unit ball in X. On B_H (recall that H is a Hilbert space) there is a group of automorphisms that, in some sense, plays the role of Möbius transforms in the unit disc. We give here the definition and a basic property that we use later.

From [32, Proposition 1] we know that, given $a \in B_H$, the linear operator $\gamma_a : H \to H$ defined by

$$\gamma_a(x) := \frac{1}{1+v(a)} a\langle x, a\rangle + v(a)x\,,$$

where $v(a) = \sqrt{1-\|a\|^2}$, satisfies $\|\gamma_a(x)\| \le \|x\|$ for all $x \in H$ and $\gamma_a(a) = a$. Once we have this, for each $a \in B_H$ we can define an automorphism $\alpha_a : B_H \to B_H$ by doing

$$\alpha_a(x) = \gamma_a\Big(\frac{a-x}{1-\langle x, a\rangle}\Big)\,. \tag{2}$$

This satisfies $\alpha_a(0) = a$, $\alpha_a(a) = 0$, and $\alpha_a^{-1} = \alpha_a$ (the first two follow by direct computation, and the third one proceeding as in [32, Proposition 1]). The following result follows from [32, (9')]; we include a proof for the sake of completeness.

Lemma 2.1. *For each $0 < r < 1$ there is $0 < \rho < 1$ such that*

$$\alpha_a(rB_H) \subseteq \rho B_H, \tag{3}$$

for every $a \in rB_H$.

Proof. For $x \in B_H$ with $\|x\| < r$, we put $y := \alpha_a(x)$. Straightforward (though long) computation (see [32, (2)]) yields

$$1 - \|y\|^2 = \frac{(1-\|a\|^2)(1-\|x\|^2)}{|1-\langle x, a\rangle|^2}.$$

Since

$$|1-\langle x, a\rangle|^2 \le (1+\|x\|\|a\|)^2 \le (1+r)^2,$$

we deduce $1-\|y\|^2 \ge (1-r^2)^2(1+r)^{-2} = (1-r)^2$, which gives the conclusion for $\rho := \sqrt{1-(1-r)^2}$. □

A mapping $P : X \to Y$ between two Banach spaces X and Y is a (continuous) *m-homogeneous polynomial* if there is a continuous m-linear mapping $L : X \times \cdots \times X \to Y$ so that $P(x) = L(x, \ldots, x)$ for every $x \in X$. We write $\mathcal{P}(^m X)$ for the space of all m-homogeneous polynomials $P : X \to \mathbb{C}$, which endowed with the norm $\|P\| = \sup_{\|x\|\le 1} \|P(x)\|$ is a Banach space.

We refer the reader to [28,29] for general theory of functional analysis and Banach space theory, to [10,12,31] for the theory of holomorphic functions on Banach spaces and to [2,19] for topics related with linear dynamics.

3. Stable and B_X-stable orbits

Given an open set $U \subseteq X$, following [7] a self map $f : U \to U$ is said to have *stable orbits* if for every compact subset K of U there is a compact subset $L \subset U$ such that $f^n(K) \subseteq L$ for every $n \in \mathbb{N}$ or equivalently, if $\overline{\bigcup_{n=0}^{\infty} f^n(K)}$ is compact in U for every compact set $K \subseteq U$. This property was already used in [7] or [4] to characterise power boundedness and/or mean ergodicity of weighted composition operators.

We introduce now a sort of 'bounded type' counterpart. A set $A \subseteq U$ is U-bounded if it is bounded and has positive distance to the boundary of U (whenever $U = X$, the notions of 'bounded' and 'X-bounded' coincide). Then we say that f has *U-stable orbits* if for every U-bounded set $A \subset U$ there is a U-bounded set $L \subset U$ such that $f^n(A) \subseteq L$ for every $n \in \mathbb{N}$ (equivalently, $\bigcup_{n=0}^{\infty} f^n(A)$ is U-bounded for every U-bounded set $A \subseteq U$).

Remark 3.1. The orbit $\{f^n(x) \colon n \in \mathbb{N}\}$ of each point $x \in U$ is relatively compact if f has stable orbits and U-bounded if f has U-stable orbits.

The notion of a function having B_X-stable orbits (we only deal with the case $U = B_X$) seems to be new. However, it is not hard to find functions with this property. In fact, the following well known version of the Schwarz lemma gives immediate examples.

Lemma 3.2. *Let $\varphi : B_X \to B_X$ be holomorphic so that $\varphi(0) = 0$. Then $\|\varphi(x)\| \leq \|x\|$ for every $x \in B_X$.*

Proof. It is enough to apply the classical Schwarz lemma to the family of functions

$$\Big\{[\lambda \in \mathbb{D} \mapsto x^*\big(\varphi(\lambda x/\|x\|)\big)] \colon x^* \in X^*,\ \|x^*\| \leq 1,\ 0 < \|x\| < 1\Big\}. \quad \square$$

Proposition 3.3. *Let $\varphi : B_X \to B_X$ be a holomorphic mapping such that $\varphi(0) = 0$, then φ has B_X-stable orbits.*

Proof. Lemma 3.2 clearly implies $\|\varphi^n(x)\| \leq \|x\|$ for all $n \in \mathbb{N}$ and all $x \in B_X$ and, therefore, for each $0 < r < 1$ we have

$$\varphi^n(rB_X) \subseteq rB_X,$$

for all $n \in \mathbb{N}$. This gives the claim. $\square$

As a consequence, every continuous homogeneous polynomial $P : X \to X$ (in particular every linear operator) with $\|P\| \leq 1$ has B_X-stable orbits.

Example 3.4. If X is either c_0 or ℓ_p with $1 \leq p \leq \infty$ we consider the forward and backward shifts operators $F, B : X \to X$ defined as

$$F(x_1, x_2, \dots) = (0, x_1, x_2, \dots) \text{ and } B(x_1, x_2, \dots) = (x_2, x_3, \dots). \tag{4}$$

Both are linear and clearly have norm less or equal 1, hence have B_X-stable orbits. It is not difficult to see that B has stable orbits (just using the characterisation of compact sets in c_0 or in ℓ_p; see for instance [11, p. 6]). For the forward shift, however, we have that the set

$$\left\{F^n\left(\frac{e_1}{2}\right) : n \in \mathbb{N}\right\} = \left\{\frac{e_n}{2} : n > 1\right\}$$

is not relatively compact and, by Remark 3.1, F does not have stable orbits.

We may also consider the mapping $\phi : B_X \to B_X$ defined as

$$\phi(x_1, x_2, \dots) = \big(\tfrac{x_1+1}{2}, 0, 0, \dots\big).$$

Note that $\phi^n(0) = \big(\sum_{i=1}^{n} \frac{1}{2^i}, 0, 0, \dots\big)$ and, therefore,

$$\lim_{n\to\infty} \|\phi^n(0)\| = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = 1.$$

Hence ϕ has neither stable nor B_{c_0}-stable orbits.

We do not know so far whether or not having stable orbits implies having B_X-stable orbit. However, if $T : X \to X$ is continuous and linear and has stable orbits, then it is power bounded (because $\{T^n x\}_n$ is bounded for every $x \in X$), and a simple computation shows that, then T has X-stable orbits.

3.1. The Hilbert-space case

If H is a Hilbert space, for each $a \in B_H$ the automorphism $\alpha_a : B_H \to B_H$ defined in (2) satisfies $\alpha_a^{-1} = \alpha_a$. Hence

$$\bigcup_{n=0}^{\infty} \alpha_a^n(A) = A \cup \alpha_a(A)\,,$$

for every $A \subseteq B_H$. If A is compact, $\alpha_a(A)$ is again compact, and if A is B_H-bounded, by Lemma 2.1, so also is $\alpha_a(A)$. This shows that α_a has both stable and B_H-stable orbits.

Using these automorphisms, in the case of Hilbert spaces we can extend Proposition 3.3 showing that every holomorphic function with a fixed point has B_H-stable orbits.

Lemma 3.5. *If $\varphi : B_H \to B_H$ has stable orbits (respectively B_H-stable orbits), then the mapping $\psi = \alpha_a \circ \varphi \circ \alpha_a$ has stable orbits (respectively B_H-stable orbits) for every $a \in B_H$.*

Proof. If $K \subseteq B_H$ is compact, then $\alpha_a(K)$ is compact and, having φ stable orbits, we can find a compact set $L \subseteq B_H$ so that $\varphi^n(\alpha_a(K)) \subseteq L$ for each $n \in \mathbb{N}$. Then $\alpha_a(L) \subseteq B_H$ is compact and $\alpha_a\big(\varphi^n(\alpha_a(K))\big) \subseteq \alpha_a(L)$. Since $\psi^n = \alpha_a \circ \varphi^n \circ \alpha_a$ (because $\alpha_a^2 = \mathrm{id}$), ψ has stable orbits.

The argument if φ has B_H-stable orbits is exactly the same, using that by Lemma 2.1 $\alpha_a(A)$ is B_H-bounded for every B_H-bounded A. □

Proposition 3.6. *Let $\varphi : B_H \to B_H$ be a holomorphic mapping with a fixed point. Then φ has B_H-stable orbits.*

Proof. Take $a \in B_H$ with $\varphi(a) = a$. The holomorphic function $\overline{\varphi} = \alpha_a \circ \varphi \circ \alpha_a : B_H \to B_H$ satisfies $\overline{\varphi}(0) = \alpha_a(\varphi(\alpha_a(0))) = \alpha_a(\varphi(a)) = \alpha_a(a) = 0$. Then, by Proposition 3.3 the function $\overline{\varphi}$ has B_H-stable orbits, and Lemma 3.5 gives the conclusion. □

4. The space of holomorphic functions

Given a Banach space X, we define $H(B_X)$ as the space of all holomorphic functions $f : B_X \to \mathbb{C}$, endowed with the topology τ_0 of uniform convergence on compact sets. This is a locally convex Hausdorff space.

Remark 4.1. If $\varphi : B_X \to B_X$ is holomorphic, then the composition operator $C_\varphi : H(B_X) \to H(B_X)$ is clearly well defined (and continuous). On the other hand, if C_φ is well defined, then $x' \circ \varphi$ is holomorphic for every $x' \in X'$ and, by Dunford's theorem (see e.g. [10, Theorem 15.45]), φ is holomorphic. Then, there is no restriction to assume that φ is holomorphic.

Remark 4.2. Suppose that the composition operator C_φ is topologizable. Given any $f \in H(B_X)$, a straightforward computation using (1) with f^m (for $m \in \mathbb{N}$) and taking the m-root shows that for every compact set $K \subseteq B_X$ there is some compact $L \subseteq B_X$ so that

$$\sup_{x\in K}|f(\varphi^n(x))| \le \frac{1}{a_n^{1/m}}\sup_{x\in L}|f(x)|\,.$$

Letting $m \to \infty$ yields

$$\sup_{x\in K}|f(\varphi^n(x))| \le \sup_{x\in L}|f(x)|\,,$$

and in particular C_φ is power bounded. Our aim now is to show that in fact this implication can be reversed and characterised in terms of the symbol.

Following [33] and [9], given a family $\mathcal{F}$ of $\mathbb{C}$-valued holomorphic functions defined on an open set U, the $\mathcal{F}$-hull of $A \subseteq U$ is denoted

$$\widehat{A}_{\mathcal{F}} = \{x \in U : |f(x)| \le \sup_{y\in A}|f(y)|,\ \text{for all } f \in \mathcal{F}\}\,. \tag{5}$$

Stable orbits of the symbol is the property that characterises the power boundedness of the composition operator.

Theorem 4.3. *Let $\varphi : B_X \to B_X$ be holomorphic. The following assertions are equivalent:*

(a) φ has stable orbits on B_X.
(b) $C_\varphi : H(B_X) \to H(B_X)$ is power bounded.
(c) $\left(\frac{1}{n}C_\varphi^n\right)_n$ is equicontinuous in $L(H(B_X))$.
(d) $C_\varphi : H(B_X) \to H(B_X)$ is topologizable.

Proof. (a) ⇒ (b) If φ has stable orbits, given a compact set $K \subseteq B_X$ there is a compact set $L \subseteq B_X$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$. Hence

$$\sup_{x\in K}|C_\varphi^n(f)(x)| = \sup_{x\in K}|f(\varphi^n(x))| \le \sup_{x\in L}|f(x)|,$$

for all $f \in H(B_X)$ and $n \in \mathbb{N}$. So the sequence $(C_\varphi^n)_n$ is equicontinuous and C_φ is power bounded.

(b) ⇒ (c) Suppose now that C_φ is power bounded, then for each compact set $K \subseteq B_X$ we can find $c > 0$ and a compact set $L \subseteq B$ so that

$$\sup_{x\in K}|C_\varphi^n(f)(x)| \le c\sup_{x\in L}|f(x)|$$

for every $f \in H(B_X)$ and $n \in \mathbb{N}$. This obviously implies

$$\sup_{x\in K}\left|\tfrac{1}{n}C_\varphi^n(f)(x)\right| \le c\sup_{x\in L}|f(x)|$$

for every f and n, and $\left(\frac{1}{n}C_\varphi^n\right)_n$ is equicontinuous.

(c) ⇒ (d) follows just taking $a_n = \frac{1}{cn}$ in (1).

(d) ⇒ (a) Fix some compact set $K \subseteq B_X$. Since C_φ is topologizable, we can find some compact set $W \subseteq B_X$, and $(a_n)_{n\in\mathbb{N}}$ with $a_n > 0$ such that,

$$\sup_{x\in K}|f(\varphi^n(x))| \le \frac{1}{a_n}\sup_{x\in W}|f(x)|\,, \tag{6}$$

for all $f \in H(B_X)$ and $n \in \mathbb{N}$. By [31, Corollary 10.7 and Theorem 11.4], the set $L = \widehat{W}_{H(B_X)}$ (recall (5)) is compact and contains W. We see that $\varphi^n(K) \subseteq L$ for every n. Suppose that this is not the case and take $x_0 \in K$ and $n_0 \in \mathbb{N}$ so that $\varphi^{n_0}(x_0) \notin L$. Then there is $f \in H(B_X)$ such that $|f(\varphi^{n_0}(x_0))| > \sup_{y \in W} |f(y)|$, and there exists $m \in \mathbb{N}$ such that

$$\sup_{y \in W} \frac{|f(y)|^m}{|f(\varphi^{n_0}(x_0))|^m} < a_{n_0} ,$$

and this clearly contradicts (6) (taking $g = f^m$). □

Proposition 4.4. *Let $\varphi : B_X \to B_X$ be holomorphic. If $C_\varphi : H(B_X) \to H(B_X)$ is power bounded, then it is also uniformly mean ergodic.*

Proof. From [31, Proposition 9.16] we know that every bounded subset of $H(B_X)$ is relatively compact, therefore $H(B_X)$ is semi-Montel and, in particular, semi-reflexive. Then, as a consequence of [6, p. 917] (see also [22, Proposition 3.1]) we have that every power bounded operator is uniformly mean ergodic. □

5. The space of holomorphic functions of bounded type

If X and Y are Banach spaces and $U \subseteq X$ and $V \subseteq Y$ are open sets, a function $f : U \to V$ is of bounded type if it sends U-bounded sets to V-bounded sets. We consider the space $H_b(B_X)$ of all holomorphic functions $f : B_X \to \mathbb{C}$ of bounded type, endowed with the topology τ_b of uniform convergence on B_X-bounded sets. This is a Fréchet space.

If $\varphi : B_X \to B_X$ is holomorphic of bounded type, then clearly $C_\varphi : H_b(B_X) \to H_b(B_X)$ is well defined. On the other hand, we observe that $X' \subseteq H_b(B_X)$ because every functional is trivially Fréchet differentiable. In fact, X' is a complemented subspace of $H_b(B_X)$, as we explain below in the proof of Proposition 5.4. So, if the composition operator is well defined (as a self map on $H_b(B_X)$), then the argument in Remark 4.1 shows that φ has to be holomorphic. Furthermore, [18, Proposition 3] shows that φ is of bounded type.

Our first goal in this section is to characterise the power boundedness of composition operators on $H_b(B_X)$. As in Remark 4.2, if the composition operator C_φ is topologizable, then for every B_X-bounded set U there is some B_X-bounded set V such that

$$\sup_{x \in U} |f(\varphi^n(x))| \leq \sup_{x \in V} |f(x)| ,$$

and C_φ is power bounded. We go further.

Lemma 5.1. *If U is an absolutely convex open set on a Banach space X, then $\widehat{A}_{H_b(U)}$ is U-bounded for every U-bounded set A.*

Proof. The polar set of A is a subset of X' and it is contained in $H_b(U)$. Then a straightforward computation shows that $\widehat{A}_{H_b(U)}$ is contained in the bipolar of A, which by the Bipolar Theorem coincides with $\overline{\text{co}}(A)$ (the closure of the absolutely convex hull of A). Since U is absolutely convex, [8, Remark, p. 527] gives that $\overline{\text{co}}(A)$ is U-bounded, which completes the proof. □

With exactly the same proof as in Theorem 4.3, replacing 'compact' by 'B_X-bounded' we have the following.

Theorem 5.2. *Let $\varphi : B_X \to B_X$ be a holomorphic mapping. The following assertions are equivalent*

(a) φ *has* B_X*-stable orbits.*
(b) $C_\varphi : H_b(B_X) \to H_b(B_X)$ *is power bounded.*
(c) $\left(\frac{1}{n}C_\varphi^n\right)_n$ *is equicontinuous in* $L(H_b(B_X))$*.*
(d) $C_\varphi : H_b(B_X) \to H_b(B_X)$ *is topologizable.*

We now show that in this case every mean ergodic composition operator is power bounded, and there are power bounded operators that are not mean ergodic.

Proposition 5.3. *Let* $\varphi : B_X \to B_X$ *a holomorphic mapping. If* $C_\varphi : H_b(B_X) \to H_b(B_X)$ *is mean ergodic, then* C_φ *is power bounded.*

Proof. The mean ergodicity immediately gives that the sequence $(\frac{1}{n}C_\varphi^n)$ tends to zero (pointwise), so it is pointwise bounded. Since $H_b(B_X)$ is barrelled (because it is a Fréchet space), it is also equicontinuous on $H_b(B_X)$. This, in view of Theorem 5.2, gives the conclusion. □

We want to find now composition operators that are power bounded but not mean ergodic. The shifts defined in (4) provide us with such examples.

Proposition 5.4. *The composition operators* $C_B : H_b(B_{c_0}) \to H_b(B_{c_0})$ *and* $C_F : H_b(B_{\ell_1}) \to H_b(B_{\ell_1})$ *are power bounded but not mean ergodic.*

Proof. We already noted in Example 3.4 that B has B_{c_0}-stable orbits which, in view of Theorem 5.2, shows that C_B is power bounded.

We now see that C_B is not mean ergodic. We begin by observing that $H_b(B_X)$ contains a complemented copy of X' for every Banach space X. Indeed, given a holomorphic $f : B_X \to \mathbb{C}$, we denote its differential at 0 (that belongs to X') by $df(0)$. Then a simple computation shows that the mappings $P : H_b(B_X) \to X'$ and $J : X' \to H_b(B_X)$ defined by $P(f) = df(0)$ and $J(u) = u|_{B_X}$ give our claim.

We consider now the restriction of C_B to $J(\ell_1)$ (recall that $c_0' = \ell_1$) and we have, for each $u \in \ell_1$ and $x \in c_0$,

$$\begin{aligned}\langle C_B u, x\rangle = u(B(x)) = \langle u, B(x)\rangle = \langle (u_1, u_2, u_3, \dots), (x_2, x_3, x_4, \dots)\rangle \\ = u_1x_2 + u_2x_3 + u_3x_4 + \dots = \langle (0, u_1, u_2, u_3, \dots), (x_1, x_2, x_3, x_4, \dots)\rangle = \langle Fu, x\rangle .\end{aligned}$$

Thus $F = P \circ C_B \circ J$, which is not mean ergodic on ℓ_1 (see, for instance, [6]). This implies that C_B is not mean ergodic on $H_b(B_{c_0})$.

For the forward shift, Example 3.4 showed that F has B_{ℓ_1}-stable orbits. Essentially the same argument as before shows that the restriction of C_F to $\ell_1' = \ell_\infty$ is the backward shift B, which is not mean ergodic. This yields the conclusion. □

We look now for sufficient conditions for a given power bounded composition operator to be mean ergodic (and up to some point, even to reverse the implication in Proposition 5.3). Before we need the following lemma. The argument of the proof is essentially the one in [26, Chapter 2, Theorem 1.1] (see also [6, page 908]); since our setting is slightly different we sketch the proof here for the sake of completeness.

Lemma 5.5. *Let* E *be a locally convex Hausdorff space,* $T \in L(E)$ *be power bounded and* $x \in E$*. If* $y \in E$ *is a* $\sigma(E, E')$*-cluster point of* $(T_{[n]}x)_{n\in\mathbb{N}}$*, then* $\lim_{n\to\infty} T_{[n]}x = y$*.*

Proof. Fix $p \in \Gamma_E$. Since T is power bounded we can find $q \in \Gamma_E$ so that $p(T^n z) \le q(z)$ for every $z \in E$. If y is a $\sigma(E, E')$-cluster point of $(T_{[n]}x)_{n\in\mathbb{N}}$ then it belongs to the $\sigma(E, E')$-closure of the set which (note that

$(T_{[n]}x)_{n\in\mathbb{N}} \subseteq \text{co}(T^n x)_{n\in\mathbb{N}_0})$ is contained in the $\sigma(E,E')$-closure of $\text{co}(T^n x)_{n\in\mathbb{N}_0}$. But as a consequence of the Hahn-Banach theorem, for convex sets the $\sigma(E,E')$-closure coincides with the closure. So, $y \in \overline{\text{co}}(T^n x)_{n\in\mathbb{N}_0}$, and for a given $\varepsilon > 0$ we can find $z \in \text{co}(T^n x)_{n\in\mathbb{N}_0}$ so that $q(z-y) < \varepsilon$, so that we can write

$$p(y - T_{[n]}x) \leq p(y - T_{[n]}z) + p(T_{[n]}z - T_{[n]}x)\,. \tag{7}$$

Note that $z = \sum_{k=0}^{m} \lambda_k T^k x$ for some $0 \leq \lambda_k \leq 1$ with $\sum_{k=0}^{m} \lambda_k = 1$. We define $S = \sum_{k=0}^{m} \lambda_k T^k \in L(E)$. Hence

$$p(T_{[n]}Sx - T_{[n]}x) \leq \sum_{k=0}^{m} \lambda_k p(T_{[n]}T^k x - T_{[n]}x)\,.$$

If $n \geq m \geq k$, we have

$$p(T_{[n]}T^k x - T_{[n]}x) \leq \frac{1}{n}\Big(\sum_{j=0}^{k-1} p(T^j x) + \sum_{j=n}^{n+k-1} p(T^j x)\Big) \leq \frac{2k}{n} q(x). \tag{8}$$

With this we can estimate the second addend in the right-hand term of (7). In order to control the first one it is enough to see that $y = Ty$ since, if this is the case then $y = T_{[n]}y$ and $p(y - T_{[n]}z) \leq q(y-z)$. Given $x' \in E'$, we have

$$\begin{aligned}|\langle y - Ty, x'\rangle| &\leq |\langle y - T_{[m]}x, x'\rangle| + |\langle T_{[m]}x - TT_{[m]}x, x'\rangle| + |\langle Ty - TT_{[m]}x, x'\rangle| \\ &\leq |\langle y - T_{[m]}x, x'\rangle| + |\langle T_{[m]}x - TT_{[m]}x, x'\rangle| + |\langle y - T_{[m]}x, T'x'\rangle|\,.\end{aligned}$$

Since y is a $\sigma(E,E')$-cluster point, we can choose m so that the first and third term are arbitrarily small. On the other hand, (8) implies that $T_{[m]}x - TT_{[m]}x$ tends to 0 as $m \to \infty$ and, then, so also does the second term. This shows that $y = Ty$ and completes the proof. □

We denote by $\mathcal{P}(X)$ the algebra of all continuous polynomials on X (these are finite sums of homogeneous polynomials), and by $\sigma(X,\mathcal{P}(X))$ the coarsest topology making all $P \in \mathcal{P}(X)$ continuous. This is a Hausdorff topology satisfying $\|\cdot\| \succeq \sigma(X,\mathcal{P}(X)) \succeq \sigma(X,X^*)$, and the concepts of (relatively) countably compact subset, (relatively) sequentially compact subset and (relatively) compact subset all agree with respect to this topology [15].

Proposition 5.6. *Let $\varphi : B_X \to B_X$ be holomorphic, having B_X-stable orbits and such that $\varphi(A)$ is relatively $\sigma(X,\mathcal{P}(X))$-compact for every B_X-bounded set A. Then $C_\varphi : H_b(B_X) \to H_b(B_X)$ is mean ergodic.*

Proof. From Theorem 5.2 we have that the composition operator C_φ is power bounded, and the equicontinuity of $(C_\varphi^n)_{n\in\mathbb{N}}$ gives that the set $\{(C_\varphi)_{[n]}(f) : n \in \mathbb{N}\}$ is bounded for every $f \in H_b(B_X)$. Now, by [16, Theorem 2.9] C_φ maps bounded sets of $H_b(B_X)$ into relatively $\sigma(H_b(B_X), H_b(B_X)')$-compact sets, so for every $f \in H_b(B_X)$ the set

$$C_\varphi\big(\{(C_\varphi)_{[n]}(f)\}_n\big) = \Big\{\frac{1}{n}\sum_{k=1}^{n} C_\varphi^k(f) : n \in \mathbb{N}\Big\}$$

is relatively $\sigma(H_b(B_X), H_b(B_X)')$-compact and, therefore it has a $\sigma(H_b(B_X), H_b(B_X)')$-cluster point. Our aim now is to see that $\{(C_\varphi)_{[n]}(f)\}_n$ has a $\sigma(H_b(B_X), H_b(B_X)')$-cluster point which, using Lemma 5.5, implies that the sequence $((C_\varphi)_{[n]}(f))_{n\in\mathbb{N}}$ converges in $H_b(B_X)$ for all $f \in H_b(B_X)$, and C_φ is mean ergodic.

Note that

$$(C_\varphi)_{[n]}(f) = \frac{1}{n}(f - C_\varphi^n(f)) + \frac{1}{n}\sum_{k=1}^{n} C_\varphi^k(f)$$

for every n. The fact that C_φ is power bounded implies that $\left(\frac{1}{n}(\mathrm{id}(f) - C_\varphi^n(f))\right)_{n\in\mathbb{N}}$ tends to 0 as $n \to \infty$, and this gives that $\{(C_\varphi)_{[n]}(f)\}_n$ has a $\sigma(H_b(B_X), H_b(B_X)')$-cluster point, as we wanted. □

Corollary 5.7. *Let X be a Banach space such that every B_X-bounded set is relatively $\sigma(X, \mathcal{P}(X))$-compact. Then $C_\varphi : H_b(B_X) \to H_b(B_X)$ is power bounded if and only if C_φ is mean ergodic.*

An example of a Banach space which satisfies such a property is the Tsirelson space T^*: it is known that T^* is reflexive and the polynomials on T^* are weakly sequentially continuous [12, p. 121]. Hence, any sequence in the unit ball of T^* has a weakly convergent subsequence, which converges in the topology $\sigma(T^*, \mathcal{P}(T^*))$. Since $\sigma(T^*, \mathcal{P}(T^*))$ is angelic [15, p. 150], the unit ball is also relatively $\sigma(T^*, \mathcal{P}(T^*))$-compact.

We find now conditions to ensure that a given composition operator is uniformly mean ergodic. Here C_0 denotes the composition operator defined by the constant function 0 (i.e. $C_0(f) = f(0)$ for every f).

Theorem 5.8. *Let $\varphi : B_X \to B_X$ be holomorphic so that for every $0 < t < 1$ there exists $0 < \rho < t$ such that*

$$\varphi(tB_X) \subseteq \rho B_X\,. \tag{9}$$

Then

$$C_{\varphi^n} \to C_0,$$

in the topology of bounded convergence on $H_b(B_X)$. In particular,

$$(C_\varphi)_{[n]} \to C_0, \tag{10}$$

in the topology of bounded convergence on $H_b(B_X)$ and $C_\varphi : H_b(B_X) \to H_b(B_X)$ is uniformly mean ergodic.

Proof. Fix some $0 < t < 1$. First of all, (9) implies, on the one hand, that $\varphi^n(tB_X) \subset \rho B_X$ for every $n \in \mathbb{N}$ and, on the other hand, that $\varphi(0) = 0$. We can then apply Lemma 3.2 to the function $[x \rightsquigarrow \frac{1}{\rho}\varphi(tx)]$ and get

$$\|\varphi^n(x)\| \le \left(\frac{\rho}{t}\right)^n \|x\|, \tag{11}$$

for every $x \in tB_X$ and $n \in \mathbb{N}$. Now, given $f \in H_b(B_X)$, we obviously have $\|f \circ \varphi^n\|_{tB_X} \le \|f\|_{tB_X}$ for every $n \in \mathbb{N}$. We define $g : B_X \to \mathbb{D}$ by $g(x) = \frac{1}{2\|f\|_{tB_X}}\big(f(\varphi(tx)) - f(0)\big)$. This is clearly holomorphic and satisfies $g(0) = 0$. Then we can apply Lemma 3.2 to g and (11) to obtain

$$\|C_\varphi^n(f) - C_0(f)\|_{tB_X} = \sup_{x\in tB_X} |f(\varphi^n(x)) - f(0)| \le 2\|f\|_{tB_X} \sup_{x\in tB_X} \|\varphi^{n-1}(x)\| \le 2\|f\|_{tB_X} \left(\frac{\rho}{t}\right)^{n-1}.$$

This implies, for every $0 < t < 1$ and every bounded set $A \subseteq H_b(B_X)$,

$$\lim_{n\to\infty} \sup_{f\in A} \sup_{x\in tB_X} |C_{\varphi^n}(f)(x) - f(0)| = 0.$$

Hence, $C_{\varphi^n} \to C_0$ in the topology of bounded convergence. Once we have this, (10) is a straightforward consequence. □

Remark 5.9. If $\varphi : B_X \to B_X$ is holomorphic and satisfies

$$\varphi(B_X) \subseteq rB_X \text{ for some } 0 < r < 1 \text{ and } \varphi(0) = 0\,, \tag{12}$$

then, applying Lemma 3.2 to the function $[x \rightsquigarrow \frac{1}{r}\varphi(x)]$ we get $\|\varphi(x)\| \le r\|x\|$ for every $x \in B_X$, and this implies that φ satisfies (9) with $\rho = tr$.

There are, however, functions satisfying (9) but not (12). To see this just consider the restriction to B_X of any m-homogeneous polynomial (for $m > 1$) $P : X \to X$ with $\|P\| \le 1$. For a fixed $0 < t < 1$ take any $0 < \varepsilon < t - t^m$ and note that

$$\|P(tx)\| \le t^m \|x\|^m \le (t - \varepsilon)\|x\|,$$

for every $x \in B_X$. That is, every homogeneous polynomial with norm ≤ 1 satisfies (9). If $\|P\| = 1$ and attains its norm (that is, there is x_0 with $\|x_0\| = 1$ so that $\|P(x_0)\| = \|P\|$) then

$$\big\|P\big((1 - \tfrac{1}{n})x_0\big)\big\| = \Big(1 - \frac{1}{n}\Big)^m,$$

and there is no $0 < r < 1$ so that $P(B_X) \subseteq rB_X$. For a concrete example of such a polynomial just consider the 2-homogeneous one $P : \ell_2 \to \ell_2$ given by $P\big((x_n)_n\big) = (x_n^2)_n$ (in this case one can take $x_0 = e_1$).

In particular, we have that, if $m > 1$ and P is an m-homogeneous polynomial with $\|P\| \le 1$, then $C_P : H_b(B_X) \to H_b(B_X)$ is uniformly mean ergodic. For $m = 1$, that is, for linear operators, this property does not hold, as Proposition 5.13 shows.

One may also ask if in (12) we can drop the condition on the fixed point and still get (9) just assuming that $\varphi(B_X) \subseteq rB_X$ for some $0 < r < 1$. But this is not the case: fix some $x_0 \in B_X$ and consider the constant function $\varphi(x) = x_0$ for every $x \in B_X$.

5.1. Example of a composition operator which is mean ergodic but not uniformly mean ergodic in $H_b(B_{c_0})$

The following result is well known [25, §39, 4(1), p. 138].

Lemma 5.10. *Let $(T_n)_n$ be a sequence of equicontinuous operators on a locally convex space E. If (T_n) is pointwise convergent to a continuous operator T on some dense set $D \subseteq E$, then $(T_n)_n$ is pointwise convergent to T in E.*

We also need the following property [10, Theorem 15.60].

Theorem 5.11. *For each $m \in \mathbb{N}$, the set $A_m := \{x^\alpha : |\alpha| = m\}$ of monomials generates a dense subspace of $\mathcal{P}(^m c_0)$.*

Remark 5.12. Since B_{c_0} is a balanced set, the polynomials are dense on $H_b(B_{c_0})$. Therefore, by Theorem 5.11 we have that the set

$$\operatorname{span}\{x^\alpha : \alpha \in \mathbb{N}_0^{(\mathbb{N})}\}$$

is dense on $H_b(B_{c_0})$.

Proposition 5.13. *Let $F : B_{c_0} \to B_{c_0}$ be the forward shift. The composition operator $C_F : H_b(B_{c_0}) \to H_b(B_{c_0})$ is mean ergodic but not uniformly mean ergodic.*

Proof. First, we see that C_F is mean ergodic. We follow a similar scheme to that in [3, Theorem 2.2], using that span$\{x^\alpha : \alpha \in \mathbb{N}_0^{(\mathbb{N})}\}$ is a dense subspace of $H_b(B_{c_0})$ (Remark 5.12) together with Lemma 5.10. Since C_F is power bounded on $H_b(B_{c_0})$ (because it has B_{c_0}-stable orbits), $(C_F^n)_n$ is equicontinuous. Therefore, $\big((C_F)_{[n]}\big)_n$ is also equicontinuous on $H_b(B_{c_0})$. Since $C_F(1) = 1 = C_0(1)$ for any constant mapping (this is in fact true for any composition operator), it remains to see that $\big((C_F)_{[n]}(h)\big)_n$ τ_b-converges to $C_0(h)$ for every $h \in A_m$ and $m > 0$ (on these cases $C_0(h) = 0$). For $h(x) = x^\alpha$ with $|\alpha| = m$, we define $n_h = \max\{j \in \mathbb{N} : (\alpha)_j \neq 0\}$ which is a finite number. Observe that $C_F^n(h) = C_F^n(x^\alpha) = (F^n(x))^\alpha = 0$ for all $n \geq n_h$, and the claim follows.

As in the proof of Proposition 5.4, one can see that $P \circ C_F \circ J = B$, where $B : \ell_1 \to \ell_1$ is the backward shift (recall (4)). If C_F were uniformly mean ergodic on $H_b(B_{c_0})$, then $B : \ell_1 \to \ell_1$ would be uniformly mean ergodic, but this is not the case. Indeed, since $B^j x$ tends to 0 in ℓ_1 for all $x \in \ell_1$, the only possible value for the limit projection of $\frac{1}{N}\sum_{j=0}^{N-1} B^j$ is 0. But, for each $N \in \mathbb{N}$, we have

$$\sup_{\|x\|\leq 1} \Big\| \frac{1}{N} \sum_{j=0}^{N-1} B^j(x) \Big\|_{\ell_1} \geq \frac{1}{N} \Big\| \sum_{j=0}^{N-1} B^j(e_N) \Big\|_{\ell_1} = \frac{1}{N} \big\|(1, \overset{(N)}{\ldots}, 1, 0, \ldots)\big\|_{\ell_1} = 1.$$

And it is not true that

$$\lim_{N\to\infty} \Big\| \frac{1}{N} \sum_{j=0}^{N-1} B^j \Big\| = 0. \quad \square$$

5.2. The Hilbert-space case

Let us go back to (12) for a moment. If we only assume $\varphi(B_X) \subseteq rB_X$, the Earle-Hamilton fixed point theorem [13] implies that there exists a unique $a \in B_X$ such that $\varphi(a) = a$. It is then natural to ask if this is enough to ensure that the composition operator is uniformly mean ergodic. If we restrict ourselves to Hilbert spaces H we can say something in this respect. We need the following lemma.

Lemma 5.14. *Let $\varphi : B_H \to B_H$ be holomorphic so that $C_{\varphi^n} \to C_0$ in the topology of bounded convergence of $L(H_b(B_H))$. Then for every $a \in B_H$ the mapping $\overline{\varphi} = \alpha_a \circ \varphi \circ \alpha_a$ satisfies that $C_{\overline{\varphi}^n} \to C_a$ in the topology of bounded convergence of $L(H_b(B_H))$.*

Proof. Since both φ and α_a are of bounded type (see Lemma 2.1), the composition $\alpha_a \circ \varphi \circ \alpha_a$ is of bounded type and $C_{\overline{\varphi}} : H_b(B_H) \to H_b(B_H)$ is well defined. Observe now that $\overline{\varphi}^n = \alpha_a \circ \varphi^n \circ \alpha_a$ for all $n \in \mathbb{N}$ since $\alpha_a^{-1} = \alpha_a$. Then

$$C_{\overline{\varphi}^n} = C_{\alpha_a \circ \varphi^n \circ \alpha_a} = C_{\alpha_a} \circ C_{\varphi^n} \circ C_{\alpha_a} \to C_{\alpha_a} \circ C_0 \circ C_{\alpha_a} = C_{\alpha_a} \circ C_{\alpha_a(0)} = C_{\alpha_a} \circ C_a = C_a. \quad \square$$

Proposition 5.15. *Let $\varphi : B_H \to B_H$ be holomorphic such that*

$$\varphi(B_H) \subseteq rB_H \text{ for some } 0 < r < 1. \tag{13}$$

Then, for the unique $a \in B_H$ such that $\varphi(a) = a$ we have $C_{\varphi^n} \to C_a$ in the topology of bounded convergence of $L(H_b(B_H))$. In particular $(C_\varphi)_{[n]} \to C_a$, and $C_\varphi : H_b(B_H) \to H_b(B_H)$ is uniformly mean ergodic.

Proof. Define $\phi = \alpha_a \circ \varphi \circ \alpha_a : B_H \to B_H$, which clearly satisfies $\phi(0) = 0$. Also,

$$\phi(B_H) = (\alpha_a \circ \varphi \circ \alpha_a)(B_H) = (\alpha_a \circ \varphi)(B_H) \subseteq \alpha_a(rB_H),$$

and using Lemma 2.1 we can find some $0 < \varepsilon < 1$ so that

$$\phi(B_H) \subseteq (1-\varepsilon)B_H .$$

Then ϕ satisfies (12) and, by Theorem 5.8, $C_{\phi^n} \to C_0$. Since $\varphi = \alpha_a \circ \phi \circ \alpha_a$ (because $\alpha_a^{-1} = \alpha_a$), Lemma 5.14 yields the claim. □

Let us consider any analytic self-map $\varphi : B_X \to B_X$ (being X any Banach space) so that $\varphi \circ \varphi = \mathrm{id}$. Then

$$C_\varphi^n = \begin{cases} C_\varphi \text{ if } n \text{ is odd,} \\ C_{\mathrm{id}_{B_X}} = \mathrm{id}_{H_b(B_X)} \text{ if } n \text{ is even,} \end{cases}$$

and for each $k \in \mathbb{N}$ we have

$$(C_\varphi)_{[2k-1]} = \frac{1}{2k-1} \sum_{n=0}^{2k-1} C_\varphi^n = \frac{k}{2k-1}\big(C_\varphi + \mathrm{id}_{H_b(B_X)}\big),$$

and

$$(C_\varphi)_{[2k]} = \frac{1}{2k} \sum_{n=0}^{2k} C_\varphi^n = \frac{1}{2}\big(C_\varphi + \mathrm{id}_{H_b(B_X)}\big) + \frac{1}{2k}\,\mathrm{id}_{H_b(B_X)} .$$

This implies that $\lim_{n\to\infty}(C_\varphi)_{[n]} = \frac{1}{2}\big(C_\varphi + \mathrm{id}_{H_b(B_X)}\big)$ in the topology of bounded convergence of $L(H_b(B_X))$, and $C_\varphi : H_b(B_X) \to H_b(B_X)$ is uniformly mean ergodic. Note that $\alpha_a : B_H \to B_H$ (now H being a Hilbert space) satisfies this condition, so that $C_{\alpha_a} : H_b(B_H) \to H_b(B_H)$ is uniformly mean ergodic. However, α_a does not satisfy neither (9) nor (13).

6. The space of bounded holomorphic functions

We consider now the space $H^\infty(B_X)$ of all holomorphic functions $f : B_X \to \mathbb{C}$ that are bounded. With the norm $\|f\|_\infty = \sup_{x\in B_X} |f(x)|$ it becomes a Banach space. We look at composition operators $C_\varphi : H^\infty(B_X) \to H^\infty(B_X)$. If $\varphi : B_X \to B_X$, then

$$\|C_\varphi^n(f)\|_\infty = \sup_{x\in B_X} |C_\varphi^n(f)(x)| = \sup_{x\in B_X} |f(\varphi^n(x))| \le \sup_{x\in B_X} |f(x)| = \|f\|_\infty ,$$

and $\|C_\varphi^n\| \le 1$ for all $n \in \mathbb{N}$. Hence every C_φ that is well defined on $H^\infty(B_X)$ is power bounded. Since $(X', \|\cdot\|) = (X', \tau_b)$, the dual space X' is also complemented in $H^\infty(B_X)$, and the same arguments as in Proposition 5.4 give examples of composition operators $C_\varphi : H^\infty(B_X) \to H^\infty(B_X)$ which are not mean ergodic. However, X' is in general not complemented in $H(B_X)$ since $(X', \|\cdot\|) \neq (X', \tau_0)$ and these arguments do not work for $H(B_X)$.

We give now conditions on the symbol to define a uniformly mean ergodic composition operator on $H^\infty(B_X)$.

Proposition 6.1. *Let $\varphi : B_X \to B_X$ be holomorphic such that $\varphi(B_X) \subseteq rB_X$ for some $0 < r < 1$ and $\varphi(0) = 0$. Then*

$$C_{\varphi^n} \to C_0,$$

in the norm operator topology of $L(H^\infty(B_X))$. In particular, $(C_\varphi)_{[n]} \to C_0$, and $C_\varphi : H^\infty(B_X) \to H^\infty(B_X)$ is uniformly mean ergodic.

Proof. Take some $f \in H^\infty(B_X)$ with $\|f\|_\infty \leq 1$. Defining $g : B_X \to \mathbb{D}$ by $g(x) = \frac{1}{2}(f(x) - f(0))$ and using Lemma 3.2 we get

$$|f(x) - f(0)| \leq 2\|x\|$$

for every $x \in B_X$. Proceeding as in (11), we get that $\|\varphi^n(x)\| \leq r^n\|x\|$ for every $x \in B_X$ and $n \in \mathbb{N}$. This yields

$$\big|f\big(\varphi^n(x)\big) - f(0)\big| \leq 2\|\varphi^n(x)\| \leq 2r^n\|x\| .$$

Therefore

$$\|C_\varphi^n - C_0\|_{L(H^\infty(B))} = \sup_{\|f\|_\infty \leq 1} \sup_{x \in B_X} \big|f\big(\varphi^n(x)\big) - f(0)\big| \leq 2 \sup_{x \in B_X} \|\varphi^n(x)\| \leq 2r^n ,$$

which gives the claim. □

We observe that the hypothesis in Proposition 6.1 is exactly the same one as (12) in Remark 5.9. One can ask if the result also holds assuming instead (9). This is not the case. We already saw in Remark 5.9 that the mapping $P : B_{\ell_2} \to B_{\ell_2}$ given by $P\big((x_n)_n\big) = (x_n^2)_n$ satisfies (9). Then, by Theorem 5.8 the Cesàro means of C_P converge to C_0. Hence, $C_P : H_b(B_{\ell_2}) \to H_b(B_{\ell_2})$ is uniformly mean ergodic.

However, the operator $C_P : H^\infty(B_{\ell_2}) \to H^\infty(B_{\ell_2})$ is not even mean ergodic. Notice that $H^\infty(B_{\ell_2}) \subseteq H_b(B_{\ell_2})$ and τ_b is weaker than the norm topology. Then if C_P were mean ergodic, $\big((C_P)_{[n]}(f)\big)_n$ should converge in norm to $C_0(f)$ for every $f \in H^\infty(B_{\ell_2})$. Take $f \in H^\infty(B_{\ell_2})$ given by $f\big((x_n)_n\big) = x_1$ and consider $z_m = (1 - \frac{1}{m})e_1 \in B_{\ell_2}$ for each $m \in \mathbb{N}$. Then $P^k(z_m) = (1 - \frac{1}{m})^{2^k} e_1$ for every k and

$$(C_P)_{[n]}(f)(z_m) - C_0(f)(z_m) = \frac{1}{n}\sum_{k=0}^{n-1} f\big((1 - \frac{1}{m})^{2^k} e_1\big) - f(0) = \frac{1}{n}\sum_{k=0}^{n-1} \big(1 - \frac{1}{m}\big)^{2^k} .$$

Thus

$$\sup_{x \in B_{\ell_2}} |(C_P)_{[n]}(f)(x) - C_0(f)(x)| \geq \sup_{m \in \mathbb{N}} \frac{1}{n}\sum_{k=0}^{n-1} \big(1 - \frac{1}{m}\big)^{2^k} = 1 ,$$

and $\big((C_P)_{[n]}(f)\big)_n$ does not converge in norm to $C_0(f)$. This finally shows that $C_P : H^\infty(B_{\ell_2}) \to H^\infty(B_{\ell_2})$ is not mean ergodic.

The same argument as in Lemma 5.14 and Proposition 5.15 shows the following.

Proposition 6.2. *Let $\varphi : B_H \to B_H$ be analytic such that $\varphi(B_H) \subseteq rB_H$ for some $0 < r < 1$. Then, for the unique $a \in B$ such that $\varphi(a) = a$ we have that $C_{\varphi^n} \to C_a$ in the norm of $L(H^\infty(B_H))$. In particular $(C_\varphi)_{[n]} \to C_a$ and C_φ is uniformly mean ergodic.*

We have formulated our results for the open unit ball of a Banach (or Hilbert) space, mainly with the purpose of simplicity and to give a uniform presentation of our results. Our proofs, however, transfer with no extra effort, to some other, more general settings. Let us briefly point out how.

- A set U is said to be holomorphically convex if $\widehat{K}_{H(U)}$ (recall in (5)) is compact for every compact set $K \subseteq U$ (see [31, Definition 11.3]). The proof of Theorem 4.3 transfers word by word if B_X is replaced by a holomorphically convex set U.
- The proof of Proposition 3.3 works exactly in the same way if B_X is replaced by any absolutely convex open set U. Then, Theorem 5.2, as well as Propositions 5.3, 5.6 and Corollary 5.7 also hold for arbitrary absolutely convex open sets (note that [16, Theorem 2.9] also holds in this case).
- The key element in the proofs of Propositions 3.6, 5.15 and 6.2 (stated for the open unit ball of a Hilbert space) is the existence of a family of biholomorphic automorphisms on the ball (as in (2)) satisfying (3). Hilbert spaces are not the only examples of such a situation. In every C^*-algebra, for example, also such a family of automorphisms can be defined. In fact, there is a wider class of Banach spaces, known as JB^*-triples, that also have this property: if X is a JB^*-triple, then there is a family of biholomorphic automorphisms $\{\alpha_a\}_{a \in B_X}$ on B_X satisfying $\alpha_a(0) = a$, $\alpha_a(a) = 0$, $\alpha_a^{-1} = \alpha_a$. The class of JB^*-triples includes Hilbert spaces and C^*-algebras, but also wider classes such as J^*-algebras (closed subspaces of the space of operators between two Hilbert spaces $L(H_1, H_2)$ which are closed under $T \rightsquigarrow TT^*T$, being T^* the adjoint of T); the interested reader may find more information on the subject in [20,30]. Moreover, these automorphisms satisfy the corresponding analogue of Lemma 2.1 [27, Lemma 1]. So, the aforementioned results remain valid if B_H is replaced by the open unit ball B_X of a JB^*-triple (in particular a C^*-algebra) X.

Finally, we observe that some questions remain open. The first one is whether or not (13) implies that the composition operator C_φ is uniformly mean ergodic (that is, Proposition 5.15 extends to arbitrary Banach spaces). It would also be interesting to find examples of the following situations:

(a) A composition operator on $H(B_X)$ which is mean ergodic but not uniformly mean ergodic.
(b) A composition operator on $H(B_X)$ which is mean ergodic but not power bounded.
(c) A composition operator on $H^\infty(B_X)$ which is mean ergodic but not uniformly mean ergodic.

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References

[1] R. Aron, P. Galindo, M. Lindström, Compact homomorphisms between algebras of analytic functions, Stud. Math. 123 (3) (1997) 235–247.
[2] F. Bayart, E. Matheron, Dynamics of Linear Operators, Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, Cambridge, 2009.
[3] M.J. Beltrán-Meneu, M.C. Gómez-Collado, E. Jordá, D. Jornet, Mean ergodic composition operators on Banach spaces of holomorphic functions, J. Funct. Anal. 270 (12) (2016) 4369–4385.
[4] M.J. Beltrán-Meneu, M.C. Gómez-Collado, E. Jordá, D. Jornet, Mean ergodicity of weighted composition operators on spaces of holomorphic functions, J. Math. Anal. Appl. 444 (2) (2016) 1640–1651.
[5] J. Bonet, Topologizable operators on locally convex spaces, in: Topological Algebras and Applications, in: Contemp. Math., vol. 427, Amer. Math. Soc., Providence, RI, 2007, pp. 103–108.
[6] J. Bonet, B. de Pagter, W.J. Ricker, Mean ergodic operators and reflexive Fréchet lattices, Proc. R. Soc. Edinb., Sect. A 141 (5) (2011) 897–920.
[7] J. Bonet, P. Domański, A note on mean ergodic composition operators on spaces of holomorphic functions, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. RACSAM 105 (2) (2011) 389–396.
[8] P.A. Burlandy, L.A. Moraes, The spectrum of an algebra of weakly continuous holomorphic mappings, Indag. Math. (N. S.) 11 (4) (2000) 525–532.

[9] D. Carando, S. Muro, Envelopes of holomorphy and extension of functions of bounded type, Adv. Math. 229 (3) (2012) 2098–2121.
[10] A. Defant, D. García, M. Maestre, P. Sevilla-Peris, Dirichlet Series and Holomorphic Functions in High Dimensions, New Mathematical Monographs, vol. 37, Cambridge University Press, Cambridge, 2019.
[11] J. Diestel, Sequences and Series in Banach Spaces, Graduate Texts in Mathematics, vol. 92, Springer-Verlag, New York, 1984.
[12] S. Dineen, Complex Analysis on Infinite-Dimensional Spaces, Springer Monographs in Mathematics, Springer-Verlag, London, Ltd., London, 1999.
[13] C.J. Earle, R.S. Hamilton, A fixed point theorem for holomorphic mappings, in: Global Analysis, Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968, Amer. Math. Soc., Providence, R.I., 1970, pp. 61–65.
[14] P. Galindo, T.W. Gamelin, M. Lindström, Fredholm composition operators on algebras of analytic functions on Banach spaces, J. Funct. Anal. 258 (5) (2010) 1504–1512.
[15] P. Galindo, M. Lindström, R. Ryan, Weakly compact composition operators between algebras of bounded analytic functions, Proc. Am. Math. Soc. 128 (1) (2000) 149–155.
[16] P. Galindo, L. Lourenço, L.A. Moraes, Compact and weakly compact homomorphisms on Fréchet algebras of holomorphic functions, Math. Nachr. 236 (2002) 109–118.
[17] D. García, M. Maestre, P. Sevilla-Peris, Composition operators between weighted spaces of holomorphic functions on Banach spaces, Ann. Acad. Sci. Fenn., Math. 29 (1) (2004) 81–98.
[18] M. González, J.M. Gutiérrez, Schauder type theorems for differentiable and holomorphic mappings, Monatshefte Math. 122 (4) (1996) 325–343.
[19] K.-G. Grosse-Erdmann, A. Peris Manguillot, Linear Chaos, Universitext, Springer, London, 2011.
[20] L.A. Harris, A generalization of C^*-algebras, Proc. Lond. Math. Soc. (3) 42 (2) (1981) 331–361.
[21] E. Jordá, A. Rodríguez-Arenas, Ergodic properties of composition operators on Banach spaces of analytic functions, J. Math. Anal. Appl. 486 (1) (2020) 123891.
[22] D. Jornet, D. Santacreu, P. Sevilla-Peris, Mean ergodic composition operators in spaces of homogeneous polynomials, J. Math. Anal. Appl. 483 (1) (2020) 123582.
[23] T. Kalmes, Power bounded weighted composition operators on function spaces defined by local properties, J. Math. Anal. Appl. 471 (1–2) (2019) 211–238.
[24] H. Keshavarzi, Mean ergodic composition operators on $H^\infty(\mathbb{B}_n)$, arXiv:2010.06938, 2020.
[25] G. Köthe, Topological Vector Spaces. II, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Science), vol. 237, Springer-Verlag, New York-Berlin, 1979.
[26] U. Krengel, Ergodic Theorems, De Gruyter Studies in Mathematics, vol. 6, Walter de Gruyter & Co., Berlin, 1985, With a supplement by Antoine Brunel.
[27] M. Mackey, P. Sevilla-Peris, J.A. Vallejo, Composition operators on weighted spaces of holomorphic functions on JB^*-triples, Lett. Math. Phys. 76 (1) (2006) 19–26.
[28] R.E. Megginson, An Introduction to Banach Space Theory, Graduate Texts in Mathematics, vol. 183, Springer-Verlag, New York, 1998.
[29] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press, Oxford University Press, New York, 1997, Translated from the German by M.S. Ramanujan and revised by the authors.
[30] P. Mellon, Holomorphic invariance on bounded symmetric domains, J. Reine Angew. Math. 523 (2000) 199–223.
[31] J. Mujica, Complex Analysis in Banach Spaces, North-Holland Mathematics Studies, vol. 120, North-Holland Publishing Co., Amsterdam, 1986.
[32] A. Renaud, Quelques propriétés des applications analytiques d'une boule de dimension infinie dans une autre, Bull. Sci. Math. (2) 97 (1974) 129–159, 1973.
[33] D.M. Vieira, Spectra of algebras of holomorphic functions of bounded type, Indag. Math. (N. S.) 18 (2) (2007) 269–279.
[34] W. Żelazko, Operator algebras on locally convex spaces, in: Topological Algebras and Applications, in: Contemp. Math., vol. 427, Amer. Math. Soc., Providence, RI, 2007, pp. 431–442.