Noname manuscript No.
(will be inserted by the editor)

On totally nonpositive matrices associated with a triple negatively realizable

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Received: date / Accepted: date

Abstract Let $A \in \mathbb{R}^{n\times n}$ be a totally nonpositive matrix (t.n.p.) with rank r and principal rank p, that is, every minor of A is nonpositive and p is the size of the largest invertible principal submatrix of A. We introduce that a triple (n, r, p) will be called negatively realizable if there exists a t.n.p. matrix A of order n and such that its rank is r and its principal rank is p.

In this work we extend the results obtained for irreducible totally nonnegative matrices given in [9] to t.n.p. matrices. For that, we consider the sequence of the first p-indices of A and study the linear dependence relations between their rows and columns. These relations allow us to construct t.n.p. matrices associated with a triple (n, r, p) negatively realizable and a specific sequence of the first p-indices.

Keywords Totally nonpositive matrix · Principal rank · Triple negatively realizable · Linear Algebra

Mathematics Subject Classification (2010) 15A03 · 15A15 · 65F40

This research was supported by the Ministerio de Economía y Competividad under the Spanish DGI grant MTM2017-85669-P-AR.

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1 Introduction

A matrix $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ is called *totally nonpositive (negative)* if all its minors are nonpositive (negative) and it is abbreviated as t.n.p. (t.n.) see, for instance, [4–8,13,15,16,19]. These matrices can be considered as a generalization of the partially negative matrices, that is, matrices with all its principal minors negative. The partially negative matrices are called N-matrices in economic models [2,17]. N-matrices have similar applications and properties to P-matrices, which are matrices with all its principal minors positive.

If, instead, all minors of a matrix are nonnegative (positive) the matrix is called *totally nonnegative (totally positive)* and it is abbreviated as TN (TP). These classes of matrices have been studied by several authors [1,10–12,14,18] obtaining properties, the Jordan structure and characterizations by applying the Gaussian or Neville elimination with applications in algebra, geometry differential equations, economics and other fields.

There exist some relations between t.n.p. and TN matrices (or t.n. and TP matrices). Concretely, in [13] a variety of open problems and unresolved issues related to these classes of matrices are introduced. For instance, if a TN matrix can be written as the product of two t.n.p. matrices. In [9], it is proved that there exists a TN matrix associated with a realizable triple (n, r, p) and a given sequence of the first p-indices. In order to study the relation between TN matrices as the product of t.n.p. matrices, in this work we present an algorithm to construct t.n.p. matrices associated with a triple negatively realizable and a given sequence of its first p-indices.

Regarding to the nonsingular t.n.p. matrices we recall that all its entries are negative except for $-a_{11} \leq 0$ and $-a_{nn} \leq 0$. If $-a_{11} < 0$, this class of matrices has been characterized using their LDU factorization in [4] given a criteria to determine if A is t.n.p. and reducing the number of minors that are checked to decide the total nonpositivity of A. If $-a_{11} = 0$ but $-a_{nn} < 0$, we obtain a UDL factorization of A by permutation similarity. When the nonsingular t.n.p. matrix A has the entries $-a_{11} = -a_{nn} = 0$, a characterization in terms of the signs of minors with consecutive initial rows or consecutive initial columns was obtained in [15]. Moreover, in [7] A is characterized in terms of a quasi-LDU factorization, that is, a $\tilde{L}DU$ factorization where $\tilde{L}$ is a lower block triangular matrix, U is a unit upper triangular TN matrix and D is a diagonal matrix. For rectangular t.n.p. matrices, there are characterizations in [6] when $-a_{11} < 0$, and in [8] when $-a_{11} = 0$, using the full rank factorization LDU or $\tilde{L}DU$, respectively. By [8, pp. 62] we consider from now on and without loss of generality that we work with matrices which have nonzero rows and nonzero columns.

We recall that the *rank* of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by rank$(A)$, is the size of the largest invertible square submatrix of A, and its *principal rank*, denoted by p-rank(A), is the size of the largest invertible principal submatrix of A. For $n \geq 2$, A is *irreducible* if there is not a permutation matrix P such that $PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$, where O is an $(n-r) \times r$ zero matrix $(1 \le r \le n-1)$. If $n = 1$, $A = (a)$ is irreducible when $a \neq 0$.

Proposition 1 *Every t.n.p. matrix without zero rows and columns is irreducible.*

Proof Let $A = (-a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular t.n.p. matrix. It is well-known, see [4], that the entries of A are $-a_{ij} < 0$, except for $-a_{11} \le 0$ and $-a_{nn} \le 0$. In this case the result follows.

If A is a singular t.n.p. matrix with $-a_{11} < 0$, then $-a_{1r} < 0$ and $-a_{r1} < 0$ for $r = 2, \ldots, n$ because otherwise A would have a zero row or column which is a contradiction. If there exists $-a_{ij} = 0$, for $i, j = 2, 3, \ldots, n$, since A is t.n.p. then $-a_{st} = 0$ for $s = i, i+1, \ldots, n$ and $t = j, j+1, \ldots, n$. Hence,

$$A = \begin{array}{c} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & O \end{bmatrix} \begin{array}{c} i-1 \\ n-(i-1) \end{array} \\ \begin{array}{cc} j-1 & n-(j-1) \end{array} \end{array}$$

and the result follows.

Now, suppose that $-a_{11} = 0$. If $-a_{1j}$ is the first nonzero entry in the first row from the left, then the columns $2, 3, \ldots, j-1$ are a linear combination of the first one, and if $-a_{i1}$ is the first nonzero entry in the first column from the top, then the rows $2, 3, \ldots, i-1$ are a linear combination of the first one. Analogously, if $-a_{nn} = 0$ and $-a_{tn}$ is the first nonzero entry in the last column from the bottom, then the rows $t+1, t+2, \ldots, n-1$ are a linear combination of the last one and if $-a_{ns}$ is the first nonzero entry in the last row from the right, then the columns $s+1, s+2, \ldots, n-1$ are a linear combination of the last one. Since A has not zero rows and columns, it has the following structure by blocks:

$$\left[\begin{array}{ccc|ccc|ccc}
0 & \cdots & 0 & -a_{1j} & \cdots & -a_{1s} & -a_{1s+1} & \cdots & -a_{1n} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & -a_{i-1j} & \cdots & -a_{i-1s} & -a_{i-1s+1} & \cdots & -a_{i-1n} \\
\hline
-a_{i1} & \cdots & -a_{ij-1} & -a_{ij} & \cdots & -a_{is} & -a_{is+1} & \cdots & -a_{in} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
-a_{t1} & \cdots & -a_{tj-1} & -a_{tj} & \cdots & -a_{ts} & -a_{ts+1} & \cdots & -a_{tn} \\
\hline
-a_{t+11} & \cdots & -a_{t+1j-1} & -a_{t+1j} & \cdots & -a_{t+1s} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
-a_{n1} & \cdots & -a_{nj-1} & -a_{nj} & \cdots & -a_{ns} & 0 & \cdots & 0
\end{array}\right].$$

In this case, as its associated directed graph is strongly connected, A is irreducible. □

In Section 2 we study some linear dependence relations between columns and rows of a t.n.p. matrix $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ with p-rank$(A) = p$ and we extend known results for IrTN matrices given in [9]. For this, we will distinct between t.n.p. matrices with $-a_{11} < 0$ and t.n.p. matrices with $-a_{11} = 0$, $-a_{12} < 0$ and $-a_{21} < 0$.

Definition 1 Let $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ be a t.n.p matrix. If $-a_{11} < 0$, then A is called a *type-I* t.n.p. matrix. Otherwise, if $-a_{11} = 0$, A is called a *type-II* t.n.p. matrix if $-a_{12} < 0$ and $-a_{21} < 0$.

The definition of the sequence of the first p-indices of a matrix was given in [9, Definition 1]. Now, we introduce that definition for t.n.p. matrices. We follow the notation of [1], that is, for $p \in \{1, 2, \ldots, n\}$, $\mathcal{Q}_{p,n}$ denotes the totality of strictly increasing sequences of p integers chosen from $\{1, 2, \ldots, n\}$. If A is an $m \times n$ matrix, $\alpha \in \mathcal{Q}_{k,m}$, $\beta \in \mathcal{Q}_{l,n}$ then $A[\alpha|\beta]$ is by definition the $k \times l$ submatrix of A lying in the rows numbered by α and columns numbered by β. Besides $A[\alpha] := A[\alpha|\alpha]$. Furthermore, we denote by α_i the i-th component of α, for $i = 1, 2, \ldots, k$.

Definition 2 Let $A \in \mathbb{R}^{n\times n}$ be a t.n.p. matrix with p-rank$(A) = p$. If A is a type-I t.n.p. matrix, we say that the sequence of integers $\alpha = \{1, i_2, \ldots, i_p\} \in \mathcal{Q}_{p,n}$ is the *sequence of the first p-indices of* A if for $j = 2, \ldots, p$ we have

$$\begin{aligned} &\det(A[1, i_2, \ldots, i_{j-1}, i_j]) < 0, \\ &\det(A[1, i_2, \ldots, i_{j-1}, t]) = 0, \quad i_{j-1} < t < i_j. \end{aligned} \tag{1}$$

If A is a type-II t.n.p. matrix, we say that the sequence of integers $\alpha = \{1, 2, i_3, \ldots, i_p\} \in \mathcal{Q}_{p,n}$ is the *sequence of the first p-indices of* A if for $j = 3, \ldots, p$ we have (1) with $i_2 = 2$.

By Definition 2, $\alpha = \{1\}$ can be the sequence of the first p-indices of a type-I t.n.p. matrix A with p-rank$(A) = 1$. For the type-II t.n.p. matrices, the sequence of the first p-indices includes, at least, the indices $\{1, 2\}$ and therefore, does not exist a type-II t.n.p. matrix A with p-rank$(A) = 1$.

Next definition is an extension of [9, Definition 2] on a realizable triple (n, r, p) associated with irreducible TN matrices. From now on, we consider that $i_2 = 2$ is the second component of the sequence of the first p-indices of a type-II t.n.p. matrix.

Definition 3 A triple (n, r, p) is called $(1, i_2, \ldots, i_p)$-negatively realizable of the type-I (type-II) if there exists a type-I (type-II) t.n.p. matrix $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ with rank$(A) = r$, p-rank$(A) = p$, and $\{1, i_2, \ldots, i_p\}$ $(i_2 = 2)$ as the sequence of its first p-indices.

If a matrix A satisfies the conditions of Definition 3, then we say that A is a matrix associated with the triple (n, r, p) $(1, i_2, \ldots, i_p)$-negatively realizable of the type-I (type-II).

In Section 3.1 we construct Procedure 1 to obtain an upper block echelon TN matrix $V \in \mathbb{R}^{n\times n}$ with rank$(V) = r$, p-rank$(V) = p$ and the sequence of its first p-indices given by $\{1, i_2, \ldots, i_p\}$. In Section 3.2, we use the known results on t.n.p matrices and study the conditions so that a product of matrices $A = LDV$ will be a type-I (type-II) t.n.p. matrix associated with a triple $(n, r, p)(1, i_2, \ldots, i_p)$-negatively realizable, where $L \in \mathbb{R}^{n\times n}$ is a nonsingular lower triangular TN matrix for type-I (a lower block triangular matrix for type-II), D is a nonsingular diagonal matrix and V is obtained from Procedure 1. Finally in the Appendix, we present Algorithm 2 to calculate V, and Algorithms 3 and 4 to obtain type-I and type-II t.n.p. matrices, respectively.

2 Linear dependence relations between columns and rows of t.n.p. matrices

In this section we consider the sequence of the first p-indices of A to study the linear dependence relations between the rows and columns indexed by the sequence of the first p-indices of A and the remaining of its rows and columns.

Theorem 1 *Let $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ be a type-I t.n.p. matrix without zero rows or columns. Applying row elementary transformations we obtain that A is equivalent to the upper block TN matrix $\bar{U}$ given in this form,*

$$\bar{U} = \begin{bmatrix} \bar{u}_{11} & \bar{U}_{12} \\ 0 & \bar{U}_{22} \end{bmatrix}. \tag{2}$$

Proof Let A be the t.n.p. matrix with $-a_{11} < 0$,

$$A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1,n-1} & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2,n-1} & -a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ -a_{n-1,1} & -a_{n-1,2} & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -a_{n,1} & -a_{n,2} & \cdots & -a_{n,n-1} & -a_{nn} \end{bmatrix}.$$

Applying the first iteration of the Gaussian elimination method, we obtain

$$A_1 = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1,n} \\ 0 & \frac{1}{-a_{11}} det(A[1,2]) & \cdots & \frac{1}{-a_{11}} det(A[1,2|1,n]) \\ \vdots & \vdots & & \vdots \\ 0 & \frac{1}{-a_{11}} det(A[1,n-1|1,2]) & \cdots & \frac{1}{-a_{11}} det(A[1,n-1|1,n]) \\ 0 & \frac{1}{-a_{11}} det(A[1,n|1,2]) & \cdots & \frac{1}{-a_{11}} det(A[1,n]) \end{bmatrix}.$$

If we denote $\bar{u}_{ij} = \frac{1}{-a_{11}} det(A[1,i|1,j]) \geq 0$, for $i, j = 2, 3, \ldots, n$, then

$$A_1 = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & \bar{u}_{22} & \cdots & \bar{u}_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & \bar{u}_{n-1,2} & \cdots & \bar{u}_{n-1,n} \\ 0 & \bar{u}_{n2} & \cdots & \bar{u}_{nn} \end{bmatrix},$$

and multiplying by (-1) the first row of A_1 we obtain

$$\bar{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \bar{u}_{22} & \cdots & \bar{u}_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & \bar{u}_{n-1,2} & \cdots & \bar{u}_{n-1,n} \\ 0 & \bar{u}_{n2} & \cdots & \bar{u}_{nn} \end{bmatrix} = \begin{bmatrix} \bar{u}_{11} & \bar{U}_{12} \\ 0 & \bar{U}_{22} \end{bmatrix}.$$

Note that $\operatorname{rank}(\bar{U}) = \operatorname{rank}(A)$ because we have only used elementary transformations. Furthermore,

1) $\bar{u}_{1j} = a_{1j} > 0$ for $j = 1, 2, \ldots, n$. Moreover,
$\bar{u}_{ij} \geq 0$ for $i = 2, 3, \ldots, n$, $j = 1, 2, \ldots, n$.

2) $\begin{cases} \forall \gamma,\ \beta, \text{ such that } \operatorname{card}(\gamma) = \operatorname{card}(\beta) - 1 \leq n - 1, \\ \det(\bar{U}[1, \gamma|\beta]) = -\det(A_1[1, \gamma|\beta]) = -\det(A[1, \gamma|\beta]) \geq 0. \end{cases}$

3) $\begin{cases} \forall \gamma,\ \beta \text{ such that } \gamma_1, \beta_1 > 1, \text{and } \operatorname{card}(\gamma) = \operatorname{card}(\beta) \leq n - 1, \\ \det(\bar{U}[\gamma|\beta]) = \dfrac{1}{a_{11}} \det(\bar{U}[1, \gamma|1, \beta]) = \\ -\dfrac{1}{a_{11}} \det(A_1[1, \gamma|1, \beta]) = -\dfrac{1}{a_{11}} \det(A[1, \gamma|1, \beta]) \geq 0, \end{cases}$

therefore $\bar{U}$ is a TN matrix. □

Propositions 2 and 3 show the structure of the matrix $\bar{U}$ given in (2).

Proposition 2 *Let $\bar{U} \in \mathbb{R}^{n \times n}$ be the TN matrix given in (2). If $\bar{u}_{22} > 0$, then the submatrix $\bar{U}_{22}$ is irreducible.*

Proof Since $\bar{u}_{22} > 0$, we have:

1. $\bar{u}_{2j} > 0$, for $j = 3, 4, \ldots, n$. If not, there would be $\bar{u}_{2j} = 0$, with $3 \leq j \leq n$, and then

$$\det(\bar{U}[1, 2|2, j]) = \det\left(\begin{bmatrix} \bar{u}_{12} & \bar{u}_{1j} \\ \bar{u}_{22} & 0 \end{bmatrix}\right) = -\bar{u}_{22}\bar{u}_{1j} < 0,$$

which is a contradiction because $\bar{U}$ is a TN matrix.

2. $\bar{u}_{i2} > 0$ for $i = 3, 4, \ldots, n$. In other case, if there exists $\bar{u}_{i2} = 0$, with $3 \leq i \leq n$, we have

$$A[1,\, 2,\, i|1,\, 2] = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \\ -\alpha a_{11} & -\alpha a_{12} \end{bmatrix}, \quad \alpha > 0,$$

then $\det(A[2,\, i|1,\, 2]) = \det \begin{bmatrix} -a_{21} & -a_{22} \\ -\alpha a_{11} & -\alpha a_{12} \end{bmatrix} > 0$, which is a contradiction because A is a t.n.p. matrix.

From these results and taking into account that $\bar{U}$ is TN we deduce that $\bar{u}_{rs} > 0$ for $r, s = 3, 4, \ldots, n$. Thus, all entries of the submatrix $\bar{U}_{22}$ are positive and by [10, Lemma 2.2] the TN submatrix $\bar{U}_{22}$ is irreducible. □

Proposition 3 *Let $\bar{U} \in \mathbb{R}^{n\times n}$ be the TN matrix given in (2). If $\bar{u}_{22} = 0$ then, $\bar{U}_{22} = O_{(n-1)\times(n-1)}$, or (2) has the following form,*

$$\bar{U} = \begin{bmatrix} \bar{u}_{11} & \bar{U}_{12} & \bar{U}_{13} \\ O & O & O \\ O & O & \bar{U}_{33} \end{bmatrix}, \tag{3}$$

where all entries of the submatrix $\bar{U}_{33}$ are positive.

Proof Since $\bar{u}_{22} = 0$, using the fact that $\bar{U}$ is a TN matrix, we obtain that $\bar{u}_{2j} = 0$ (or $\bar{u}_{j2} = 0$) for $j = 3, 4, \ldots n$, that is, the second row (or column) of A is a linear combination of the first row (or column) of A, respectively. If the remaining elements of $\bar{U}_{22}$ are equal to zero, then $\bar{U}_{22} = O_{(n-1)\times(n-1)}$.

In another case, suppose that $\bar{u}_{ij}$ is the first nonzero entry of $\bar{U}$ from the top and from the left, $2 \leq i, j \leq n$, $(i, j) \neq (2, 2)$. Reasoning as in Proposition 2 we obtain that all entries of the submatrix $\bar{U}[i, i+1, \ldots, n|j, j+1, \ldots, n]$ are positive. That is, $\bar{U}$ has the following structure,

$$\bar{U} = \begin{bmatrix} \bar{u}_{11} & \bar{u}_{12} & \cdots & \bar{u}_{1j-1} & \bar{u}_{1j} & \cdots & \bar{u}_{1n-1} & \bar{u}_{1n} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \bar{u}_{ij} & \cdots & \bar{u}_{in-1} & \bar{u}_{in} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \bar{u}_{n-1j} & \cdots & \bar{u}_{n-1n-1} & \bar{u}_{n-1n} \\ 0 & 0 & \cdots & 0 & \bar{u}_{nj} & \cdots & \bar{u}_{nn-1} & \bar{u}_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{u}_{11} & \bar{U}_{12} & \bar{U}_{13} \\ O & O & O \\ O & O & \bar{U}_{33} \end{bmatrix}.$$

□

When a type-II t.n.p. matrix is considered, we apply the technique used in Theorem 1 from the second row of A. The result is a matrix (4) with a similar structure like the matrix (2).

Theorem 2 *Let $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ be a type-II t.n.p. matrix without zero rows or columns. Applying row elementary transformations we obtain that A is equivalent to the upper block TN matrix $\bar{U}$ given in this form,*

$$\bar{U} = \begin{bmatrix} \bar{u}_{11} & \bar{U}_{12} \\ 0 & \bar{U}_{22} \end{bmatrix}. \tag{4}$$

Proof Let A be the t.n.p. matrix with $-a_{11} = 0$, $-a_{12} < 0$ and $-a_{21} < 0$,

$$A = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1,n-1} & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2,n-1} & -a_{2n} \\ -a_{31} & -a_{32} & \cdots & -a_{3,n-1} & -a_{3n} \\ \vdots & \vdots & & \vdots & \vdots \\ -a_{n,1} & -a_{n,2} & \cdots & -a_{n,n-1} & -a_{n,n} \end{bmatrix}.$$

Now, we apply the first iteration of the Gaussian elimination method from the second row of A and denote $\bar{u}_{ij} = \frac{1}{-a_{21}} det(A[2,i|1,j])$, for $i = 3,4,\ldots,n$, $j = 2,3,\ldots,n$. Multiplying by (-1) the first two rows of the resultant matrix we obtain

$$A_1 = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ 0 & \bar{u}_{32} & \cdots & \bar{u}_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & \bar{u}_{n2} & \cdots & \bar{u}_{nn} \end{bmatrix}.$$

Finally, we permute the first two rows of A_1,

$$\bar{U} = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ 0 & a_{12} & \cdots & a_{1n} \\ 0 & \bar{u}_{32} & \cdots & \bar{u}_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & \bar{u}_{n2} & \cdots & \bar{u}_{nn} \end{bmatrix} = \begin{bmatrix} \bar{u}_{11} & \bar{U}_{12} \\ 0 & \bar{U}_{22} \end{bmatrix}.$$

Now, we prove that $\bar{U}$ is a TN matrix following the same process as in Theorem 1. Note that rank($\bar{U}$) = rank(A) because we have only used elementary transformations. Furthermore,

1) $\bar{u}_{1j} = a_{2j} > 0$ for $j = 1,2,\ldots,n$ and $\bar{u}_{2j} = a_{1j} > 0$ for $j = 2,\ldots,n$. Moreover,
$\bar{u}_{ij} = \dfrac{1}{-a_{21}} det(A[2,i|1,j]) \geq 0$ for $i = 3,4,\ldots,n$, $j = 2,3,\ldots,n$.
2) The submatrix $\bar{U}[1,3,\ldots,n|1,2,\ldots,n]$ is TN because

$$\det(\bar{U}[1,\gamma|\beta]) = -\det(A[2,\gamma|\beta]) \geq 0 \ \ \forall\gamma,\ \beta, |\gamma| = |\beta| - 1.$$

If we do not consider the first row of $\bar{U}$, that is, $\forall\gamma,\ \beta, |\gamma| = |\beta|$ with $\gamma_1 > 2$, we have that

$$\det(\bar{U}[\gamma|\beta]) = \begin{cases} \text{If } \beta_1 = 1 \rightarrow = 0. \\ \text{If } \beta_1 > 1 \rightarrow = \dfrac{1}{a_{21}} \det(\bar{U}[1,\gamma|1,\beta]) \\ \qquad\qquad\quad = -\dfrac{1}{a_{21}} \det(A[2,\gamma|1,\beta]) \geq 0. \end{cases}$$

3) Now, we consider the second row of $\bar{U}$, then

3.1) $\forall i, j = 1, 2, \ldots, n$,

$$\det(\bar{U}[1,2|i,j]) = -\det(A_1[1,2|i,j]) = -\det(A[1,2|i,j]) \geq 0.$$

3.2) $\forall \gamma$, β with $|\gamma| = |\beta| - 2$

$$\det(\bar{U}[1,2,\gamma|\beta]) = -\det(A_1[1,2,\gamma|\beta]) = -\det(A[1,2,\gamma|\beta]) \geq 0.$$

3.3) $\forall \gamma$, β with $|\gamma| = |\beta| - 1$

$$\det(\bar{U}[2,\gamma|\beta]) = \begin{cases} \text{If } \beta_1 = 1 \rightarrow = 0. \\ \text{If } \beta_1 > 1 \rightarrow = \dfrac{1}{a_{21}} \det(\bar{U}[1,2,\gamma|1,\beta]) \\ \qquad\qquad\quad = -\dfrac{1}{a_{21}} \det(A_1[1,2,\gamma|1,\beta]) \\ \qquad\qquad\quad = -\dfrac{1}{a_{21}} \det(A[1,2,\gamma|1,\beta]) \geq 0. \end{cases}$$

Therefore, $\bar{U}$ is a TN matrix. □

Propositions 4 and 5 show the structure of the matrix $\bar{U}$ given in (4).

Proposition 4 *Let $\bar{U} \in \mathbb{R}^{n\times n}$ be the TN matrix given in (4). If $\bar{u}_{32} > 0$, then the submatrix $\bar{U}_{22}$ is irreducible.*

Proof Since $\bar{u}_{32} > 0$, $\bar{u}_{2j} = a_{1j} > 0$ and $\bar{U}$ is TN, we have that $\bar{u}_{3j} > 0$, for $j = 3, 4, \ldots, n$.

Furthermore, $\bar{u}_{i2} > 0$, for $i = 4, 5, \ldots, n$. In other case, if there exists $\bar{u}_{i2} = 0$, with $4 \leq i \leq n$, we would have

$$A[2,\,3,\,i|1,\,2] = \begin{bmatrix} -a_{21} & -a_{22} \\ -a_{31} & -a_{32} \\ -\alpha a_{21} & -\alpha a_{22} \end{bmatrix}, \quad \alpha > 0$$

then $\det(A[3,\,i|1,\,2]) = \det \begin{bmatrix} -a_{31} & -a_{32} \\ -\alpha a_{21} & -\alpha a_{22} \end{bmatrix} > 0$, which is a contradiction because A is a t.n.p. matrix.

As a consequence, $\bar{u}_{rs} > 0$, for $r = 4, 5, \ldots, n$, and $s = 3, 4, \ldots, n$ because $\bar{U}$ is TN.

So, all the entries of the submatrix $\bar{U}_{22}$ are positive. By [10, Lemma 2.2] this submatrix is irreducible. □

Proposition 5 *Let $\bar{U} \in \mathbb{R}^{n\times n}$ be the TN matrix given in(4). If $\bar{u}_{32} = 0$, then the submatrix $\bar{U}[3, 4, \ldots, n|2, 3, \ldots, n] = O_{(n-2)\times(n-1)}$ or (4) has the following form,*

$$\bar{U} = \begin{bmatrix} \bar{u}_{11} & \bar{u}_{12} & \bar{U}_{13} & \bar{U}_{14} \\ 0 & \bar{u}_{22} & \bar{U}_{23} & \bar{U}_{24} \\ O & O & O & O \\ O & O & O & \bar{U}_{44} \end{bmatrix}, \tag{5}$$

where all entries of the submatrix $\bar{U}_{44}$ are positive.

Proof Since $\bar{u}_{32} = 0$, using the fact that $\bar{U}$ is a TN matrix, we obtain that $\bar{u}_{i2} = 0$ for $i = 4, 5, \ldots, n$, or $\bar{u}_{3j} = 0$ for $j = 3, 4, \ldots, n$. Let $\bar{u}_{ij}$ be the first nonzero entry of $\bar{U}$ from the top and from the left, $3 \leq i \leq n$, $2 \leq j \leq n$, $(i, j) \neq (3, 2)$. Reasoning as in Proposition 4 we obtain that all entries of the submatrix $\bar{U}[i, i+1, \ldots, n|j, j+1, \ldots, n]$ are positive. That is, $\bar{U}$ has the following structure,

$$\bar{U} = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2j-1} & a_{2j} & \cdots & a_{2n-1} & a_{2n} \\ 0 & a_{12} & \cdots & a_{1j-1} & a_{1j} & \cdots & a_{1n-1} & a_{1n} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \bar{u}_{ij} & \cdots & \bar{u}_{in-1} & \bar{u}_{in} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \bar{u}_{n-1j} & \cdots & \bar{u}_{n-1n-1} & \bar{u}_{n-1n} \\ 0 & 0 & \cdots & 0 & \bar{u}_{nj} & \cdots & \bar{u}_{nn-1} & \bar{u}_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{u}_{11} & \bar{u}_{12} & \bar{U}_{13} & \bar{U}_{14} \\ 0 & \bar{u}_{22} & \bar{U}_{23} & \bar{U}_{24} \\ O & O & O & O \\ O & O & O & \bar{U}_{44} \end{bmatrix}.$$

□

Next Proposition is an extension of [9, Proposition 1] on TN matrices to t.n.p. matrices.

Proposition 6 *Let $A = (-a_{ij}) \in \mathbb{R}^{n \times n}$ be a type-I (type-II) t.n.p. matrix with* p-rank$(A) = p$ *and let $\alpha = \{1, 2, \ldots, p\}$ be the sequence of the first p-indices of A, then* $\operatorname{rank}(A) = p$.

Proof Let $A = (-a_{ij}) \in \mathbb{R}^{n \times n}$ be a type-I t.n.p. matrix and let $\bar{U}$ be the TN matrix obtained in Theorem 1. Since $\alpha = \{1, 2, \ldots, p\}$ is the sequence of the first p-indices of A, then for $j = 2, 3, \ldots, p$,

$$\det(\bar{U}[1, 2, \ldots, j]) = -\det(A_1[1, 2, \ldots, j]) = -\det(A[1, 2, \ldots, j]) > 0,$$

that is, p-rank$(\bar{U}) \geq p$.

We suppose that p-rank$(\bar{U}) = p + 1$, then there exists $t > p$, such that, $\det(\bar{U}[1, 2, \ldots, p, t]) > 0$, hence

$$\det(A[1, 2, \ldots, p, t]) = -\det(\bar{U}[1, 2, \ldots, p, t]) < 0,$$

and p-rank$(A) \geq p + 1$, which is a contradiction. So, we conclude that p-rank$(\bar{U}) = p$, and $\alpha = \{1, 2, \ldots, p\}$ is the sequence of its first p-indices.

Note that $\{1, 2, \ldots, p-1\}$ is the sequence of the first $(p-1)$-indices of the irreducible TN matrix $\bar{U}_{22}$. Applying [9, Proposition 1] to $\bar{U}_{22}$, we obtain that $\operatorname{rank}(\bar{U}_{22}) = p-1$. As a consequence $\operatorname{rank}(\bar{U}) = p$ and the result follows because $\operatorname{rank}(A) = \operatorname{rank}(\bar{U})$.

Now, we consider that $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ is a type-II t.n.p. matrix and let U be the TN matrix obtained in Theorem 2. Since $\alpha = \{1, 2, \ldots, p\}$ is the sequence of the first p-indices of A, then $\det(A[1, 2, \ldots, j]) < 0$ for $j = 2, 3, \ldots, p$, that is,

$$\begin{aligned} \det(\bar{U}[1]) &= a_{21} > 0, \\ \det(\bar{U}[1, 2]) &= a_{21}a_{12} > 0, \end{aligned}$$

and for $j = 3, 4, \ldots, p$

$$\det(\bar{U}[1, 2, \ldots, j]) = -\det(A_1[1, 2, \ldots, j]) = -\det(A[1, 2, \ldots, j]) > 0.$$

So, p-rank$(U) \geq p$.

We suppose that p-rank$(\bar{U}) = p + 1$, reasoning like the proof of the type-I t.n.p. matrix, we conclude that p-rank$(\bar{U}) = p$, and $\alpha = \{1, 2, \ldots, p\}$ is the sequence of its first p-indices.

If $\bar{u}_{32} > 0$, by Proposition 4 we have that $\bar{U}_{22}$ is irreducible. Since $\alpha = \{1, 2, \ldots, p-1\}$ is the sequence of its first $(p-1)$-indices, applying [9, Proposition 1] to $\bar{U}_{22}$, we obtain that $\operatorname{rank}(\bar{U}_{22}) = p-1$. As a consequence $\operatorname{rank}(\bar{U}) = p$, which implies that $\operatorname{rank}(A) = p$.

In the other case, since $\det(\bar{U}[1, 2, 3]) > 0$ we obtain that $\bar{u}_{33} > 0$, and by Proposition 5 all entries of the submatrix $\bar{U}[3, 4, \ldots, n]$ are positive. Since $\alpha = \{1, 2, \ldots, p-2\}$ is the sequence of its first $(p-2)$-indices, applying [9, Proposition 1] to $\bar{U}[3, 4, \ldots, n]$, we obtain that $\operatorname{rank}(\bar{U}[3, 4, \ldots, n]) = p-2$. Then $\operatorname{rank}(\bar{U}) = p$, which implies that $\operatorname{rank}(A) = p$. □

Next Proposition is an extension of [9, Proposition 2] on TN matrices to t.n.p. matrices.

Proposition 7 *Let $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ be a type-I (type-II) t.n.p. matrix and let $\alpha = \{1, 2, \ldots, q, q+t\}$ be the sequence of the first $(q+1)$-indices of A, $1 \leq q < n-1$ and $1 < t$. Then, each row (or column) $q+1, q+2, \ldots, q+t-1$ is a linear combination of the first q rows (or columns) of A.*

Proof Let $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ be a type-I t.n.p. matrix and let $\bar{U}$ be the TN matrix obtained in Theorem 1. We suppose that $\bar{u}_{22} > 0$. Working in a similar way to the proof of Proposition 6 we obtain that $\{1, 2, \ldots, q, q+t\}$ is the sequence of the first $(q+1)$-indices of $\bar{U}$. Since $\{1, 2, \ldots, (q-1), (q-1)+t\}$ is the sequence of the first q-indices of $\bar{U}_{22}$ and $\bar{U}_{22}$ is irreducible, applying [9, Proposition 2] to this matrix we obtain that each row (or column) $q, q+1, \ldots, (q-1)+t-1$ is a linear combination of the first $q-1$ rows (or columns) of $\bar{U}_{22}$. Then, each row (or column) $q+1, q+2, \ldots, q+t-1$ is a linear combination of the first q rows (or columns) of $\bar{U}$ and the result follows.

If $\bar{u}_{22} = 0$, then $q = 1$. Let $\bar{u}_{ij}$ be the first nonzero entry of $\bar{U}$ from the top and from the left, $2 \leq i, j \leq n$, $(i, j) \neq (2, 2)$, then $q + t = 1 + t = \max\{i, j\}$ and we obtain the result from the matrix given in (3).

Now, we consider that $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ is a type-II t.n.p. matrix and let U be the TN matrix obtained in Theorem 2. If $u_{32} > 0$, in a similar way to the type-I t.n.p. matrices, we obtain that $\{1, 2, \ldots, q, q+t\}$ is the sequence of the first $(q + 1)$-indices of $\bar{U}$. Since $\{1, 2, \ldots, (q - 1), (q - 1) + t\}$ is the sequence of the first q-indices of $\bar{U}_{22}$ and by Proposition 4 $\bar{U}_{22}$ is irreducible, applying [9, Proposition 2] to this matrix we obtain that each row (or column) $q, q + 1, \ldots, (q - 1) + t - 1$ is a linear combination of the first $q - 1$ rows (or columns) of $\bar{U}_{22}$. Therefore, each row (or column) $q+1, q+2, \ldots, q+t-1$ is a linear combination of the first q rows (or columns) of $\bar{U}$ and the result follows.

If $\bar{u}_{32} = 0$ and $\bar{u}_{33} \neq 0$, then the submatrix $\bar{U}[3, 4, \ldots, n]$ is irreducible and applying [9, Proposition 2] the result follows. If $\bar{u}_{32} = 0$ and $\bar{u}_{33} = 0$, then $q = 2$ and if $\bar{u}_{ij}$ is the first nonzero entry from upper left, then $q + t = \max\{i, j\}$. In this case we obtain the result from the matrix given in (5). □

By Propositions 6 and 7 we extend [9, Theorem 10] from IrTN matrices to t.n.p. matrices, reasoning in the same way.

Theorem 3 *Let $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ be a type-I (type-II) t.n.p. matrix. If p-rank$(A) = p$, $1 \leq p < n$, then*

$$\operatorname{rank}(A^p) = p\text{-}rank(A) = p.$$

In particular, the size of the largest zero Jordan block is at most p.

As an immediate consequence of Theorem 3, we extend the relation between the order n of an irreducible TN matrix, its rank r and its principal rank p given in [9, Theorem 11].

Theorem 4 *Let $A = (-a_{ij}) \in \mathbb{R}^{n\times n}$ be a type-I (type-II) t.n.p. matrix. If p-rank$(A) = p$ and rank$(A) = r$, then*

$$p \leq r \leq n - \left\lceil \frac{n-p}{p} \right\rceil . \tag{6}$$

The maximum rank of a t.n.p. matrix A with p-rank$(A) = p$ depends on the sequence of its first p-indices, as in the case of IrTN matrices. In [9, Algorithm 1] the authors present an algorithm that calculates the maximum rank of IrTN matrices in function of its first p-indices. We can apply this algorithm to t.n.p. matrices to obtain the maximum rank and we conclude that this maximum rank reaches the upper bound given in (6) when its first p-indices are distributed along the matrix equidistantly (it is not the only case, but it is the most obvious).

3 Construct a t.n.p matrix associated with a triple (n, r, p) $(1, i_2, \ldots, i_p)$-negatively realizable

In this section we consider Definition 3 of triple $(1, i_2, \ldots, i_p)$-negatively realizable of the type-I (type-II) and we give a method to construct t.n.p. matrices associated with these triples.

In [9, Section 4] the authors prove that there exists an upper block echelon TN matrix associated with a realizable triple and a given sequence of the first p-indices. Now, in Section 3.1, a procedure to obtain this matrix is constructed. Next, in Section 3.2, we use it to calculate a type-I (type-II) t.n.p. matrix $A \in \mathbb{R}^{n\times n}$ with $\text{rank}(A) = r$ and $p\text{-rank}(A) = p$ associated with a triple (n, r, p) $(1, i_2, \ldots, i_p)$-negatively realizable of the type-I (type-II).

3.1 Constructing an upper block echelon TN matrix V

We give the following procedure to construct an upper block echelon TN matrix $V \in \mathbb{R}^{n\times n}$ with $\text{rank}(V) = r$, $p\text{-rank}(V) = p$ and the sequence of its first p-indices given by $\{1, i_2, \ldots, i_p\}$. This procedure will be programmed as Algorithm 2.

From now on and for simplicity, we use the following MatLab notation: $A(i, :)$ denotes the i-th row of A and $A(:, j)$ denotes its j-th column; $\text{ones}(n, m)$ denotes the $n\times m$ matrix of ones; $\text{triu}(\text{ones}(n, m))$ denotes the upper triangular part of $\text{ones}(n, m)$; $\text{zeros}(n, m)$ denotes the $n \times m$ zero matrix. Moreover, we recall that a matrix is an upper echelon matrix if the first nonzero entry in each row (leading entry) is to the right of the leading entry in the row above it and all zero rows are at the bottom. A matrix is upper block echelon if each nonzero block, starting from the left, is to the right of the nonzero blocks below and the zero blocks are at the bottom. A matrix is a lower (block) echelon matrix if its transpose is an upper (block) echelon matrix.

Procedure 1 *Given a triple (n, r, p), this process constructs an upper block echelon TN matrix $V \in \mathbb{R}^{n\times n}$ with $\text{rank}(V) = r$, $p\text{-rank}(V) = p$ and the sequence of its first p-indices is $H = \{1, i_2, \ldots, i_p\}$.*

- *Calculate s as the number of the consecutive first indices of the sequence H.*
- *If $s = p$ and*
  - *$n = H(p)$, then $V = \text{triu}(\text{ones}(n, n))$ or*
  - *$n > H(p)$, then $V = [\text{triu}(\text{ones}(s, n)); \text{zeros}(n - s, n)]$.*
- *If $s \neq p$ then we construct an upper block echelon matrix $\widetilde{V}$ in the following form,*

$$\widetilde{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$ *where $v^{(i)}$, $i = s+1, s+2 \ldots, n$ are the rows of matrix V_2.*

1. *Construct the* $s \times n$ *matrix* V_1,

   $V_1 = \mathrm{triu}(\mathrm{ones}(s, n))$.

2. *Construct an identity matrix of size* n, $G = \mathrm{eye}(n)$.
3. *Calculate* $q = r - p$, $i = s + 1$, $t = H(s + 1)$, $x = 1$ *and* $s = s + 1$.
4. *While* $t \leq n$ *do*

   4.1. *if* $i \leq H(s)$ *then compare* t *with the entries of* H,

   4.1.1. *if* $t \in H$ *then* $v^{(i)} = [\mathrm{zeros}(1, t-1) \;\; \mathrm{ones}(1, n-t+1)]$ *and for* $l = t, \ldots, i+1$, *step* -1, *construct* $E = \mathrm{eye}(n)$, *replace* $E(l, l-1) = -1$, $E_x = E$ *and replace* $x = x + 1$;

   4.1.2. *if* $t \notin H$ *and* $q \neq 0$ *then* $v^{(i)} = [\mathrm{zeros}(1, t-1) \;\; \mathrm{ones}(1, n-t+1)]$ *and rename* $q = q - 1$;

   4.1.3. *if* $t \notin H$ *and* $q = 0$ *then* $v^{(i)} = [\mathrm{zeros}(1, n)]$;

   4.1.4. *rename* $i = i + 1$ *and* $t = t + 1$.

   4.2 *if* $i > H(s)$ *then rename* $s = s + 1$,

   4.2.1. *while* $t < H(s)$, *rename* $t = t + 1$;

   4.2.2. *compare* t *with the entries of* H *and apply Steps from [4.1.1] to [4.1.4].*

5. *For* $j = i, \ldots, n$, *construct the rest of the rows of* V_2, $v^{(j)} = [\mathrm{zeros}(1, n)]$.
6. *Let* w *the largest value assigned to* x. *For* $x = w, \ldots, 1$, *step* -1, *obtain the matrix* $G = E_x^{-1} * G$.
7. *Finally, the matrix* $V = G * \widetilde{V}$.

Remark 1

1. The matrix $\widetilde{V}$ constructed in Procedure 1 is an upper block echelon matrix whose nonzero entries are equal to one. Thus, $\widetilde{V}$ is a TN matrix. On the other hand, the matrices E_x^{-1}, with $x = w, w-1, \ldots, 1$, are the inverse of bidiagonal matrices and it is known that they are TN matrices (see, [14]). Then, $G = E_1^{-1} E_2^{-1} \cdots E_w^{-1}$ is a TN matrix and as a consequence, $V = G\widetilde{V}$ is also TN.
2. In Procedure 1, to reach the rank $q = n - r$ we use the Step 4.1.2. In this case, the procedure uses the first consecutive rows of the matrix $\widetilde{V}$, which are not used for the p-indices. Note that it is possible to choose other rows to get the rank q.

Now, we present two examples. In Example 1, Step 4.2.1. of Procedure 1 it is not applied and the process of construction of V is explained step by step. In Example 2, Step 4.2.1. is necessary to obtain V and we only show the differences with the previous example.

Example 1 Obtain an 11×11 upper block echelon TN matrix V with $\text{rank}(V) = 7$, p-$\text{rank}(V) = 3$ and $H = \{1, 5, 9\}$ as the sequence of its first 3-indices.

In this case, $s = 1$, $p = 3$ and $s \neq p$, then the upper block echelon matrix has the structure $\widetilde{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$. Thus, we construct the first block matrices:

1. $V_1 = [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]_{1\times 11}$.
2. $G = \text{eye}(11)$.
3. $q = 4$, $i = 2$, $t = 5$, $x = 1$ and $s = 2$.
4. All results obtained in this Step are showed in the following table. $n = 11$ and $H(2) = 5$,

t	i	Step	Result
5	2	4.1.1.	$v^{(2)} = [\text{zeros}(1,4)\ \ \text{ones}(1,7)]$,
			$E = \text{eye}(11)$, $E(5,4) = E(4,3) = E(3,2) = -1$, $E_1 = E$, $x = 2$.
6	3	4.1.2.	$v^{(3)} = [\text{zeros}(1,5)\ \ \text{ones}(1,6)]$, $q = 3$.
7	4	4.1.2.	$v^{(4)} = [\text{zeros}(1,6)\ \ \text{ones}(1,5)]$, $q = 2$.
8	5	4.1.2.	$v^{(5)} = [\text{zeros}(1,7)\ \ \text{ones}(1,4)]$, $q = 1$.
9	6	4.2.	$s = 3$, $H(3) = 9$,
		4.2.2	$v^{(6)} = [\text{zeros}(1,8)\ \ \text{ones}(1,3)]$,
			$E = \text{eye}(11)$, $E(9,8) = E(8,7) = E(7,6) = -1$, $E_2 = E$.
10	7	4.1.2.	$v^{(7)} = [\text{zeros}(1,9)\ \ \text{ones}(1,2)]$, $q = 0$.
11	8	4.1.4.	$v^{(8)} = [\text{zeros}(1,11)]$, $i = 9$, $t = 12$.

5. $v^{(i)} = [\text{zeros}(1,11)]$, for $i = 9, 10, 11$.
6. $w = 2$, then, for $x = 2, 1$,
   - $x = 2$, $G = E_2^{-1} * G$
   - $x = 1$, $G = E_1^{-1} * G$
7. Finally, the matrix $V = G * \widetilde{V}$.

Then, the matrix V is constructed:

$$V = \left[\begin{array}{cccc|cccc|ccc}
\mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\
0 & 0 & 0 & 0 & \mathbf{1} & 2 & 3 & 4 & 4 & 4 & 4 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2 & 2 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Example 2 Obtain an 18×18 upper block echelon TN matrix V with $\text{rank}(V) = 14$, p-$\text{rank}(V) = 9$ and $H = \{1, 2, 4, 7, 8, 10, 11, 12, 15\}$ as the sequence of its first 9-indices.

In this case, $s = 2$, $p = 9$ and $s \neq p$, then we obtain the matrix $\widetilde{V}$ using Procedure 1.

1. $V_1 = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}_{2\times 18}$.
2. $G = \text{eye}(18)$.
3. $q = 5$, $i = 3$, $t = 4$, $x = 1$ and $s = 3$.
4. In this Step we only explain the differences with respect to Example 1.

   When $t = 6 \leq n = 18$ and $i = 5 > H(3) = 4$, then by [4.2] $s = s + 1 = 4$ and $H(4) = 7$.

   As $t = 6 < H(4) = 7$, then we rename $t = 7$ and continue applying the algorithm.
5. $v^i = [\text{zeros}(1, 18)]$, for $i = 17, 18$.
6. $w = 7$, then, for $x = 7, 6, 5, 4, 3, 2, 1$ we calculate $G = E_x^{-1} * G$
7. Finally, the matrix $V = G * \widetilde{V}$, being

$$G = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&2&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&2&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&2&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}, \quad \widetilde{V} = \begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

and

$$V = \left[\begin{array}{ccc|ccc|c|cc|c|c|ccc|cccc}
\mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \mathbf{1} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 3 & 4 & 4 & 4 & 4 & 4 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2 & 3 & 3 & 3 & 3 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2 & 2 & 2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

In the Appendix, we present Algorithm 2 associated with Procedure 1 to calculate the upper block echelon TN matrix $V \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(V) = r$, p-$\operatorname{rank}(V) = p$ and the sequence of its p-indices given by $\{1, i_2, \ldots, i_p\}$. Using Algorithm 2, the results of Examples 1 and 2 are obtained directly.

3.2 Constructing the t.n.p. matrix $A = LDV$

Our main result is given in Theorem 5 where we construct a type-I (type-II) t.n.p. matrix $A \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(A) = r$ and p-$\operatorname{rank}(A) = p$ associated with a triple (n, r, p) $(1, i_2, \ldots, i_p)$-negatively realizable of the type-I (type-II). For that, we consider the TN matrix $V \in \mathbb{R}^{n\times n}$ constructed in Section 3.1 and prove the following Lemma.

Lemma 1 *Consider a lower echelon TN matrix $Q \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(Q) = r$ and all entries that are not trivially zero are positive. Then, for all $\delta > 0$ it is possible to construct an $n \times n$ nonsingular lower triangular TN matrix $Q(\delta)$, such that $\lim_{\delta\to 0} Q(\delta) = Q$.*

Proof Suppose that we can apply to Q the Neville elimination process with no pivoting until the k-th iteration. Then, we have the matrix

$$E_{(k)}E_{(k-1)}\ldots E_{(2)}E_{(1)}Q = Q_k = \begin{bmatrix} I_k & O \\ O & Q_{k_{22}} \end{bmatrix}$$

with some entries in its $k+1$ column equal to zero.

Now, to apply the $(k+1)$-th iteration of Neville elimination with no pivoting, we change the k+1 column of Q_k in the following way:

- Denote $h = 1$.
- If the first nonzero entry of the column is in the j-th row, with $j > k+1$, replace the nonzero entries from position $(k+1, k+1)$ to position $(j-1, k+1)$, by $\delta^{(h-1)+j-s}$ with $s = k+1, k+2, \ldots, j-1$.
- From the first nonzero entry to the last, replace the zero entries of that column by δ^h. If the zero and nonzero pattern is repeated in columns consecutive to $k+1$, replace the zero entries of those columns so that to apply Neville elimination process, the initial zero pattern to the right of the column remains. For example, suppose that we have obtained the following submatrix after one iteration of Neville elimination

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 3 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 1 & 2 & 5 & 1 & 2 \end{bmatrix}.$$

Now, to apply again Neville elimination we replace the zeros in positions $(4,2)$ and $(5,2)$ by δ^h. Since the zero and nonzero pattern of column 2 is repeated in consecutive columns 3 and 4, we replace these zeros in the following form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 3 & 0 & 0 \\ 0 & \delta^h & \delta^h & 3\delta^h/2 & 0 & 0 \\ 0 & \delta^h & \delta^h & 3\delta^h/2 & 0 & 0 \\ 0 & 1 & 2 & 5 & 1 & 0 \end{bmatrix}.$$

Applying Neville elimination we obtain the following submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & 1 & 7/2 & 1 & 0 \end{bmatrix},$$

in which the zero and nonzero pattern in columns 3 and 4 remains.
- The zero entries bellow the last nonzero entry of the column are not replaced.
- Call the new matrix $\bar{Q}_k(\delta)$ and apply the Neville elimination process without interchange of rows obtaining $E_{(k+1)}(\delta)\bar{Q}_k(\delta) = Q_{k+1}(\delta)$.

From $Q_{k+1}(\delta)$ we construct $\bar{Q}_{k+1}(\delta)$ changing its k+2 column in the following way:

- $h = h+1$.

- If the column is a zero column, then replace the zero entry of the main diagonal by δ^r, with $r = n - j + h$, where j is the column that is changing. Apply the Neville elimination process with $E_{(k+2)}(\delta) = I$, obtaining $Q_{k+2}(\delta) = \bar{Q}_{k+1}(\delta)$.
- If the column is a zero column with a nonzero entry in its main diagonal, denote this matrix as $\bar{Q}_{k+1}(\delta)$ and apply the Neville elimination process with $E_{(k+2)}(\delta) = I$, obtaining $Q_{k+2}(\delta) = \bar{Q}_{k+1}(\delta)$.
- If in the $k+2$ column there are zero and nonzero entries, replace the nonzero entries as similar way as the first case and denote the obtained matrix as $\bar{Q}_{k+1}(\delta)$. Applying the Neville elimination process without interchange of rows we obtain $E_{k+2}(\delta)\bar{Q}_{k+1}(\delta) = Q_{k+2}(\delta)$.

The process is repeated until to arrive to the matrix $\bar{Q}_{n-1}(\delta)$. Then, the matrix

$$Q(\delta) = E_{(1)}^{-1}E_{(2)}^{-1}\cdots E_{(k)}^{-1}E_{(k+1)}^{-1}(\delta)\cdots E_{(n-2)}^{-1}(\delta)E_{(n-1)}^{-1}(\delta)\bar{Q}_{n-1}(\delta)$$

is a lower triangular TN matrix such that

$$Q(\delta) = Q + \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & p_{ij}(\delta) & \\ 0 & & & \end{array}\right], \tag{7}$$

where $p_{ij}(\delta)$ are polynomials in δ with nonnegative coefficients such that

$$\lim_{\delta\to 0} p_{ij}(\delta) = 0.$$

Therefore, we have that $\lim_{\delta\to 0} Q(\delta) = Q$. □

Example 3 Given the 6×6 lower echelon TN matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \end{bmatrix},$$

we construct a lower triangular nonsingular TN matrix using the method described in Lemma 1.

$$Q=\begin{bmatrix}1&0&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\1&1&0&0&0&0\\1&2&1&0&0&0\\1&2&2&1&0&0\end{bmatrix}\xrightarrow{E_{(1)}} Q_1=\begin{bmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&1&1&0&0&0\\0&0&1&1&0&0\end{bmatrix}\xrightarrow[h=1]{column\ 2}$$

$$\bar{Q}_1(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&1&0&0&0&0\\0&\delta&0&0&0&0\\0&1&1&0&0&0\\0&0&1&1&0&0\end{bmatrix}\xrightarrow{E_{(2)}(\delta)} Q_2(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\end{bmatrix}\xrightarrow[h=2]{column\ 3}$$

$$\bar{Q}_2(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&0&\delta^3&0&0&0\\0&0&\delta^2&0&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\end{bmatrix}\xrightarrow{E_{(3)}(\delta)} Q_3(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&0&\delta^3&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\end{bmatrix}\xrightarrow[h=3]{column\ 4}$$

$$\bar{Q}_3(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&0&\delta^3&0&0&0\\0&0&0&\delta^4&0&0\\0&0&0&\delta^3&0&0\\0&0&0&1&0&0\end{bmatrix}\xrightarrow{E_{(4)}(\delta)} Q_4(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&0&\delta^3&0&0&0\\0&0&0&\delta^4&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}\xrightarrow[h=4]{column\ 5}$$

$$\bar{Q}_4(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&0&\delta^3&0&0&0\\0&0&0&\delta^4&0&0\\0&0&0&0&\delta^5&0\\0&0&0&0&0&0\end{bmatrix}\xrightarrow{E_{(5)}(\delta)=I} Q_5(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&0&\delta^3&0&0&0\\0&0&0&\delta^4&0&0\\0&0&0&0&\delta^5&0\\0&0&0&0&0&0\end{bmatrix}\xrightarrow[h=5]{column\ 6}$$

$$\bar{Q}_5(\delta)=\begin{bmatrix}1&0&0&0&0&0\\0&\delta&0&0&0&0\\0&0&\delta^3&0&0&0\\0&0&0&\delta^3&0&0\\0&0&0&0&\delta^5&0\\0&0&0&0&0&\delta^5\end{bmatrix}$$

The lower triangular TN matrix is

$$Q(\delta) = E_{(1)}^{-1}E_{(2)}^{-1}(\delta)E_{(3)}^{-1}(\delta)E_{(4)}^{-1}(\delta)E_{(5)}^{-1}(\delta)\bar{Q}_5(\delta)$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \delta & 0 & 0 & 0 & 0 \\ 1 & \delta+1 & \delta^3 & 0 & 0 & 0 \\ 1 & 2\delta+1 & \delta^4+\delta^3+\delta^2 & \delta^4 & 0 & 0 \\ 1 & 2\delta+2 & \delta^4+2\delta^3+\delta^2+\delta+1 & \delta^4+2\delta^3+\delta^2 & \delta^5 & 0 \\ 1 & 2\delta+2 & \delta^4+2\delta^3+\delta^2+\delta+2 & \delta^4+3\delta^3+2\delta^2+1 & 2\delta^5+\delta^2 & \delta^5 \end{bmatrix},$$

such that
$$\lim_{\delta\to 0} Q(\delta) = Q.$$

Remark 2 Note that if the TN matrix is a block lower echelon matrix, then the process to obtain a nonsingular lower triangular TN matrix is the same. If we have a block upper echelon matrix, then we transpose and apply Lemma 1.

Theorem 5 *Let $V = (v_{ij}) \in \mathbb{R}^{n\times n}$ be an upper echelon TN matrix with nonnegative entries above the diagonal, with $v_{1n} > 0$ and $\operatorname{rank}(V) = r$.*

1. *If $L \in \mathbb{R}^{n\times n}$ is a lower triangular TN matrix with ones along the main diagonal and positive entries below it, and $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix $D = diag(-d_1, d_2, \dots, d_n)$ with $d_i > 0$, $i = 1, 2, \dots, n$. If $A = LDV$ with $-a_{nn} \le 0$, then A is a type-I t.n.p. matrix with $\operatorname{rank}(A) = r$.*
2. *If $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix $D = diag(-d_1, -d_2, d_3, \dots, d_n)$ with $d_i > 0$, $i = 1, 2, \dots, n$ and $L \in \mathbb{R}^{n\times n}$ is the block lower triangular matrix*
$$L = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \textit{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
where the entries in the first column of $\tilde{L}_{21}$ are positive, in the second one are nonpositive, $\tilde{L}_{22}$ is a lower triangular TN matrix with ones along the main diagonal and positive entries below it, and such that
$$det(\tilde{L}[\alpha|1, 2, \dots, k]) \le 0, \forall\alpha \in \mathcal{Q}_{k,n},\ k = 2, 3, \dots, n.$$
If $A = LDV$ with $-a_{nn} \le 0$, then A is a type-II t.n.p. matrix with $\operatorname{rank}(A) = r$.

Proof 1. First consider the case $-a_{nn} < 0$. Since V is an upper echelon TN matrix we obtain the transpose and apply the process given in Lemma 1 to obtain a nonzero upper triangular TN matrix $V(\delta)$ with a structure similar to (7),
$$V(\delta) = V + \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & p_{ij}(\delta) & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|c} v_{11} & V_{12} \\ \hline O & V_{22}(\delta) \end{array}\right].$$

Thus,

$$A(\delta) = LDV(\delta) = \left[\begin{array}{c|c} 1 & O \\ \hline L_{21} & L_{22} \end{array}\right] \left[\begin{array}{c|c} -d_1 & O \\ \hline O & D_{22} \end{array}\right] \left[\begin{array}{c|c} v_{11} & V_{12} \\ \hline O & V_{22}(\delta) \end{array}\right]$$
$$= \begin{bmatrix} -d_1 v_{11} & -d_1 V_{12} \\ -d_1 L_{21} v_{11} & -d_1 L_{21} V_{12} + L_{22} D_{22} V_{22}(\delta) \end{bmatrix}.$$

It is not difficult to see that $A(\delta)(n,n) = -a_{nn} + q_{nn}(\delta)$, where $q_{nn}(\delta)$ is a polynomial in δ with nonnegative coefficients such that $\lim_{\delta\to 0} q_{nn}(\delta) = 0$. Since $-a_{nn} < 0$, there exists δ_0 such that $A(\delta)(n,n) < 0,\ \forall\delta < \delta_0$. Then, $A(\delta)$ is a t.n.p. matrix for all $\delta < \delta_0$ and

$$\det(A[\alpha|\beta]) = \lim_{\delta\to 0} \det(A(\delta)[\alpha|\beta]) \le 0, \quad \forall\alpha,\beta \in \mathcal{Q}_{k,n},\ k = 1,2,\ldots,n.$$

Hence, A is a type-I t.n.p. matrix and obviously with $\text{rank}(A) = r$.

Now, suppose that $-a_{nn} = 0$ and consider the following matrix

$$B = L \begin{bmatrix} -d_1 - x & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} V$$
$$= A + \begin{bmatrix} -xv_{11} & -xv_{12} & \ldots & -xv_{1n} \\ -xv_{11}l_{21} & -xv_{12}l_{21} & \ldots & -xv_{1n}l_{21} \\ \vdots & \vdots & & \vdots \\ -xv_{11}l_{n1} & -xv_{12}l_{n1} & \ldots & -xv_{1n}l_{n1} \end{bmatrix},$$

whose (n,n) entry is equal to $-xl_{n1}v_{1n} < 0$, because $l_{n1} > 0$ and $v_{1n} > 0$. Applying the previous case we obtain that B is a t.n.p. matrix for all $x > 0$. Therefore, $\forall\alpha,\beta \in \mathcal{Q}_{k,n},\ k \in \{1,2,\ldots,n\}$, $\det(B[\alpha|\beta])$ is a first degree polynomial, i.e.

$$\det(B[\alpha|\beta]) = c_{\alpha,\beta}x + t_{\alpha,\beta} \le 0,\ \forall x > 0,$$

which implies that

$$c_{\alpha,\beta},\ t_{\alpha,\beta} \le 0,\ \forall\alpha,\beta \in \mathcal{Q}_{k,n},\ k \in \{1,2,\ldots,n\}.$$

Since,

$$\det(A[\alpha|\beta]) = \lim_{x\to 0^+} \det(B[\alpha|\beta]) = t_{\alpha,\beta} \le 0,$$
$$\forall\alpha,\beta \in \mathcal{Q}_{k,n},\ k \in \{1,2,\ldots,n\},$$

we conclude that A is a type-I t.n.p. matrix with rank equal to r.

2. The proof is similar to the previous case but $A(\delta)$ is factorized as

$$\begin{aligned} A(\delta) &= LDV(\delta) \\ &= \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline L_{31} & L_{32} & L_{33} \end{array}\right] \left[\begin{array}{cc|c} -d_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ \hline O & O & D_{33} \end{array}\right] \left[\begin{array}{cc|c} v_{11} & v_{12} & V_{13} \\ 0 & v_{22} & V_{23}(\delta) \\ \hline O & O & V_{33}(\delta) \end{array}\right]. \end{aligned}$$

□

From Procedure 1 we calculate an upper block echelon TN matrix $V \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(V) = r$, p-$\operatorname{rank}(V) = p$ and $\{1, i_2, \ldots, i_p\}$ as the sequence of its first p-indices. Now, we give a procedure to construct a t.n.p. matrix $A \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(A) = r$, p-$\operatorname{rank}(A) = p$ and with the same sequence of the first p-indices. In this procedure we use the following MatLab notation: $tril(ones(n, n))$ denotes the lower triangular part of $ones(n, n)$; $diag(v)$ when v is a vector of n components, returns a square matrix of order n, with the elements of v on the principal diagonal.

Procedure 2 *Given a triple (n, r, p)-negatively realizable, this process constructs a t.n.p. matrix $A \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(A) = r$, p-$\operatorname{rank}(A) = p$ and with $\{1, i_2, \ldots, i_p\}$ as the sequence of its first p-indices.*

First, we use Procedure 1 and construct an upper block echelon TN matrix $V = (v_{ij}) \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(V) = r$, p-$\operatorname{rank}(V) = p$ and $\{1, i_2, \ldots, i_p\}$ as the sequence of its first p-indices. Then,

1. *For the type-I t.n.p. matrices:*
   - *(a) Construct a lower triangular TN matrix $L = tril(ones(n, n))$.*
   - *(b) Construct $D = \operatorname{diag}(-d_1, 1, \ldots, 1)$, with $d_1 \geq \sum_{j=2}^{i_p} v_{jn}$.*
   - *(c) $A = L * D * V$.*

2. *For the type-II t.n.p. matrices:*
   - *(a) Construct the following TN matrix $L \in \mathbb{R}^{n\times n}$,*

$$L = \begin{bmatrix} L_{11} & O \\ L_{21} & L_{22} \end{bmatrix}, \quad L_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad L_{21} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} \tag{8}$$

   and $L_{22} = tril(ones(n-2, n-2))$.
   - *(b) Construct $D = \operatorname{diag}(-d_1, -1, 1, \ldots, 1)$, with $d_1 \geq \sum_{j=2}^{i_p} v_{jn}$.*
   - *(c) $A = L * D * V$.*

Proposition 8 *Let $A = LDV$ be a matrix constructed by Procedure 2. Then A is a t.n.p. matrix, with $\operatorname{rank}(A) = r$, p-$\operatorname{rank}(A) = p$ and $\{1, i_2, \ldots, i_p\}$ as the sequence of its first p-indices.*

Proof By Theorem 5, the matrix $A = LDV$ constructed by Procedure 2 is a t.n.p. matrix if $-a_{nn} \leq 0$. For the type-I t.n.p. matrices we have that

$$-a_{nn} = [1\ 1\ \cdots\ 1]\,\mathrm{diag}(-d_1, 1, 1, \ldots, 1)\begin{bmatrix} 1 \\ v_{2n} \\ \vdots \\ v_{i_p n} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -d_1 + \sum_{j=2}^{i_p} v_{jn} \leq 0,$$

and for the type-II t.n.p. matrices we have that

$$-a_{nn} = [1\ \ -1\ \ 1\ \cdots\ 1]\,\mathrm{diag}(-d_1, -1, 1, \ldots, 1)\begin{bmatrix} 1 \\ 1 \\ v_{3n} \\ \vdots \\ v_{i_p n} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -d_1 + \sum_{j=2}^{i_p} v_{jn} \leq 0.$$

Therefore, A is a t.n.p. matrix. Obviously, $\mathrm{rank}(A) = r$ because L and D are nonsingular matrices.

The matrix V is computed by Procedure 1 with $\{1, i_2, \ldots, i_p\}$ as the sequence of its first p-indices. Then, applying [9, Proposition 3] we obtain that p-$\mathrm{rank}(A) = p$ and the sequence of its first p-indices is $\{1, i_2, \ldots, i_p\}$, where $i_2 = 2$ if A is a type-II t.n.p. matrix. □

Example 4 Obtain a type-I (type-II) t.n.p. matrix associated with the triple $(18, 14, 9)$-negatively realizable and $\alpha = \{1, 2, 4, 7, 8, 10, 11, 12, 15\}$ as the sequence of its first 9-indices.

From Example 2 we consider the matrix V, and following Procedure 2 we have two cases:

- A type-I t.n.p.matrix. As $\sum_{j=2}^{15} v_{j,18} = 32$ we choose, for instance, $d_1 = -32$ and obtain:

$A = \text{tril}(\text{ones}(18,18)) * \text{diag}([-32, 1, 1 \ldots, 1]) * V =$

$$\begin{bmatrix}
-32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 \\
-32 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & -31 \\
-32 & -31 & -31 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 \\
-32 & -31 & -31 & -29 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 \\
-32 & -31 & -31 & -29 & -28 & -28 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 \\
-32 & -31 & -31 & -29 & -28 & -28 & -26 & -25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -22 & -21 & -21 & -21 & -21 & -21 & -21 & -21 & -21 & -21 & -21 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -18 & -18 & -18 & -18 & -18 & -18 & -18 & -18 & -18 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -17 & -16 & -16 & -16 & -16 & -16 & -16 & -16 & -16 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -13 & -12 & -12 & -12 & -12 & -12 & -12 & -12 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -9 & -8 & -8 & -8 & -8 & -8 & -8 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -5 & -5 & -5 & -5 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -4 & -4 & -4 & -4 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -3 & -2 & -2 & -2 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -2 & 0 & 0 & 0 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -2 & 0 & 0 & 0 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -2 & 0 & 0 & 0 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -2 & 0 & 0 & 0
\end{bmatrix}.$$

• A type-II t.n.p.matrix. In this case, we choose, for instance, $d_1 = -32$, then L is given by (8) and obtain:

$A = L * \text{diag}([-32, -1, 1 \ldots, 1]) * V =$

$$\begin{bmatrix}
0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 & -32 \\
-32 & -31 & -31 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 \\
-32 & -31 & -31 & -29 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 & -28 \\
-32 & -31 & -31 & -29 & -28 & -28 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 & -27 \\
-32 & -31 & -31 & -29 & -28 & -28 & -26 & -25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -22 & -21 & -21 & -21 & -21 & -21 & -21 & -21 & -21 & -21 & -21 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -18 & -18 & -18 & -18 & -18 & -18 & -18 & -18 & -18 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -17 & -16 & -16 & -16 & -16 & -16 & -16 & -16 & -16 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -13 & -12 & -12 & -12 & -12 & -12 & -12 & -12 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -9 & -8 & -8 & -8 & -8 & -8 & -8 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -5 & -5 & -5 & -5 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -4 & -4 & -4 & -4 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -3 & -2 & -2 & -2 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -2 & 0 & 0 & 0 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -2 & 0 & 0 & 0 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -2 & 0 & 0 & 0 \\
-32 & -31 & -31 & -29 & -28 & -28 & -25 & -21 & -19 & -16 & -12 & -8 & -6 & -5 & -2 & 0 & 0 & 0
\end{bmatrix}.$$

In the Appendix we present Algorithms 3 and 4 associated with Procedure 2 to obtain the type-I and type-II t.n.p. matrices, respectively. Using these algorithms, the results of Example 4 are obtained directly.

Appendix

In this section, we construct Algorithm 2 associated with Procedure 1 to obtain an upper block echelon TN matrix $V \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(V) = r$, p-$\operatorname{rank}(V) = p$ and the sequence of its first p-indices given by $\{1, i_2, \ldots, i_p\}$. To apply Algorithm 2 we introduce Algorithm 1 given in [9, Algorithm 1] in order to know the maximum rank of a matrix depending on the sequence of its first p-indices.

Algorithm 1 $r_{\max} = \operatorname{maxrank}(n, p, H = [1, i_2, \ldots, i_p])$

```
W = [H n+1]; k = p; s = 0;
for j = p : -1 : 2 do
    f = W(j) - W(j-1) - 1;
    c = W(j+1) - W(j) - 1 + s;
    if f <= c then
        k = k + f;
        s = c - f;
    else
        k = k + c;
        s = 0;
    end if
end for
r_max = k
```

Algorithm 2 $V = \mathrm{TPV}(n, r, p, H)$

```
W = [H, n + 1];
w = maxrank(n,p,W)
if r > w | r < p then
    It is not a realizable triple
else
    c = H − [1 : p];  g = c > 0;  h = g ∗ ones(p, 1);  s = p − h;
    if s == p then
        if n == H(p) then
            V = triu(ones(n, n));
        else
            V = [triu(ones(s, n)); zeros(n − s, n)];
        end if
        T = [s + 1, H(s + 1)];
        for j = s + 2 : H(s + 1) do
            [a, b] = size(T);
            T = [T; j T(a, b) + 1];
        end for
        for t = s + 1 : p − 1 do
            for j = H(t) + 1 : H(t + 1) do
                [a, b] = size(T);
                if T(a, b) < n then
                    T = [T; j, T(a, b) + 1];
                    while T(a + 1, b) < H(t + 1) do
                        T(a + 1, b) = T(a + 1, b) + 1;
                    end while
                end if
            end for
        end for
        [a, b] = size(T);
        S = [0 0];
        for j = 1 : a do
            m = ismember(T(j, b), H);
            if m == 1 then
                S = [S; T(j, 1) T(j, 2)];
            end if
        end for
        [s1 s2] = size(S); S = S(2 : s1, :);
        Q = [0 0];
        for j = 1 : a do
            m = ismember(T(j, b), H);
            if m == 0 then
                Q = [Q; T(j, 1) T(j, 2)];
            end if
        end for
        [q1 q2] = size(Q); Q = Q(2 : q1, :);
        for j = 1 : s1 − 1 do
            V(S(j, 1), S(j, 2) : n) = 1;
        end for
        z = r − p;
        for j = 1 : z do
            V(Q(j, 1), Q(j, 2) : n) = 1;
        end for
        G = eye(n, n); x = 0;
        for j = p : −1 : s + 1 do
            E = eye(n, n); x = x + 1;
            for i = H(j) : −1; S(s1 − x, 1) + 1 do
                E(i, i − 1) = −1;
            end for
            G = E^{−1} ∗ G;
        end for
        V = G ∗ V;
    end if
end if
```

Finally, we consider a triple $(n, r, p)(1, i_2, \ldots, i_p)$-negatively realizable of the type-I (type-II), and we construct Algorithm 3 (Algorithm 4) to obtain a type-I (type-II) t.n.p. matrix A associated with the given triple. These algorithms are based on Procedure 2.

Algorithm 3 $A = TNPI(n, r, p, H = [1, i_2, \ldots, i_p])$

1: $V = TPV(n, r, p, H)$;
2: $t = [0 \text{ ones}(1, n-1)]$;
3: $D = eye(n, n)$;
4: $D(1, 1) = -t * V(:, n)$;
5: $A = tril(ones(n, n)) * D * V$;

Algorithm 4 $A = TNPII(n, r, p, H = [1, 2, i_3, \ldots, i_p])$

1: $V = TPV(n, r, p, H)$;
2: $t = [0 \text{ ones}(1, n-1)]$;
3: $D = eye(n, n)$;
4: $D(1, 1) = -t * V(:, n);\ D(2, 2) = -1$;
5: $L = [[0\ 1; 1\ 0]\ \text{zeros}(2, n-2); \text{ones}(n-2, 1) - \text{ones}(n-2, 1)\ \text{tril}(\text{ones}(n-2, n-2))]$;
6: $A = L * D * V$;

As we have seen in the proof of Proposition 8 to obtain a t.n.p. matrix by Procedure 2 we only need that $d_1 \geq \sum_{j=2}^{i_p} v_{jn}$. As a consequence, from Algorithms 3 and 4, we can obtain different type-I and type-II t.n.p. matrices associated with the given triple by changing the value of $D(1, 1)$ and whenever $D(1, 1) \leq -t * V(:, n)$.

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