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Additional Information

Irreducible totally nonnegative matrices with a prescribed Jordan structure ☆

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Abstract

Let $A \in \mathbb{R}^{n \times n}$ be an irreducible totally nonnegative matrix with rank r and principal rank p, that is, all minors of A are nonnegative, r is the size of the largest invertible square submatrix of A and p is the size of its largest invertible principal submatrix. A triple (n, r, p) is called realizable if there exists an $n \times n$ irreducible totally nonnegative matrix A with rank r and principal rank p.

In this work we present a method to construct irreducible totally nonnegative matrices associated with a realizable triple (n, r, p) and with a prescribed Jordan canonical form corresponding to its zero eigenvalue.

Keywords: Totally nonnegative matrix, irreducible matrix, principal rank, Jordan canonical form.
AMS classification: 15A03, 15A15, 65F40

1. Introduction

A matrix $A \in \mathbb{R}^{n \times n}$ is called totally nonnegative if all its minors are nonnegative and it is abbreviated as TN, see for instance [1, 7, 8, 9, 10]. Numerous researchers have worked with TN matrices due to its theoretical importance as well as its wide applications in statistics, economics, biology, combinatorics, dynamic systems, approximation theory and computer-aided geometric design [10]-[18].

We recall that A is an *irreducible* matrix if there is not a permutation matrix P such that $PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$, where O is an $(n-r) \times r$ zero matrix $(1 \le r \le n-1)$. In [9, p. 87] the authors denoted by ITN the irreducible TN matrices.

The rank of A, denoted by rank(A), is the size of the largest invertible square submatrix of A. The principal rank of A, denoted by p-rank(A), is the size of

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the largest invertible principal submatrix of A. If there exists an ITN matrix $A \in \mathbb{R}^{n\times n}$ with $\text{rank}(A) = r$ and p-$\text{rank}(A) = p$, then the triple (n, r, p) is called *realizable* [8, p. 709], and A is considered as an ITN matrix *associated with* the triple (n, r, p). Furthermore, the nonzero eigenvalues of A are positive and distinct [8, Theorem 3.3]. In fact, if $\lambda_1, \ldots, \lambda_p, \ldots \lambda_n$ are the eigenvalues of A, we have

$$\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0, \text{ and } \lambda_{p+1} = \lambda_{p+2} = \ldots = \lambda_n = 0, \tag{1}$$

that is, if A is an associated matrix with a realizable triple (n, r, p) then p is the number of nonzero eigenvalues of A and $n - p$ is the algebraic multiplicity of its zero eigenvalue. Moreover, since $\text{rank}(A^p) = p$ the size of the zero Jordan blocks of A is at most p. So, the matrix A has $n - r$ zero Jordan blocks whose sizes are given by the Segre characteristic of A relative to its zero eigenvalue [19]. This sequence, denoted by $S = (s_1, s_2, \ldots, s_{n-r})$, satisfies

$$\begin{cases} s_1 \leq \min\{r - p + 1, p\} \\ s_i \leq s_{i-1}, \quad i = 2, \ldots, n - r \\ \sum_{i=1}^{n-r} s_i = n - p. \end{cases} \tag{2}$$

The Weyr characteristic $W = (w_1, w_2, w_3, \ldots, w_{s_1})$ of A, relative to its zero eigenvalue, is the conjugated sequence of the Segre characteristic of A associated with the same eigenvalue [19], that is, $w_i := \text{Card}\{k : s_k \geq i\}$, for $i = 1, 2, \ldots, s_1$. In this case this sequence satisfies

$$\begin{cases} w_1 = \dim \text{Ker}(A) = n - r \\ w_i \leq w_{i-1}, \quad i = 2, 3, \ldots, s_1 \\ \sum_{j=1}^{i} w_j = \dim \text{Ker}(A^i) \\ \sum_{j=1}^{s_1} w_j = \dim \text{Ker}(A^{s_1}) = n - p. \end{cases} \tag{3}$$

The problem to characterize completely all possible Jordan structures of A has been studied by several authors (see for instance [6, 8, 9]) extending the classical result of Gantmacher and Krein who introduced the total positivity and studied the eigenvalues of the oscillatory matrices. Otherwise, the relationship between the Jordan structure of two matrices sufficiently close have been studied in the matrix literature, deflation problems, the pole assignment problem and the stability in control problems (see [3] and references therein).

Using Number Theory, the authors in [6] calculate the number of zero Jordan canonical forms of ITN matrices associated with a realizable triple (n, r, p). Furthermore, they obtain all these zero Jordan canonical forms by using full rank factorization and the Flanders theorem.

If we consider a realizable triple (n, r, p) and one zero Jordan canonical form associated with (n, r, p), that is, a prescribed Segre characteristic S satisfying the relations given in (2), an important question deals with how to construct an ITN matrix A associated with (n, r, p) and exactly with the prescribed Segre characteristic S corresponding to its zero eigenvalue. In Section 3 we present a method for constructing that matrix A. Previously in Section 2, we present Procedure 1 and the corresponding algorithm to construct an upper echelon TN matrix $U \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(U) = r$, p-$\mathrm{rank}(U) = p$ and $n - r$ zero Jordan blocks whose sizes are given by the Segre characteristic $S = (s_1, s_2, \ldots, s_{n-r})$ satisfying (2). In Procedure 1 we use the following nonsingular TN matrices "Min" of size $q \times q$ (see [2, 10]),

$$M_q = [\min\{i,j\}]_q = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & q-1 & q-1 \\ 1 & 2 & 3 & \cdots & q-1 & q \end{bmatrix}_{q\times q} . \quad (4)$$

This matrix that we called M_q is the inverse of a tridiagonal matrix and it is also ITN.

In [5] the authors use the principal rank to study the dependence relations between rows and columns of an ITN matrix and they introduce the sequence of the first p-indices of linearly independent rows and columns of A. They consider the notation given in [1], that is, for $k, n \in \mathbb{N}$, $1 \le k \le n$, $\mathcal{Q}_{k,n}$ denotes the set of all increasing sequences of k natural numbers less than or equal to n. If $A \in \mathbb{R}^{n\times n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathcal{Q}_{k,n}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathcal{Q}_{k,n}$, $A[\alpha|\beta]$ denotes the $k\times k$ submatrix of A lying in rows α_i and columns β_i, $i = 1, 2, \ldots, k$. The principal submatrix $A[\alpha|\alpha]$ is abbreviated as $A[\alpha]$.

Definition 1 (Definition 1 of [5]). *Let $A \in \mathbb{R}^{n\times n}$ be a matrix with p-rank$(A) = p$. We say that the sequence of integers $\alpha = \{i_1, i_2, \ldots, i_p\} \in \mathcal{Q}_{p,n}$ is the* sequence of the first p-indices of A *if for $j = 2, \ldots, p$ we have*

$$\det(A[i_1, i_2, \ldots, i_{j-1}, i_j]) \neq 0,$$
$$\det(A[i_1, i_2, \ldots, i_{j-1}, t]) = 0, \quad i_{j-1} < t < i_j.$$

In the next section we introduce a procedure to construct an upper echelon TN matrix $U \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(U) = r$, p-$\mathrm{rank}(U) = p$, $n - r$ zero Jordan blocks of sizes $S = (s_1, s_2, \ldots, s_{n-r})$ satisfying (2), and U is partitioned into $s_1 + 1$ blocks. Furthermore, we obtain the sequence of the first p-indices of U. This sequence is used in Theorem 2 of Section 3 to prove that U and the ITN matrix $A = LU$ have the same zero Jordan structure, where L is a lower triangular matrix and A is calculated by Algorithm 2.

2. An upper echelon TN matrix U with a prescribed zero Jordan structure

We present a procedure to construct an upper echelon matrix $U \in \mathbb{R}^{n\times n}$, we recall [4, Section 1] that U is an upper echelon matrix if it satisfies the following conditions:

1. The first nonzero entry in each row is called the leading entry for that row.
2. Each leading entry is to the right of the leading entry in the row above it.
3. Rows with all zero elements, if any, are below rows having a nonzero element.

Procedure 1. *This procedure obtains an upper echelon TN matrix $U \in \mathbb{R}^{n\times n}$ with rank$(U) = r$, p-rank$(U) = p$ and $n-r$ zero Jordan blocks of sizes $S = (s_1, s_2, \ldots, s_{n-r})$ satisfying (2). Moreover, the procedure also gives the sequence of the first p-indices of U.*

1. *Construct the block U_1 of size $n_1 \times n$, with $n_1 = p+1-s_1$ in the following form,*
$$U_1 = [\ U_{11}\ U_{12}\ U_{13}\ \cdots\ U_{1,s_1-1}\ U_{1,s_1}\ U_{1,s_1+1}\ ],$$
with
$$U_{11} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n_1\times n_1}$$
and
$$[\ U_{12}\ U_{13}\ \cdots\ U_{1,s_1-1}\ U_{1,s_1}\ U_{1,s_1+1}\ ] = U_{11}(:, n_1) * \mathrm{ones}(1, n-n_1).$$
Note that $\mathrm{rank}(U_1) = \mathrm{rank}(U_{11}) = n_1$. *So, the first n_1-indices of U are $\alpha = \{1, 2, \ldots, n_1\}$.*

2. *Obtain the conjugated sequence $W = (w_1, w_2, \ldots, w_{s_1})$ of S and define the sequence $R = (0, r_2, \ldots, r_{s_1})$, such that $r_i = w_i$ for $i = 2, 3, \ldots, s_1$.*

3. *Construct the block U_2 of size $(r_2+1)\times n$, using the matriz given in (4) in the following form,*
$$U_2 = [\ \mathrm{zeros}(r_2+1, n_1+r_2)\ U_{23}\ U_{24}\ \cdots\ U_{2,s_1-1}\ U_{2,s_1}\ U_{2,s_1+1}\ ],$$
where $U_{23} = M_{r_2+1}$ and
$$[\ U_{24}\ \cdots\ U_{2,s_1}\ U_{2,s_1+1}\ ] = U_{23}(:, r_2+1) * \mathrm{ones}(1, n-n_1-2r_2-1).$$
Using this construction we have that $\mathrm{rank}(U_2) = \mathrm{rank}(U_{23}) = r_2+1$ *and $n_1 + r_2 + 1$ is added to the list of the first p-indices. Therefore, the first (n_1+1)-indices of U are $\{1, 2, \ldots, n_1, n_1+r_2+1\}$.*

4. *Construct the blocks U_i of size $(r_{i-1}+1)\times n$, for $i=3,4,\ldots,s_1-1$, using the matrix given in (4) in the following form,*

$$U_i = \left[\ \text{zeros}\left(r_{i-1}+1, n_1+r_2+i-2+\sum_{j=2}^{i-1} r_j\right)\ U_{i,i+1}\ U_{i,i+2}\ \cdots\ U_{i,s_1}\ U_{i,s_1+1}\ \right],$$

- *if $r_i = r_{i-1}$, then*

$$[\,U_{i,i+1}U_{i,i+2}\ \cdots\ U_{i,s_1}\ U_{i,s_1+1}\,] = [\,M_{r_i+1}\ M_{r_i+1}(:,r_i{+}1)\cdots\ M_{r_i+1}(:,r_i{+}1)],$$

- *if $r_i < r_{i-1}$, then*

$$[\ U_{i,i+1}\ U_{i,i+2}\ \cdots\ U_{i,s_1}\ U_{i,s_1+1}\ ] =$$
$$= \left[\begin{array}{c|cccc} M_{r_i+1} & M_{r_i+1}(:,r_i+1) & M_{r_i+1}(:,r_i+1) & \cdots & M_{r_i+1}(:,r_i+1)] \\ \hline M_{r_i+1}(r_i+1,:) & r_i+1 & r_i+1 & \cdots & r_i+1 \\ \vdots & \vdots & \vdots & & \vdots \\ M_{r_i+1}(r_i+1,:) & r_i+1 & r_i+1 & \cdots & r_i+1 \end{array}\right],$$

with $\operatorname{rank}(U_i) = \operatorname{rank}(U_{i,i+1}) = r_i+1,\ i=3,4,\ldots,s_1$. *Moreover $n_1+1+r_2+\sum_{j=2}^{i-1}(r_j+1)$ is added to the sequence of the first p-indices of U.*

5. *Construct the block U_{s_1}, of size $(r_{s_1-1}+1)\times n$, using the matrix given in (4) in the following form,*

$$U_{s_1} = \left[\ \text{zeros}\left(r_{s_1-1}+1, n_1+r_2+s_1-2+\sum_{j=2}^{s_1-1} r_j\right)\quad U_{s_1,s_1+1}\ \right],$$

where $\operatorname{size}(U_{s_1,s_1+1}) = (r_{s_1-1}+1)\times((n-r)-r_2+r_{s_1}+1)$, *and*

- *if $r_{s_1} = r_{s_1-1}$, then*

$$U_{s_1,s_1+1} = [\ M_{r_{s_1}+1}\ M_{r_{s_1}+1}(:,r_{s_1}+1)\cdots\ M_{r_{s_1}+1}(:,r_{s_1}+1)],$$

- *if $r_{s_1} < r_{s_1-1}$, then*

$$U_{s_1,s_1+1} = \left[\begin{array}{c|ccc} M_{r_{s_1}+1} & M_{r_{s_1}+1}(:,r_{s_1}+1) & \cdots & M_{r_{s_1}+1}(:,r_{s_1}+1)] \\ \hline M_{r_{s_1}+1}(r_{s_1}+1,:) & r_{s_1}+1 & \cdots & r_{s_1}+1 \\ \vdots & \vdots & & \vdots \\ M_{r_{s_1}+1}(r_{s_1}+1,:) & r_{s_1}+1 & \cdots & r_{s_1}+1 \end{array}\right],$$

$\operatorname{rank}(U_{s_1}) = \operatorname{rank}(U_{s_1,s_1+1}) = r_{s_1}+1$ *and $n_1+1+r_2+\sum_{j=2}^{s_1-1}(r_j+1)$ is added to the sequence of the first p-indices of U.*

6. *Finally, construct the last block U_{s_1+1} equal to $O_{((n-r)-r_2+r_{s_1})\times n}$.*

7. The matrix U is

$$U=\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \vdots \\ U_{s_1-1} \\ U_{s_1} \\ U_{s_1+1} \end{bmatrix}=\begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & \cdots & U_{1,s_1-1} & U_{1,s_1} & U_{1,s_1+1} \\ O & O & U_{23} & U_{24} & U_{25} & \cdots & U_{2,s_1-1} & U_{2,s_1} & U_{2,s_1+1} \\ O & O & O & U_{34} & U_{35} & \cdots & U_{3,s_1-1} & U_{3,s_1} & U_{3,s_1+1} \\ O & O & O & O & U_{45} & \cdots & U_{4,s_1-1} & U_{4,s_1} & U_{4,s_1+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ O & O & O & O & O & \cdots & O & U_{s_1-1,s_1} & U_{s_1-1,s_1+1} \\ O & O & O & O & O & \cdots & O & O & U_{s_1,s_1+1} \\ O & O & O & O & O & \cdots & O & O & O \end{bmatrix}.$$

whose partition into blocks by rows is

$$n_1,\ r_2+1,\ r_2+1,\ r_3+1,\ \cdots,\ r_{s_1-1}+1,\ (n-r)-r_2+r_{s_1},$$

and the partition into blocks by columns is

$$n_1,\ r_2,\ r_2+1,\ r_3+1,\ \cdots,\ r_{s_1-1}+1,\ (n-r)-r_2+r_{s_1}+1.$$

Note that the elements of both partitions are positive and each sum is equal to n, therefore the matrix U is well defined.

Next result proves that $U\in\mathbb{R}^{n\times n}$ constructed by applying Procedure 1 has $\operatorname{rank}(U)=r$, p-$\operatorname{rank}(U)=p$ and $n-r$ zero Jordan blocks of sizes $S=(s_1,s_2,\ldots,s_{n-r})$.

Theorem 1. *Consider the matrix U obtained by applying Procedure 1. Then, the following conditions hold:*

1. *U is a TN matrix with $\operatorname{rank}(U)=r$ and p-$\operatorname{rank}(U)=p$.*
2. *U has $n-r$ zero Jordan blocks of sizes $S=(s_1,s_2,\ldots,s_{n-r})$.*

Proof.

1. By construction, the matrix U is a TN matrix. Moreover, using conditions given in (3) we have that,

$$\begin{aligned}\operatorname{rank}(U) &= \operatorname{rank}(U_{11})+\sum_{i=2}^{s_1}\operatorname{rank}(U_{i,i+1})=p+1-s_1+\sum_{i=2}^{s_1}r_i+(s_1-1)\\ &= p+\sum_{i=2}^{s_1}w_i=n-(n-r)=r.\end{aligned}$$

Finally, from the sequence of the first p-indices obtained in Procedure 1 and the structure of U, it is easily checked that p-$\operatorname{rank}(U)=p$.

2. Taking into account the relationship between the blocks of U, by similarity and permutation similarity we can transform U into the matrix

$$T = \left[\begin{array}{c|cccccccc} T_{11} & O & O & O & O & \cdots & O & O & O \\ \hline O & O & T_{23} & O & O & \cdots & O & O & O \\ O & O & O & T_{34} & O & \cdots & O & O & O \\ O & O & O & O & T_{45} & \cdots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ O & O & O & O & O & \cdots & O & T_{s_1-1,s_1} & O \\ O & O & O & O & O & \cdots & O & O & T_{s_1,s_1+1} \\ O & O & O & O & O & \cdots & O & O & O \end{array}\right],$$

where the block partition by rows and columns is

$$p,\ r_2,\ r_2,\ r_3,\ \ldots,\ r_{s_1-2},\ r_{s_1-1},\ r_{s_1}+(n-r)-r_2.$$

The block T_{11} is nonsingular, of size $p \times p$, with a unique Jordan block associated with the eigenvalue $\lambda = 1$, and with the following structure

$$T_{11} = \begin{array}{c} \left[\begin{array}{ccccc|cccc} 1 & 1 & \cdots & 1 & 1 & \star & \star & \cdots & \star \\ 0 & 1 & \cdots & 1 & 1 & \star & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \star & \star & \cdots & \star \\ 0 & 0 & \cdots & 0 & 1 & \star & \star & \cdots & \star \\ \hline 0 & 0 & \cdots & 0 & 0 & 1 & \star & \cdots & \star \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & \star \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\ \vdots\\1\\1\end{array}}\right\} n_1 \\ \left.\vphantom{\begin{array}{c}1\\1\\ \vdots\\1\end{array}}\right\} p-n_1 \end{array}.$$

The block $T_{23} = I_{r_2 \times r_2}$ and for $i = 3, 4, \ldots, s_1 - 1$, the blocks $T_{i,i+1}$ have the following form

$$T_{i,i+1} = \begin{bmatrix} I_{r_i \times r_i} \\ O \end{bmatrix} \begin{array}{c} r_i \\ r_{i-1} - r_i \end{array}$$

and finally,

$$T_{s_1,s_1+1} = \begin{array}{cc} \begin{bmatrix} I_{r_{s_1} \times r_{s_1}} & O \\ O & O \end{bmatrix} & \begin{array}{l} r_{s_1} \\ r_{s_1-1} - r_{s_1}. \end{array} \\ \begin{array}{cc} r_{s_1} & (n-r)-r_2 \end{array} & \end{array}$$

Obviously, $\lambda = 1$ is an eigenvalue of T with unique Jordan block of size p. The other eigenvalue is $\lambda = 0$. From the structure of the blocks $T_{i,i+1}$

and using (3) we have

$$\begin{aligned}
\text{rank}(T) &= p + r_2 + r_3 + \cdots + r_{s_1-1} + r_{s_1} \\
&= p + (n-p) - w_1 = n - (n-r) = r \;\rightarrow\; \dim \text{Ker}(T) = n - r \\
\text{rank}(T^2) &= p + r_3 + \cdots + r_{s_1-1} + r_{s_1} = p + (n-p) - w_1 - r_2 \\
&= n - (n-r) - r_2 = r - r_2 \;\rightarrow\; \dim \text{Ker}(T^2) = n - r + r_2 \\
&\vdots \\
\text{For } k &= 3, 4, \ldots, s_1 - 1 \\
\text{rank}(T^k) &= p + r_{k+1} + \cdots + r_{s_1-1} + r_{s_1} = p + (n-p) - w_1 - r_2 - \cdots - r_k \\
&= r - r_2 - r_3 - \cdots - r_k \;\rightarrow\; \dim \text{Ker}(T^k) = n - r + \sum_{i=2}^{k} r_i.
\end{aligned}$$

Finally, $\text{rank}(T^{s_1}) = p \;\rightarrow\; \dim \text{Ker}(T^{s_1}) = n - p$.

Then, we have the following sequence

$$\begin{aligned}
&(\dim \text{Ker}(T), \dim \text{Ker}(T^2) - \dim \text{Ker}(T), \dim \text{Ker}(T^3) - \dim \text{Ker}(T^2), \\
&\dim \text{Ker}(T^4) - \dim \text{Ker}(T^3), \ldots, \dim \text{Ker}(T^{s_1}) - \dim \text{Ker}(T^{s_1-1})) \\
&= (n-r, r_2, r_3, \ldots, r_{s_1-1}, r_{s_1}) = (n-r = w_1, w_2, w_3, \ldots, w_{s_1-1}, w_{s_1})
\end{aligned}$$

whose conjugated sequence $S = (s_1, s_2, \ldots, s_{n-r})$ gives the sizes of the zero Jordan blocks of T.

Since U is similar to T, both have the same Jordan structure. Then, the result holds. □

In the following example we obtain a matrix U using Procedure 1.

Example 1. *Obtain a 15×15 upper echelon TN matrix U with rank$(U) = 11$, p-rank$(U) = 7$ and with 4 zero Jordan blocks of sizes $S = (3, 3, 1, 1)$.*

By Procedure 1 we obtain the upper echelon TN matrix

$$
U = \left[\begin{array}{ccccc|cc|ccc|ccccc}
\boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 & 3 & 3 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
$$

and the sequence $\alpha = [1, 2, 3, 4, 5, 8, 11]$ of its first 7-indices (see Definition 1).

The matrix T similar to U that appears in Theorem 1 is given by

$$
T = \left[\begin{array}{ccccccr|cc|cc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
$$

whose partition in blocks by rows and columns is $7, 2, 2, 4$. From this partition and the number of the blocks of T we obtain that its Weyr characteristic relative to zero is $W = (4, 2, 2)$. Then, its Segre characteristic is $S = (3, 3, 1, 1)$.

Remark 1. *Let us recall that Koev [14, 15] computes accurate eigenvalues of a TN matrix A starting from its bidiagonal factorization. In fact, if we consider the TN matrix U obtained in Example 1, the command:*

$$(B, C) = STNBD(U)$$

computes the bidiagonal decomposition of the matrix U by performing Neville elimination. The bidiagonal factorization is stored in $[B, C]$. Then,

$$(z, jb) = \mathit{STNEigenValues}(B, C)$$

computes the eigenvalues of the TN matrix U with high relative accuracy and computing the zero eigenvalues exactly. The vector z stores the eigenvalues of U, and the vector jb stores the sizes of the zero Jordan blocks of U, that is, jb is the Segre characteristic of U associated with the zero eigenvalue. For the matrix U obtained in Example 1 we have,

$$\begin{aligned} z &= [1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0], \\ jb &= [3,\ 3,\ 1,\ 1]. \end{aligned}$$

Finally, we present the Algorithm 1 associated with Procedure 1 to obtain the upper echelon matrix U and its first p-indices.

Algorithm 1 $[U, a] = \mathrm{TPU}(n, r, p, S)$

```
b = size(S,2); W(1) = 0; T = S − ones(1,b); = zones(b,1);
for j = 2 : S(1) do
    x=T > j-2;
    W(j)=x*z;
end for
for i = 1 : W(2) + 1 do
    for t = 1 : W(2) + 1 do
        M(i,t) = min(i,t);
    end for
end for
n1 = p + 1 − S(1); k = n1 + 1 + W(2); p2 = n − k − W(2); a = [1 : n1, k]
U = triu(ones(n1,n)); U = [U; zeros(W(2) + 1, n1 + W(2)), M, M(:, W(2) + 1) ∗
    ones(1, p2)];
for j = 3 : S(1) do
    k = k + 1 + W(j − 1); f = n − k − W(j);
    U1 = [zeros(W(j−1)+1, k−1)]; U2 = [M(1 : W(j)+1, W(j)+1)∗ones(1, f)];
    if W(j) == W(j − 1) then
        U = [U; U1, M(1 : W(j) + 1, 1 : W(j) + 1), U2];
    else
        M1 = [M(1 : W(j)+1, 1 : W(j)+1); ones(W(j−1)−W(j), 1)∗M(W(j)+1, 1 :
            W(j) + 1)];
        M2 = [U2; (W(j) + 1) ∗ ones(W(j − 1) − W(j), f)];
        U = [U; U1, M1, M2];
    end if
    a = [a, k];
end for
s = S(1);
U = [U; zeros(W(s) + (n − r) − W(2), n)];
```

3. An ITN matrix with a prescribed zero Jordan structure

In this section we construct an ITN matrix A associated with a realizable triple (n, r, p) and with a prescribed zero Jordan structure given by the Segre characteristic $S = (s_1, s_2, \ldots, s_{n-r})$.

Taking into account that in Section 2 we have presented an algorithm that computes an upper echelon TN matrix $U \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(U) = r$, p-rank$(U) = p$ and $n-r$ zero Jordan blocks of sizes $S = (s_1, s_2, \ldots, s_{n-r})$ satisfying (2), now we can consider the following algorithm to obtain the desired matrix A. This algorithm computes an ITN matrix $A \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(A) = r$, p-rank$(A) = p$ and with $n-r$ zero Jordan blocks of sizes $S = (s_1, s_2, \ldots, s_{n-r})$ satisfying (2).

Algorithm 2 $A = \mathrm{Matrix}A(n, r, p, S)$

```
[U, I] = TPU(n, r, p, S);
L = tril(ones(n))
A = L * U
```

Now we give some properties of the matrix A computed in the Algorithm 2.

Proposition 1. *The matrix A constructed by applying Algorithm 2 satisfies the following properties:*

1. *A is ITN.*
2. *$\mathrm{rank}(A) = r$.*
3. *A and U have the same sequence of first p-indices and the same principal rank.*

Proof.

1. By Theorem 1, U is a TN matrix. Then A is an ITN matrix because it is a product of TN matrices and
$$a_{ij} = l^{(i)} u_j > 0, \quad \forall\, i, j = 1, 2, \ldots, n.$$
2. Since L is a nonsingular matrix and by Theorem 1, $\mathrm{rank}(U) = r$, then $\mathrm{rank}(A) = \mathrm{rank}(U) = r$.
3. By [5, Procedure 2 and Proposition 3] we obtain that A has the same p-rank and the same sequence of the first p-indices as the matrix U. □

Remark 2. *Note that the matrix $A = LU$ can be partitioned into blocks in the same form as the matrix U, constructed using Procedure 1. Thus,*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1,s_1} & A_{1,s_1+1} \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2,s_1} & A_{2,s_1+1} \\ A_{31} & A_{32} & A_{33} & A_{34} & \cdots & A_{3,s_1} & A_{3,s_1+1} \\ A_{41} & A_{42} & A_{43} & A_{44} & \cdots & A_{4,s_1} & A_{4,s_1+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ A_{s_1,1} & A_{s_1,2} & A_{s_1,3} & A_{s_1,4} & \cdots & A_{s_1,s_1} & A_{s_1,s_1+1} \\ A_{s_1+1,1} & A_{s_1+1,2} & A_{s_1+1,3} & A_{s_1+1,4} & \cdots & A_{s_1+1,s_1} & A_{s_1+1,s_1+1} \end{bmatrix}$$

and the following relations are obtained,

$$[A_{12}\ A_{13}\ \cdots\ A_{1,s_1}\ A_{1,s_1+1}] = A_{11}(:, n_1) * ones(1, n - n_1).$$

Moreover,

$$\begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \\ \vdots & \vdots \\ A_{s_1,1} & A_{s_1,2} \\ A_{s_1+1,1} & A_{s_1+1,2} \end{bmatrix} = ones(n - n_1, 1) * [\, A_{11}\ A_{12}\,](n_1, :).$$

For $i = 2, \ldots, s_1$

$$[A_{i,i+2}\ A_{i,i+3}\ \cdots\ A_{i,s_1}\ A_{i,s_1+1}] = A_{i,i+1}(:, r_i+1) * ones(1, n - n_1 - r_2 - \sum_{j=2}^{i}(r_j+1)).$$

and

$$\begin{bmatrix} A_{i+1,i+1} \\ A_{i+2,i+1} \\ \vdots \\ A_{s_1+1,i+1} \end{bmatrix} =$$

$$= \quad ones\left(n - n_1 - (r_2+1) - \sum_{j=3}^{i}(r_{j-1}+1), 1\right) * [\, A_{i,i+1}\ A_{i,i+2}\ \ldots\ A_{i,i+s_1+1}\,](r_i+1, :).$$

From these relations we have,

$$\begin{aligned} &rank(A_{11}) = rank(U_{11}) = n_1 \\ &rank(A_{23}) = rank(U_{23}) = r_2 + 1 \\ &rank(A_{i,i+1}) = rank(U_{i,i+1}) = r_i + 1 \quad i = 3, 4, \ldots, s_1. \end{aligned}$$

Since the r nonzero eigenvalues of U are equal to 1 and those of A are pairwise distinct, $\lambda_1 > \lambda_2 > \ldots > \lambda_r > 0$ (see (1)), then A and U are not similar matrices but we will prove that the zero Jordan structure of both matrices is the same.

Theorem 2. *Consider the ITN matrix $A = LU$ computed by applying Algorithms 1 and 3 and associated with the realizable triple (n, r, p). Then, the matrices A and U have the same zero Jordan structure.*

Proof. Suppose that $S = (s_1, s_2, \ldots, s_{n-r})$ are the sizes of the zero Jordan blocks of matrix U. From Algorithm 1 we obtain the matrix U and the sequence of the first p-indices of U

$$\begin{gathered}(1,\, 2,\, \ldots,\, n_1,\, n_1 + r_2 + 1,\, n_1 + r_2 + 1 + (r_2 + 1),\, n_1 + r_2 + 1 + (r_2 + 1) + (r_3 + 1),\\ n_1 + r_2 + 1 + (r_2 + 1) + (r_3 + 1) + (r_4 + 1),\, \ldots,\, n_1 + r_2 + 1 + \textstyle\sum_{j=2}^{s_1-1}(1 + r_j)).\end{gathered}$$

From Proposition 1 this sequence is also the sequence of the first p-indices of A. By [5, Theorem 1] the matrix A is similar to the following block matrix

$$B = \begin{bmatrix} B_{11} & O \\ O & B_{22} \end{bmatrix} = \left[\begin{array}{c|cccccc} B_{11} & O & O & O & \cdots & O & O \\ \hline O & O & B_{23} & B_{24} & \cdots & B_{2,s_1} & B_{2,s_1+1} \\ O & O & O & B_{34} & \cdots & B_{3,s_1} & B_{3,s_1+1} \\ O & O & O & O & \cdots & B_{4,s_1} & B_{4,s_1+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & O & \cdots & B_{s_1-1,s_1} & B_{s_1-1,s_1+1} \\ O & O & O & O & \cdots & O & B_{s_1,s_1+1} \\ O & O & O & O & \cdots & O & O \end{array}\right]$$

where $B_{11} \in \mathbb{R}^{p \times p}$ is a nonsingular matrix and from (1) all its eigenvalues are positive and pairwise distinct. B_{22} has the following partition into blocks by rows and columns,

$$r_2,\ r_2, r_3,\ r_4,\ \ldots,\ r_{s_1-1},\ r_{s_1} + (n - r) - r_2,$$

where $\operatorname{rank}(B_{i,i+1}) = r_i$, for $i = 2, 3, \ldots, s_1$, and $\operatorname{rank}(B_{22}) = r - p$. By similarity we can transform B into the matrix

$$Y = \begin{array}{@{}c@{}l@{}} \left[\begin{array}{c|cccccccc} B_{11} & O & O & O & O & \cdots & O & O & O \\ \hline O & O & Y_{23} & O & O & \cdots & O & O & O \\ O & O & O & Y_{34} & O & \cdots & O & O & O \\ O & O & O & O & Y_{45} & \cdots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ O & O & O & O & O & \cdots & O & Y_{s_1-1,s_1} & O \\ O & O & O & O & O & \cdots & O & O & Y_{s_1,s_1+1} \\ O & O & O & O & O & \cdots & O & O & O \end{array}\right] & \begin{array}{l} p \\ r_2 \\ r_2 \\ r_3 \\ \vdots \\ r_{s_1-2} \\ r_{s_1-1} \\ r_{s_1} + (n-r) - r_2 \end{array} \end{array}$$

where for $i = 2, 3, \ldots, s_1$, the blocks $Y_{i,i+1}$ are equal to the blocks $T_{i,i+1}$ obtained in Theorem 1. Therefore, $Y_{23} = I_{r_2 \times r_2}$ and for $i = 3, 4, \ldots, s_1 - 1$,

$$Y_{i,i+1} = \begin{array}{@{}c@{}l@{}} \begin{bmatrix} I_{r_i \times r_i} \\ \\ O \end{bmatrix} & \begin{array}{l} r_i \\ \\ r_{i-1} - r_i \end{array} \end{array}$$

and for s_1,

$$Y_{s_1,s_1+1} = \begin{array}{c} \left[\begin{array}{cc} I_{r_{s_1}\times r_{s_1}} & O \\ O & O \end{array}\right] \begin{array}{l} r_{s_1} \\ (n-r)-r_2. \end{array} \\ \begin{array}{cc} r_{s_1} & (n-r)-r_2 \end{array} \end{array}$$

Therefore, the matrices Y and T have the same zero Jordan structure and, by similarity, we conclude that U and A have also the same zero Jordan structure. $\square$

Example 2. *Given the realizable triple* $(15, 11, 7)$*, obtain an associated ITN matrix* A *with* 4 *zero Jordan blocks of sizes* $S = (3, 3, 1, 1)$*.*

We use the previous results given in Remark 2 and Theorem 2 and apply Algorithm 2 to calculate

$$A = \left[\begin{array}{ccccc|cc|ccc|ccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
\hline
1 & 2 & 3 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 8 & 10 & 11 & 11 & 11 & 11 & 11 & 11 \\
\hline
1 & 2 & 3 & 4 & 5 & 5 & 5 & 8 & 10 & 11 & 12 & 12 & 12 & 12 & 12 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 8 & 10 & 11 & 13 & 14 & 14 & 14 & 14 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 8 & 10 & 11 & 14 & 16 & 17 & 17 & 17 \\
\hline
1 & 2 & 3 & 4 & 5 & 5 & 5 & 8 & 10 & 11 & 14 & 16 & 17 & 17 & 17 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 8 & 10 & 11 & 14 & 16 & 17 & 17 & 17 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 8 & 10 & 11 & 14 & 16 & 17 & 17 & 17 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 8 & 10 & 11 & 14 & 16 & 17 & 17 & 17
\end{array}\right].$$

Remark 3. *As in Remark 1, we can apply to the matrix* A *the following two algorithms given in [14, 15],*

$$\begin{array}{l} (B, C) = STNBD(A); \\ (z, jb) = STNEigenValues(B, C) \end{array}$$

then the vector z *stores the eigenvalues of* A*, and the vector* jb *stores the sizes of the zero Jordan blocks of* A*, that is,* jb *is the Segre characteristic of* A *associated with the zero eigenvalue,*

$$\begin{aligned} z &= [115.1808,\ 13.5603,\ 3.3630,\ 1.6027,\ 0.6485,\ 0.3702,\ 0.2746,\ 0,\ 0,\ 0, \\ &\quad\ 0,\ 0,\ 0,\ 0,\ 0], \\ jb &= [3,\ 3,\ 1,\ 1]. \end{aligned}$$

Note that, if we consider a realizable triple (n, r, p), applying the algorithms given in [6] we compute all the Segre characteristics relative to the zero eigenvalue of the ITN matrices associated with (n, r, p). Now, for each Segre characteristic, we apply Algorithm 2 to construct an ITN matrix A associated with (n, r, p). For instance, if we have the realizable triple $(15, 11, 7)$, we obtain the following Segre characteristics,

$$\begin{array}{l} S_1 = (5\ 1\ 1\ 1) \\ S_2 = (4\ 2\ 1\ 1) \\ S_3 = (3\ 3\ 1\ 1) \\ S_4 = (3\ 2\ 2\ 1) \\ S_5 = (2\ 2\ 2\ 2) \end{array}$$

and the corresponding ITN matrix for each sequence. For instance, the ITN matrix A_1 associated with $(15, 11, 7)$ and with the Segre characteristic $S_1 = (5\ 1\ 1\ 1)$ is constructed by Algorithm 2 as follows,

$$A_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 2 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
1 & 2 & 3 & 3 & 5 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 11 & 12 & 12 & 12 & 12 & 12 & 12 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 11 & 12 & 13 & 13 & 13 & 13 & 13 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 11 & 12 & 14 & 15 & 15 & 15 & 15 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 11 & 12 & 14 & 15 & 15 & 15 & 15 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 11 & 12 & 14 & 15 & 15 & 15 & 15 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 11 & 12 & 14 & 15 & 15 & 15 & 15 \\
1 & 2 & 3 & 3 & 5 & 6 & 8 & 9 & 11 & 12 & 14 & 15 & 15 & 15 & 15
\end{bmatrix}.$$

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