A multistep Steffensen-type method for solving nonlinear systems

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Abstract

This paper is devoted to the semilocal analysis of a high order frozen Steffensen-type method. The methods are free of bilinear operators and derivatives, which constitutes the main limitation of the classical high order iterative schemes. Although the methods are more demanding, a semilocal convergence analysis is presented using weaker conditions than classical Steffensen's method.

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1 Introduction

Newton's method is the most usual tool to approximate the solution of a nonlinear equation $F(x) = 0$ ($F : D \subseteq X \to X$, X is a Banach space and D is a convex subset of X). Steffensen's method should be considered as a very good alternative, if we are not interested in the computation of derivatives, but having the same order convergence. Moreover, our aim is to study iterative methods with a generic number of steps k with the purpose of offering an alternative to choose an iterative method with the desired order of convergence taking into account simultaneously their efficiency. Some papers have been published for this purpose in the unidimensional case proposing optimal derivative free iterative methods, see for instance, [4] and [11]. But we concentrate in the most general case, we deal with Banach spaces with the aim of setting a semilocal convergence study.

In this paper we study the following k-step method that increases the order of a uniparametric type-Steffensen's method:

$$\begin{array}{rcl} For\ n & = & 1, 2, \dots \\ x_n^{(0)} & = & x_n, \\ x_n^{(1)} & = & x_n^{(0)} + \alpha \Gamma_n F(x_n), \\ x_n^{(2)} & = & x_n^{(1)} - \Gamma_n F(x_n^{(1)}), \\ & \vdots & \\ x_n^{(k)} & = & x_n^{(k-1)} - \Gamma_n F(x_n^{(k-1)}), \\ x_{n+1} & = & x_n^{(k)} \end{array} \tag{1}$$

where $\Gamma_n = [x_n, x_n + F(x_n); F]^{-1} \in L(X, X)$.

An advantage of these methods is that, as the matrix that appears at each sub-iteration is the same, the number of associated linear systems that we need to solve is smaller. This happens because, at each sub-iteration, only one LU decomposition is computed. In most cases, the computational cost of solving a linear system is more expensive than that of the evaluations of the operator. The maximum efficiency for a family of Newton-like methods with frozen derivatives, that is the number k of sub-iterations, depends on the problem, but it can be computed before solving it [3]. Moreover, this method is free of derivatives.

These schemes are generalizations of the third order iterative method

$$\begin{aligned} y_n &= x_n + \alpha \Gamma_n F(x_n), \\ x_{n+1} &= y_n - \Gamma_n F(y_n). \end{aligned} \tag{2}$$

studied in [1].

———————>>>>>>>>>>>>>>En este trabajo se utiliza el mismo gamma??????. Hacer algún comentario más....

The objective of the present paper is to generalize this study to the new family of iterative methods. Although the methods are more demanding, we are able to obtain a new semilocal convergence analysis using weaker conditions.

The fact of obtaining the semilocal convergence study for a generic number of steps is an important task that requires a more intricate deployment of conditions and of course a non trivial development for obtaining the whole process. However the final result can be very useful having into account that it offers a possibility of taking the iterative method that fits with the needs of a problem and compare with similar procedures of different convergences order.

———————>>>>>>>>>>>>>>Mejorar un poco la introducción

This paper is organized as follows. In Section 2 we study the semilocal convergence for the family (1), by using omega conditions for the divided differences and constructing adequate functions for bounding the iterates. Section 3 is devoted to develop an application for nonlinear systems of equations with maximum efficiency. We consider a special case of a nonlinear conservative system and approximate its solution by using different approximations by divided differences. Finally we give some concluding remarks.

Along the paper, let $U(v,\rho)$ and $\bar{U}(v,\rho)$ stand, respectively for the open and closed balls in X with center $v \in X$ and of radius $\rho > 0$.

2 Semilocal convergence study for Banach spaces

It is convenient for the semilocal convergence of our method to introduce some parameters and scalar functions.

Let $\gamma_0 > 0, \theta > 0, \eta > 0, \alpha \in \mathbb{R}$ be parameters.

Define

$$\begin{aligned} R^* &:= \sup\{t \geq 0 : \bar{U}(x_0,t)\} \subseteq D\}, \\ A &:= \{(s,t) : s \in [0,R^*], t \in [0,(1+\theta)R^*]\}. \end{aligned}$$

Let also $\omega_0 : A \to [0, \frac{1}{\gamma_0})$, $\omega : \mathbb{R}_+ \bigcup\{0\} \times \mathbb{R}_+ \bigcup\{0\} \to \mathbb{R}_+ \bigcup\{0\}$ be continuous and non-decreasing functions.

Moreover, define

$$\begin{aligned} R_0 &:= \sup\{(s,t) \in A : \gamma_0\omega_0(s,t) < 1\}, \\ b_0 &:= |\alpha|\gamma_0\omega_0(\gamma_0\eta|\alpha|,\eta) + |\alpha+1|, \\ \eta_0 &:= \gamma_0(b_0+|\alpha|)\eta, \\ \gamma &:= \gamma(s) = \gamma_0 max\{b_0, \omega_0(s,s+\eta), \sqrt{b_0\omega_0(\eta_0,(1+|\alpha|\gamma_0)\eta}\}, \\ \delta_0 &:= \delta_0(s) = \delta_{0,k}(s) = \omega_0(s,s+\eta)\gamma^{k-1}, \quad \text{for} \quad k = 1,2,\ldots, \\ \gamma_1 &:= \gamma_1(s) = \frac{\gamma_0}{1-\gamma_0\omega_0(s,(1+\theta)s)}, \\ \lambda_0 &:= |\alpha|\gamma_1\delta_0, \\ b_1 &:= |\alpha|\delta_0 + \omega(2s, 2s+\delta_0\eta)\}, \\ \lambda &:= \lambda(s) = max\{b_1, \gamma_1 b_1, \gamma_1(\omega(2s,2s+\delta_0\eta))\}. \end{aligned}$$

The semilocal convergence analysis is based on the following conditions:

(A.1) $F : D \subseteq X \to X$ is a nonlinear operator with a divided difference

$$[\cdot,\cdot;F] : D \times D \to L(X,X)$$

satisfying

$$[x, y; F](x - y) = F(x) - F(y)$$

for each $x, y \in D$.

There exists $x_0 \in D$ such that

$$\Gamma_0 = [x_0, x_0 + F(x_0)]^{-1} \in L(X, X).$$

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$$[x, y; F](x - y) = F(x) - F(y)$$

for each $x, y \in D$, with $x \neq y$.

(**A**.**2**) There exists $x_0 \in D$ such that $\Gamma_0 = [x_0, x_0 + F(x_0)]^{-1} \in L(X, X)$ and, for each $x, y \in D$,

$$||[x, y; F] - [x_0, x_0 + F(x_0); F)]|| \leq \omega_0(||x - x_0||, ||y - x_0 - F(x_0)||).$$

(**A**.**3**) For each $x, y, v, w \in U := D \bigcap U(x_0, R_0)$

$$||[x, y; F] - [v, w; F)]|| \leq \omega(||x - v||, ||y - w||).$$

(**A**.**4**) There exist $\theta > 0, \gamma_0 > 0, \eta > 0$ such that for each $x \in U$

$$\begin{aligned} ||[x, x_0; F]|| &\leq \theta, \\ ||\Gamma_0|| &\leq \gamma_0, \\ ||F(x_0)|| &\leq \eta. \end{aligned}$$

(**A**.**5**) For each $s \in [0, R_0]$

$$\begin{aligned} \gamma &= \gamma(s) < 1, \\ \lambda &= \lambda(s) < 1, \end{aligned}$$

(**A**.**6**) Equation

$$\left(\frac{\lambda}{1-\lambda} + \frac{\gamma}{1-\gamma} + |\alpha|\gamma_0 + \lambda_0\right)\eta - t = 0$$

has at least one positive zero. Denote by R the smallest such zero, and $R < R_0$.

(**A**.**7**)

$$\bar{U}(x_0, R_1) \subset D,$$

where $R_1 = (1+\theta)R_0 + \eta$.

We have the following result for the family (1).

Theorem 1 *Suppose that the conditions $(A.1) - (A.7)$ hold. Then, method (1) is well defined, remains in $U(x_0, R)$ and converges to a solution x^* of the equation $F(x) = 0$ in $\bar{U}(x_0, R)$.*

Proof:

We shall show sequence $\{x_n\}$ is complete and remains in $\bar{U}(x_0, R)$. Let $x \in \bar{U}(x_0, R_0)$, then we have that

$$||x_+F(x) - x_0|| \leq ||x - x_0|| + ||[x, x_0; F](x - x_0)|| + ||F(x_0)|| \leq (1+\theta)R_0 + \eta = R_1,$$

so, $x + F(x) \in D$.

By conditions $(A.1) - (A.2)$, iterates $x_0^{(0)}, x_0^{(1)}, \ldots, x_0^{(k)}$ are well defined. We can write by the first substep of method (1) that

$$\begin{aligned} F(x_0^{(1)}) &= F(x_0^{(1)}) - F(x_0^{(0)}) - \Gamma_0^{-1}(x_0^{(1)} - x_0^{(0)}) + (\alpha+1)F(x_0^{(0)}), \\ &= ([x_0^{(1)}, x_0^{(0)}; F] - [x_0^{(0)}, x_0^{(0)} + F(x_0^{(0)}); F])(x_0^{(1)} - x_0^{(0)}) + (\alpha+1)F(x_0^{(0)}). \end{aligned}$$

Notice that

$$||x_0^{(0)} - (x_0^{(0)} + F(x_0^{(0)}))|| = ||F(x_0^{(0)})|| \leq \eta < R,$$

and

$$||x_0^{(1)} - (x_0^{(0)}|| = ||\alpha\Gamma_0 F(x_0)|| \leq |\alpha|||\Gamma_0||||F(x_0)|| \leq |\alpha|\gamma_0\eta < R,$$

so, $x_0^{(0)} + F(x_0^{(0)}) \in U(x_0, R)$ and $X - 0^{(1)} \in U(x_0, R)$.

Thus, using $(A.2)-(A.4)$, we get in turn that

$$\begin{aligned}
||F(x_0^{(1)})|| &\leq \omega_0(||x_0^{(1)}-x_0^{(0)}||,||F(x_0^{(0)})||)||x_0^{(1)}-x_0^{(0)}||+|\alpha+1|||F(x_0^{(0)})|| \\
&\leq \omega_0(||\Gamma_0|||\alpha|||F(x_0^{(0)})||,||F(x_0^{(0)})||)||\Gamma_0|||\alpha|||F(x_0^{(0)})||+|\alpha+1|||F(x_0^{(0)})|| \\
&\leq \omega_0(\gamma_0|\alpha|\eta,\eta)\gamma_0|\alpha+1|\eta+|\alpha|\eta=b_0\eta,
\end{aligned}$$

so

$$\begin{aligned}
||x_0^{(2)}-x_0^{(1)}|| &= ||\Gamma_0 F(x_0^{(1)})|| \\
&\leq ||\Gamma_0||||F(x_0^{(1)})|| \\
&\leq \gamma_0 b_0\eta \leq \gamma\eta
\end{aligned} \tag{3}$$

by the definition of γ, and

$$\begin{aligned}
||x_0^{(2)}-x_0|| &\leq ||x_0^{(2)}-x_0^{(1)}||+||x_0^{(1)}-x_0|| \\
&\leq \gamma_0 b_0\eta+\gamma_0|\alpha|\eta=\eta_0<R,
\end{aligned}$$

by the definition of η_0 and $(A.6)$, so, $x_0^{(2)} \in U(x_0,R)$.

Similarly, for the second substep of (1) we can write

$$\begin{aligned}
F(x_0^{(2)}) &= F(x_0^{(2)})-F(x_0^{(1)})-\Gamma_0^{-1}(x_0^{(2)}-x_0^{(1)}), \\
&= ([x_0^{(2)},x_0^{(1)};F]-[x_0^{(0)},x_0^{(0)}+F(x_0^{(0)});F])(x_0^{(2)}-x_0^{(1)}),
\end{aligned}$$

leading by the definition of γ to

$$\begin{aligned}
||F(x_0^{(2)})|| &\leq \omega_0(||x_0^{(2)}-x_0^{(0)}||,||x_0^{(1)}-x_0^0-F(x_0^0)||)||x_0^{(2)}-x_0^{(1)}|| \\
&\leq \omega_0(\eta_0,|\alpha|\gamma_0\eta+\eta)\gamma_0 b_0\eta \leq \frac{\gamma^2\eta}{\gamma_0},
\end{aligned}$$

so

$$\begin{aligned}
||x_0^{(3)}-x_0^{(2)}|| &= ||\Gamma_0 F(x_0^{(2)})|| \\
&\leq ||\Gamma_0||\ ||F(x_0^{(2)})|| \\
&\leq \gamma_0\omega_0(\eta_0,(1+|\alpha|\gamma_0)\eta)\gamma_0 b_0\eta \leq \gamma^2\eta,
\end{aligned}$$

and

$$\begin{aligned}
||x_0^{(3)}-x_0|| &\leq ||x_0^{(3)}-x_0^{(2)}||+||x_0^{(2)}-x_0|| \\
&\leq \omega_0(\eta_0,(1+|\alpha|\gamma_0)\eta)\gamma_0^2 b_0\eta+\eta_0<\gamma^2\eta+\eta_0<R,
\end{aligned}$$

so, by $(A.6)$, $x_0^{(3)} \in U(x_0, R)$.

Moreover, we have again by the definition of γ that

$$\begin{aligned}||F(x_0^{(3)})|| &\leq \omega_0(||x_0^{(3)} - x_0||, ||x_0^{(2)} - x_0 - F(x_0^{(0)})||)||x_0^{(3)} - x_0^{(2)}|| \\ &\leq \omega_0(R, R+\eta)\gamma^2\eta \leq \frac{\gamma^3}{\gamma_0}\eta,\end{aligned}$$

so

$$||x_0^{(4)} - x_0^{(3)}|| \leq \gamma_0\omega_0(R, R+\eta) \leq \gamma^3\eta$$

and

$$\begin{aligned}||x_0^{(4)} - x_0|| &\leq ||x_0^{(4)} - x_0^{(3)}|| + ||x_0^{(3)} - x_0^{(2)}|| + ||x_0^{(2)} - x_0^{(1)}|| + ||x_0^{(1)} - x_0|| \\ &\leq \gamma^3\eta + \gamma^2\eta + \gamma\eta + |\alpha|\gamma_0\eta \\ &= \gamma\eta\frac{1-\gamma^3}{1-\gamma} + |\alpha|\gamma_0\eta < \frac{\gamma\eta}{1-\gamma} + |\alpha|\gamma_0\eta < R,\end{aligned}$$

so, $x_0^{(4)} \in U(x_0, R)$.

Then, in an analogous way

$$||F(x_0^{(i)})|| \leq \frac{\gamma^i}{\gamma_0}\eta, ||x_0^{(k)} - x_0^{k-1}|| \leq \gamma^{k-1}\eta, \quad \text{for} \quad i = 1, 2, \ldots k,$$

and

$$||x_0^{(k)} - x_0|| \leq (\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0)\eta < R.$$

Hence, $x_1 = x_0^{(k)} \in U(x_0, R)$ and is well defined.

We can write

$$\begin{aligned}F(x_1) &= F(x_0^{(k)}) - F(x_0^{(k-1)}) - \Gamma_0^{-1}(x_0^{(k)} - x_0^{(k-1)}) \\ &= ([x_0^{(k)}, x_0^{(k-1)}; F] - \Gamma_0^{-1})(x_0^{(k)} - x_0^{(k-1)}),\end{aligned}$$

so

$$\begin{aligned}||F(x_1)|| &\leq \omega_0(||x_0^{(k)} - x_0^{(0)}||, ||x_0^{(k-1)} - x_0^{(0)} - F(x_0^{(0)})||)||x_0^{(k)} - x_0^{(k-1)}|| \\ &\leq \omega_0(R, R+\eta)\gamma^{k-1}\eta = \delta_0\eta.\end{aligned}$$

Suppose that $x_m \in U(x_0, R)$. Next, we show that $\Gamma_m^{-1} \in L(X, X)$. We have in turn the estimate

$$\begin{aligned}||\Gamma_0||\ ||\Gamma_m^{-1} - \Gamma_0^{-1}|| &\leq \gamma_0\omega_0(||x_m - x_0||, ||x_m + F(x_m) - x_0 - F(x_0)||) \\ &\leq \gamma_0\omega_0(R, R + ||F(x_m) - F(x_0)||) \\ &\leq \gamma_0\omega_0(R, R + ||[x_m, x_0; F||\ ||x_m - x_0||) \\ &\leq \gamma_0\omega_0(R, R + (1+\theta)R) < 1,\end{aligned}$$

since $R < R_0$.

It follows from the preceding estimate and the Banach lemma on invertible operators [5] that $\Gamma_m^{-1} \in L(X,X)$ and

$$||\Gamma_m|| \leq \frac{\gamma_0}{1-\gamma_0\omega_0(R,(1+\theta)R)} = \gamma_1.$$

By the definition of the method (1), we have that

$$\begin{aligned} ||x_1^{(1)} - x_1^{(0)}|| &\leq |\alpha|||\Gamma_1||||F(x_1^{(0)})|| \\ &\leq |\alpha|\gamma_1\delta_0\eta = \lambda_0\eta. \end{aligned}$$

Then, we can write

$$\begin{aligned} F(x_1^{(1)}) &= F(x_1^{(1)}) - F(x_1^{(0)}) - \Gamma_1^{-1}(x_1^{(1)} - x_1^{(0)}) + (\alpha+1)F(x_1^{(0)}) \\ &= ([x_1^{(1)}, x_1^{(0)}; F] - [x_1, x_1 + F(x_1); F])(x_1^{(1)} - x_1^{(0)}) + (\alpha+1)F(x_1^{(0)}), \end{aligned}$$

leading to

$$\begin{aligned} ||F(x_1^{(1)})|| &\leq \omega(\|x_1^{(1)} - x_1\|, \|x_1^{(0)} - x_1 + F(x_1)\|)\|x_1^{(1)} - x_1^{(0)}\| + |\alpha+1|\|F(x_1^{(0)})\| \\ &\leq \omega(2R, 2R + \delta_0\eta)\lambda_0\eta + |\alpha+1|\delta_0\eta = b_1\eta, \end{aligned}$$

so

$$\begin{aligned} ||x_1^{(2)} - x_1^{(1)}|| &= ||\Gamma_1 F(x_1^{(1)})|| \\ &\leq \gamma_1 b_1 \eta = \lambda\eta. \end{aligned}$$

Notice that we have

$$||x_1 - x_0|| = ||x_0^{(k)} - x_0|| \leq \left(\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0\right)\eta < R,$$

$$\begin{aligned} ||x_1^{(1)} - x_0|| &\leq ||x_1^{(1)} - x_1^{(0)}|| + ||x_1^{(0)} - x_0|| \\ &\leq \lambda_0\eta + ||x_1 - x_0|| \leq \lambda_0\eta + (\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0)\eta < R \end{aligned}$$

and

$$\begin{aligned} ||x_1^{(2)} - x_0|| &\leq ||x_1^{(2)} - x_1^{(1)}|| + ||x_1^{(1)} - x_0|| \\ &\leq \lambda\eta + \lambda_0\eta + ||x_1 - x_0|| \leq \lambda\eta + \lambda_0\eta + \left(\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0\right)\eta < R \end{aligned}$$

so $x_1, x_1^{(1)}, x_1^{(2)} \in U(x_0, R)$.

Similarly, we have that

$$\begin{aligned} ||F(x_1^{(2)})|| &= \|F(x_1^{(2)}) - F(x_1^{(1)}) - \Gamma_1^{-1}(x_1^{(2)} - x_1^{(1)})\| \\ &= \|([x_1^{(2)}, x_1^{(1)}; F] - [x_1, x_1 + F(x_1); F])(x_1^{(2)} - x_1^{(1)})\| \\ &\leq \omega(\|(x_1^{(2)} - x_0) + (x_0 - x_1)\|, \|(x_1^{(1)} - x_0) + (x_0 - x_1) - F(x_1)\|)\|(x_1^{(2)} - x_1^{(1)})\| \\ &\leq \omega(2R, 2R + \delta_0\eta)\lambda\eta \leq \frac{\lambda^2\eta}{\gamma_1}, \end{aligned}$$

leading to

$$\begin{aligned} ||x_1^{(3)} - x_1^{(2)}|| &= ||\Gamma_1 F(x_1^{(2)})|| \\ &\leq \gamma_1\omega(\lambda\eta, b_1\eta)\lambda\eta \leq \lambda^2\eta \end{aligned}$$

and

$$\begin{aligned} ||x_1^{(3)} - x_0|| &\leq ||x_1^{(3)} - x_1^{(2)}|| + ||x_1^{(2)} - x_0|| \\ &\leq \lambda^2\eta + \lambda\eta + \lambda_0\eta + (\frac{\gamma}{1-\gamma} + |\alpha|\gamma_0)\eta < R, \end{aligned}$$

so, $x_1^{(3)} \in U(x_0, R)$.

Therefore, we get in an analogous way that

$$||F(x_1^{(i)})|| \leq \frac{\lambda^i}{\gamma_1}\eta, ||x_1^{(k)} - x_1^{(k-1)}|| \leq \lambda^{k-1}\eta$$

and

$$x_1^{(i)} \in U(x_0, R), \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Notice that in view of the estimates on consecutive distances and the definition of λ and γ_1, we deduce that sequence $\{x_n\}$ is complete in a Banach space X and then it converges to some $x^* \in \bar{U}(x_0, R)$.

Finally, notice that sequence $\{F(x_n)\}$ is bounded from above by sequence $\{||x_n - x_{n-1}||\}$, so

$$||F(x^*)|| = \lim_{n\to\infty} ||F(x_n)|| \leq \lim_{n\to\infty} ||x_n - x_{n-1}|| = 0.$$

Hence, we deduce that $F(x^*) = 0$. □

Concerning the uniqueness of the solution, we have the following result.

Theorem 2 *Suppose the hypotheses of Theorem 1 hold. Then, the point x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x_0, R_2)$ where*

$$R_2 = \sup\{t \in [R, R^*] : \gamma_0\omega_0(t, R + \eta) < 1\}.$$

Proof

The existence of the solution of equation $F(x) = 0$, $x^* \in \bar{U}(x_0, R)$ has been shown in Theorem 1.

Let $y^* \in \bar{U}(x_0, R_1)$ be a solution of equation $F(x) = 0$.

Using $(A.2)$ and $(A.4)$, we get in turn for $M = [y^*, x^*; F]$:

$$\begin{aligned} ||\Gamma_0(M - \Gamma_0^{-1})|| &\leq \gamma_0\omega_0(||y^* - x_0||, ||x^* - x_0 - F(x_0)||) \\ &\leq \gamma_0\omega_0(R_2, R + \eta) < 1. \end{aligned}$$

It follows that $M^{-1} \in L(X, X)$. Then, from the identity

$$0 = F(y^*) - F(x^*) = M(y^* - x^*),$$

we conclude that $y^* = x^*$. □

Remark 1 *The convergence of these type of methods usually involves a stronger condition than $(A.3)$ in the literature given by*

$$||[x, y; F] - [v, w; F]|| \leq \omega_1(||x - v||, ||y - w||)$$

for each $x, y, v, w \in D$, where ω_1 is a function like ω.

Notice than in general for each pair (s, t)

$$\omega(s, t) \leq \omega_1(s, t)$$

since $U \subseteq D$ and

$$\omega_0(t, s) \leq \omega_1(t, s).$$

Moreover, the latest inequality have been used by us to refine convergence results for other simpler methods. The same is now true, if we use the first of the preceding inequalities. Notice that $(A.2)$, i. e., the function ω_0 and the definition of γ_0 help us to define R_0 which is turn helps us define function ω. This way the iterates are being located in U which is a more precise location than D used in earlier studies.

3 Application for nonlinear systems of equations

The main goal of this section is to solve a nonlinear system of equations, given by

$$F(x) = 0, \tag{4}$$

where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinear operator with D a non-empty open domain. We look for approximating a solution of (4) with maximum efficiency by means an iterative process of the family (1) and coosing α and k appropiately. We are going to choose α value and the steps number that we will perform with the family of iterative processes (1) so that maximum efficiency is reached. To do this, we consider the Computational Efficiency index [8], given by:

$$CE = \rho^{1/\mu},$$

where the R-order of convergence and the operational cost of doing an step of the algorithm (1) are denoted by ρ and μ, respectively. Once we have chosen α value and the number of steps making optimum efficiency of the family of iterative processes (1), then, from Theorem 1, we solve the nonlinear system raised previously, (4).

3.1 R-order of convergence

According to Traub in [10], it is known that we can obtain one-point iterative methods with a higher R-order of convergence from one-point iterative methods of the form

$$\begin{cases} x_0 \in D, \\ x_{n+1} = \mathcal{G}(x_n), \quad n \geq 0, \end{cases} \tag{5}$$

if we use the following modification of (5):

$$\begin{cases} x_0 \in D, \\ y_n = \mathcal{G}(x_n), \quad n \geq 0, \\ x_{n+1} = y_n - [F'(x_n)]^{-1} F(y_n), \end{cases} \tag{6}$$

if we suppose that method (5) has an R-order of convergence of at least ρ, then we remember [10] that method (6) has an R-order of convergence of at least $\rho + 1$.

In different papers have been studied similar uniparametric methods that we consider here but using derivatives, see [7], where it is studied a Newton-type method, and it is proved order the convergence four for $\alpha = \pm 1$ and order three for any value of $\alpha \neq \pm 1$. In [2] it is consider a k-step methods similar to this work but using derivatives and an interesting dynamical study is performed. In our study we analyze these methods when we approximate the derivatives by divided differences. The fact is that for $\alpha = 1$ the resulting iterative methods preserve the order of convergence for any divided differences but for $\alpha = -1$ the order of convergence is preserved only if we use an approximation of second order for the derivative $F'(x_n)$.

Specifically, we know that if we consider an approximation of second order for the derivative $F'(x_n)$ by means a first order divided difference $[x_n, x_n + F(x_n); F]$, see [6], it follows that the convergence order of an iterative process it is preserved. In this case we have that a

$$\begin{cases} x_0 \in D, \\ y_n = \mathcal{G}(x_n), \quad k \geq 0, \\ x_{n+1} = y_n - [x_n, x_n + F(x_n); F]^{-1} F(y_n), \end{cases} \tag{7}$$

has an R-order of convergence of at least $\rho + 1$ too.

From the previous results we can calculate the R-order of convergence of family of iterative processes (1) for different α values.

In first place, for $\alpha = -1$, if we consider $k = 1$ then we obtain the Steffensen's method:

$$\begin{cases} x_0 \in D, \\ x_{n+1} = x_n - [x_n, x_n + F(x_n); F]^{-1} F(x_n), \end{cases}$$

which has quadratic convergence [10], i. e., R-order at least 2. Then, applying recursively the Traub's result [10], we obtain that, for $\alpha = -1$ and k steps, the family of iterative processes (1) has R-order of convergence $k + 1$.

In second place, for $\alpha = 1$, if we consider $k = 2$ then we obtain the iterative process given by

$$\begin{cases} x_0 \in D, \\ y_n = x_n + [x_n, x_n + F(x_n); F]^{-1} F(x_n) \\ x_{n+1} = y_n - [x_n, x_n + F(x_n); F]^{-1} F(y_n). \end{cases}$$

As it is know [2], this iterative process has R-order of convergence at least three when it uses derivatives and by [6] the order is preserved. Now, as before, applying recursively the Traub's result [10], we obtain that the family of iterative processes (1) has R-order of convergence $k+1$ too.

To finish our study of the R-order of the family of iterative processes (1), if we consider $\alpha \in \mathbb{R} - \{-1, 1\}$, as for $k = 1$ we have an iterative process with at least R-order of convergence 1, applying Traub's result, we obtain that the family of iterative processes (1) has R-order of convergence k.

3.2 Operational cost

From now on, for computing the operational cost of doing an iteration of the algorithm (1), we note that the practical application of these iterative processes is performed from the following algorithm, depending on the chosen number of steps.

$$\left\{\begin{array}{l} x_n = x_n^{(0)}, \\ [x_n, x_n + F(x_n); F](x_n^{(1)} - x_n^{(0)}) = \alpha F(x_n^{(0)}), \\ [x_n, x_n + F(x_n); F](x_n^{(2)} - x_n^{(1)}) = -F(x_n^{(1)}), \\ \qquad \vdots \\ [x_n, x_n + F(x_n); F](x_n^{(k-1)} - x_n^{(k-2)}) = -F(x_n^{(k-2)}), \\ [x_n, x_n + F(x_n); F](x_n^{(k)} - x_n^{(k-1)}) = -F(x_n^{(k-1)}) \\ x_{n+1} = x_n^{(k)}, \quad n \geqslant 0, \end{array}\right. \tag{8}$$

In order to compute the operational cost of doing an iteration of this algorithm, we have $m(m-1)(2m-1)/6$ products and $m(m-1)/2$ quotients in the LU decomposition for the $[x_n, x_n + F(x_n); F]$ matrix and $m(m-1)$ products and m quotients in the resolution of two triangular linear systems. Taking into account that after k steps we have solved two triangular linear systems k times and only one LU decomposition , we obtain the following operational cost of doing an iteration of this algorithm with k steps for a nonlinear system of m equations:

$$\mu(k, m) = \frac{1}{3}(m^3 + 3km^2 - m) \tag{9}$$

3.3 Efficiency and Dynamics

From the previous study, we have obtained :

$$CE(k,m)=\begin{cases}(k+1)^{\frac{3}{m^3+3km^2-m}} & \text{if} \quad \alpha=-1,\\ (k+1)^{\frac{3}{m^3+3km^2-m}} & \text{if} \quad \alpha=1,\\ k^{\frac{3}{m^3+3km^2-m}} & \text{if} \quad \alpha\neq -1,1.\end{cases}$$

Then, obviously, for $\alpha=-1,1$ we obtain the maximum efficiency.

CREO QUE SE PODRIAN CALCULAR LAS CUENCAS DE ATRACCION O ALGUN DIBUJO DE LA DINAMICA PARA LOS CASOS 1 Y -1 CON K=2,5,10,15 , PARA PODER DIFERENCIAR UN POCO LAS DOS SITUACIONES...??????????????????

3.4 Numerical example

Now, we consider the special case of a nonlinear conservative system described by the equation

$$\frac{d^2x(t)}{dt^2}+\Psi(x(t))=0 \tag{10}$$

with the boundary conditions

$$x(0)=x(1)=0. \tag{11}$$

After that, we use a discretization process to transform problem (10)–(11) into a finite-dimensional problem and look for an approximated solution of this problem when a particular function $\Psi(u)$ is considered. So, we transform problem (10)–(11) into a system of nonlinear equations by approximating the second derivative by a standard numerical formula.

Now, we introduce the points $t_j=jh$, $j=0,1,\dots,m+1$, where $h=\frac{1}{m+1}$ and m is an appropriate integer. A scheme is then designed for the determination of numbers x_j, it is hoped, approximate the values $x(t_j)$ of the true solution at the points t_j. A standard approximation for the second derivative at these points is

$$x_j'' \approx \frac{x_{j-1} - 2x_j + x_{j+1}}{h^2}, \quad j = 1, 2, \dots, m.$$

A natural way to obtain such a scheme is to demand that the x_j satisfy at each interior mesh point t_j the difference equation

$$x_{j-1} - 2x_j + x_{j+1} + h^2\Psi(x_j) = 0. \tag{12}$$

Since x_0 and x_{m+1} are determined by the boundary conditions, the unknowns are $x_1, x_2, \dots, x_m$.

A further discussion is simplified by the use of matrix and vector notation. Introducing the vectors

$$\mathbf{x} = (x_1, x_2, \dots, x_m)^t, \qquad v_{\mathbf{x}} = (\Psi(x_1), \Psi(x_2), \dots, \Psi(x_m))^t$$

and the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix},$$

the system of equations, arising from demanding that (12) holds for $j = 1, 2, \dots, m$, can be written compactly in the form

$$F(\mathbf{x}) \equiv A\mathbf{x} + h^2 v_{\mathbf{x}} = 0, \tag{13}$$

where F is a function from $\mathbb{R}^m$ into $\mathbb{R}^m$.

From now on, the focus of our attention is to solve a particular system of form (13). We choose $m = 20$ and the infinity norm. For this size of systems we plot the graphic of efficiency, see Figure 1 having that the best case is between 5 and 6 we use iterative methods with values of $k = 2, 3, 4, 5$.

If we now choose the law $\Psi(u) = 1 + u^3$ for the heat generation in problem (10)–(11), then the vector $v_{\mathbf{x}}$ of (13) is given by

$$v_{\mathbf{x}} = (v_1, v_2, \dots, v_{20})^t, \qquad v_i = 1 + x_i^3, \qquad i = 1, 2, \dots, 20. \tag{14}$$

Then, we apply iterative method 1 to solve this problem by using different divided difference operators $[x, y; F]_{ij}$, $i, j = 1, \dots, n$, defined as follows:

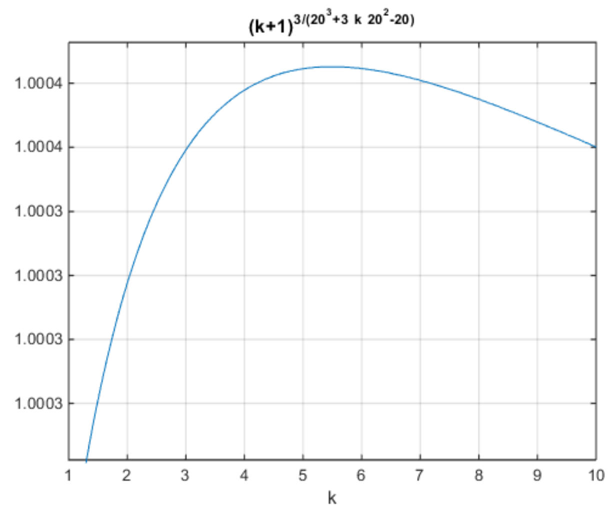

Figure 1: Efficiencies for m=20

$$\frac{1}{y_j - x_j}\Big(F_i(y_1,\ldots,y_{j-1},y_j,x_{j+1},\ldots,x_n) - F_i(y_1,\ldots,y_{j-1},x_j,x_{j+1},\ldots,x_n)\Big),$$

this is the classical first order approximation of the jacobian $F'(x)$ and will be denoted in the numerical experience as $dd1$ and $[x-F(x), x+F(x); F]$ as approximation of second order for the derivative, we denote in the numerical results as $dd2$.

In order to obtain the numerical results we have used variable arithmetic precision with 100 digits, with different number of steps k, considering $\alpha = -1, 1/2, 1, 2$. By taking starting guess $x_0 = (1\cdots, 1)$ in Table 1 one can check the computational convergence order denoted by p, the number of iterations needed, denoted by it, in order to reach the stopping criterion $\|x_{n+1} - x_n\| < 10^{-30}$. Finally we include in the numerical experience the norm value of the function at the approximation of the solution, $\|F(x_{n+1})\|$. As it can be seen at the solution for the parameter $\alpha = -1$ and divided differences of order one, $dd1$, the convergence order fells down one unit.

So, we perform a new computational experience for cases where maximum order is reached, these are $\alpha = 1$ with divided differences given by $dd1$ and $dd2$ and $\alpha = -1$ with divided differences given by $dd2$, with the aim of studying the computational time, denoted by CT, for reaching the solution under the

I.M.		**dd1**				**dd2**			
k	α	$\lVert x_{n+1} - x_n \rVert$	it	p	$\lVert F(x_{n+1}) \rVert$	$\lVert x_{n+1} - x_n \rVert$	it	p	$\lVert F(x_{n+1}) \rVert$
2	2	7.5e-41	6	1.9	1.1e-84	6.4e-41	6	1.9	8.1e-85
	1/2	3.8e-38	7	1.9	1.2e-78	6.7e-38	7	1.9	3.5e-78
	1	5.6e-50	5	2.9	1.2e-108	3.4e-85	5	2.9	8.1e-109
	-1	3.8e-35	7	1.9	2.7e-74	5.6e-68	7	2.9	4.4e-106
3	2	1.8e-61	5	2.9	4.3e-63	4.7e-61	5	2.9	1.1e-62
	1/2	7.2e-53	5	3.0	1.7e-54	7.1e-53	5	2.9	1.7e-54
	1	5.1e-38	4	3.9	8.0e-40	5.1e-38	4	3.9	1.2e-39
	-1	6.6e-60	5	3.0	1.5e-61	4.9e-88	5	4.0	1.1e-89
4	2	3.4e-42	4	3.9	8.2e-44	5.1e-42	4	3.9	1.2e-43
	1/2	1.3e-31	4	4.1	3.2e-33	1.7e-31	4	4.1	3.2e-33
	1	6.9e-62	4	5.0	5.4e-64	6.9e-62	4	5.0	1.6e-63
	-1	1.7e-36	4	4.2	1.1e-58	1.7e-36	4	5.0	4.2e-38
5	2	6.3e-71	4	5.0	1.4e-72	1.2e-70	4	5.0	3.1e-72
	1/2	2.8e-61	4	4.9	6.7e-63	3.6e-61	4	4.9	8.5e-63
	1	1.1e-103	4	5.9	9.9e-105	3.7e-103	4	5.9	8.8e-105
	-1	4.1e-63	4	5.0	8 9.8e-65	2.5e-72	4	5.9	6.74e-74

Table 1: Numerical results for different values of α.

criterion established before. We also obtain the total operational cost, TOC, multiplying the value obtained in (9) by the number of iterations performed, and the total computational efficiency, defined by $TCE = (k+1)^{\frac{1}{TOC}}$. As can be seen in Table 2, when we analyzed deeply the cases for maximum efficiency, that is when the difference divided used allow us to preserve the convergence order, we notice that although the operational cost for an iteration of this k-step method gives us maximum efficiency for k around 5 and 6, in this example if we have into account the total number of iterations performed the maximum efficiency it is obtained for $k = 8$ and $\alpha = 1$, similar results is obtained for divided differences $dd1$ and $dd2$, moreover although $dd2$ perform one more functional evaluation by iteration similar computational times are obtained.

I.M.		**dd1**				**dd2**			
k	α	it	TCC	CT	TCE	it	TCC	CT	TCE
2	1	5	17300	7.32	1.000064	5	17300	7.23	1.000064
	-1					7	24220	7.25	1.000045
3	1	4	15440	5.82	1.000090	6	23160	8.77	1.000060
	-1					5	19300	7.28	1.000072
4	1	4	17040	5.98	1.000094	4	17040	6.03	1.000094
	-1					4	17040	5.94	1.000094
5	1	4	18640	6.24	1.000096	5	18640	6.25	1.000096
	-1					4	18640	6.12	1.000096
7	1	4	21840	7.02	1.000095	4	21840	6.41	1.000095
	-1					4	21840	6.33	1.000095
8	1	3	17580	5.88	**1.000125**	3	17580	6.06	1.000125
	-1					3	23440	7.33	1.000094
9	1	3	18780	5.04	1.000123	3	18780	4.89	1.000123
	-1					3	18780	4.94	1.000123
10	1	3	19980	5.03	1.000120	3	19980	5.2	1.000120
	-1					3	19980	5.08	1.000120

Table 2: Comparing results for values of maximum efficiency.

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