Proceedings

of the

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada

Gijón (Asturias), Spain

June 14-18, 2021

SeMA
Sociedad Española de Matemática Aplicada

Universidad de Oviedo

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Universidad de Oviedo
Servicio de Publicaciones de la Universidad de Oviedo
Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)
Tel. 985 10 95 03 Fax 985 10 95 07
http: www.uniovi.es/publicaciones
servipub@uniovi.es

ISBN: 978-84-18482-21-2

Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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A method to construct irreducible totally nonnegative matrices with a given Jordan canonical form

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Abstract

Let $A \in \mathbb{R}^{n\times n}$ be an irreducible totally nonnegative matrix (ITN), that is, A is irreducible with all its minors nonnegative. A triple (n, r, p) is called *realizable* if there exists an ITN matrix $A \in \mathbb{R}^{n\times n}$ with $\text{rank}(A) = r$ and $p\text{-rank}(A) = p$ (recall that $p\text{-rank}(A)$ is the size of the largest invertible principal submatrix of A). Each ITN matrix A associated with a realizable triple (n, r, p) has p positive and distinct eigenvalues, and for the zero eigenvalue it is verified that $n - r$ and $n - p$ are the geometric and the algebraic multiplicity, respectively. Moreover, since $\text{rank}(A^p) = p$, A has $n - r$ zero Jordan blocks whose sizes are given by the Segre characteristic, $S = (s_1, s_2, \dots, s_{n-r})$, with $s_i \le p$, $i = 1, 2, \dots, n - r$.

We know the number of zero Jordan canonical forms of ITN matrices associated with a realizable triple (n, r, p) and all these zero Jordan canonical forms. The following important question that we present in this talk deals with how to construct an ITN matrix A associated with (n, r, p) and exactly with one of these Segre characteristic S corresponding to the zero eigenvalue.

1. Introduction

A matrix $A \in \mathbb{R}^{n\times n}$ is called totally nonnegative if all its minors are nonnegative and it is abbreviated as TN. The wide study of these matrices is due to the large number of applications in different branches of science, see for instance [1, 7–18]. Now, we recall some basic concepts that we will use throughout the paper:

1. The rank of A, denoted by $\text{rank}(A)$, is the size of the largest invertible square submatrix of A. The principal rank of A, denoted by $p\text{-rank}(A)$, is the size of the largest invertible principal submatrix of A. It is clear that

$$0 \le p\text{-rank}(A) \le \text{rank}(A) \le n$$

2. The characteristic polynomial of a matrix A is given by

$$q_A(\lambda) = \det(\lambda I - A) = \lambda^n + \sum_{k=1}^{n} (-1)^k \left(\sum_{\forall \alpha \in Q(\text{k,n})} \det(A[\alpha]) \right) \lambda^{n-k}$$

where Q(k,n) denotes the set of all increasing sequences of k natural numbers less than or equal to n, for $k, n \in \mathbb{N}$, $1 \le k \le n$, see [1]. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in Q_{k,n}$ and $\beta = (\beta_1, \beta_2, \dots, \beta_k) \in Q_{k,n}$, $A[\alpha|\beta]$ denotes the $k \times k$ submatrix of A lying in rows α_i and columns β_i, $i = 1, 2, \dots, k$. The principal submatrix $A[\alpha|\alpha]$ is abbreviated as $A[\alpha]$.

If A is TN and $p\text{-rank}(A) = p$, the minors of the same order have the same sign or are zero, then there are no cancelations in the summands and then,

$$\begin{aligned} q_A(\lambda) &= \lambda^{n-p} \left(\lambda^p + \textstyle\sum_{k=1}^{p} (-1)^k \left(\sum_{\forall \alpha \in Q(\text{k,n})} \det(A[\alpha]) \right) \lambda^{p-k} \right) \\ &= \lambda^{n-p} \left(\lambda^p - c_1 \lambda^{p-1} + \cdots + (-1)^p c_p \right). \end{aligned}$$

Then, if A is a TN matrix with $\text{rank}(A) = r$ and $p\text{-rank}(A) = p$, has p nonzero eigenvalues and the algebraic and geometric multiplicities of the zero eigenvalue are equal to $n - p$ and $n - r$, respectively.

3. A matrix $A \in \mathbb{R}^{n\times n}$, with $n \ge 2$, is an *irreducible* matrix if there is not a permutation matrix P such that $PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$, where O is an $(n - r) \times r$ zero matrix $(1 \le r \le n - 1)$. If $n = 1$, $A = (a)$ is irreducible when $a \ne 0$.

Fallat, Gekhtman and Johnson in [8] characterize the irreducible TN matrices as follows: a TN matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is irreducible if and only if $a_{ij} > 0$ for all i, j such that $|i - j| \le 1$ and they represent this class of matrices by ITN.

4. If there exists an ITN matrix $A \in \mathbb{R}^{n\times n}$ with rank$(A) = r$ and p-rank$(A) = p$, then the triple (n, r, p) is called *realizable* [8, p. 709], and A is considered as an ITN matrix *associated with* the triple (n, r, p). In order to a triple (n, r, p) be realizable it is necessary that $p \le r \le n - \left\lceil \frac{n-p}{p} \right\rceil$.
5. If A is an associated matrix with a realizable triple (n, r, p) then, its p nonzero eigenvalues are positive and distinct ([8, Theorem 3.3]). That is, if $\lambda_1, \dots, \lambda_p, \dots \lambda_n$ are the eigenvalues of A, we have

$$\lambda_1 > \lambda_2 > \dots > \lambda_p > 0, \text{ and } \lambda_{p+1} = \lambda_{p+2} = \dots = \lambda_n = 0, \tag{1.1}$$

Moreover, since the algebraic multiplicity of the zero eigenvalue is $n - p$ and rank$(A^p) = p$, the size of the zero Jordan blocks of A is at most p.

Taking into account the above results, given a realizable triple (n, r, p) the following questions arises in a natutal way:

First question: *How many different zero Jordan canonical forms are associated with a realizable triple (n, r, p)?*

As we have seen, the ITN matrices associated with a realizable triple (n, r, p) verify that the algebraic multiplicity of the zero eigenvalue is $n-p$, the geometric one is $n-r$ and the size of the Jordan blocks is maximum p. Therefore, this problem is equivalent to the following *in how many ways can we distribute $n-p$ marbles in $n-r$ bags, knowing that all bags must have at least one marble and that at most each bag will fit p marbles.*

In [6], by using Number Theory, the authors calculated this number (represented by $p^{(p)}_{n-r}(n-p)$) and they gived an algorithm to obtain it. For example, if we have the triple realizable $(19, 14, 8)$ applying this algorithm we have that $p^{(8)}_5(11) = 10$.

Second question: Since we konw the number of different zero Jordan canonical forms associated with a realizable triple (n, r, p), then *what are these zero Jordan forms?*

In [6], using properties and the full rank LU factorization of ITN matrices and the Flanders Theorem the authors give and Procedure and the corresponding algorithm to compute the specific different zero Jordan canonical forms. For example, for the realizable triple $(19, 14, 8)$ we have obtained that there are 10 different zero Jordan canonical forms and applying the new algorithm we obtain these specific zero Jordan structures, all of them have 5 zero Jordan blocks of differents sizes. These structures are,

7	1	1	1	1
6	2	1	1	1
5	3	1	1	1
5	2	2	1	1
4	4	1	1	1
4	3	2	1	1
4	2	2	2	1
3	3	3	1	1
3	3	2	2	1
3	2	2	2	2

Remark 1.1 The sizes of the zero Jordan blocks of a matrix A are known as the Segre characteristic of A relative to its zero eigenvalue. Given an ITN matrix A associated to a realizable triple (n, r, p), if we represent this Segre sequence by $S = (s_1, s_2, \dots, s_{n-r})$ then, it is satisfied that

$$\begin{array}{ll} (1) & s_1 \le \min\{r-p+1, p\} \\ (2) & s_i \le s_{i-1}, \ \ i = 2, 3, \dots, n-r \\ (3) & \sum_{i=1}^{n-r} s_i = n-p \end{array} \tag{1.2}$$

Associated to the Segre characteristic $S = (s_1, s_2, \dots, s_{n-r})$ we have the Weyr characteristic of A relative to the zero eigenvalue $W = (w_1, w_2, \dots, w_{s_1})$, where $w_i = \text{Car}\{k \ : \ s_k \ge i\}$ for $i = 1, 2, \dots, s_1$ and

$$\begin{array}{ll} (1) & w_1 = \dim \text{Ker}(A) = n-r \\ (2) & w_i \le w_{i-1}, \ \ i = 2, 3, \dots, s_1 \\ (3) & \sum_{j=1}^{i} w_j = \dim \text{Ker}(A^i) \\ (4) & \sum_{j=1}^{s_1} w_j = \dim \text{Ker}(A^{s_1}) = n-p \end{array} \tag{1.3}$$

Thirth question: Finally, knowing the number of zero Jordan canonical forms and the specific structures associated with a realizable triple (n, r, p), the following question is the main goal of this work, *how to construct an ITN matrix associated with a realizable triple (n, r, p) and with $n - r$ zero Jordan blocks whose sizes are given by the Segre characteristic $S = (s_1, s_2, \ldots, s_{n-r})$ satisfying (1.2).*

To answer this question, in the next section we first described a procedure that allow us to construct an upper block echelon matrix $U \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(U) = r$, p-$\operatorname{rank}(U) = p$ and $n - r$ zero Jordan blocks whose sizes are given by the Segre characteristic S satisfying (1.2). After that, from U we will obtain the desired ITN matrix A associated with the realizable triple (n, r, p) and with the same zero Jordan structure that U as $A = LU$, where L is a lower triangular matrix with all its nonzero entries equal to 1.

2. Constructing an upper block echelon TN matrix U with a zero Jordan canonical form

In this section we describe a procedure to construct an upper block echelon matrix $U \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(U) = r$, p-$\operatorname{rank}(U) = p$ and $n - r$ zero Jordan blocks whose sizes are given by the Segre characteristic S satisfaying (1.2).

We recall that a matrix is upper block echelon if each nonzero block, starting from the left, is to the right of the nonzero blocks below and the zero blocks are at the bottom. A matrix is a lower (block) echelon matrix if its transpose is an upper (block) echelon matrix. In the Procedure 1 we use the nonsingular ITN matrix

$$V_q = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 2 & 2 & \ldots & 2 & 2 \\ 1 & 2 & 3 & \ldots & 3 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \ldots & q-1 & q-1 \\ 1 & 2 & 3 & \ldots & q-1 & q \end{bmatrix} = [\min\{i, j\}]_{q\times q}$$

and the following MatLab notation: $A(i, :)$ denotes the i-th row of A and $A(:, j)$ denotes its j-th column; $\operatorname{ones}(n, m)$ denotes the $n \times m$ matrix of ones; $\operatorname{triu}(\operatorname{ones}(n, m))$ denotes the upper triangular part of $\operatorname{ones}(n, m)$; $\operatorname{zeros}(n, m)$ denotes the $n \times m$ zero matrix.

Note that if $r = p$ the algebraic and geometric multiplicity of the zero eigenvalue Of U is the same, therefore U has $n - r$ zero Jordan blocks of size 1×1. In this case is easy to see that the matrix U can be the following

$$U = \begin{bmatrix} \operatorname{triu}(\operatorname{ones}(p, n)) \\ \operatorname{zeros}(n-p, n) \end{bmatrix}.$$

If $p < r$ we construct a matrix U by blocks as follows

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} & \ldots & U_{1,s_1-1} & U_{1,s_1} & U_{1,s_1+1} \\ O & O & U_{23} & U_{24} & \ldots & U_{2,s_1-1} & U_{2,s_1} & U_{2,s_1+1} \\ O & O & O & U_{34} & \ldots & U_{3,s_1-1} & U_{3,s_1} & U_{3,s_1+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ O & O & O & O & \ldots & O & U_{s_1-1,s_1} & U_{s_1-1,s_1+1} \\ O & O & O & O & \ldots & O & O & U_{s_1,s_1+1} \\ O & O & O & O & \ldots & O & O & O \end{bmatrix}.$$

Each block and its size are given in the following procedure.

Procedure 1. Given a realizable triple (n, r, p) and the Segre characteristic $S = (s_1, s_2, \ldots, s_{n-r})$ satisfying (1.2), this procedure obtains an upper block echelon matrix $U \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(U) = r$, p-$\operatorname{rank}(U) = p$ and $n - r$ zero Jordan blocks whose sizes are given by S.

Step 1. Obtain the conjugated sequence of S, $W = (w_1, w_2, \ldots, w_{s_1})$ and from W define $R = (0, r_2, \ldots, r_{s_1})$, with $r_i = w_i,\ i = 2, 3, \ldots, s_1$.

Step 2. Calculate $n_1 = p + 1 - s_1$ and construct

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} & \ldots & U_{1,s_1} & U_{1,s_1+1} \end{bmatrix} = \operatorname{triu}(\operatorname{ones}(n_1, n))]$$

Step 3. Construct $U_{23} = V_{r_2+1}$ and

$$\begin{aligned} &\begin{bmatrix} O & O & U_{23} & U_{24} & \ldots & U_{2,s_1} & U_{2,s_1+1} \end{bmatrix} \\ &= [\operatorname{zeros}(r_2+1, n_1+r_2)\ U_{23}\ U_{23}(:, r_2+1) * \operatorname{ones}(1, n-n_1-2r_2-1)] \end{aligned}$$

Step 4. For $i = 3, 4, \ldots, s_1$

4.1. If $r_i = r_{i-1}$ construct $U_{i,i+1} = V_{r_i+1}$.

4.2. If $r_i < r_{i-1}$ construct $U_{i,i+1} = \begin{bmatrix} V_{r_i+1} \\ \text{ones}(r_{i-1} - r_i, 1) * V_{r_i+1}(r_i + 1, :) \end{bmatrix}$

After, in both cases,

$$\begin{bmatrix} O & \ldots & O & U_{i,i+1} & U_{i,i+2} & \ldots & U_{i,s_1} & U_{i,s_1+1} \end{bmatrix}$$
$$= [\text{zeros}(r_{i-1} + 1, n_1 + r_2 + \textstyle\sum_{j=2}^{i-1}(r_j + 1)) \;\; U_{i,i+1} \;\; U_{i,i+1}(:, r_i + 1) * \text{ones}(1, n - n_1 - r_2 - \sum_{j=2}^{i}(r_j + 1))]$$

Step 5. Finally, the last block is equal to

$$\text{zeros}\left(n - n_1 - (r_2 + 1) - \sum_{i=2}^{s_1-1}(r_i + 1),\; n\right)$$

□

In the answer to the second question we have seen that the realizable triple $(19, 14, 8)$ has associated 10 different zero Jordan canonical forms, being $(4, 3, 2, 1, 1)$ one of them. In the following example we construct an upper block echelon TN matrix U with this Jordan canonical form using Procedure 1.

Example 2.1 Obtain a 19×19 upper block echelon TN matrix U, with $\text{rank}(U) = 14$, p-$\text{rank}(U) = 8$ and with 5 zero Jordan blocks of sizes $S = (4, 3, 2, 1, 1)$.

Since $r \neq p$ and $s_1 = 4$, following Procedure 1 we construct an upper block TN matrix

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} \\ O & O & U_{23} & U_{24} & U_{25} \\ O & O & O & U_{34} & U_{35} \\ O & O & O & O & U_{45} \\ O & O & O & O & O \end{bmatrix}.$$

Step 1. The conjugated sequence of S is $W = (5, 3, 2, 1)$ and then, $R = (0, 3, 2, 1)$.

Step 2. $n_1 = p + 1 - s_1 = 5$ and

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} \end{bmatrix} = \text{triu}(\text{ones}(5, 19)) =$$

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}.$$

Step 3. $U_{23} = V_4$ and

$$\begin{bmatrix} O & O & U_{23} & U_{24} & U_{2,5} \end{bmatrix} = [\text{zeros}(4, 8) \;\; U_{23} \;\; U_{23}(:, 4) * \text{ones}(1, 7)] =$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&2&2&2&2&2&2&2&2&2&2 \\ 0&0&0&0&0&0&0&0&1&2&3&3&3&3&3&3&3&3&3 \\ 0&0&0&0&0&0&0&0&1&2&3&4&4&4&4&4&4&4&4 \end{bmatrix}.$$

Step 4. For $i = 3$, since $r_3 < r_2$, construct

$$U_{34} = \begin{bmatrix} V_3 \\ \text{ones}(1, 1) * V_3(3, :) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

and

$$\begin{bmatrix} O & O & O & U_{34} & U_{35} \end{bmatrix} = [\text{zeros}(4,12)\ U_{34}\ U_(34)(:,3) * \text{ones}(1,4)] =$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&2&2&2&2&2&2 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&2&3&3&3&3&3 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&2&3&3&3&3&3 \end{bmatrix}.$$

Now, for $i = 4$ since $r_4 < r_3$, construct

$$U_{45} = \begin{bmatrix} V_2 \\ \text{ones}(1,1) * V_2(2,:) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} O & O & O & O & U_{45} \end{bmatrix} = [\text{zeros}(3,15)\ U_{45}\ U_(45)(:,2) * \text{ones}(1,2)] =$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&2&2 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&2&2 \end{bmatrix}$$

Step 5. The last zero block is

$$\left[\text{zeros}\left(n - n_1 - (r_2+1) - \sum_{i=2}^{s_1-1}(r_i+1),\ n\right)\right] = [\text{zeros}(3,19)]\,.$$

Therefore, the matrix U is

$$\left[\begin{array}{ccccc|ccc|cccc|ccc|cccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ \hline
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&1&2&2&2&2&2&2&2&2&2&2 \\
0&0&0&0&0&0&0&0&1&2&3&3&3&3&3&3&3&3&3 \\
0&0&0&0&0&0&0&0&1&2&3&4&4&4&4&4&4&4&4 \\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&2&2&2&2&2&2 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&2&3&3&3&3&3 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&2&3&3&3&3&3 \\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&2&2 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&2&2 \\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].$$

The following result proves that the matrix U constructed by Procedure 1 verifies the desired properties.

Theorem 2.2 [6, *Theorem* 1]
Consider the matrix U constructed by Procedure 1. *Then the following properties hold*:

1. *U is a TN matrix with rank$(U) = r$ and p-rank$(U) = p$.*
2. *U has $n - r$ zeros Jordan blocks whose sizes are given by the sequence $S = (s_1, s_2, \ldots, s_{n-r})$.*

3. Construct an ITN matrix with a prescribed zero Jordan structure

In this section we construct an ITN matrix A associated with the realizable triple (n, r, p) and with a zero Jordan canonical form associated with this triple. For that, we use the procedure given in the previous section to construct an upper block echelon TN matrix U of size $n \times n$, with $\text{rank}(U) = r$, $p\text{-rank}(U) = p$ and with a zero Jordan canonical form associated with this triple. Now, we give the following procedure to compute the matrix A.

Procedure 2. Given a realizable triple (n, r, p) and the Segre characteristic $S = (s_1, s_2, \ldots, s_{n-r})$ satisfying (1.2), this procedure obtains an ITN matrix $A \in \mathbb{R}^{n\times n}$, associated with this triple and with $n - r$ zero Jordan blocks whose sizes are given by S.

Step 1. Apply Procedure 1 to construct the upper block matrix U.

Step 2. Construct the lower triangular TN matriz $L = \text{tril}(\text{ones}(n, n))$.

Step 3. Obtain $A = L * U$.

□

The following result proves that the matrix A satisfies the prescribed conditions.

Theorem 3.1 [6, *Proposition* 1, *Theorem* 2]
The matrix A constructed by Procedure 2 satisfies the following conditions:

1. *A is a ITN matrix.*
2. *$\text{rank}(A) = r$.*
3. *$p\text{-rank}(A) = p$.*
4. *Matrices A and U have the same zero Jordan structure.*

Example 3.2 Construct a 19×19 ITN matrix A, associated with the realizable triple $(19, 14, 8)$ and with 5 zero Jordan blocks of sizes $S = (4, 3, 2, 1, 1)$.

Using the matrix U obtained in Example 2.1 and following Procedure 2, we have

$$A = \text{tril}(\text{ones}(n, n)) * U =$$
$$\left[\begin{array}{ccccc|ccc|cccc|ccc|cccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2\\
1&2&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3\\
1&2&3&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4\\
1&2&3&4&5&5&5&5&5&5&5&5&5&5&5&5&5&5&5\\
\hline
1&2&3&4&5&5&5&5&6&6&6&6&6&6&6&6&6&6&6\\
1&2&3&4&5&5&5&5&7&8&8&8&8&8&8&8&8&8&8\\
1&2&3&4&5&5&5&5&8&10&11&11&11&11&11&11&11&11&11\\
1&2&3&4&5&5&5&5&9&12&14&15&15&15&15&15&15&15&15\\
\hline
1&2&3&4&5&5&5&5&9&12&14&15&16&16&16&16&16&16&16\\
1&2&3&4&5&5&5&5&9&12&14&15&17&18&18&18&18&18&18\\
1&2&3&4&5&5&5&5&9&12&14&15&18&20&21&21&21&21&21\\
1&2&3&4&5&5&5&5&9&12&14&15&19&22&24&24&24&24&24\\
\hline
1&2&3&4&5&5&5&5&9&12&14&15&19&22&24&25&25&25&25\\
1&2&3&4&5&5&5&5&9&12&14&15&19&22&24&26&27&27&27\\
1&2&3&4&5&5&5&5&9&12&14&15&19&22&24&27&29&29&29\\
\hline
1&2&3&4&5&5&5&5&9&12&14&15&19&22&24&27&29&29&29\\
1&2&3&4&5&5&5&5&9&12&14&15&19&22&24&27&29&29&29\\
1&2&3&4&5&5&5&5&9&12&14&15&19&22&24&27&29&29&29
\end{array}\right]$$

Acknowledgements

This work has been supported by the Ministerio de Economía y Competitividad under the Spanish DGI grant MTM2017-85669-P-AR.

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