5

A new three-steps iterative method for solving nonlinear systems

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1 Introduction

The problem of solving equations and systems of nonlinear equations is among the most important in theory and practice, not only of applied mathematics, but also in many branches of science, engineering, physics, computer science, astronomy, finance,... A glance at the literature shows a high level of contemporary interest. The search for solutions of systems of nonlinear equations is an old, frequent and important problem for many applications in mathematics and engineering (for example, see [1–3]).

This work deals with the approximation of a solution ξ of a system of nonlinear equations $F(x) = 0$, where $F: D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a sufficiently differentiable function on the convex set $D \subset \mathbb{R}^n$. The most commonly used techniques are iterative methods, where, from an initial estimate, a sequence is built converging to the solution of the problem under some conditions. Although not as many as in the case of equations, some publications have appeared in the recent years, proposing different iterative methods for solving nonlinear systems. They have made several modifications to the classical methods to accelerate the convergence and to reduce the number of operations and functional evaluations per step of the iterative method. Newton's method is the most used iterative technique for solving this kind of problems, whose iterative expression is

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \quad k = 0, 1, \ldots, \tag{1}$$

where $F'(x)$ denotes the Jacobian matrix associated to function F.

Let $\{x^{(k)}\}_{k\geq 0}$ be a sequence in $\mathbb{R}^n$ which converges to ξ, then the convergence is called of order p with $p \geq 1$, if there exists $M > 0$ ($0 < M < 1$ if $p = 1$) and k_0 such that

$$\|x^{(k+1)} - \xi\| \leq M\|x^{(k)} - \xi\|^p, \forall k \geq k_0,$$

or

$$\|e^{(k+1)}\| \leq M\|e^{(k)}\|^p, \forall k \geq k_0, \text{ where } e^{(k)} = x^{(k)} - \xi.$$

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Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently Fréchet differentiable in D, for $\xi + h \in \mathbb{R}^n$ lying in a neighborhood of a solution ξ of $F(x) = 0$, applying Taylor expansion and assuming that the Jacobian matrix $F'(\xi)$ is non singular, we have

$$F(\xi + h) = F'(\xi)\left[h + \sum_{q=2}^{p-1} C_q h^q\right] + O(h^p) \tag{2}$$

where $C_q = (1/q!)[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q \geq 2$. We take into account that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(\xi) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\xi)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$. We can also express F' as

$$F'(\xi + h) = F'(\xi)\left[I + \sum_{q=2}^{p-1} qC_q h^{q-1}\right] + O(h^{p-1}), \tag{3}$$

where I is the identity matrix and $qC_q h^{q-1} \in \mathcal{L}(\mathbb{R}^n)$.

If $X = \mathbb{R}^{n\times n}$ denotes the Banach space of real square matrices of size$n \times n$, we can define $H : X \to X$ such that its Fréchet derivative satisfies:

(a) $H'(u)(v) = H_1uv$, where $H' : X \to \mathcal{L}(X)$ and $H_1 \in \mathbb{R}$,

(b) $H''(u, v)(v) = H_2uvw$, where $H'' : X \times X \to \mathcal{L}(X)$ and $H_2 \in \mathbb{R}$.

By using different techniques: composition of known methods, Jacobian "frozen", weight matrix function procedure, etc. several Newton-type methods of different orders have been designed for improving Newton's scheme. One of the first algorithms was Jarratt's method [4] whose iterative expression is

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= x^{(k)} - [6F'(y^{(k)}) - 2F'(x^{(k)})]^{-1}(3F'(y^{(k)}) - F'(x^{(k)}))[F'(x^{(k)})]^{-1}F(x^{(k)}). \end{aligned} \tag{4}$$

More recently, other authors have constructed different methods for solving nonlinear systems. For example, Cordero et al. in [5] design a three-steps iterative method of order six, by combining Newton and Jarratt's schemes; Behl et al. in [6] also construct a iterative method of order six, with two Jacobian matrix in its iterative expression.

In order to compare the different methods, we analyze the computational effort that they involve, in terms of functional evaluations d and amount of products and quotients op. By using this information, we are going to use two the multidimensional extension of the efficiency index defined by Ostrowski in [7] as $I = p^{1/d}$ and the computational efficiency index CI defined in [5] as $CI = p^{1/(d+op)}$, where p is the order of convergence, d is the number of functional evaluations per iteration and op is the number of products-quotients per iteration.

In this work, a new class of iterative methods for solving nonlinear systems of equations is presented. This family is developed by using a weight function procedure getting 6th-order of convergence. We present the convergence result and an study of the efficiency of our method in comparison with other known ones.

2 The proposed scheme: convergence order and efficiency

Our proposed family, denoted as PS6, is designed by using Newton's scheme and the weight function procedure. Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a real sufficiently differentiable function, H a matrix weight function that should be chosen and the three step iterative method

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - H(t^{(k)})[F'(x^{(k)})]^{-1}F(y^{(k)}), \\ x^{(k+1)} &= z^{(k)} - H(t^{(k)})[F'(x^{(k)})]^{-1}F(z^{(k)}), \end{aligned} \tag{5}$$

being $t^{(k)} = I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]$.

Theorem 1 *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open neighborhood D of such that $F(\xi) = 0$, and $H : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ a sufficiently differentiable matrix function. Let us assume that $F'(x)$ is nonsingular at ξ and $x^{(0)}$ is an initial estimatione close enough to ξ. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained from expression (5) converges to ξ with order 6 if the function H satisfies $H_0 = I$, $H_1 = 2$ and $|H_2| < \infty$, where $H_0 = H(0)$ and I is the identity matrix. The error equation is*

$$\begin{aligned} e^{(k+1)} &= \frac{1}{4}\Big[120C_2^5 - 22H_2C_2^5 + H_2^2C_2^5 - 24C_2^2C_3C_2 + 2H_2C_2^2C_3C_2 \\ &\quad +4C_3^2C_2 - 20C_3C_2^3 + 2H_2C_3C_2^3\Big]{e^{(k)}}^6 + O({e^{(k)}}^7), \end{aligned}$$

where $C_q = \frac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q = 2, 3, ...$

For comparing the efficiency of this family and other known ones, we choose the weight function $H(t) = I + 2t + \frac{1}{2}H_2t^2$, that satisfies the conditions of previous result. In order to calculate de efficiency index I, we recall that the number of functional evaluations of F. F' and first order divided difference $[\cdot, \cdot, F]$ at certain iterates is n, n^2 and $n(n-1)$, respectively. The comparison of efficiency index for the different methods is shown in Table 1. $n.F$, $n.F'$ and $n.[\cdot, \cdot; F]$ denote the number of functional evaluations F, Jacobian matrix F' and divided difference $[\cdot, \cdot; F]$, respectively, per iteration. FE is the number of scalar functions per iteration.

Method	$n.F$	$n.F'$	$n.[\cdot, \cdot; F]$	FE	I
$\text{PS6}_{\{H_2\neq 0\}}$	3	1	1	$2n^2 + 2n$	$6^{1/(2n^2+2n)}$
$\text{PS6}_{\{H_2=0\}}$	3	1	1	$2n^2 + 2n$	$6^{1/(2n^2+2n)}$
Newton	1	1	0	$n^2 + n$	$2^{1/(n^2+n)}$
Jarratt	1	2	0	$2n^2 + n$	$4^{1/(2n^2+n)}$

Table 1: Efficiency indices of the new and known methods

For calculating the computational efficiency index CI, we take in account that the number of products-quotients required for solving a linear system by Gaussian elimination is $\frac{1}{3}n^3 + n^2 - \frac{1}{3}n$

where n is the system size. If required the solution by using LU decomposition of m linear systems with the same matrix of coefficients, then is necessary $\frac{1}{3}n^3+mn^2-\frac{1}{3}n$ products-quotients operations. In addition, n^2 products are necessary for a matrix-vector multiplication and n^2 quotients for first order divided differences. The notation LS($[F'(x)]^{-1}$) and LS($Others$) is the number of lineal systems with $[F'(x)]$ as the matrix of coefficients and with others matrix coefficients, respectively. The comparison of computational efficiency index for the new and known methods is shown in Table 2.

Method	FE	LS($[F'(x)]^{-1}$)	LS($Others$)	$M \times V$	CI
$PS6_1$=$PS6_{\{H_2\neq 0\}}$	$2n^2+2n$	7	0	4	$6^{1/((1/3)n^3+13n^2+(5/3)n)}$
$PS6_2$=$PS6_{\{H_2=0\}}$	$2n^2+2n$	5	0	2	$6^{1/((1/3)n^3+9n^2+(5/3)n)}$
Newton	n^2+n	1	0	0	$2^{1/((1/3)n^3+3n^2+(2/3)n)}$
Jarratt	$2n^2+n$	1	1	1	$4^{1/((2/3)n^3+5n^2+(1/3)n)}$

Table 2: Computational efficiency index of the new and known methods

It is easy to observe that for any value of n, $n \geq 2$, we have

$$CI_{PS6_2} > CI_{PS6_1} > CI_{Newton} > CI_{Jarratt},$$

so, the best method under this point of view is $PS6_2$.

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