

A general class of arbitrary order iterative methods for computing generalized inverses[☆]

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ABSTRACT

A family of iterative schemes for approximating the inverse and generalized inverse of a complex matrix is designed, having arbitrary order of convergence p . For each p , a class of iterative schemes appears, for which we analyze those elements able to converge with very far initial estimations. This class generalizes many known iterative methods which are obtained for particular values of the parameters. The order of convergence is stated in each case, depending on the first non-zero parameter. For different examples, the accessibility of some schemes, that is, the set of initial estimations leading to convergence, is analyzed in order to select those with wider sets. This wideness is related with the value of the first non-zero value of the parameters defining the method. Later on, some numerical examples (academic and also from signal processing) are provided to confirm the theoretical results and to show the feasibility and effectiveness of the new methods.

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1. Introduction

Computing the inverse matrix of large-scale nonsingular complex matrices is essential in some engineering applications and it has binding computational requirements. In a wide variety of topics, such as signal and image processing [1–4], cryptography [5,6], control system analysis [7,8], and others, the inverse or different generalized inverses must be computed in order to solve the involved issues.

There are several procedures to address this problem, divided into two classes: the direct schemes such as Gaussian elimination with partial pivoting, which require big amount of calculations and memory for large scale problems, and the iterative schemes like the Schulz-type iterations, in which an approximation of the inverse matrix is obtained up to the desired precision.

In the last decade, many iterative schemes of different orders have been designed for approximating the inverse or some generalized inverse (Moore-Penrose inverse, Drazin inverse, etc.) of a complex matrix A . In this manuscript, we focus on constructing a new class of iterative methods, free of inverse operators and with arbitrary order of convergence, for finding the inverse of a non-singular complex matrix. We also analyze the proposed schemes for computing the Moore-Penrose

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inverse of complex rectangular matrices. The designed family depends on several real parameters, which by taking particular values provide us numerous known methods constructed by other authors.

Recently, some authors have presented a unified approach of the computation of different generalized inverses by means of the calculation of outer inverses $A_{T,S}^{(2)}$ for appropriate choice of T and S (see, for example, [9,10]).

Perhaps, the most common scheme to compute the inverse A^{-1} of a non-singular complex matrix A is the Schulz's method given in 1933, with iterative expression

$$X_{k+1} = X_k(2I - AX_k), \quad k = 0, 1, \dots \tag{1}$$

where I is the identity matrix with the same size of A . Schulz in [11] demonstrated the convergence of sequence $\{X_k\}_{k \geq 0}$, obtained from (1), to the inverse A^{-1} is guaranteed if the eigenvalues of matrix $I - AX_0$ are lower than 1. Taking into account that the residuals $E_k = I - AX_k$, $k = 0, 1, \dots$ satisfy $\|E_{k+1}\| \leq \|E_k\|^2$, expression (1) has quadratic convergence. In general, in the Schulz-type methods it is common to use as initial approach $X_0 = \alpha A^*$ or $X_0 = \alpha A$, where $0 < \alpha < 2/\rho(A^*A)$, where A^* is the conjugate transpose of A and $\rho(\cdot)$ the spectral radius. In this paper, we use in the case of inverses and also in generalized inverses, the initial estimation $X_0 = \beta A^*/\|A\|^2$. We will also analyze the set of values of the parameter β guaranteeing the convergence.

Li et al. in [12] investigated the family of iterative methods

$$X_{k+1} = X_k \left(\nu I - \frac{\nu(\nu-1)}{2} AX_k + \frac{\nu(\nu-1)(\nu-2)}{3!} (AX_k)^2 - \dots + (-1)^{\nu-1} (AX_k)^{\nu-1} \right), \quad \nu = 2, 3, \dots$$

with $X_0 = \alpha A^*$. They proved the convergence of ν -order of $\{X_k\}_{k \geq 0}$ to the inverse of matrix A . This class was used by Chen et al. in [13] and by Li et al. in [14] for approximating the Moore-Penrose inverse.

Soleymani et al. in [15] also proposed a fourth-order iterative scheme for calculating the inverse and the Moore-Penrose inverse, with iterative expression

$$X_{k+1} = \frac{1}{2} X_k (9I - AX_k (16I - AX_k (14I - AX_k (6I - AX_k))))), \quad k = 0, 1, \dots$$

On the other hand, Stanimirović et al. in [16] proposed the following scheme of order eleven for computing the generalized outer inverse $A_{T,S}^{(2)}$

$$X_{k+1} = X_k (I + (R_k + R_k^2)(I + (R_k^2 + R_k^4)(I + R_k^4))), \quad k = 0, 1, \dots$$

being $R_k = I - AX_k$, $k = 0, 1, \dots$

Kaur et al. in [17], by using also the hyperpower iterative method, designed the following scheme of order five for obtaining the weighted Moore-Penrose inverse

$$X_{k+1} = X_k (5I - 10AX_k + 10(AX_k)^2 - 5(AX_k)^3 + (AX_k)^4), \quad k = 0, 1, \dots$$

These papers are some of the manuscripts that have been published to approximate the inverse of a non-singular matrix or some of the generalized inverses of arbitrary matrices. In this paper, we design a parametric family of iterative methods with arbitrary order of convergence that contains many of the methods constructed up to date. For each fixed value of the order of convergence, we still have a class of iterative methods depending on several parameters. We are able to select the most stable members of these classes in terms of the wideness of the set of initial estimations converging to the solution (defined as basin of convergence). The best members are chosen to be compared, both dynamical and numerically, with some known methods.

The rest of this manuscript is organized as follows. Section 2 is devoted to the construction of the proposed class of iterative schemes, proving its convergence to the inverse of a nonsingular complex matrix, with arbitrary order of convergence. In Section 3, it is proven that the same family of iterative methods is able to converge to the Moore-Penrose inverse of a complex matrix of size $m \times n$. Some particular cases of this class are found in Section 4, corresponding to existing methods proposed by different authors. Section 5 is devoted to study the dependence on the initial estimations of the methods of the same class, with a fixed value of p , and also the effect of the kind of initial estimation in the wideness of the set of converging initial estimations. A wide range of numerical test are also found in Section 6, checking the robustness and applicability of the proposed methods, compared with other known ones on different kinds of matrices. Finally, some conclusions are stated.

2. A class of iterative schemes for matrix inversion

In this section, we are going to present a parametric family of iterative schemes for obtaining approximated inverses and to prove their order of convergence. Firstly, we define the following polynomial matrix.

Definition 1. Let $B \in \mathbb{C}^{m \times m}$ be a complex square matrix and $p > 0$ a positive integer number. We define the polynomial matrix $G_p(B)$ as

$$G_p(B) = \sum_{j=1}^p (-1)^{j-1} C_p^j B^{j-1} = C_p^1 I - C_p^2 B + C_p^3 B^2 - \dots + (-1)^{p-1} C_p^p B^{p-1},$$

where C_p^j is the combinatorial number $C_p^j = \binom{p}{j} = \frac{p!}{j!(p-j)!}$.

The following result is a technical lemma whose proof is straightforward by using mathematical induction with respect to p .

Lemma 1. Let $p > 0$ be a positive integer and $B \in \mathbb{C}^{m \times m}$. Then $(I - B)^p = I - BG_p(B)$.

Let $A \in \mathbb{C}^{m \times m}$ be a nonsingular matrix and $p > 1$ a positive integer. Let $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ be a set of real parameters such that $\alpha_i \in [0, 1]$, for $i = 1, 2, \dots, p - 1, \alpha_p \in]0, 1]$ and $\sum_{i=1}^p \alpha_i = 1$.

We assume a sequence of complex matrices $\{X_0, X_1, \dots, X_n, \dots\}$, of size $m \times m$, satisfying following two conditions:

- (a) $\|I - AX_0\| = \gamma_0 < 1$,
- (b) $I - AX_{k+1} = \sum_{i=1}^p \alpha_i (I - AX_k)^i$.

We consider the family of iterative schemes described as

$$X_{k+1} = X_k \sum_{i=1}^p \alpha_i G_i(AX_k), \quad k = 0, 1, \dots \tag{2}$$

For each positive integer p , $p > 1$, we have a different class of iterative methods, whose order of convergence depends on the value of parameters α_i , $i = 1, 2, \dots, p$.

In the following results, we prove the convergence of these schemes to the inverse of A .

Proposition 1. Let $A \in \mathbb{C}^{m \times m}$ be a nonsingular matrix and $p > 1$ a positive integer. Let us consider the sequence of complex matrices constructed as

$$X_{k+1} = X_k \sum_{i=1}^p \alpha_i G_i(AX_k), \quad k = 0, 1, \dots,$$

where $\alpha_i \in [0, 1]$, for $i = 1, 2, \dots, p - 1, \alpha_p \in]0, 1]$ and $\sum_{i=1}^p \alpha_i = 1$. Then, condition

$$I - AX_{k+1} = \sum_{i=1}^p \alpha_i (I - AX_k)^i,$$

is equivalent to

$$X_{k+1} = X_k \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (AX_k)^{j-1} \right). \tag{3}$$

Proof. From Lemma 1 we obtain:

$$\begin{aligned} I - AX_{k+1} &= \sum_{i=1}^p \alpha_i (I - AX_k)^i \Leftrightarrow I - AX_{k+1} = \sum_{i=1}^p \alpha_i (I - AX_k G_i(AX_k)) \\ &\Leftrightarrow I - AX_{k+1} = I - \sum_{i=1}^p \alpha_i AX_k G_i(AX_k) \\ &\Leftrightarrow AX_{k+1} = AX_k \sum_{i=1}^p \alpha_i G_i(AX_k) \\ &\Leftrightarrow X_{k+1} = X_k \sum_{i=1}^p \alpha_i G_i(AX_k) \end{aligned}$$

Therefore, the result is proven. \square

By mathematical induction it is also easy to prove the following result.

Proposition 2. Let us consider sequence $\{X_k\}_{k \geq 0}$ obtained from expression (2). If $\|I - AX_0\| < 1$, then

$$\|I - AX_k\| < 1, \quad k = 1, 2, \dots$$

Therefore, the main result regarding matrix inversion can be stated.

Theorem 1. Let $A \in \mathbb{C}^{m \times m}$ be a nonsingular matrix and an initial guess $X_0 \in \mathbb{C}^{m \times m}$. Let $\alpha_1, \dots, \alpha_p$ be nonnegative real numbers such that $\alpha_i \in [0, 1]$, $\alpha_p \neq 0$ and $\sum_{i=1}^p \alpha_i = 1$. If $\|I - AX_0\| < 1$, then sequence $\{X_k\}_{k \geq 0}$, obtained by (2), converges to A^{-1} with convergence order q for any $p > 1$, where $q = \min_{i=1,2,\dots,p} \{i \mid \alpha_i \neq 0\}$.

Proof. By using previous condition (b), we have

$$\|I - AX_{k+1}\| = \left\| \sum_{i=1}^p \alpha_i (I - AX_k)^i \right\| \leq \sum_{i=1}^p \alpha_i \|I - AX_k\|^i = \sum_{i=q}^p \alpha_i \|I - AX_k\|^i,$$

but, from Proposition 2 we can assure

$$\|I - AX_{k+1}\| \leq \sum_{i=q}^p \alpha_i \|I - AX_k\|^q = \|I - AX_k\|^q.$$

Thus,

$$\|I - AX_{k+1}\| \leq \|I - AX_k\|^q. \tag{4}$$

By using Equation (4) and mathematical induction, we prove

$$\|I - AX_{k+1}\| \leq \|I - AX_0\|^{q^{k+1}}.$$

Therefore, as $k \rightarrow \infty$:

$$\|I - AX_{k+1}\| \rightarrow 0 \Rightarrow I - AX_{k+1} \rightarrow \mathbf{0} \Rightarrow \{X_n\}_{k \geq 0} \rightarrow A^{-1},$$

with order q . \square

In the next section, we extend family (2) for approximating the Moore-Penrose inverse of any complex rectangular matrix.

3. A class of iterative schemes for computing Moore-Penrose inverse

Let A be a $m \times n$ complex matrix. The Moore-Penrose inverse of A (also called pseudoinverse), denoted by A^\dagger , is the unique $n \times m$ matrix X satisfying

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

This generalized inverse plays an important role in several fields, such as eigenvalue problems and the linear least square problems [18]. It can be obtained, explicitly, from the singular value decomposition of A but, with a high computational cost. Therefore, it is interesting to have efficient iterative methods to approximate this matrix. In this section, we prove how family (2) allows us to compute the pseudoinverse with the same order of convergence that in the previous section, where the inverse of a square matrix was calculated. First, we establish the following technical result, that is proven by mathematical induction (similar to Lemma 2.1 of [13]), although other authors state similar results in the context of outer inverses (see, for example, [9]).

Lemma 2. Let us consider $X_0 = \alpha A^*$, where $\alpha \in \mathbb{R}$, and sequence $\{X_k\}_{k \geq 0}$ generated by family (2). For any $k \geq 0$, it is held that

$$(X_k A)^* = X_k A, \quad (AX_k)^* = AX_k, \quad X_k A A^\dagger = X_k, \quad A^\dagger A X_k = X_k. \tag{5}$$

Proof. For $n = 0$, the first two equations can readily be verified, and we only give a verification to the last two equations:

$$X_0 A A^\dagger = \alpha A^* A A^\dagger = \alpha A^* (A A^\dagger)^* = \alpha A^* (A^\dagger)^* A^* = \alpha (A A^\dagger A)^* = \alpha A^* = X_0,$$

and

$$A^\dagger A X_0 = (A^\dagger A)^* (\alpha A^*) = \alpha A^* (A^\dagger)^* A^* = \alpha (A A^\dagger A)^* = \alpha A^* = X_0.$$

Let us assume now that the conclusion holds for k . We now show that it continues to be held for $k + 1$. Using the iterative procedure in Equation (2), we have

$$\begin{aligned} (X_{k+1} A)^* &= \left(\left[X_n \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (AX_k)^{j-1} \right) \right] A \right)^* \\ &= \left(\sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (X_k A)^j \right) \right)^* \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (X_k A)^j \right) \\
 &= \left[X_k \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (AX_k)^{j-1} \right) \right] A = X_{k+1} A,
 \end{aligned}$$

where the fourth equality uses that $(X_k A)^* = X_{k+1} A$. Thus, the first equality of (5) holds for $k + 1$. Second equality can be proven in a similar way.

For the third equality of (5), we use the assumption that $X_k A A^\dagger = X_k$ and the iterative procedure in (2):

$$\begin{aligned}
 X_{k+1} A A^\dagger &= \left[X_k \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (AX_k)^{j-1} \right) \right] A A^\dagger \\
 &= \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (X_k A)^{j-1} X_k A A^\dagger \right) \\
 &= X_k \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (AX_k)^{j-1} \right) = X_{k+1}.
 \end{aligned}$$

Hence the third equality of (5) holds for $k + 1$. The fourth equality can similarly be proven, and the desired result follows. □

Now, some technical results are presented.

Lemma 3 ([13]). Let $A \in \mathbb{C}^{m \times n}$. Let $X_0 = \alpha A^*$ be, where $\alpha < \frac{1}{\sigma_1^2}$ and σ_1 is the largest singular value of A . Then $\|(X_0 - A^\dagger)A\| < 1$.

The following result is straightforward by using Lemma 2.

Lemma 4. Let $A \in \mathbb{C}^{m \times n}$ and $\{X_k\}_{k \geq 0}$ be the sequence generated by (2). Let us consider $E_k = X_k - A^\dagger$. Then,

1. $\|X_k - A^\dagger\| \leq \|E_k A\| \|A^\dagger\|$,
2. $(I - A^\dagger A)E_k A = \mathbf{0}$.

Lemma 5. Let $A \in \mathbb{C}^{m \times n}$ and $\{X_k\}_{k \geq 0}$ be the sequence generated by (2). We denote $E_k = X_k - A^\dagger$. Then

$$E_{k+1} A = \sum_{i=1}^p \alpha_i (-1)^{i-1} (E_k A)^i, \quad k = 0, 1, \dots \tag{6}$$

Proof. By using the fact that $\sum_{i=1}^p \alpha_i = 1$, we have

$$\begin{aligned}
 E_{k+1} A &= \left[X_k \sum_{i=1}^p \alpha_i G_i (AX_k) \right] A - A^\dagger A \\
 &= \left[X_k \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (AX_k)^{j-1} \right) \right] A - A^\dagger A \\
 &= \sum_{i=1}^p \alpha_i \left(\sum_{j=1}^i (-1)^{j-1} C_i^j (X_k A)^j \right) - A^\dagger A \\
 &= \sum_{i=1}^p \alpha_i \left(-I + \sum_{j=1}^i (-1)^{j-1} C_i^j (X_k A)^j \right) + I - A^\dagger A \\
 &= - \sum_{i=1}^p \alpha_i (I - X_k A)^i + I - A^\dagger A.
 \end{aligned}$$

Now, taking into account that $X_k A = A^\dagger A + E_k A$, $(I - A^\dagger A)^q = I - A^\dagger A$ and $(I - A^\dagger A)E_k A = \mathbf{0}$ (from Lemma 4), we get

$$- \sum_{i=1}^p \alpha_i (I - X_k A)^i + I - A^\dagger A = - \sum_{i=1}^p \alpha_i (I - A^\dagger A - E_k A)^i + I - A^\dagger A$$

$$\begin{aligned}
 &= -\left(\sum_{i=1}^p \alpha_i [(I - A^\dagger A)^i - C_i^1 (I - A^\dagger A)^{i-1} E_k A + C_i^2 (I - A^\dagger A)^{i-2} (E_k A)^2 + \dots \right. \\
 &\quad \left. + (-1)^{i-1} C_i^{i-1} (I - A^\dagger A) (E_k A)^{i-1} + (-1)^i C_i^i (E_k A)^i\right] + I - A^\dagger A \\
 &= -\left(\sum_{i=1}^p \alpha_i [(I - A^\dagger A) - C_i^1 (I - A^\dagger A) E_k A + C_i^2 (I - A^\dagger A) (E_k A)^2 + \dots \right. \\
 &\quad \left. + (-1)^{i-1} C_i^{i-1} (I - A^\dagger A) (E_k A)^{i-1} + (-1)^i C_i^i (E_k A)^i\right] + I - A^\dagger A \\
 &= -\left(\sum_{i=1}^p \alpha_i [(I - A^\dagger A) + (-1)^i C_i^i (E_k A)^i]\right) + I - A^\dagger A \\
 &= \sum_{i=1}^p \alpha_i (-1)^{i-1} (E_k A)^i.
 \end{aligned}$$

□

Finally, we can state the following result.

Theorem 2. Let $A \in \mathbb{C}^{m \times n}$ and $q = \min_{i=1,2,\dots,p} \{\alpha_i \mid \alpha_i \neq 0\}$. Then, sequence $\{X_k\}_{k \geq 0}$ generated by (2) converges to the Moore-Penrose inverse A^\dagger with q th-order provided that $X_0 = \alpha A^*$, where $\alpha < \frac{1}{\sigma_1^2}$ is a constant and σ_1 is the largest singular value of A .

Proof. Therefore,

$$\|X_{k+1} - A^\dagger\| \leq \|E_k A\|^q \|A^\dagger\|. \tag{7}$$

Now, from (7) and by applying mathematical induction, we prove

$$\|X_{k+1} - A^\dagger\| \leq \|E_0 A\|^{q^{k+1}} \|A^\dagger\|.$$

Therefore, as $n \rightarrow \infty$:

$$\|X_{k+1} - A^\dagger\| \rightarrow 0 \Rightarrow X_{k+1} - A^\dagger \rightarrow \mathbf{0} \Rightarrow \{X_k\}_{k \geq 0} \rightarrow A^\dagger,$$

with order q . □

4. Proposed family and previous known methods

The proposed class of iterative schemes in (2) is a generalization of other methods constructed with different techniques. Some of them are described below.

- For any $p > 1$, if $\alpha_1 = \dots = \alpha_{p-1} = 0$ and $\alpha_p = 1$, then we obtain the method proposed by Li and Li. (see Eq. (2.3) in [12] for inverse case and Eq. (2.1) in [13] for pseudoinverse one). Recall that method proposed by Li and Li generalizes the Newton-Schultz ($p = 2$) and Chebyshev method ($p = 3$).
- On the other hand, if $\alpha_i = 0$ for $i = 1, 2, \dots, 8$, $\alpha_9 = \alpha_{12} = 1/8$ and $\alpha_{10} = \alpha_{11} = 3/8$, then we get the method proposed by Soleymani and Stanimirovic (see Eq. (12) in [19]).
- Also expression (2) gives us the method proposed by Toutounian and Soleymani (see Eq. (18) in [15]), if $\alpha_4 = 1/2$ and $\alpha_5 = 1/2$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
- When the only not null parameter is $\alpha_7 = 1$, then we obtain method proposed by Soleymani (see Eq. (18) in [20]).
- In a similar way, if the only parameter different from zero is $\alpha_6 = 1$, then the method proposed by Soleymani, Stanimirovic and Zaka (see Eq. (2.4) in [21]) is obtained.
- When $\alpha_i = 0$ for $i = 1, 2, \dots, 7$, $\alpha_8 = \alpha_{10} = 1/4$ and $\alpha_9 = 2/4$, the resulting scheme is that proposed by Soleymani in Eq. (9) in [22].
- The method proposed by Soleymani et al in [23], Eq. (10), appears if $\alpha_1 = \dots = \alpha_8 = 0$, $\alpha_9 = 7/9$ and $\alpha_{10} = 2/9$.
- The scheme proposed by Razavi, Kerayechian, Gachpazan and Shateyi, (see Eq. (16) in [24]) is obtained if we choose $\alpha_1 = \dots = \alpha_9 = 0$, $\alpha_{10} = \alpha_{12} = 1/4$ and $\alpha_{11} = 1/2$ in Eq. (2).
- When the first eight parameters are null, $\alpha_9 = 343/729$, $\alpha_{10} = 294/729$, $\alpha_{11} = 84/729$ and $\alpha_{12} = 8/729$ then we get method proposed by Al-Fhaid et al in [25], Eq. (5).
- If $\alpha_7 = 9/16$, $\alpha_8 = 6/16$, $\alpha_9 = 1/16$ and the rest of parameters are zero, then the scheme proposed by Soleymani is found (see Eq. (3.1) in [26]).
- When $p = 12$ and the only parameters different from zero are $\alpha_9 = \alpha_{12} = 1/8$ and $\alpha_{10} = \alpha_{11} = 3/8$, therefore the method proposed by Liu and Cai. (see Eq. (4) in [27]) is obtained.
- If $\alpha_2 = 0$, $\alpha_1 = 1 - \alpha$ and $\alpha_3 = \alpha$, where $\alpha \in (0, 1]$, then we find the method proposed by Srivastava and Gupta in [28], Eq. (6).

5. Dependence on initial estimations

A key aspect of an iterative method is its order of convergence and also the set of initial guesses that converge to the estimated solution. When a class of iterative schemes is defined, all with the same order of convergence, a good behavior in terms of wider basins of convergence can make the difference. In this section, we use some graphical tools from complex discrete dynamics in order to deduce which scheme should be used within a class of iterative procedures with the same order of convergence.

Several known methods are used to this purpose, that are included in our proposed class: some of them are the extension for matrix equations of classical methods; the rest have been developed by other researchers in the last years and have been deduced as elements of our proposed class in Section 4.

So, the methods used are second-order Newton-Schulz (NS) scheme, with iterative expression (see [11])

$$X_{k+1} = 2X_k - X_kAX_k, \quad k \geq 0,$$

Chebyshev's scheme (CH), with third-order of convergence, deduced by Amat et al. in [29],

$$X_{k+1} = X_k(3I - AX_k(3I - AX_k)), \quad k \geq 0.$$

On the other hand, seventh-order scheme (SS) defined by Soleymani and Stanimirovich in [19] is also considered,

$$X_{k+1} = -1/8X_k(-7I + 9Y_k - 5Y_k^2 + Y_k^3)(12I - 42Y_k + 103Y_k^2 - 156Y_k^3 + 157Y_k^4 - 104Y_k^5 + 43Y_k^6 - 10Y_k^7 + Y_k^8), \quad k \geq 0,$$

where $Y_k = AX_k$.

It is also interesting to compare the results with the fourth-order procedure by Toutonian and Soleymani (TS) [15], whose iterative expression is

$$X_{k+1} = (1/2)X_k(9I - Y_k(16I - Y_k(14I - Y_k(6I - Y_k)))), \quad k \geq 0,$$

where $Y_k = AX_k$. Also, third-order Homeier's method (HM) is expressed by Li et al. as (see [30])

$$X_{k+1} = X_k(I + (1/2)(I - AX_k)(I + (2I - AX_k)^2)), \quad k \geq 0,$$

and also Midpoint scheme (MP) has third-order of convergence and the iterative expression (see [30]) provided by Li et al,

$$X_{k+1} = (I + 1/4(I - X_kA)(3I - X_kA)^2)X_k, \quad k \geq 0.$$

Let us remark that the last two methods do not belong to the family of proposed methods.

To observe which is the performance of proposed and existing methods by using different initial estimations, we use the dynamical planes. A dynamical plane is obtained by iterating a method on a wide set of initial estimations. To generate it, we used a mesh of 400×400 points as values of parameter $\beta \in \mathbb{C}$, used to define the initial estimation of the process X_0 . We paint in orange the values of β whose related starting points converge to the inverse (or pseudoinverse) of matrix A , with a tolerance of 10^{-3} . Those points whose orbit (set of consecutive iterates) achieve the maximum of 80 iterations are painted in black.

Firstly, let us find an estimation of the inverse of Toeplitz matrix

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

We firstly employ the initial estimation $X_0 = \beta I$, being I the identity matrix of the size of the matrix to be inverted. In order to satisfy condition $\|AX_0 - I\| < 1$, it is necessary that $|\beta - 1| < 1$. In Fig. 1, it can be observed that the set of converging initial values is narrow. However, it is wider in HM and MP cases than in NS or CH ones. In this case, we have worked with a the mesh in $[0, 1] \times [-0.5, 0.5]$ corresponding to real and imaginary part of a complex value of β .

Now, let us change the initial estimation. If we consider $X_0 = \beta \frac{A_1^T}{\|A_1\|^2}$, then the set of converging initial values is wider than in previous case (see Fig. 2). Nevertheless, the widest ones correspond again to HM and MP methods. This is the reason why we have constructed the mesh in $[-1, 4] \times [-2.5, 2.5]$.

Let us observe the case $p = 2$ in expression (2), where, fixing $\alpha_2 = 1 - \alpha_1$, a class of first-order (at least) iterative schemes including Newton-Schulz's method (for $\alpha_1 = 0$) is obtained,

$$X_{k+1} = X_k(\alpha_1 + (1 - \alpha_1)(2I - AX_k)).$$

In Fig. 3, it can be observed that the basin of convergence (set of values of β that, taken as initial estimation, allows the process to converge) is wider for higher values of $|\beta|$. However, the color of these basins are brighter for lower values of $|\beta|$; the reason is that the order of convergence is only linear as $\alpha_1 \neq 0$ and it is slower for higher values of β .

Let us consider now a class of methods defined by the general expression of the family (2) with $p = 3$, $\alpha_1 = 0$ and $\alpha_2 + \alpha_3 = 1$,

$$X_{k+1} = X_k(\alpha_2(2I - AX_k) + (1 - \alpha_2)(3I - 3(AX_k) + (AX_k)^2)).$$

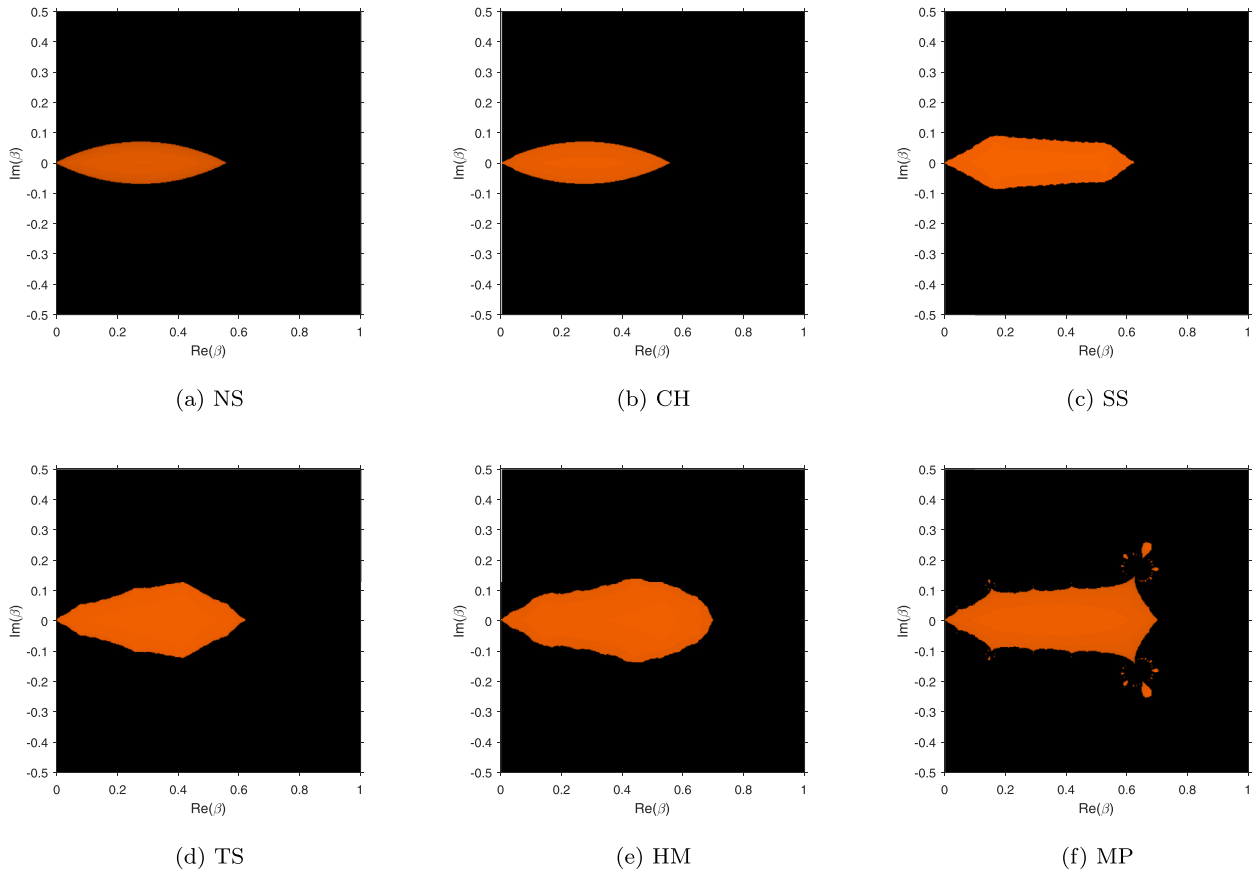


Fig. 1. Dynamical planes corresponding to $X_0 = \beta I$ of different methods on Toeplitz matrix A_1 .

The members of this class have, at least, second order of convergence if $\alpha_2 \neq 0$ and third-order (the corresponding iterative scheme is Chebyshev's one) if $\alpha_2 = 0$. As the initial estimation taken is $X_0 = \beta \frac{A_1^T}{\|A_1\|^2}$, in Fig. 4, we notice that also the wideness of the basin of attraction of the inverse matrix is bigger for higher absolute values of β but this amplitude is lower than in the case of the members of the family for $p = 2$, for the same value of β . That is, the order is higher but the set of converging values of β is lower.

Now, let us to compare the performance of the methods with other kind of matrices by using command `gallery('leslie',3)` in Matlab, a 3×3 matrix appearing in population models,

$$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By using this matrix and $X_0 = \beta \frac{A_2^T}{\|A_2\|^2}$, the set of converging initial values is painted in orange color for Newton-Schulz's method and also several second-order schemes of family corresponding to $p = 3$ and $\alpha_1 = 0$. Specifically, for $\alpha_2 = 0.3$, $\alpha_2 = 0.5$, $\alpha_2 = 0.7$ and $\alpha_2 = 0.8$ and $\alpha_3 = 1 - \alpha_2$. In Fig. 5 it is observed that the basin of attraction increases its wideness as well as parameter grows, being $\alpha_2 \leq 0.7$. Moreover, the set of converging values of β is connected meanwhile $\alpha_2 \leq 0.7$ and disconnected in other cases.

If the matrix is ill-conditioned, the role of the convergence and dependence on initial estimations is even more important. In Fig. 6, the inverse of a Hilbert matrix of size 3×3 ,

$$A_3 = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix},$$

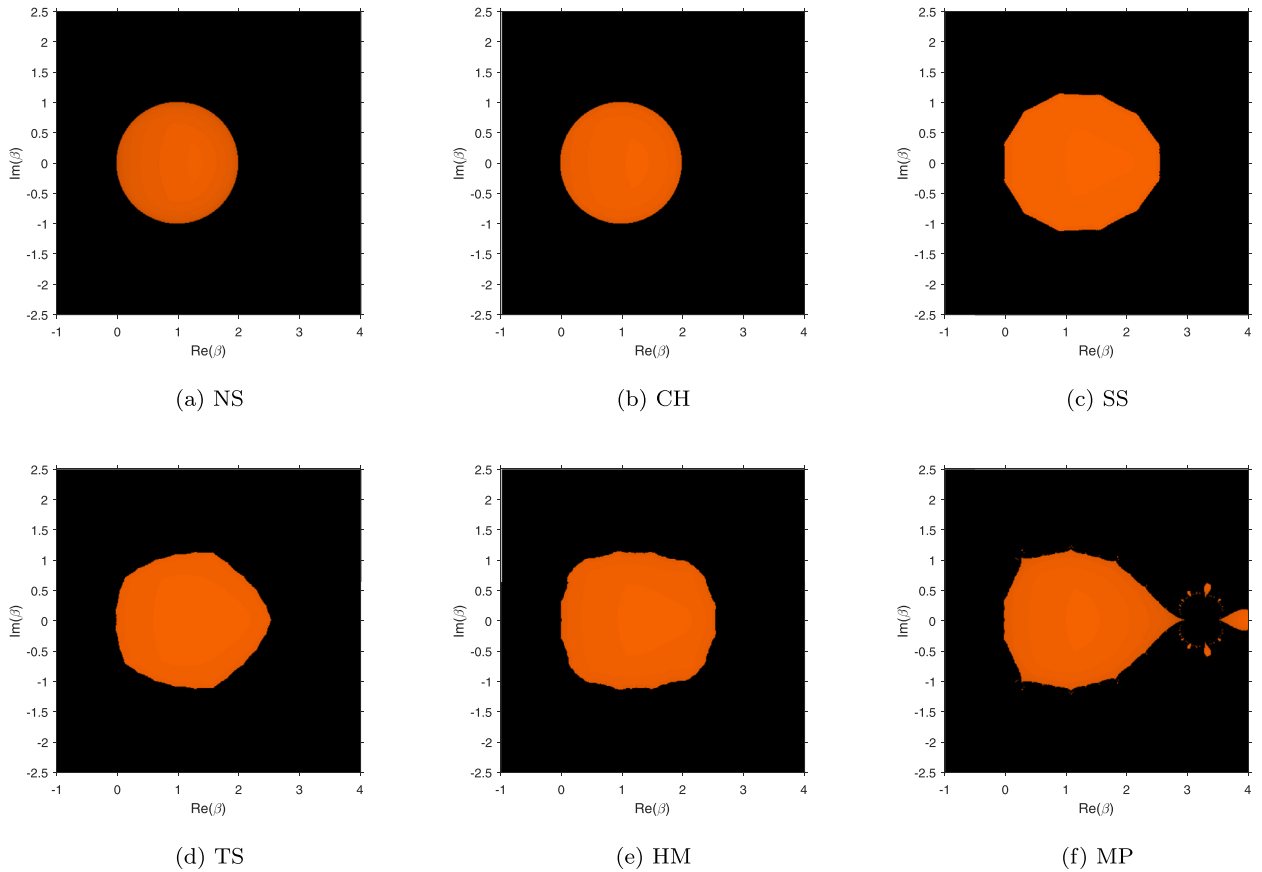


Fig. 2. Dynamical planes corresponding to $X_0 = \beta \frac{A_1^T}{\|A_1\|^2}$ on Toeplitz matrix A_1 .

is estimated with $X_0 = \beta \frac{A_3^T}{\|A_3\|^2}$. Let us notice that the color of the basins of convergence is darker than in previous cases. This means that the convergence is slower, due to stability problems. However, each scheme has the same wideness and shape than in previous cases, were the conditioning of the involved matrices were good.

Now, pseudoinverse calculation is made for the low-rank matrix

$$A_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 4 & 6 & 8 \end{pmatrix}.$$

In Fig. 7, it can be observed that the performance of the iterative schemes for estimating the pseudoinverse matrix is the same as in previous non-singular square matrices, showing the best behavior the method $\alpha_2 = 0.7$, in terms of the size of the set of converging initial estimations. In case of first-order class of iterative methods, it has been also observed that the performance of the procedures (as far as it depends on the initial estimation) is the same as in the inverse calculations.

6. Numerical performance

In this section, we check the performance of the proposed schemes, in comparison with other known ones, on small and large-scale matrices. These came from academical problems and also from applied signal processing problems. Among the academical problems, we use the Hilbert matrix as example of ill-conditioned matrix. These numerical tests have been made with Matlab R2018b, by using double precision arithmetics. The convergence is checked by means of the stopping criterium of the residual, $\|AX_k - I\| < 10^{-6}$ and a maximum of 200 iterations. In all cases, the initial estimation taken is $X_0 = \beta \frac{A^T}{\|A\|^2}$, being A the matrix whose inverse we are estimating, as we have seen in the previous section that this estimation, with values of β close to 0.7, gives the best results.

In Tables 1, 2, 3, elements $\alpha_1 = 0$, $\alpha_1 = 0.6$ and $\alpha_1 = 0.8$ for the class $p = 2$ (all of them with $\alpha_2 = 1 - \alpha_1$) are compared with the members of class $p = 3$ corresponding to $\alpha_1 = 0$ and $\alpha_2 = 0$, $\alpha_2 = 0.6$ and $\alpha_2 = 0.8$, where $\alpha_3 = 1 - \alpha_2$. The

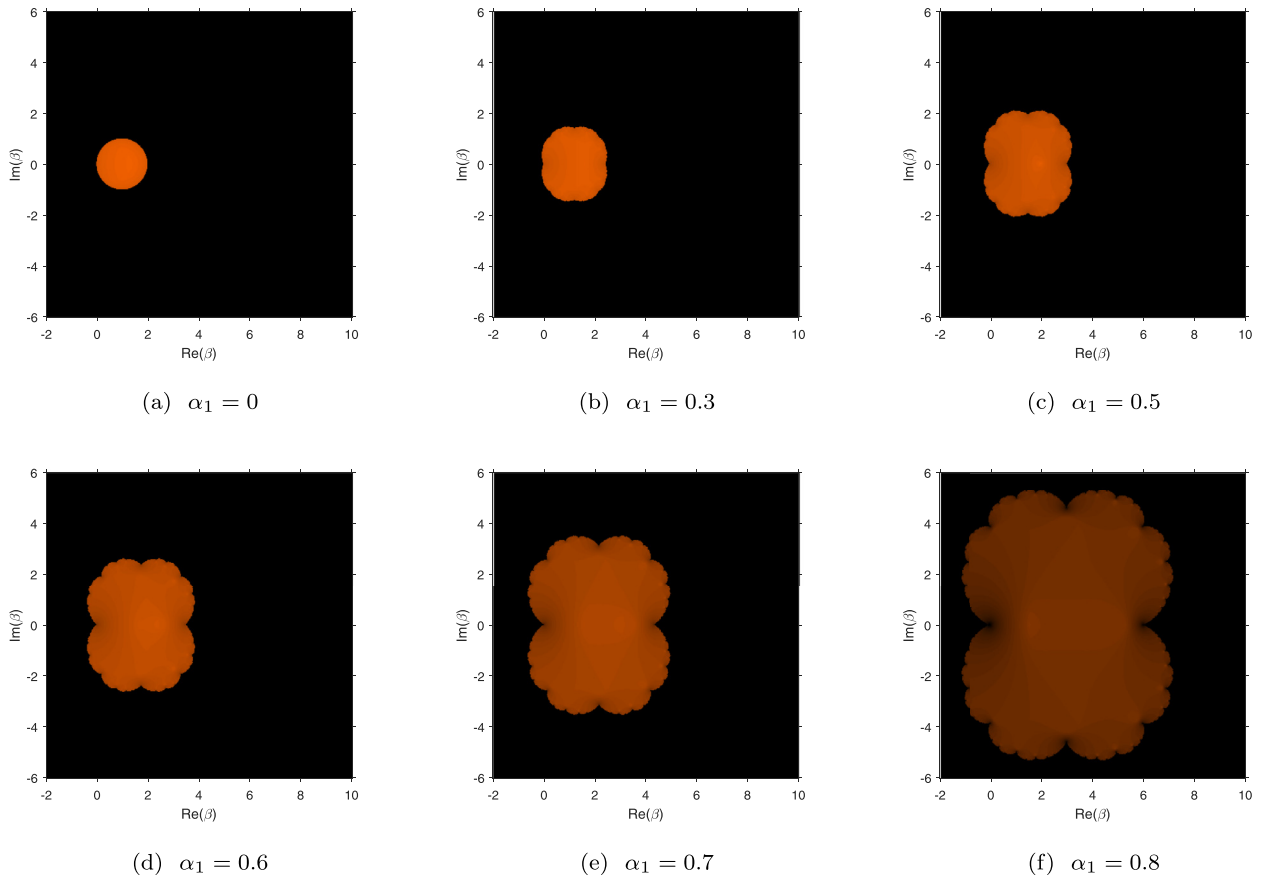


Fig. 3. Dynamical planes corresponding to first-order class with $\alpha_1 \in [0, 1]$ and $X_0 = \beta \frac{A_1^T}{\|A_1\|^2}$ on matrix A_1 .

Table 1
Numerical results for Toeplitz matrix A_1 .

β	$p = 2$						$p = 3, \alpha_1 = 0$					
	$\alpha_1 = 0$		$\alpha_1 = 0.6$		$\alpha_1 = 0.8$		$\alpha_2 = 0$		$\alpha_2 = 0.6$		$\alpha_2 = 0.8$	
	it	res	it	res	it	res	it	res	it	res	it	res
1	5	2.3e-10	28	9.3e-7	63	9.3e-7	3	7.5e-9	4	1.2e-7	5	2.5e-12
1.5	5	2.3e-10	26	6.8e-7	58	8.6e-7	3	7.5e-9	4	3.7e-10	4	2.2e-7
2	nc	-	25	6.9e-7	58	9.9e-7	nc	-	4	1.2e-7	6	1.5e-9
2.5	nc	-	24	8.1e-7	59	8.8e-7	nc	-	3	2.1e-8	nc	-
3	nc	-	28	9.4e-7	59	9.1e-7	nc	-	5	4.8e-8	nc	-
3.5	nc	-	nc	-	59	8.8e-7	nc	-	nc	-	nc	-
4	nc	-	nc	-	58	9.9e-7	nc	-	nc	-	nc	-
4.5	nc	-	nc	-	59	8.5e-7	nc	-	nc	-	nc	-
5	nc	-	nc	-	59	8.8e-7	nc	-	nc	-	nc	-
5.5	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
6	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-

comparison is made through the number of iterations needed to converge (it) and the residual $\|AX_k - I\|$, denoted by (res). If the method does not converge (typically giving "NaN"), it is denoted by "nc" in the column of iterations; if the scheme simply needs more than 200 iterations to converge, it is denoted by > 200 .

Table 1 shows the results obtained for matrix A_1 introduced in the previous section. In Table 2 the numerical results correspond to a Leslie matrix of size 100×100 , and in Table 3 the results generated by a Hilbert matrix of size 5×5 are shown.

From the results appearing in Table 1, it can be deduced that, among second-order schemes, the best convergence results are those corresponding to the method of class $p = 2$ with $\alpha_1 = 0$ and $\alpha_2 = 0.8$. It converges for a much wider set of initial estimations, although the number of iterations needed is high. In terms of the number of iterations needed to converge,

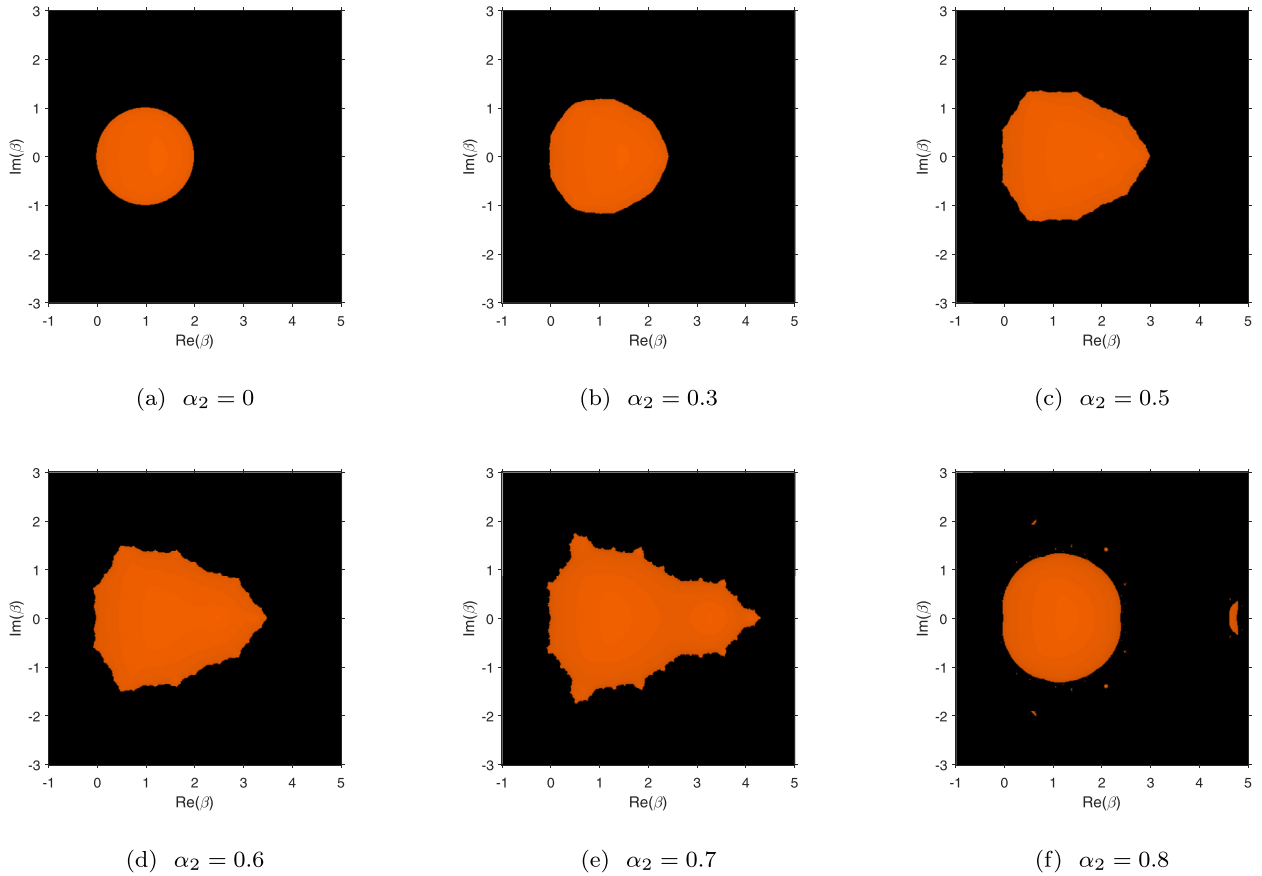


Fig. 4. Dynamical planes corresponding to second-order class ($p = 3$) with $\alpha_1 = 0$, $\alpha_2 \in [0, 1[$ and $X_0 = \beta \frac{A_1^T}{\|A_1\|^2}$ on matrix A_1 .

Table 2
Numerical results for a Leslie matrix of size 100×100 .

β	$p = 2$						$p = 3, \alpha_1 = 0$					
	$\alpha_1 = 0$		$\alpha_1 = 0.6$		$\alpha_1 = 0.8$		$\alpha_2 = 0$		$\alpha_2 = 0.6$		$\alpha_2 = 0.8$	
	it	res	it	res	it	res	it	res	it	res	it	res
1	18	6.9e-12	55	8.5e-7	113	9.2e-7	11	2.9e-8	14	6.4e-7	16	2.4e-10
1.5	17	4.2e-9	54	7.7e-7	111	8.8e-7	11	4.8e-12	14	2.4e-9	15	1.4e-7
2	53	3.7e-11	53	8.3e-7	107	9.6e-7	33	8.0e-11	14	1.3e-11	15	1.4e-9
2.5	nc	-	52	9.8e-7	108	9.2e-7	nc	-	13	3.9e-7	nc	-
3	nc	-	52	7.5e-7	107	9.2e-7	nc	-	13	3.7e-8	nc	-
3.5	nc	-	nc	-	106	9.5e-7	nc	-	26	1.1e-12	nc	-
4	nc	-	nc	-	106	8.1e-7	nc	-	nc	-	nc	-
4.5	nc	-	nc	-	105	8.7e-7	nc	-	nc	-	nc	-
5	nc	-	nc	-	104	9.6e-7	nc	-	nc	-	14	1.2e-12
5.5	nc	-	nc	-	104	8.5e-7	nc	-	nc	-	nc	-
6	nc	-	nc	-	> 200	-	nc	-	nc	-	nc	-

the best results are those provided by the element of family $p = 3$ with $\alpha_1 = 0$, $\alpha_2 = 0.6$ (and $\alpha_3 = 1 - \alpha_2$); the process converges in a very reduced number of iterations, with lower residual error than classical Newton-Schulz method. Regarding the results obtained by using classical Chebyshev's scheme, we remark that it does not improve the data from the best of proposed schemes, in spite of having third-order of convergence.

In **Table 2**, we notice that for large-scale (100×100) Leslie matrix, the numerical results obtained by $p = 2$, $\alpha_1 = 0$ and $\alpha_2 = 0.6$ show convergence to the inverse matrix even when parameter β of the initial estimation is not close to zero. However, in these cases the number of iterations is very high. Regarding the lowest number of iterations needed to converge, the best method corresponds to $p = 3$, $\alpha_1 = 0$ and $\alpha_2 = 0.6$ as it holds low number of iterations and high value of β .

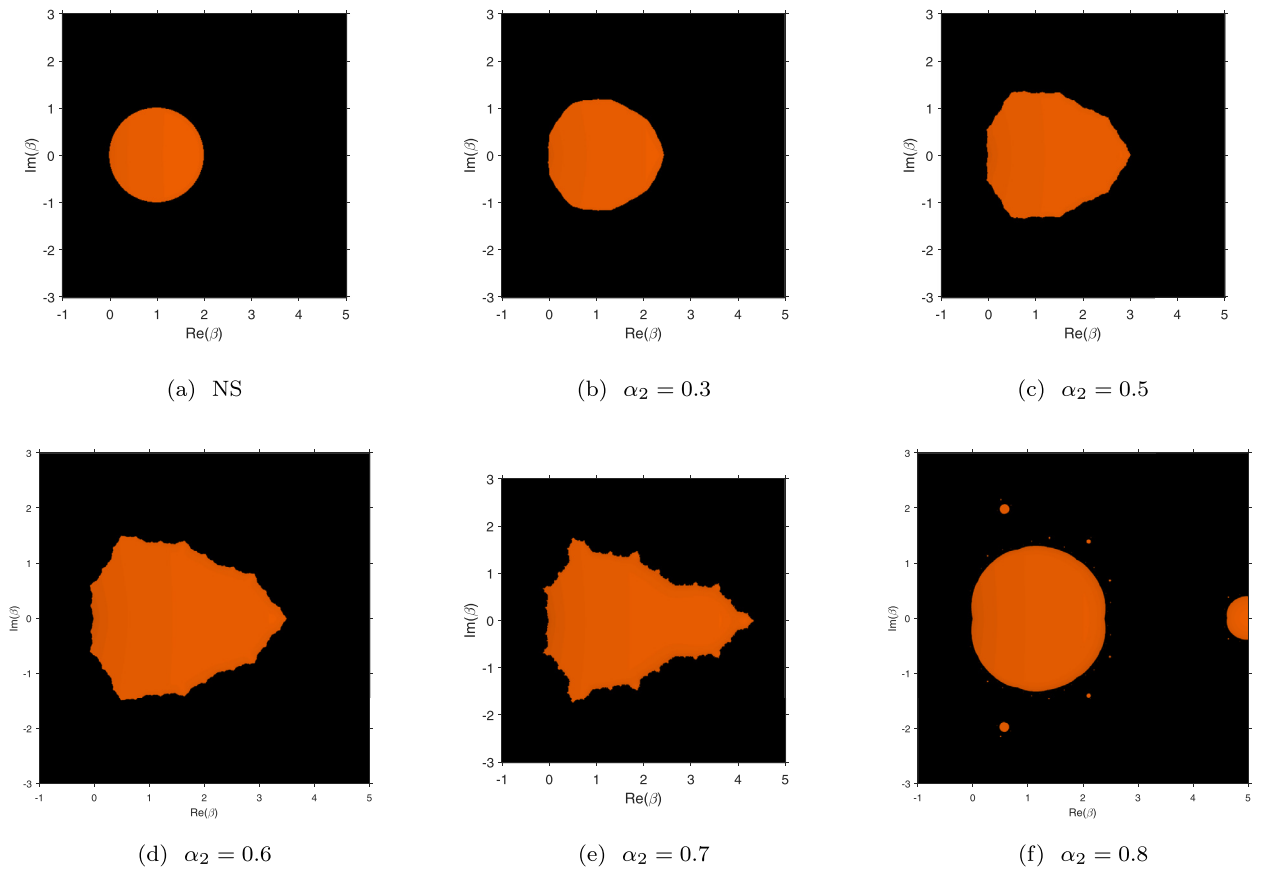


Fig. 5. Dynamical planes corresponding to some second-order methods with $X_0 = \beta \frac{A_2^T}{\|A_2\|^2}$ on Leslie matrix A_2 .

Table 3
Numerical results for a Hilbert matrix of size 5×5 .

β	$p = 2$						$p = 3, \alpha_1 = 0$					
	$\alpha_1 = 0$		$\alpha_1 = 0.2$		$\alpha_1 = 0.4$		$\alpha_2 = 0$		$\alpha_2 = 0.6$		$\alpha_2 = 0.8$	
	it	res	it	res	it	res	it	res	it	res	it	res
1	42	3.9e-9	54	5.7e-7	72	4.3e-7	27	2.3e-11	34	9.5e-11	37	1.07e-7
1.5	41	4.9e-7	53	9.3e-7	71	4.8e-7	26	5.1e-8	33	1.5e-7	37	9.8e-11
2	56	5.7e-8	53	4.2e-7	70	6.9e-7	nc	-	33	2.3e-9	36	3.9e-7
2.5	nc	-	nc	-	70	4.5e-7	nc	-	33	4.8e-11	nc	-
3	nc	-	nc	-	nc	-	nc	-	33	1.3e-11	nc	-
3.5	nc	-	nc	-	nc	-	nc	-	32	2.1e-8	nc	-
4	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
4.5	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
5	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
5.5	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-
6	nc	-	nc	-	nc	-	nc	-	nc	-	nc	-

Table 3 corresponds to a test on a 5×5 Hilbert matrix. It is clear that the number of iterations is high due to the bad conditioning of the matrix. Nevertheless, the performance is, in general similar to previous cases.

Now, we compare our proposed method of family $p = 3, \alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 0.7$ (denoted by PM3_{0.7} with other known ones, introduced in the previous section and denoted by SS, TS, HM and MP. In Table 4, we present the results obtained for these schemes on a Toeplitz matrix of size 100×100 , by using the initial estimation $X_0 = \beta \frac{A^*}{\|A\|^2}$, tolerance 10^{-6} and a maximum of 200 iterations. We notice that our scheme gets convergence to the inverse matrix in a very similar execution time (e-t) than other methods with higher order of convergence and moreover, it converges with initial estimations with bigger values of β than the rest of procedures.

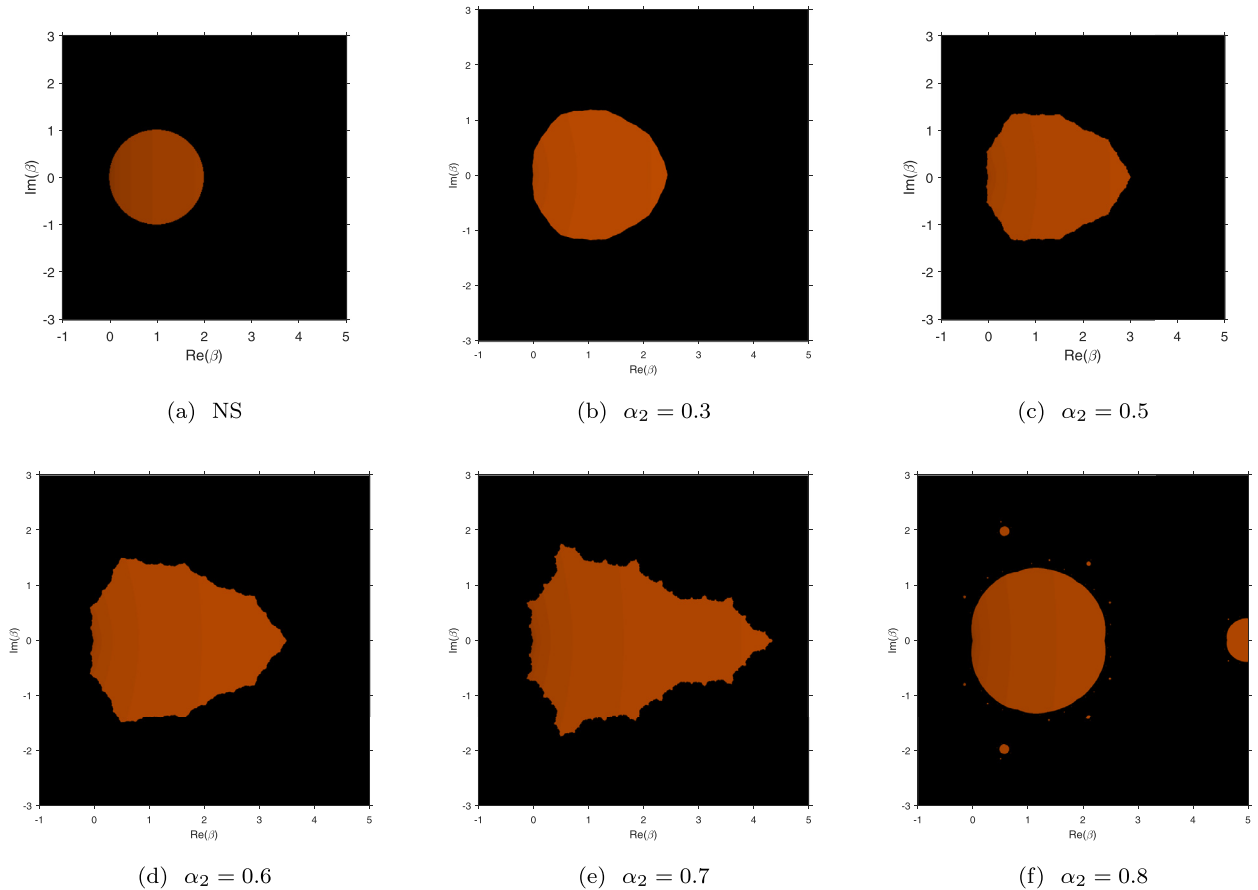


Fig. 6. Dynamical planes corresponding to some second-order methods with $X_0 = \beta \frac{A_3^T}{\|A_3\|^2}$ on Hilbert matrix A_3 .

Table 4
Numerical results for a Toeplitz matrix of size 100×100 .

β	SS			PM3 _{0.7}			TS			HM			MP		
	it	res	e-t	it	res	e-t	it	res	e-t	it	res	e-t	it	res	e-t
0.1	4	3.8e-14	0.028	10	3.1e-10	0.025	6	4.8e-15	0.044	7	1.4e-15	0.027	7	6.9e-12	0.022
0.5	3	4.4e-14	0.029	8	6.4e-10	0.026	4	4.2e-7	0.026	5	3.6e-8	0.028	6	1.1e-15	0.022
0.9	3	4.1e-14	0.031	7	2.6e-8	0.028	4	6.2e-12	0.034	5	1.7e-13	0.027	5	1.3e-10	0.031
1	3	3.9e-14	0.037	7	4.8e-9	0.032	4	3.9e-13	0.029	5	8.7e-15	0.038	5	1.1e-11	0.031
1.5	3	3.5e-14	0.029	7	1.1e-12	0.026	4	7.3e-15	0.032	4	1.8e-7	0.022	5	9.9e-16	0.026
2	2	2.7e-8	0.020	6	2.1e-8	0.028	3	8.6e-7	0.024	4	1.3e-9	0.027	4	5.6e-8	0.022
2.5	3	6.4e-14	0.024	6	3.3e-10	0.024	4	1.9e-10	0.027	5	1.1e-9	0.017	6	7.8e-15	0.028
3	nc	-	-	6	1.5e-11	0.027	nc	-	-	nc	-	-	nc	-	-
3.5	nc	-	-	6	1.4e-11	0.023	nc	-	-	nc	-	-	nc	-	-
4	nc	-	-	7	1.5e-11	0.028	nc	-	-	nc	-	-	nc	-	-
4.5	nc	-	-	nc	-	-	nc	-	-	nc	-	-	nc	-	-

The rest of the section is devoted to several test matrices for which we approximate its pseudoinverse. Table 5 includes the results of pseudoinverse calculation made for the low-rank matrix A_4 defined in Section 4. In this case, Chebyshev's method performs better than the most of methods under analysis, as it needs a very low number of iterations to converge to the pseudo-inverse, although $p = 2$ can converge even with values of $\beta = 6$ or higher.

A similar performance is observed in Table 6, where we use a random matrix of size 300×200 .

Finally, we present the last example corresponding to a random matrix with no full rank. Let us consider matrix

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \in \mathbb{R}^{303 \times 204},$$

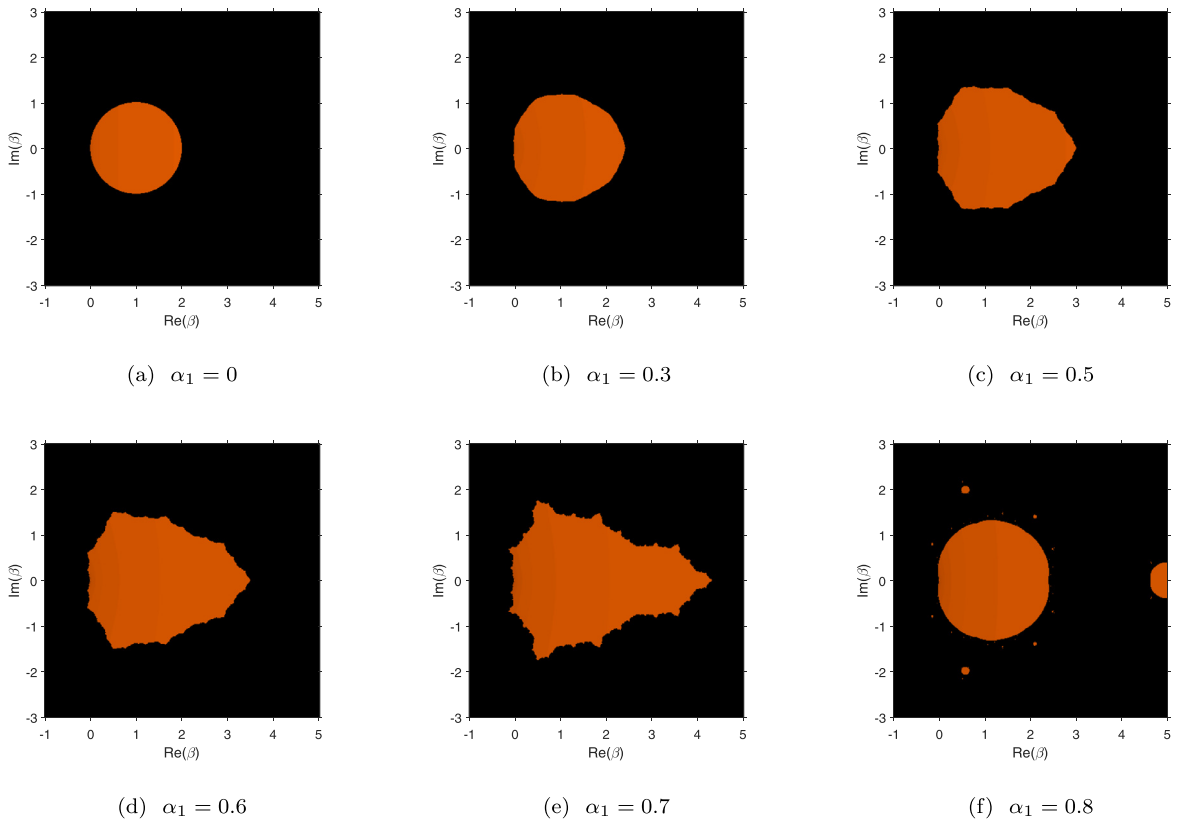


Fig. 7. Dynamical planes corresponding to second-order class with $\alpha_1 \in [0, 1[$ and $X_0 = \beta \frac{A_4^T}{\|A_4\|^2}$ on matrix A_4 .

Table 5

Numerical results for the estimation of the pseudo-inverse of A_4 with $X_0 = \beta \frac{A_4^T}{\|A_4\|^2}$.

β	$p = 2$						$p = 3, \alpha_1 = 0$					
	$\alpha_1 = 0$		$\alpha_1 = 0.6$		$\alpha_1 = 0.8$		$\alpha_2 = 0$		$\alpha_2 = 0.6$		$\alpha_2 = 0.8$	
	it	res	it	res	it	res	it	res	it	res	it	res
1	13	4.9e-10	43	6.9e-7	86	8.2e-7	11	7.6e-10	12	1.2e-10	9	2.9e-15
1.5	12	1.0e-7	42	6.2e-7	83	9.7e-7	11	2.0e-13	11	8.7e-8	8	3.5e-8
2	12	4.6e-10	41	6.7e-7	82	8.5e-7	10	1.3e-8	11	7.2e-10	8	1.1e-10
2.5	nc	-	40	7.9e-7	81	8.1e-7	10	3.5e-10	nc	-	nc	-
3	nc	-	39	9.9e-7	80	8.0e-7	10	1.0e-11	nc	-	nc	-
3.5	nc	-	39	7.9e-7	79	9.3e-7	10	6.2e-10	nc	-	nc	-
4	nc	-	nc	-	78	8.8e-7	nc	-	nc	-	nc	-
4.5	nc	-	nc	-	77	9.5e-7	nc	-	nc	-	nc	-
5	nc	-	nc	-	77	8.3e-7	nc	-	nc	-	nc	-
5.5	nc	-	nc	-	76	9.2e-7	nc	-	nc	-	nc	-
6	nc	-	nc	-	76	8.3e-7	nc	-	nc	-	nc	-

where B is a random matrix of size 300×200 , $B = rand(300, 200)$, C is a matrix of zeros, $C = zeros(300, 4)$, D is a matrix of ones, $D = ones(3, 200)$ and E is the matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 \end{pmatrix}.$$

In this case, we use as stopping criterium $\|X_{k+1} - X_k\| < 10^{-6}$ and the last value of $\|X_{k+1} - X_k\|$ appears in the column called 'res'.

The results corresponding to this example are shown in Table 7. When this table is observed, we notice that the best global results are from MP and our proposed method PM30.7. Both schemes unify very wide areas of converging initial estimations with low execution times.

Table 6

Numerical results for the estimation of the pseudoinverse of a random matrix of size 300×200 with $X_0 = \beta \frac{A^T}{\|A\|^2}$.

β	$p = 2$						$p = 3, \alpha_1 = 0$					
	$\alpha_1 = 0$		$\alpha_1 = 0.6$		$\alpha_1 = 0.8$		$\alpha_2 = 0$		$\alpha_2 = 0.6$		$\alpha_2 = 0.8$	
	it	res	it	res	it	res	it	res	it	res	it	res
1	19	5.4e-8	56	6.7e-7	109	9.7e-7	13	4.9e-15	16	1.2e-9	18	5.7e-14
1.5	19	1.3e-11	55	6.0e-7	107	9.3e-7	12	4.3e-8	16	4.2e-13	17	3.8e-10
2	18	5.4e-8	54	6.5e-7	106	8.1e-7	12	1.5e-10	15	2.1e-8	17	6.8e-13
2.5	nc	-	53	7.7e-7	104	9.7e-7	nc	-	15	6.1e-10	nc	-
3	nc	-	52	9.7e-7	103	9.7e-7	nc	-	15	2.1e-11	nc	-
3.5	nc	-	52	7.7e-7	103	8.0e-7	nc	-	15	3.5e-8	nc	-
4	nc	-	nc	-	102	8.5e-7	nc	-	nc	-	nc	-
4.5	nc	-	nc	-	101	9.2e-7	nc	-	nc	-	nc	-
5	nc	-	nc	-	101	8.1e-7	nc	-	nc	-	nc	-
5.5	nc	-	nc	-	100	9.0e-7	nc	-	nc	-	nc	-
6	nc	-	nc	-	100	8.1e-7	nc	-	nc	-	nc	-

Table 7

Numerical results for non-square matrix without full rank of size 303×204 .

β	SS			PM3 _{0.7}			TS			HM			MP		
	it	res	e-t	it	res	e-t	it	res	e-t	it	res	e-t	it	res	e-t
0.1	8	3.6e-14	0.308	19	4.9e-7	0.266	11	8.4e-9	0.211	13	3.7e-8	0.240	14	3.9e-11	0.244
0.5	7	3.3e-14	0.266	17	9.7e-7	0.273	10	1.2e-9	0.245	12	6.0e-11	0.264	13	1.8e-15	0.236
0.9	7	3.3e-14	0.289	17	1.2e-10	0.266	10	4.1e-15	0.231	12	1.9e-15	0.238	12	1.1e-9	0.235
1	7	3.6e-14	0.276	17	1.5e-11	0.262	10	4.1e-15	0.242	11	6.4e-7	0.244	12	1.3e-10	0.248
1.5	7	3.2e-14	0.273	16	2.9e-8	0.266	9	7.9e-7	0.245	11	1.2e-9	0.240	12	3.7e-15	0.237
2	6	3.7e-7	0.265	16	2.4e-10	0.245	9	1.0e-8	0.248	11	3.1e-12	0.250	11	5.4e-7	0.240
2.5	6	1.1e-8	0.258	16	1.4e-12	0.258	9	1.5e-10	0.233	11	9.8e-5	0.242	11	1.9e-8	0.250
3	nc	-	-	15	2.0e-7	0.230	nc	-	-	nc	-	-	11	6.7e-10	0.264
3.5	nc	-	-	15	2.3e-8	0.231	nc	-	-	nc	-	-	24	6.7e-12	0.314
4	nc	-	-	15	2.8e-9	0.265	nc	-	-	nc	-	-	11	9.6e-13	0.228
4.5	nc	-	-	nc	-	-	nc	-	-	nc	-	-	nc	-	-

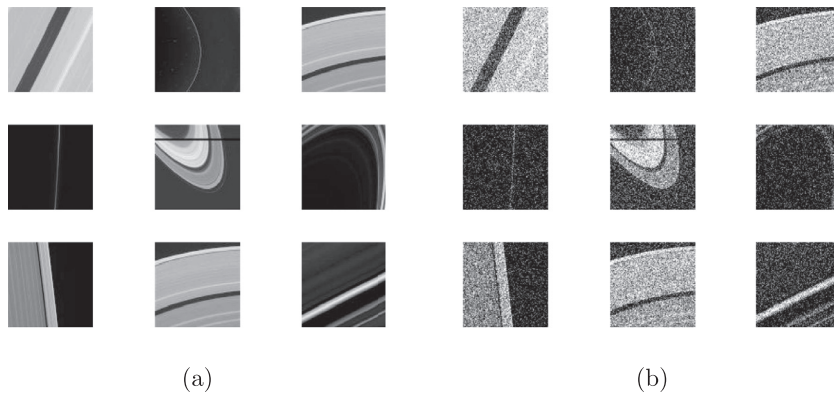


Fig. 8. (a) Some randomly selected images of Saturn. (b) Noisy versions of images in (a).

Now, we are going to apply the proposed schemes on a non-academic problem dealing with image processing.

6.1. Illustrative example: image denoising

This example illustrates an application of proposed method to an image denoising problem [31–33]. Specifically, we consider the problem of compression, filtering and decomposition of a noisy image $\bar{C} \in \mathbb{R}^{90 \times 90}$ estimation on the basis of the set of training images $\mathcal{A} = \{A^{(1)}, \dots, A^{(s)}\}$, where $A^{(j)} \in \mathbb{R}^{90 \times 90}$, for $j = 1, \dots, s$. Our training set \mathcal{A} consists of $s = 2000$ different satellite images from Saturn (see Fig. 8(a) for 9 sample images). Images in \mathcal{A} were taken from the NASA Solar System Exploration Database [34].

It is assumed that instead of images in \mathcal{A} , we observed their noisy version $\mathcal{C} = \{C^{(1)}, \dots, C^{(s)}\}$, where $C^{(j)} \in \mathbb{R}^{90 \times 90}$, for $j = 1, \dots, s$. Each $C^{(j)}$ is simulated as $C^{(j)} = A^{(j)} + 0.2N^{(j)}$, where $N^{(j)} \in \mathbb{R}^{90 \times 90}$ is a random matrix generated from a normal distribution with zero-mean and standard deviation 1. For each image $A^{(j)}$, matrix $N^{(j)}$ simulates noise. (see Fig. 8(b) for 9

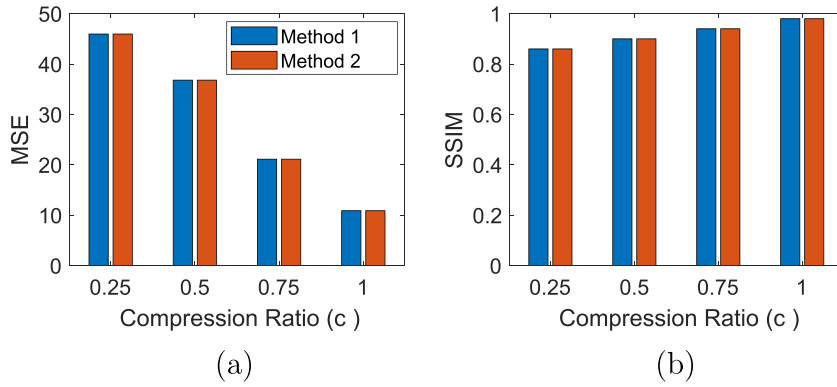


Fig. 9. Bar graphs of Method 1 and Method 2. Compression ratio c_r versus: (a) MSE and (b) SSIM between Methods 1 and 2.

sample images). It is assumed that noisy image \bar{C} does not necessarily belong to \mathcal{C} , but it is “similar” to one of them, i.e., there is $C^{(j)} \in \mathcal{C}$ such that

$$C^{(j)} \in \arg \min_{C^{(i)} \in \mathcal{C}} \|C^{(i)} - \bar{C}\|_{fr} \leq \delta,$$

for a given $\delta \geq 0$.

Here, we vectorize matrices $A^{(j)}$ and $C^{(j)}$, i.e., convert each matrix into a column vector by stacking the columns of the matrix. Let $vec : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{mn}$ be the vectorization transform. We write $a_j = vec(A^{(j)}) \in \mathbb{R}^{8100}$ and $c_j = vec(C^{(j)}) \in \mathbb{R}^{8100}$. If $A = [a_1 \ a_2 \ \dots \ a_s] \in \mathbb{R}^{8100 \times 2000}$ and $C = [c_1 \ c_2 \ \dots \ c_s] \in \mathbb{R}^{8100 \times 2000}$, the goal of the image denoising problem is find $\hat{X} \in \mathbb{R}^{8100 \times 8100}$ that gives a small reconstruction error for the training set, i.e., find \hat{X} such that

$$\|A - \hat{X}C\|_{fr}^2 = \min_{X \in \mathbb{R}^{8100 \times 8100}} \|A - XC\|_{fr}^2. \tag{8}$$

Here, $\mathbb{R}_r^{m \times n}$ denotes the set of all real $m \times n$ matrices of rank at most $r \leq \min\{m, n\}$. A solution of problem (8), given by Friedland and Torokhti [35], is

$$\hat{X} = [AC^{\dagger}C]_r C^{\dagger}, \tag{9}$$

where $[M]_r$ is the r -truncated SVD of M . In this example, we use the proposed method to compute the pseudoinverse, with $C_0 = \frac{C^T}{\|C\|^2}$, $p = 2$, $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$.

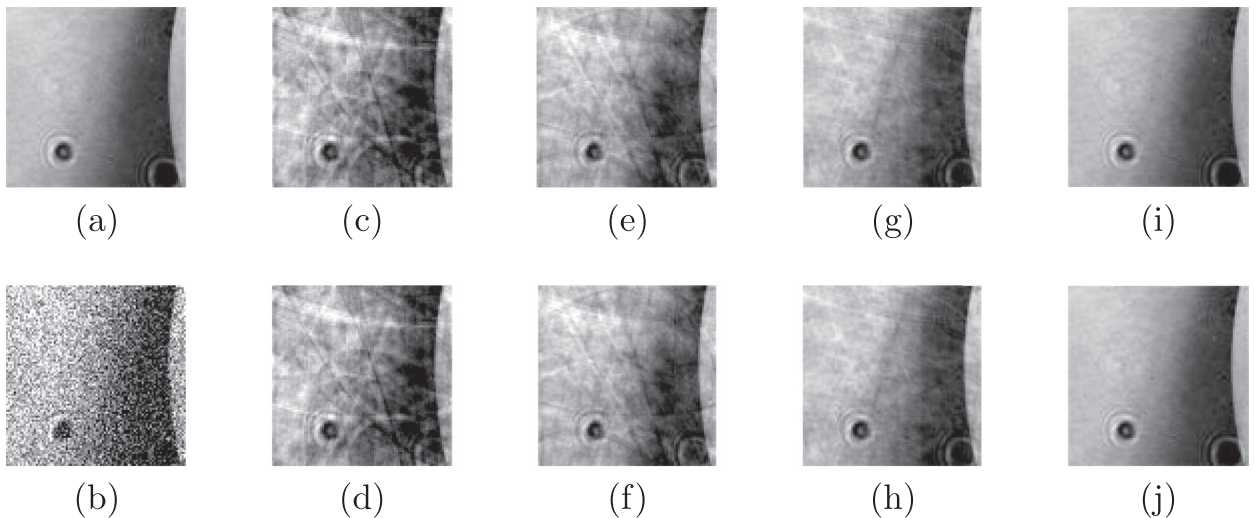


Fig. 10. Illustration of the estimation of noisy image \bar{C} by Methods 1 and 2. (a) Source image. (b) Noisy observed image \bar{C} . (c)-(d) Estimation using Methods 1 and 2, respectively, for $c_r = 0.25$. (e)-(f) Estimation using Methods 1 and 2, respectively, for $c_r = 0.5$. (g)-(h) Estimation using Methods 1 and 2, respectively, for $c_r = 0.75$. (i)-(j) Estimation using Methods 1 and 2, respectively, for $c_r = 1$.

The compression ratio in problem (8) is given by $c_r = \frac{r}{\min\{p,q\}}$. Fig. 9 presents the execution MSE and structural similarity index¹ (SSIM) between Method 1, which compute (9) using Matlab command `pinv`, and Method 2, which compute (9) using proposed method, for compression ratio $c_r \in \{0.25, 0.5, 0.75, 1\}$, i.e., $r \in \{2025, 4050, 6075, 8100\}$. In Fig. 10, we show the estimates of \bar{C} using both methods. The numerical results obtained in Figs. 9 and 10 clearly demonstrate the advantages of the proposed method. It is clear that both methods achieve the same MSE.

7. Conclusions

In this paper, we have developed a parametric family of iterative methods for computing inverse and pseudoinverse of a complex matrix, having arbitrary order of convergence. Moreover, we have shown in Theorems 1 and 2 that the order of the suggested method in (2) depends on the first non-zero parameter. Moreover, among all the procedures with the same order of convergence, those with the first non-zero parameter between 0.6 and 0.8 show the best performance, in terms of stability.

The proposed parametric family in (2) is a generalization of other methods which are obtained for particular values of the parameters. The numerical experiments show the feasibility and effectiveness of the new methods, for both nonsingular and rectangular matrices with or without full rank and arbitrary size. Indeed, the proposed methods have proven their applicability by solving image denoising problems with success.

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¹ The structural similarity (SSIM) index is a method for predicting the perceived quality of digital images (and videos). The resultant SSIM index is a decimal value between -1 and 1, and value 1 is only reachable in the case of two identical sets of data. More details are available in [36].

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