Probabilistic analysis of a class of impulsive linear random differential equations via density functions

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Abstract

An important class of non-homogeneous first-order linear random differential equations subject to an infinite sequence of square impulses with random intensity is studied. In applications, these equations are useful to model the dynamics of a population with periodic harvesting and migration under uncertainties. The solution is explicitly obtained via the first probability density function assuming an arbitrary joint density for all model parameters. Probabilistic stability analysis is carried out through the densities of the random sequences of minima and maxima. All the theoretical results are fully illustrated through two numerical examples.

Keywords: Random differential equations; Probability density function; Stochastic periodic jumps; Probabilistic stability.

1. Introduction

The study of differential equations whose right-hand side is discontinuous has theoretical interest by itself, but also plays a main role in applications, particularly, in control theory to account for punctual or periodic changes [1]. In the setting of differential equations with uncertainties, one mainly distinguishes two classes of equations, namely, stochastic differential equations (SDEs) and random differential equations (RDEs) [2, Sec. 4.7]. The former are driven by stochastic processes whose trajectories are highly irregular (e.g., nowhere differentiability, as the Wiener process), and in their formulation uncertainty is set via specific patterns (Gaussian in the case of Wiener process). This may limit their applicability in real-world scenarios as recently reported [3, 4]. Complementary, RDEs assume more regularity in the model inputs (which are assumed to be random variables and/or stochastic processes), but permitting more flexibility when assigning them probability distributions, which in turns results they can be better accommodated to describe uncertainties in a wider range of applications [2]. In the setting of SDEs with discontinuous right-hand side, numerous interesting contributions have been made, mainly, in the case of the so called impulsive functional SDEs including the analysis of their stability [5, 6, 7, 8]. In contrast, the corresponding study for RDEs is scarcer or it has been made limiting the type of probabilistic distributions for the model inputs. For instance, in [9] one performs an interesting analysis for a new class of linear random impulsive differential equations involving the Bernoulli distribution. The aim of this paper is to advance in the realm of RDEs whose right-hand side is discontinuous without restricting the probability distributions assigned to model inputs which, in fact, as it shall be seen down below, can be arbitrary, even considering the most complex case that all model inputs can have a joint distribution (so, probabilistic independence is not assumed). Specifically, we will tackle the study of non-homogeneous linear RDEs of exponential growth/decay controlled (or driven) by an infinite sequence of square pulses of time duration, τ,

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$$\dot{x}(t) = \alpha x(t) + \beta - \gamma x(t) \sum_{n=1}^{\infty} \left(H(t - (nT - \tau)) - H(t - nT) \right), \quad x(0) = x_0, \tag{1}$$

where $H(t)$ is the Heaviside function: $H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$ For the sake of generality, we will assume that all model inputs (the initial condition, x_0, the growth/decay rate, α, the migration rate, β, and the intensity of the infinite square wave train, γ) are absolutely continuous random variables defined in a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and having a joint probability density function (p.d.f.), $f_{x_0,\alpha,\beta,\gamma} := f_{x_0,\alpha,\beta,\gamma}(x_0, \alpha, \beta, \gamma)$. Additionally, the parameters $T > 0$ and $\tau > 0$, stand for the period and duration of the square wave, respectively. Initial value problem (IVP) formulated in (1) is useful to model the random dynamics of a population subject to periodic harvesting (whose intensity of jump randomly changes according to γ) and fluctuating migration (described by β). For instance, this model has been applied to describe population dynamics of the diamondback moth in a broccoli crop with periodic applications of a biological pesticide, in the deterministic context [10]. Let us remark, that models whose right-hand-side is discontinuous or that periodically changes, appear vastly in classic applications related to electronic engineering, but usually as second order differential equations. To the best of our knowledge, the study of (1) is a first attempt to deal with this class of RDEs.

As a main difference with respect to what happens in the deterministic context, solving a RDE consists not only in obtaining its solution, which is a stochastic process, but also its main probabilistic properties, mainly the mean and the variance. However, a more ambitious goal is to determine the so called the finite distributions (*fidis*) of the solution [11]. Let us recall that, given a stochastic process $\{X(t) : t \in \mathcal{T} \subset \mathbb{R}\}$, for every finite set $\{t_1, t_2, \ldots, t_n\} \subset \mathcal{T}$, the *fidis* associated to $X(t)$ are the n-th joint probability distribution functions, denoted by $F_n := F_n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)$, corresponding to the random vector $\mathbf{X} = (X_1 = X(t_1), X_2 = X(t_2), \ldots, X_n = X(t_n))$. In the case that $\mathbf{X}$ is also absolutely continuous, the *fidis* of $X(t)$ can be equivalently given via the n-th probability density functions, f_n, which can be obtained by differentiating $F_n, f_n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = \partial_n F_n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)/\partial x_1 \cdots \partial x_n$. It is particularly important the first p.d.f. (1-p.d.f.), $f_1(x, t)$, since from it one can calculate any one-dimensional moment, $\mathbb{E}[(x(t))^k] = \int_{-\infty}^{\infty} x^k f_1(x, t)\, \mathrm{d}x, \quad k = 1, 2, \ldots$, (including the mean and the variance), but also any higher moment as well as the probability that the solution lies within any specific interval of interest, $\mathbb{P}\left[\{\omega \in \Omega : x(t; \omega) \in [\hat{x}_1, \hat{x}_2]\}\right] = \int_{\hat{x}_1}^{\hat{x}_2} f_1(x, t)\, \mathrm{d}x$, for t arbitrary but fixed. This latter is particularly relevant in real-world applications [2, 11]. Therefore, the widest meaning of solving a RDE stands for computing its 1-p.d.f. Some recent contributions dealing with the computation of the 1-p.d.f. in theoretical and practical settings for different classes of RDEs include ordinary RDEs [12], partial RDEs [13], delay RDEs [14], among others. We have already mentioned the advantages of applying RDEs in dealing with real-world problems, however, as it shall see later, some challenges appear when applying the RVT technique to determine the 1-p.d.f. of the random solution, particularly, defining an appropriate mapping whose Jacobian is different from zero, and also, computing multidimensional integrals.

In this paper, we address this important goal including the analysis of the probabilistic stability of the solution. To conduct our study we will extensively apply the Random Variable Transformation (RVT) technique [11] taking advantage of some recent results established, by some of the authors, for the deterministic formulation of IVP (1).

The deterministic solution of model (1) is given by [10],

$$x(t) = \begin{cases} \left(x_0 + \phi(\alpha,\beta,\gamma;T,\tau)\right) \mathrm{e}^{t\alpha - n\tau\gamma} - \dfrac{\beta}{\alpha}, & nT \leq t < (n+1)T - \tau, \\ \left(x_0 + \psi(\alpha,\beta\gamma;T,\tau)\right) \mathrm{e}^{t(\alpha-\gamma)+(n+1)(T-\tau)\gamma} - \dfrac{\beta}{\alpha - \gamma}, & (n+1)T - \tau \leq t < (n+1)T, \end{cases} \tag{2}$$

where we have introduced the following variables to shorten the notation,

$$\phi := \phi(\alpha,\beta,\gamma;T,\tau) = \frac{\beta}{\alpha} + \frac{\beta\gamma}{\alpha(\alpha-\gamma)} \mathrm{e}^{-T\alpha}(\mathrm{e}^{\tau\alpha} - \mathrm{e}^{\tau\gamma}) \frac{\mathrm{e}^{-n(T\alpha-\tau\gamma)} - 1}{\mathrm{e}^{-(T\alpha-\tau\gamma)} - 1}, \tag{3}$$

$$\psi := \psi(\alpha,\beta,\gamma;T,\tau)) = \frac{\beta}{\alpha} + \frac{\beta\gamma}{\alpha(\alpha-\gamma)} \left(\mathrm{e}^{-T\alpha}(\mathrm{e}^{\tau\alpha} - \mathrm{e}^{\tau\gamma}) \frac{\mathrm{e}^{-n(T\alpha-\tau\gamma)} - 1}{\mathrm{e}^{-(T\alpha-\tau\gamma)} - 1} + \mathrm{e}^{-\alpha(T-\tau)-n(T\alpha-\tau\gamma)} \right). \tag{4}$$

It can be checked that $x(t)$, given by (2), is continuous at $(n+1)T-\tau$. These expressions will be useful to later determine an explicit expressions of the 1-p.d.f. of the random solution. Now, we introduce further results obtained in [10] that will be applied later to carry out the probabilistic stability analysis. To this end, we compute the sequences of points at the times, nT and $(n+1)T-\tau$,

$$A_n := x(nT) = -\frac{\beta}{\alpha} + \left(x_0 + \frac{\beta}{\alpha}\right)\mathrm{e}^{n(T\alpha-\tau\gamma)} + \frac{\beta\gamma}{\alpha(\alpha-\gamma)}(\mathrm{e}^{\tau(\alpha-\gamma)}-1)\frac{1-\mathrm{e}^{n(T\alpha-\tau\gamma)}}{1-\mathrm{e}^{T\alpha-\tau\gamma}}, \tag{5}$$

$$B_n := x((n+1)T-\tau) = -\frac{\beta}{\alpha} + \mathrm{e}^{(T-\tau)\alpha}\left(x_0 + \frac{\beta}{\alpha}\right)\mathrm{e}^{n(T\alpha-\tau\gamma)} + \frac{\beta\gamma\mathrm{e}^{(T-\tau)\alpha}}{\alpha(\alpha-\gamma)}(\mathrm{e}^{\tau(\alpha-\gamma)}-1)\frac{1-\mathrm{e}^{n(T\alpha-\tau\gamma)}}{1-\mathrm{e}^{T\alpha-\tau\gamma}}. \tag{6}$$

These sequences are important because they will allow us to study the stochastic process from the discrete point of view. Observe, that these sequences are convergent, as $n\to\infty$, when $T\alpha-\tau\gamma<0$ holds, i.e.

$$A_\infty := \lim_{n\to\infty} A_n = -\frac{\beta}{\alpha} + \frac{\beta\gamma}{\alpha(\alpha-\gamma)}\left(\frac{\mathrm{e}^{(\alpha-\gamma)\tau}-1}{1-\mathrm{e}^{\alpha T-\gamma\tau}}\right), \quad B_\infty := \lim_{n\to\infty} B_n = -\frac{\beta}{\alpha} + \frac{\beta\gamma\mathrm{e}^{\alpha(T-\tau)}}{\alpha(\alpha-\gamma)}\left(\frac{\mathrm{e}^{(\alpha-\gamma)\tau}-1}{1-\mathrm{e}^{\alpha T-\gamma\tau}}\right). \tag{7}$$

Let us remark, that in this case, A_n and B_n become sequences of local minima and maxima, and the solution (2) presents an oscillatory behavior that is bounded by A_∞ and B_∞, in the long term.

2. Random solution for the model

In this section, we present our main results: the computation of the 1-p.d.f. of the random solution of the randomized version of IVP (1), the p.d.f.'s for the stochastic sequences of minima and maxima, and a formula for the probability of convergence. All these results are stated via explicit expressions and under the general hypothesis that model inputs have a joint p.d.f., $f_{x_0,\alpha,\beta,\gamma}$. As previously advanced, our study will be based on the RVT method.

2.1. 1-p.d.f. for the random solution

Theorem 1. *Let $(x_0,\alpha,\beta,\gamma)$ be an absolutely continuous random vector defined in a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with p.d.f. $f_{x_0,\alpha,\beta,\gamma}$. Then, the 1-p.d.f. of the random solution of IVP (1) is given by*

$$f_1(x,t) = \begin{cases} \displaystyle\int_{\mathbb{R}^3} f_{x_0,\alpha,\beta,\gamma}\left(\left(x+\frac{\beta}{\alpha}\right)\mathrm{e}^{-\alpha t+n\tau\gamma} - \phi, \alpha,\beta,\gamma\right)\mathrm{e}^{-\alpha t+n\tau\gamma}\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma, & nT\le t<(n+1)T-\tau, \\ \displaystyle\int_{\mathbb{R}^3} f_{x_0,\alpha,\beta,\gamma}\left(\left(x+\frac{\beta}{\alpha-\gamma}\right)\mathrm{e}^{-t(\alpha-\gamma)-(n+1)(T-\tau)\gamma} - \psi, \alpha,\beta,\gamma\right)\mathrm{e}^{-t(\alpha-\gamma)-(n+1)(T-\tau)\gamma}\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma, & (n+1)T-\tau\le t<(n+1)T, \end{cases} \tag{8}$$

where $x\in\mathbb{R}$, $\phi := \phi(\alpha,\beta,\gamma;T,\tau)$ and $\psi := \psi(\alpha,\beta,\gamma;T,\tau)$ are defined in (3) and (4), respectively. Furthermore, $f_1(x,t)$ is continuous at $t=(n+1)T-\tau$.

Proof. To determine the 1-p.d.f. of the solution, $x(t)$, to IVP (1), we will apply the RVT technique choosing an adequate transformation based on the piecewise function (2), [11, Ch. 2]. For both pieces, let us consider that $g:\mathbb{R}^4\to\mathbb{R}^4$, $v=g(u)$, $v:=(v_1,v_2,v_3,v_4)$, $u=:(x_0,\alpha,\beta,\gamma)$, where u and v will be defined, down below, in each piece.

Case 1: Consider a fixed t such as $nT\le t<(n+1)T-\tau$ and denote $x:=x(t)$. Define

$$v_1 := (x_0+\phi(\alpha,\beta,\gamma;T,\tau))\,\mathrm{e}^{\alpha t-n\tau\gamma} - \frac{\beta}{\alpha}, \quad v_2 := \alpha, \quad v_3 := \beta, \quad v_4 := \gamma.$$

The inverse transformation, h, of g, is obtained solving for x_0, α, β and γ: $h(v_1,v_2,v_3,v_4) = (h_1(v_1,v_2,v_3,v_4),v_2,v_3,v_4)$, where $h_1(v_1,v_2,v_3,v_4) = (v_1+\frac{v_3}{v_2})\,\mathrm{e}^{-v_2 t+n\tau v_3} - \phi(v_2,v_3,v_4;T,\tau)$, and its Jacobian is $Jh = \frac{\partial h_1}{\partial v_1} = \mathrm{e}^{-v_2 t+n\tau v_3}\neq 0$.

Then, according to the RVT theorem, the joint p.d.f. of $(v_1,v_2,v_3,v_4)=(x,\alpha,\beta,\gamma)$ is

$$f_{x,\alpha,\beta,\gamma}(x,\alpha,\beta,\gamma) = f_{x_0,\alpha,\beta,\gamma}\left(\left(x+\frac{\beta}{\alpha}\right)\mathrm{e}^{-\alpha t+n\tau\gamma} - \phi(\alpha,\beta\gamma;T,\tau),\alpha,\beta,\gamma\right)\mathrm{e}^{-\alpha t+n\tau\gamma}.$$

By marginalizing with respect to α, β, and γ, we obtain the first part of the 1-p.d.f., $f_1(x,t)$, given in (8).

Case 2. Let us consider t fixed, such as, $(n+1)T-\tau \le t < (n+1)T$ and denote $x := x(t)$. In this case, let us define

$$v_1 := (x_0 + \psi(\alpha,\beta,\gamma;T,\tau))\,\mathrm{e}^{t(\alpha-\gamma)+(n+1)(T-\tau)\gamma} - \frac{\beta}{\alpha-\gamma}, \quad v_2 := \alpha, \quad v_3 := \beta, \quad v_4 := \gamma.$$

As in Case 1, we compute the inverse transformation of g, h, by solving for x_0, α, β and γ, which in terms of (v_1, v_2, v_3, v_4) leads to $h(v_1, v_2, v_3, v_4) = (h_1(v_1, v_2, v_3, v_4), v_2, v_3, v_4)$, where

$$h_1(v_1,v_2,v_3,v_4) = \left(v_1 + \frac{v_3}{v_2-v_4}\right)\mathrm{e}^{t(v_2-v_4)+(n+1)(T-\tau)v_4} - \psi(v_2,v_3,v_4;T,\tau),$$

and its Jacobian is $Jh = \frac{\partial h_1}{\partial v_1} = \mathrm{e}^{t(v_2-v_4)+(n+1)(T-\tau)v_4} \neq 0$. Then, according to the RVT method, the p.d.f. of the random vector $(v_1, v_2, v_3, v_4) = (x, \alpha, \beta, \gamma)$ is given by

$$f_{x,\alpha,\beta,\gamma}(x,\alpha,\beta,\gamma) = f_{x_0,\alpha,\beta,\gamma}\left(\left(x+\frac{\beta}{\alpha-\gamma}\right)\mathrm{e}^{-t(\alpha-\gamma)-(n+1)(T-\tau)\gamma} - \psi(\alpha,\beta,\gamma;T,\tau),\alpha,\beta,\gamma\right)\mathrm{e}^{-t(\alpha-\gamma)-(n+1)(T-\tau)\gamma}.$$

Again, marginalizing with respect to α, β, and γ one directly obtains the second piece of the 1-p.d.f., $f_1(x,t)$, given in (8). Finally, the continuity of $f_1(x,t)$ at $t = (n+1)T-\tau$ is an evident consequence of the continuity of $x(t)$ at that point. This completes the proof.

□

Since minima and maxima sequences are computed by evaluating the random solution at $t = nT$ and $t = (n+1)T-\tau$ for n fixed (see (5)), then the p.d.f. of such sequences are straightforwardly deduced from Theorem 1 taking into account the continuity of $f_1(x,t)$ at $t = (n+1)T-\tau$.

Corollary 1.1. Assume the hypotheses of Theorem 1. Then, the p.d.f.'s for the random sequences of local minima and maxima, given in (5) and (6), are given by

$$f_{A_n}(x) = f_1(x,nT) = \int_{\mathbb{R}^3} f_{x_0,\alpha,\beta,\gamma}\left(\left(x+\frac{\beta}{\alpha}\right)\mathrm{e}^{-n(\alpha T+\gamma\tau)} - \phi(\alpha,\beta,\gamma;T,\tau),\alpha,\beta,\gamma\right)\mathrm{e}^{-n(\alpha T+\gamma\tau)}\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma, \tag{9}$$

$$f_{B_n}(x) = f_1(x,(n+1)T-\tau) = \int_{\mathbb{R}^3} f_{x_0,\alpha,\beta,\gamma}\left(\left(x+\frac{\beta}{\alpha}\right)\mathrm{e}^{-\alpha(T-\tau)-n(\alpha T-\gamma\tau)} - \phi(\alpha,\beta,\gamma;T,\tau),\alpha,\beta,\gamma\right)\mathrm{e}^{-\alpha(T-\tau)-n(\alpha T-\gamma\tau)}\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma. \tag{10}$$

In the same way, it is easily derived the 1-p.d.f.'s for the homogeneous case by substituting $\beta = 0$ in equations (8) of Theorem 1, (9) and (10) of Corollary 1.1, but now in terms of the p.d.f., $f_{x_0,\alpha,\gamma}$, which is obtained by marginalizing the data $f_{x_0,\alpha,\beta,\gamma}$, with respect to random variable β.

Corollary 1.2. For the homogeneous case, $\beta(\omega) = 0$, $\omega \in \Omega$ with probability one (w.p. 1), the 1-p.d.f. (8) becomes

$$f_1(x,t) = \begin{cases} \displaystyle\int_{\mathbb{R}^2} f_{x_0,\alpha,\gamma}\left(x\mathrm{e}^{-\alpha t+n\tau\gamma},\alpha,\gamma\right)\mathrm{e}^{-\alpha t+n\tau\gamma}\,\mathrm{d}\alpha\,\mathrm{d}\gamma, & nT \le t < (n+1)T-\tau, \\ \displaystyle\int_{\mathbb{R}^2} f_{x_0,\alpha,\gamma}\left(x\mathrm{e}^{-t(\alpha-\gamma)-(n+1)(T-\tau)\gamma},\alpha,\gamma\right)\mathrm{e}^{-t(\alpha-\gamma)-(n+1)(T-\tau)\gamma}\,\mathrm{d}\alpha\,\mathrm{d}\gamma, & (n+1)T-\tau \le t < (n+1)T. \end{cases} \tag{11}$$

Also, the p.d.f.'s of the maximum and minimum sequences of Corollary 1.1 can be straightforwardly determined from the above p.d.f., $f_1(x,t)$,

$$f_{A_n}(x) = f_1(x,nT) = \int_{\mathbb{R}^2} f_{x_0,\alpha,\gamma}\left(x\mathrm{e}^{-n(\alpha T+\gamma\tau)},\alpha,\gamma\right)\mathrm{e}^{-n(\alpha T+\gamma\tau)}\,\mathrm{d}\alpha\,\mathrm{d}\gamma, \tag{12}$$

$$f_{B_n}(x) = f_1(x,(n+1)T-\tau) = \int_{\mathbb{R}^2} f_{x_0,\alpha,\gamma}\left(x\mathrm{e}^{-\alpha(T-\tau)-n(\alpha T-\gamma\tau)},\alpha,\gamma\right)\mathrm{e}^{-\alpha(T-\tau)-n(\alpha T-\gamma\tau)}\,\mathrm{d}\alpha\,\mathrm{d}\gamma. \tag{13}$$

2.2. P.d.f. of the limits of the sequences of minima and maxima

Let us recall that sequences of minima and maxima, given in (5) and (6), converge to random variables A_∞ and B_∞, equations (7), when $T\alpha - \gamma\tau < 0$, as $n \to \infty$. Next, we determine the p.d.f. of those random variables.

Theorem 2. *Let T and τ be fixed times such as $0 < \tau < T$. Then, under the hypotheses of Theorem 1 the p.d.f.'s of the steady states, A_∞ and B_∞, given in (7), are*

$$f_{A_\infty}(a) = \int_{\mathbb{R}^2} f_{\alpha,\beta,\gamma}\left(\alpha, a\frac{-\alpha(\alpha-\gamma)(1-\mathrm{e}^{\alpha T-\gamma\tau})}{\alpha(1-\mathrm{e}^{\alpha T-\gamma\tau})+\gamma\mathrm{e}^{-\gamma\tau}(\mathrm{e}^{\alpha T}-\mathrm{e}^{\alpha\tau})}, \gamma\right)\left|\frac{-\alpha(\alpha-\gamma)(1-\mathrm{e}^{\alpha T-\gamma\tau})}{\alpha(1-\mathrm{e}^{\alpha T-\gamma\tau})+\gamma\mathrm{e}^{-\gamma\tau}(\mathrm{e}^{\alpha T}-\mathrm{e}^{\alpha\tau})}\right| \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma,\ a \in \mathbb{R}, \tag{14}$$

$$f_{B_\infty}(b) = \int_{\mathbb{R}^2} f_{\alpha,\beta,\gamma}\left(\alpha, \frac{-b\alpha(\alpha-\gamma)(1-\mathrm{e}^{\alpha T-\gamma\tau})}{\alpha(1-\mathrm{e}^{\alpha T-\gamma\tau})+\gamma(\mathrm{e}^{(T-\tau)\alpha}-1)}, \gamma\right)\left|\frac{-\alpha(\alpha-\gamma)(1-\mathrm{e}^{\alpha T-\gamma\tau})}{\alpha(1-\mathrm{e}^{\alpha T-\gamma\tau})+\gamma(\mathrm{e}^{(T-\tau)\alpha}-1)}\right| \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma,\ b \in \mathbb{R}. \tag{15}$$

Proof. For f_{A_∞}. Define the transformation $g : \mathbb{R}^3 \to \mathbb{R}^3$, such as, $u = g(v)$, $u := (\alpha, \beta, \gamma)$, $v := (v_1, v_2, v_3)$, and $v_1 := A_\infty = -\frac{\beta}{\alpha} + \frac{\beta\gamma}{\alpha(\alpha-\gamma)}\left(\frac{\mathrm{e}^{(\alpha-\gamma)\tau}-1}{1-\mathrm{e}^{\alpha T-\gamma\tau}}\right)$, $v_2 := \alpha$ and $v_3 := \gamma$. In order to apply the RVT, we determine the inverse of g by solving for α, β and γ:

$$\beta = v_1\left(-\frac{1}{\alpha} + \frac{\gamma}{\alpha(\alpha-\gamma)}\frac{\mathrm{e}^{(\alpha-\gamma)\tau}-1}{1-\mathrm{e}^{\alpha T-\gamma\tau}}\right)^{-1}, \quad \alpha = v_2, \quad \gamma = v_3.$$

The Jacobian of $h = g^{-1}$, $Jh = \frac{-\alpha(\alpha-\gamma)(1-\mathrm{e}^{\alpha T-\gamma\tau})}{\alpha(1-\mathrm{e}^{\alpha T-\gamma\tau})+\gamma\mathrm{e}^{-\gamma\tau}(\mathrm{e}^{\alpha T}-e^{\alpha\tau})}$. Applying the RVT technique, we find that the p.d.f. is

$$f_{\alpha,\beta,\gamma}\left(\alpha, a\frac{-\alpha(\alpha-\gamma)(1-\mathrm{e}^{\alpha T-\gamma\tau})}{\alpha(1-\mathrm{e}^{\alpha T-\gamma\tau})+\gamma\mathrm{e}^{-\gamma\tau}(\mathrm{e}^{\alpha T}-\mathrm{e}^{\alpha\tau})}, \gamma\right)\left|\frac{-\alpha(\alpha-\gamma)(1-\mathrm{e}^{\alpha T-\gamma\tau})}{\alpha(1-\mathrm{e}^{\alpha T-\gamma\tau})+\gamma\mathrm{e}^{-\gamma\tau}(\mathrm{e}^{\alpha T}-\mathrm{e}^{\alpha\tau})}\right|.$$

Marginalizing with respect to α, β, and γ, we obtain (14). For f_{B_∞}, an analogous reasoning using B_∞ yields (15). □

2.3. Probabilistic stability analysis

Before expression (7), we have already pointed out that convergence of random sequences A_n and B_n to A_∞ and B_∞ as $n \to \infty$ happens when $T\alpha - \gamma\tau < 0$. Since α and γ are random variable the foregoing condition of convergence holds with certain probability that is determined in the following result.

Theorem 3. *Let T and τ be fixed such as $0 < \tau < T$ and define $\xi := T\alpha - \gamma\tau$ being α and γ absolutely continuous random variables with a joint p.d.f., $f_{\alpha,\gamma}$. Then, the probability that $\xi < 0$ is given by*

$$\mathbb{P}[\xi < 0] = \frac{1}{T}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f_{\alpha,\gamma}\left(\frac{\xi+\gamma\tau}{T}, \gamma\right) \mathrm{d}\gamma\, \mathrm{d}\xi. \tag{16}$$

Proof. Define $g : \mathbb{R}^2 \to \mathbb{R}^2$, $u = g(v)$, where $u := (\alpha, \gamma)$ and $v := (v_1, v_2)$ with $v_1 := \xi = T\alpha - \gamma\tau$, $v_2 := \gamma$. To compute the inverse function of g, h, we solve for α and β, $\alpha = \frac{v_1+v_2\tau}{T}$, $\gamma = v_2$. Then, $h(v_1, v_2) = \left(\frac{v_1+v_2\tau}{T}, v_2\right)$, and its Jacobian is $Jh = \frac{1}{T} \neq 0$. According to RVT theorem the p.d.f. for $(v_1, v_2) = (\xi, \gamma)$ is $f_{\xi,\gamma} = f_{\alpha,\gamma}\left(\frac{\xi+\gamma\tau}{T}, \gamma\right)\frac{1}{T}$. To obtain the p.d.f. for ξ, we marginalize with respect to γ, $f_\xi(\xi) = \frac{1}{T}\int_{\mathbb{R}} f_{\alpha,\gamma}\left(\frac{\xi+\gamma\tau}{T}, \gamma\right)\mathrm{d}\gamma$. From this p.d.f., we easily obtain (16). □

Remark 1. In the homogeneous case, corresponding to $\beta(\omega) = 0$, $\omega \in \Omega$ with probability one (w.p. 1), Theorem 2 does not longer apply since A_∞ and B_∞ are zero w.p. 1. If $\mathbb{P}[\xi < 0] = 1$, minima and maxima sequences, A_n and B_n, tend to zero (w.p. 1) as $n \to \infty$. It means that $f_{A_n}(x)$ and $f_{B_n}(x)$ will tend to a Dirac delta function, $\delta(x)$, as $n \to \infty$. Notice that the probability in Theorem 3 is still valid by suppressing the dependence on β in the p.d.f.

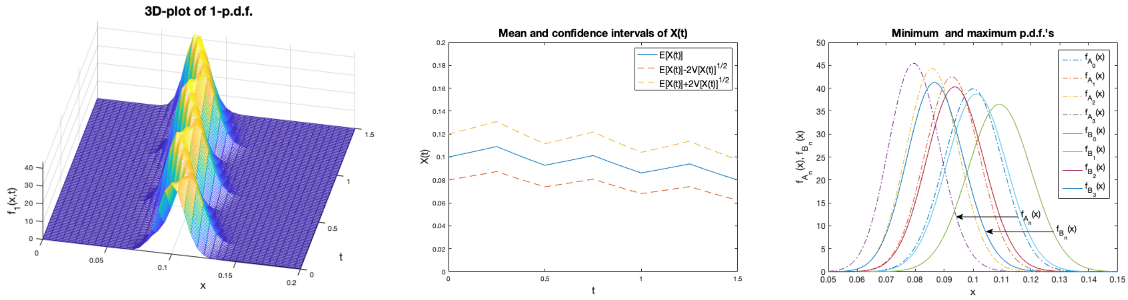

Figure 1: Homogeneous case. Left panel: 3D graphical representation of the 1-p.d.f., $f_1(x,t)$, obtained from expression (8), with three impulse applications with period $T = 1/2$ and impulse duration $\tau = T/2$. Central panel: Mean plus/minus two standard deviations. Right panel: P.d.f. of the random minima and maxima sequences (9) and (10). Example 3.1.

3. Numerical examples

In this section we illustrate the results previously established for the random IVP (1), in both, homogeneous (Example 3.1) and non-homogeneous (Example 3.2) cases. It is important to point out that the computation of 1-p.d.f., $f_1(x,\ t)$, requires multidimensional integration, which may fail to converge when applying quadrature rules. Alternatively, $f_1(x,\ t)$ can be represented via the expectation operator $\mathbb{E}[\cdot]$ and then Monte Carlo simulations can be applied. This strategy works when x_0, α and β are independent random variables. In our case, we assume that this assumption holds. In particular,

$$f_1(x,t) = \int_{\mathbb{R}^3} f_{x_0}(w(x;\alpha,\beta,\gamma;t,T,\tau)) f_\alpha(\alpha) f_\beta(\beta) f_\gamma(\gamma) \,|\, z(\alpha,\gamma;t,T,\tau) \,|\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma = \mathbb{E}\left[f_{x_0}(w(x;\alpha,\beta,\gamma;t,T,\tau)) \,|\, z(\alpha,\gamma;t,T,\tau) \,| \right],$$

being $w := w(x;\alpha,\beta,\gamma;t,T,\tau)$ and $z := z(\alpha,\gamma;t,T,\tau)$ given functions that can easily identified from (8). For example, in the first piece, $nT \le t < (n+1)T$, $w = (x_0 + \beta/\alpha)\,\mathrm{e}^{-\alpha t + n\tau\gamma} - \phi$ and $z = \mathrm{e}^{-\alpha t + n\tau\gamma}$.

Example 3.1. Homogeneous case. Let us consider IVP (1) with $\beta = 0$ w.p. 1. We assume three complete cycles of period $T = 1/2$ and duration $\tau = 1/4$ of the square wave train. We consider that the random variables x_0, α, and γ are independent and normally distributed, $x_0 \sim \mathrm{N}(0.1;\sqrt{0.0001})$, $\alpha \sim \mathrm{N}(0.35;\sqrt{0.030625})$ and $\gamma \sim \mathrm{N}(1;\sqrt{0.0025})$. For this set of distributions, equation (5) and (6), correspond to sequences of minima and maxima, respectively. In Figure 1, we show the 3D graphical representation of $f_1(x,t)$ given in Theorem 1 (left panel), the mean and confidence intervals (central panel) using the so called $2\sigma-$rule, and the 2D evolution of the p.d.f. for the sequences of minima and maxima according to Corollary 1.1 (right panel). They have been computed by expressions (8), $\mathbb{E}[x(t)] \pm 2\sqrt{\mathbb{V}[x(t)]}$, and (9) and (10), respectively. On the left panel, we can observe the 1-p.d.f. evolution over the time. Note that the 1-p.d.f. shifts to the right when the impulse is off, and to left when the impulse is on. This is in full agreement with the oscillatory behaviour of the expectation and the confidence intervals of the random solution showed on the central panel. On the right panel, we show p.d.f.'s for the sequences of minima (dotted line) and maxima (solid line). Each cycle has a minimum and a maximum p.d.f., which are close together. They are moving to the left and will converge to a Dirac delta in the long term. On the other hand, the probability of convergence computed by (16) is 0.9999, truncated to four figures.

Example 3.2. Non-homogeneous case. Let us consider IVP (1) with β an absolutely continuous random variable. We keep the same conditions and distributions as in Example 3.1. For the random variable β, we take $\beta \sim \mathrm{N}(0.1;\sqrt{0.0001})$. In the same manner as in Figure 1, we show the 1-p.d.f., mean and confidence intervals, and the p.d.f. for the random sequences of minima and maxima, on left, central and right panels, respectively. However, we have also depicted the p.d.f.'s of the limits of the minima and maxima sequences, (14) and (15), solid and dashed black lines. In general, the behavior of the 1-p.d.f. is similar to Example 3.1. However, the expectation of random solution and their confidence intervals increase (central panel). Note that the 1-p.d.f. of the random solution will oscillate between the p.d.f.'s of the minima and maxima limits, in the long term. It goes from one p.d.f. to the other in a time equal to $T/2$. Finally, the probability of convergence computed using (16) is 0.9999.

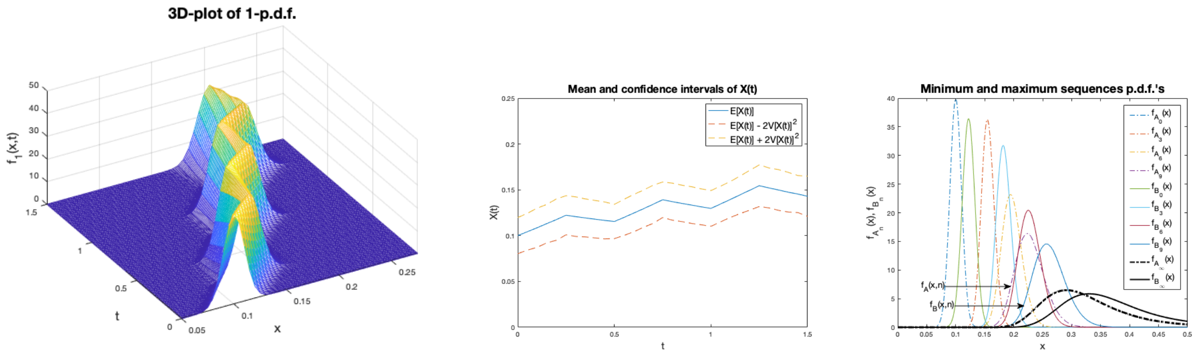

Figure 2: Non-homogeneous case. Left panel: 3D Graphical representation of the 1-p.d.f., $f_1(x,t)$, obtained from (8), with three impulse applications with period $T = 1/2$ and impulse duration $\tau = T/2$. Central panel: Mean plus/minus two standard deviations. Right panel: P.d.f. of the random sequences of minima and maxima (9) and (10). Example 3.2.

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Figure 1 left

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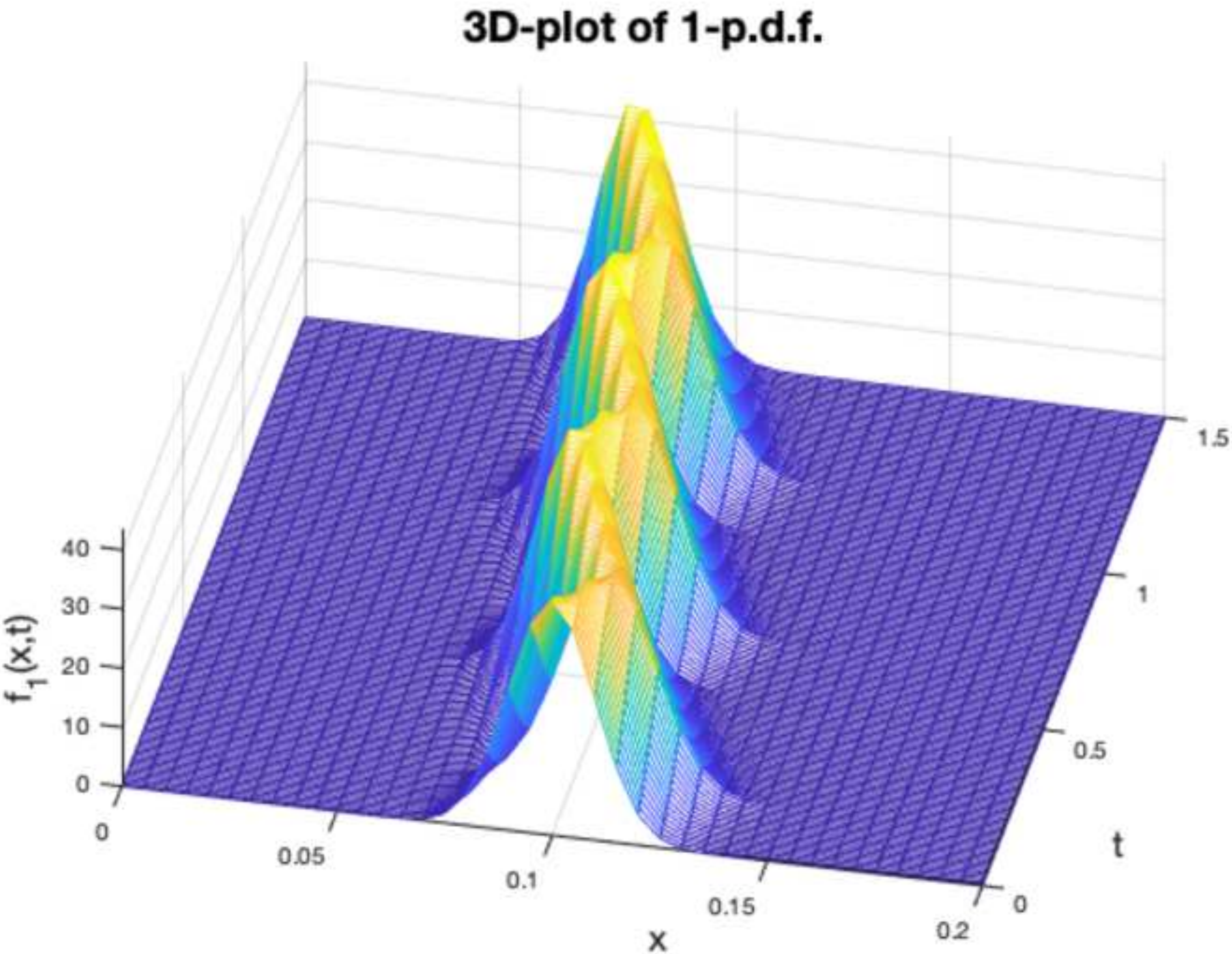

Figure 1 center

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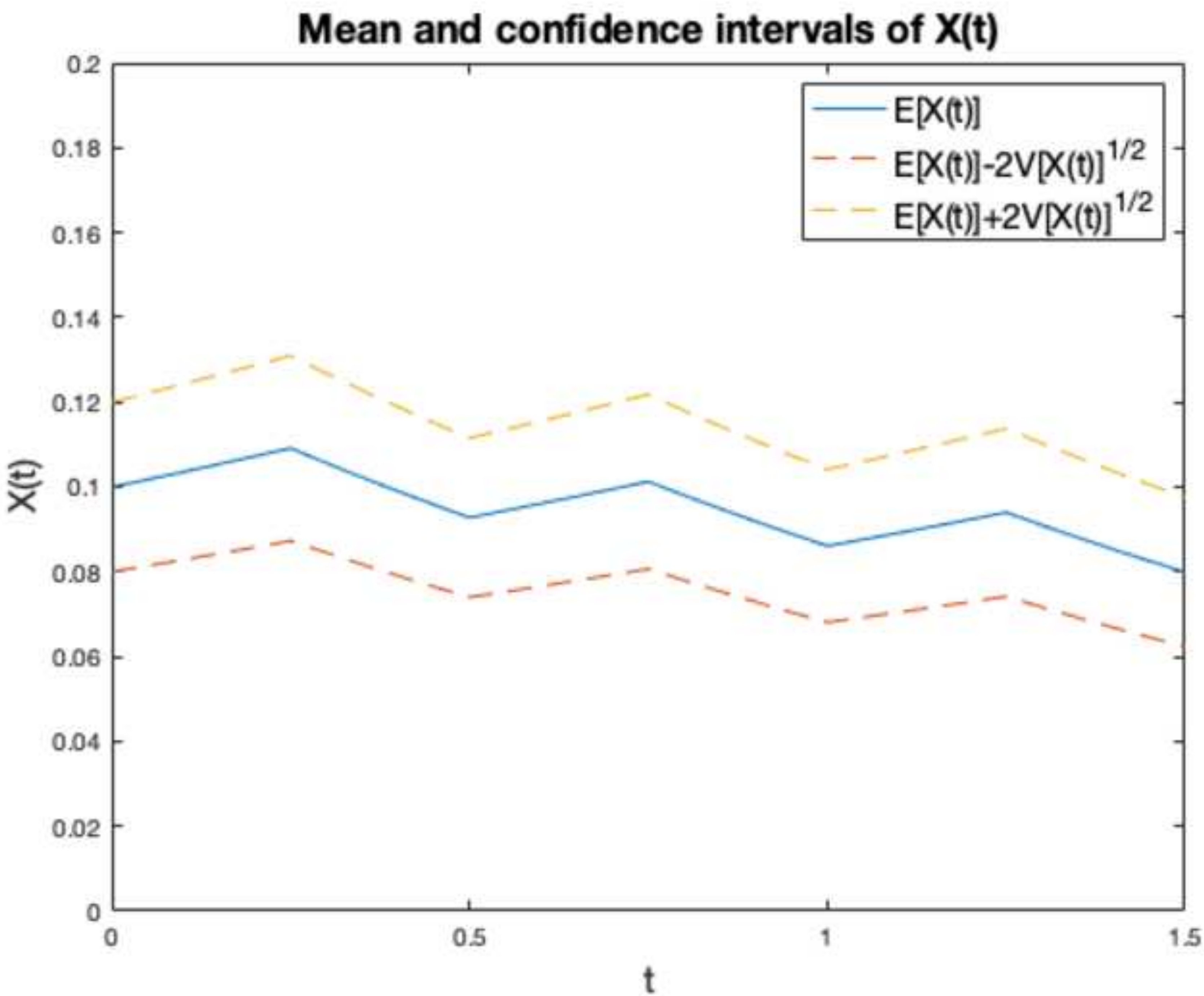

Figure 1 right

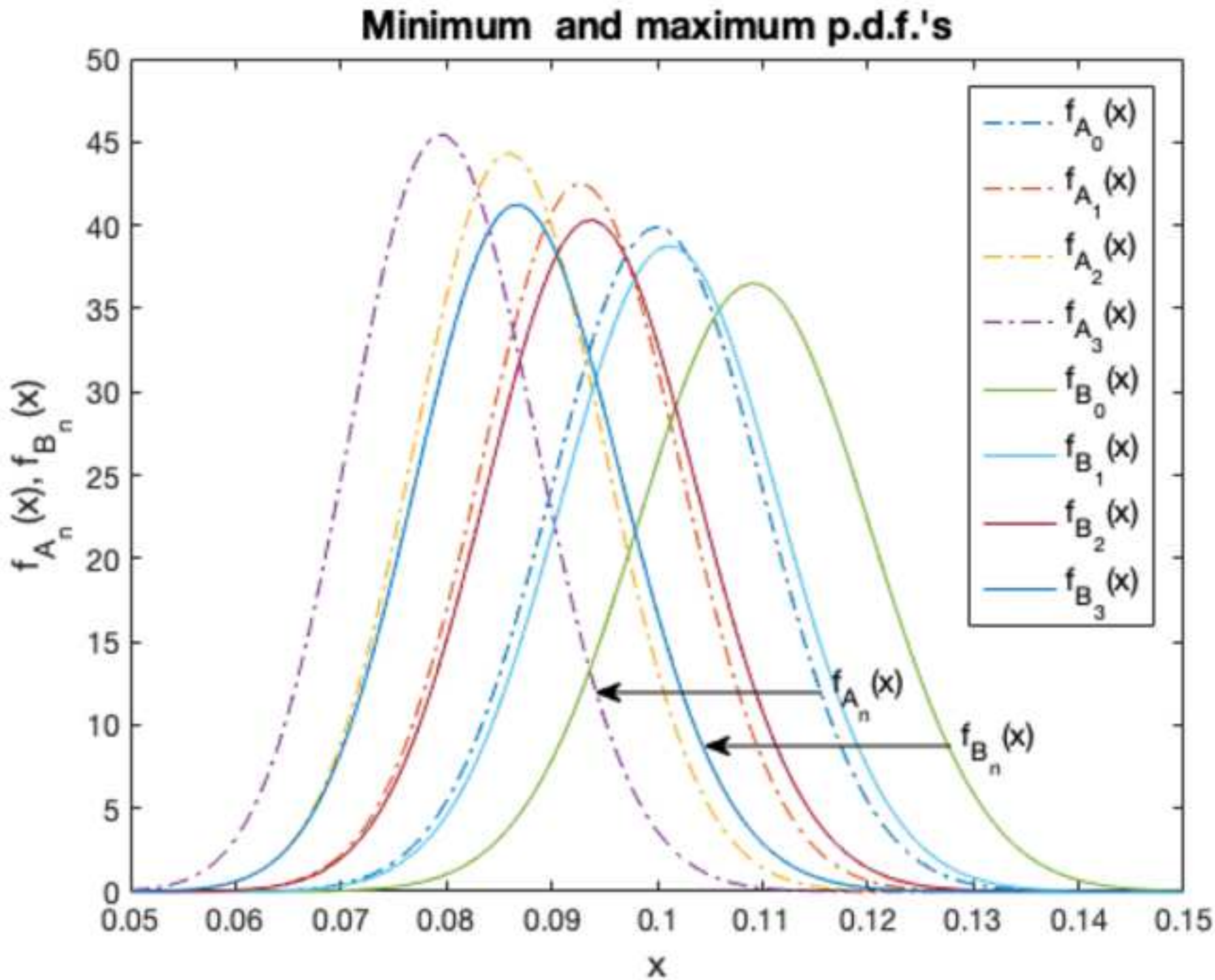

Figure 2 left

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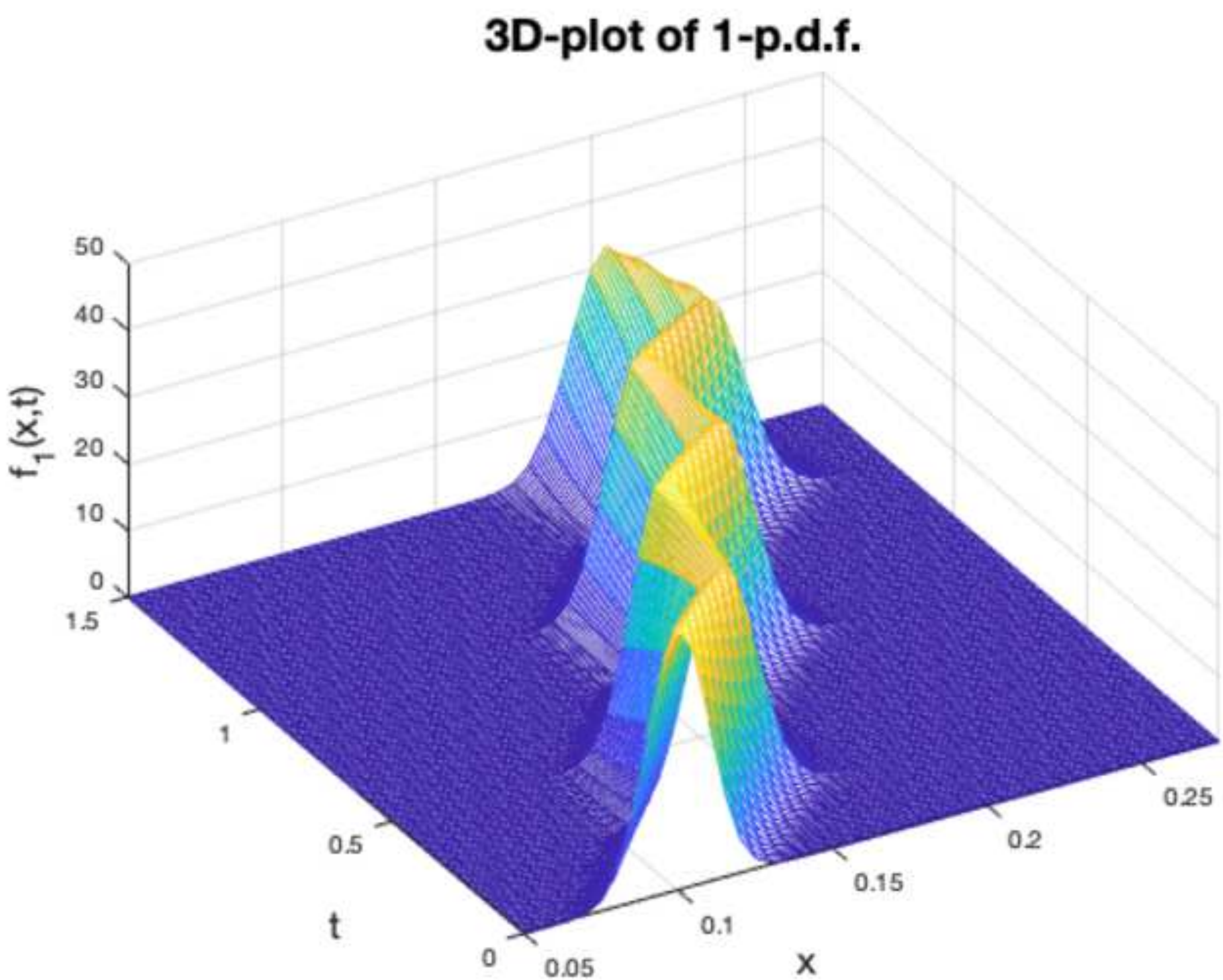

Figure 2 center

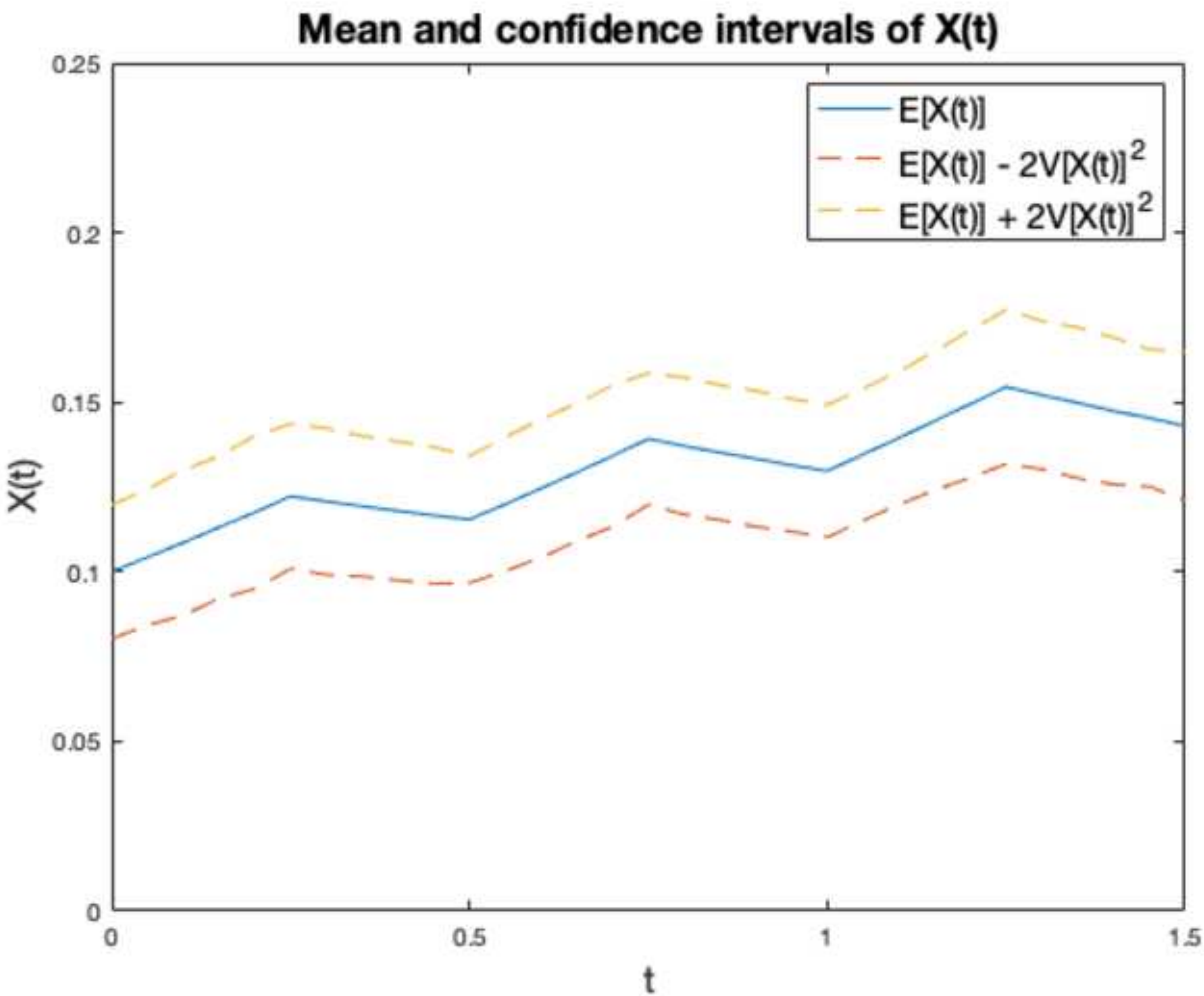

Figure 2 right