Hernández-Verón, MA.; Martínez Molada, E.; Singh, S. (2021). On the Chandrasekhar integral equation. Computational and Mathematical Methods. 3(6):1-14. https://doi.org/10.1002/cmm4.1150

On the Chandrasekhar integral equation

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Abstract

This study is devoted to solve the Chandrasekhar integral equation that it is used for modeling problems in theory of radiative transfer in a plane-parallel atmosphere, and others research areas like the kinetic theory of gases, neutron transport, traffic model, the queuing theory among others.

First of all, we transform the the Chandrasekhar integral equation into a nonlinear Hammerstein-type integral equation with the corresponding Nemystkii operator and the proper non-separable kernel. Them, we approximate the kernel in order to apply an iterative scheme. This procedure it is solved in two different ways.

First one, we solve a nonlinear equation with separable kernel and define an adequate nonlinear operator between Banach spaces that approximates the first problem. Second one, we introduce an approximation for the inverse of the Fréchet derivative that appears in the Newton's iterative scheme for solving non linear equations.

Finally, we perform a numerical experience in order to compare our results with previous ones published showing that are competitive.

keywords: Integral equation, non-separable kernel, non-linear equation, Newton iterative scheme, convergence order, domain of existence of solution, domain of uniqueness of solution.

2010 Mathematics Subject Classification: 45G10, 47H99, 65H10, 65J15, 65G49.

This research was partially supported by Ministerio de Economía y Competitividad under grant PGC2018-095896-B-C21-C22 and by the project EEQ/2018/000720 under Science and Engineering Research Board.

1 Introduction

A great variety of applied problems in the field of mathematics, and other sciences like physicist, chemistry, biology, statistics and so on, can be modeled by ordinary differential equations, partial derivative equations and nonlinear integral equations. It is well known, that to obtain the analytical solution of these type of equations requires a complex procedure that in the majority of cases do not result useful for getting the solution, for this reason the branch of Numerical analysis that studies the way of obtaining numerical methods in order to approximate with great precision the solution of these problems has suffered in last years a deep development. One important branch of these numerical methods are iterative methods to solve nonlinear equations and systems.

In this work we deal with the the Chandrasekhar integral equation type that appears in the theory of radiative transfer in a plane-parallel atmosphere. Chandrasekhar's remarkable work [6] in 1950s formulated the radioactive transferring process and derived the integral equation for the scattering function and transmitted functions, which was promptly developed as a major scientific subject across astrophysics and mathematics. Besides the radiative transfer theory, the so-called quadratic integral equations are extensively applicable to many research areas: the kinetic theory of gases, neutron transport, traffic model and the queuing theory [1].

The purpose of this work deals with equations of this type that are given by the following expression:

$$x(s) = 1 + x(s) \int_0^1 \frac{s}{s+t} \psi(t) x(t) dt, \tag{1}$$

where $x(s)$ is a continuous function in $[0, 1]$, the kernel $\frac{s}{s+t}$ is continuous and nonnegative in $[0, 1] \times [0, 1]$ and it is nonseparable, $\psi(t)$ is a real function and we will denote the solution by $x^*(s)$.

First of all, we will transform equation (1) into a nonlinear Hammerstein-type integral equation of the form

$$x(s) = f(s) + \lambda \int_a^b \mathcal{K}(s, t)[\mathcal{N}(x)](t) dt, \tag{2}$$

where $\mathcal{N}$ is a Nemystkii operator [11], $\mathcal{N} : \Omega \subseteq \mathcal{C}[a, b] \longrightarrow \mathcal{C}[a, b]$, with $[\mathcal{N}(x)](t) = N(x(t))$, being $N : \mathbb{R} \longrightarrow \mathbb{R}$ a derivable scalar function, $f : [a, b] \longrightarrow \mathbb{R}$ a continuous function and $\mathcal{K} : [a, b] \times [a, b] \longrightarrow R$ a continuous function in both arguments, λ a real parameter and x is a solution to be determined. $\mathcal{C}[a, b]$ denotes the space of continuous real functions in $[a, b]$. For this, we consider

$$y(s) = \frac{1}{x(s)}, \tag{3}$$

then (1) can be rewritten as

$$y(s) = 1 - \int_0^1 \frac{s}{s+t} \psi(t) \frac{1}{y(t)} dt. \tag{4}$$

In our case, the Chandrasekhar integral equation (1) has a kernel $\mathcal{K}(s,t) = \dfrac{s}{s+t}\psi(t)$ which is non-separable. Along the paper, we take

$$F(y)(s) = 1 - \int_0^1 \frac{s}{s+t}\psi(t)\frac{1}{y(t)}dt, \tag{5}$$

with $F : \Omega \subseteq \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$, where Ω is a nonempty convex domain in $\mathcal{C}[0,1]$. Observe that looking for a fixed point of the function (5) is equivalent to solving $G(y) = 0$, where $G : \Omega \subseteq \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$ and

$$G(y)(s) = y(s) - F(y)(s) = [(I - F)(y)](s). \tag{6}$$

It is known ([7], [9]) that if the kernel $\mathcal{K}(s,t)$ of the integral equation given in (2) is non-separable, the choice of the iterative scheme to approximate a solution is restricted if the chosen iterative scheme using inverse of Fréchet's first derivative operators. So, our main goal in this paper is to approximate a solution of equation (3), which kernel is non-separable, by means of an iterative scheme. To achieve this aim, we consider two different procedures. In first place, we consider an efficient modification of the Newton method and, in second place, an iterative scheme ad hoc for this problem is obtained.

If we pay attention to the iterative schemes that can be applied for approximating a solution $y^* \in \mathcal{C}[0,1]$ of (3), the best-known iterative scheme with quadratical convergence is Newton's method, whose algorithm is the following:

$$\begin{cases} y_0 \text{ given in } \Omega, \\ y_{n+1} = y_n - [G'(y_n)]^{-1}G(y_n), \quad n = 0, 1, 2 \dots \end{cases} \tag{7}$$

It is well known that if we compose Newton's method with itself twice, but taking into account the derivative frozen, we obtain an iterative scheme of order three. This is a classical result obtained by Traub, [14]. Moreover, being an iterative scheme of third order, it does not increase the expensive computation of derivatives because this iterative scheme only uses the same first derivative in each step. For this, it is easy to check that this iterative scheme is more efficient than Newton's method [14]. So, we consider a two-steps iterative scheme

$$\begin{cases} y_0 \text{ given in } \Omega, \\ z_n = y_n - [G^{'}(y_n)]^{-1}G(y_n) \\ y_{n+1} = z_n - [G^{'}(y_n)]^{-1}G(z_n), n \geq 0. \end{cases}$$

Notice that this iterative scheme is the frozen two steps Newton method [9] applied to the equation $G(y)(s) = y(s) - F(y)(s) = 0$. So, the algorithm of this iterative scheme is

$$\begin{cases} y_0 \text{ given in } \Omega, \\ z_n = y_n - [I - F^{'}(y_n)]^{-1}(y_n - F(y_n)) \\ y_{n+1} = z_n - [I - F^{'}(y_n)]^{-1}(z_n - F(z_n)), n \geq 0. \end{cases} \tag{8}$$

Thus, the first procedure that we apply to approximate a solution of (3) is to approximate this equation by means of another that has a separable kernel and thus be able to directly apply iterative scheme (8).

The second procedure we use in this paper is to construct an iterative scheme of fixed point type for approximating a fixed point of F, and therefore approximate a solution of (3). Considering the iterative scheme (8), more efficient and faster than Newton's method, we propose to build ad hoc an iterative scheme, as in [10], that allows us to approximate a solution of the Chandrasekhar equation for a fixed value of the characteristic real function $\psi(t)$. So, we consider the frozen two-step Newton-type method, given by the following algorithm:

$$\begin{cases} y_0 \text{ given in } \Omega, \\ z_n = y_n - A(y_n)^{-1}(y_n - F(y_n)) \\ y_{n+1} = z_n - A(y_n)^{-1}(z_n - F(z_n)), n \geq 0. \end{cases} \tag{9}$$

Notice that in the iterative scheme (9) we work directly with the operator F. In this case we approximate de inverse of the derivative, that is, $(I - F'(y))^{-1}$ by means of $A(y)^{-1}$. Thus, if we want to approximate a solution of the equation (3), with $\mathcal{K}(s,t) = \dfrac{s}{s+t}\psi(t)$ a non-separable kernel, we consider the operator $A : \Omega \subseteq \mathcal{C}[0,1] \longrightarrow \mathcal{L}(\mathcal{C}[0,1], \mathcal{C}[0,1])$, given by

$$[A(y)(z)](s) = z(s) - \int_0^1 \widetilde{\mathcal{K}}(s,t)\frac{(-1)}{y(t)^2})z(t)dt, \tag{10}$$

where $\widetilde{\mathcal{K}}(s,t)$ is a separable kernel with $\widetilde{\mathcal{K}}(s,t) = \sum_{i=1}^{m} \alpha_i(s)\beta_i(t)$, being $\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(\theta, s, t)$, and $\mathcal{L}(\mathcal{C}[0,1], \mathcal{C}[0,1])$ is the set of linear operators in $\mathcal{C}[0,1]$. Notice that, the operator $A(y)$ is an approximation of the operator $I - F'(y)$. Thus, even though the equation $G(y) = 0$ has a non-separable kernel. we can apply this iterative scheme, obtaining a good approximation of a solution of (3), which is a solution of the previous equation (1) for different values of the characteristic real function $\psi(t)$.

The paper is organized a follows. In section 1 we introduce the Chandrasekhar equation and transform it in a nonlinear equation between Banach spaces in order to approximate its solution by using iterative methods. Section 2 is devoted to describe how we manage to apply iterative methods to find the solution of our problem. For this purpose we have to approximate the non-separable kernel adequately. We also construct the algorithm for applying the iterative scheme. After that in Section 3, we obtain numerical results and compare it with ones obtained in previous paper showing the advantages of this study. Finally, Section 4 deals with the conclusion of this work.

2 Solving the Chandrasekhar integral equation

The Chandrasekhar integral equation appears in various physical problems (radioactive transfer, kinetic of gases, etc.) [16], [17]. We consider the space $C[0,1]$ of the continuous

functions in $[0,1]$ and endowed with the maximum norm $\|y\| = \max_{t\in[0,1]} |y(t)|$, $xy \in C[0,1]$. Our aim it is to find a function $x \in C[0,1]$ verifying

$$x(s) = 1 + x(s)\int_0^1 \frac{s}{s+t}\psi(t)x(t)dt.$$

As we have seen previously, the Chandrasekhar integral equation (1) can be rewritten as

$$y(s) = 1 - \int_0^1 \frac{s}{s+t}\psi(t)\frac{1}{y(t)}dt.$$

This equation can be expressed as an equation of type (2) considering the Nemystkii operator $[\mathcal{N}(y)](t) = \dfrac{1}{y(t)}$, $f(s) = 1$, $\mathcal{K}(s,t) = \dfrac{s}{s+t}\psi(t)$, $\lambda = -1$ and y is a solution to be determined.

Now, in this section the nonlinear integral equation (3) is solved using the iterative schemes (8) and (9).

2.1 Solving the Chandrasekhar integral equation using the iterative scheme (8)

The first procedure is to approximate integral equation (3) by another integral equation of the form

$$y(s) = 1 - \int_p^q \widetilde{\mathcal{K}}(s,t)\,\frac{1}{y(t)}dt, \quad s \in [p,q]. \tag{11}$$

where $\widetilde{\mathcal{K}}$ is a separable kernel which is an approximation of $\mathcal{K}(s,t) = \dfrac{s}{s+t}\psi(t)$ given by

$$\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{E}(\epsilon,s,t), \text{ with } \widetilde{\mathcal{K}}(s,t) = \sum_{i=1}^{m+1}\alpha_i(s)\beta_i(t), \tag{12}$$

for the real functions α_j and β_j in $[0,1]$ and $\epsilon, s, t \in [0,1]$. To achieve this approach (12) there are different procedures. For example, if $\mathcal{K}$ is sufficiently derivable in any of its arguments, we can apply Taylor's development in that argument. Other possible procedures may be their approximation using Bernstein polynomials or other interpolatory approximations. All these approximations verify that their error $\mathcal{E}$ tends to zero when m tends to infinity.

Notice that the proximity between the solutions of both integral equations (3) and (11), will mark the accuracy of the solution obtained by applying the iterative process (8) to solve $\widetilde{F}(y) = 0$ with

$$\widetilde{F}(y)(s) = 1 - \int_0^1 \widetilde{\mathcal{K}}(s,t)\frac{1}{y(t)}dt \quad s \in [0,1]. \tag{13}$$

regarding the solution of $F(y) = 0$. In the situation of (15), we can explicitly calculate $[I - \widetilde{F}'(y_n)]^{-1}$ and by means of the iterative process (8) to approximate a solution of $\widetilde{F}(y) = 0$ with cubical convergence.

2.1.1 Approximating the non-separable kernel

In this case, for approximating the non-separable kernel, we use the following Taylor approximation:

$$\frac{1}{s+t} = 1 - (s+t-1) + (s+t-1)^2 - (s+t-1)^3) + ...,$$

that allow us to express the kernel as follows:

$$\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(\theta,s,t), \text{ with } \widetilde{\mathcal{K}}(s,t) = \psi(t)\sum_{i=1}^{m+1}\alpha_i(s)\beta_i(t) \text{ and } \mathcal{R}(\theta,s,t) = \psi(t)\frac{(s+t-1)^{m+2}}{\theta^{m+3}} \tag{14}$$

with $\theta \in (0,1)$.

For example, if we consider $m = 2$, the real functions $\alpha_i(s)$ and $\beta_i(t)$ are:

$$\alpha_1(s) = s(1-s+s^2),\ \alpha_2(s) = s(-1+2s),\ \alpha_3(s) = s,$$
$$\beta_1(t) = 1,\ \beta_2(t) = t-1,\ \beta_3(t) = (t-1)^2.$$

For $m = 3$ the real functions are:

$$\alpha_1(s) = s(1-s+s^2-s^3),\ \alpha_2(s) = s(-1+2s-3s^3),\ \alpha_3(s) = s(1-3s), \alpha_4 = -s,$$
$$\beta_1(t) = 1,\ \beta_2(t) = t-1,\ \beta_3(t) = (t-1)^2,\ \beta_4(t) = (t-1)^3.$$

For $m = 4$ the real functions are:

$$\alpha_1(s) = s(1-s+s^2-s^3+s^4),\ \alpha_2(s) = s(-1+2s-3s^2+4s^3),\ \alpha_3(s) = s(1-3s+6s^2), \alpha_4 = s(-1+4s), \alpha_5 =$$
$$\beta_1(t) = 1,\ \beta_2(t) = t-1,\ \beta_3(t) = (t-1)^2,\ \beta_4(t) = (t-1)^3, \beta_5(t) = (t-1)^4.$$

For $m = 5$ the real functions are:

$$\alpha_1(s) = s(1-s+s^2-s^3+s^4-s^5),\ \alpha_2(s) = s(-1+2s-3s^2+4s^3-5s^4), \tag{15}$$
$$\alpha_3(s) = s(1-3s+6s^2-10s^3), \alpha_4 = s(-1+4s-10s^2), \alpha_5 = s(1-5s), \alpha_6 = -s, \tag{16}$$

$$\beta_1(t) = 1,\ \beta_2(t) = t-1,\ \beta_3(t) = (t-1)^2,\ \beta_4(t) = (t-1)^3, \beta_5(t) = (t-1)^4, \beta_6(t) = (t-1)^5.$$

We can observe in Figure 1 and Figure 2 the graphic representation of one of these approximations and the proper nonseparable kernel.

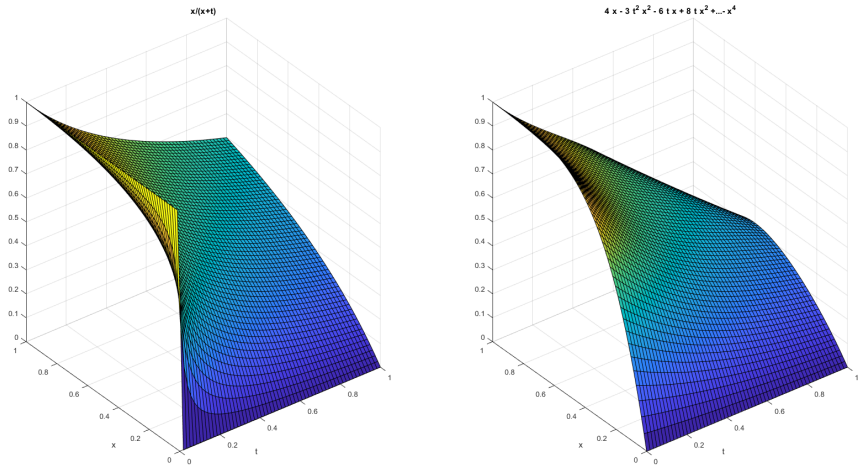

Figure 1: Approximation of the kernel with $m = 3$.

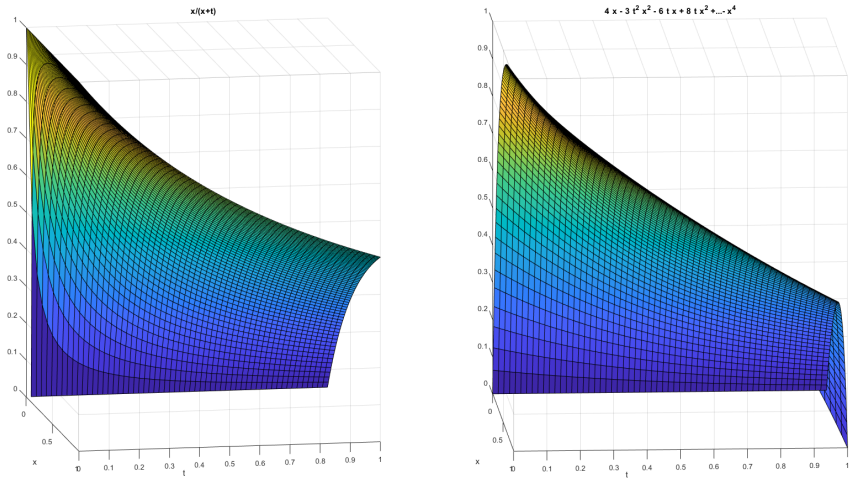

Figure 2: Approximation of the kernel with $m = 3$ from different point of view.

2.1.2 Calculating the inverse operator

Obviously, the operator $\widetilde{F}$ given in (15) is a Frèchet differentiable operator and verifies, for $y, z \in \mathcal{C}[0,1]$,

$$[\widetilde{F}'(y)z](s) = -\int_0^{1b} \mathcal{K}(s,t)[\mathcal{N}'(y)z](t)dt = -\int_0^1 \widetilde{\mathcal{K}}(s,t)\frac{(-1)}{y(t)^2}z(t)dt.$$

When we want to apply iterative scheme (8) in the infinite dimensional case, we consider the problem of the construction of the operator $[I - \widetilde{F}'(y_n)]^{-1}$ at each step. Then, as $\widetilde{\mathcal{K}}(s,t)$ is a separable kernel (14):

$$\widetilde{\mathcal{K}}(s,t) = \sum_{i=1}^{m} \alpha_i(s)\psi(t)\beta_i(t),$$

if we denote $I_j = \int_0^1 \psi(t)\beta_j(t)\frac{(-1)}{y(t)^2}z(t)\,dt$, we have

$$[(I - \widetilde{F}'(y))(z)](s) = w(s) = z(s) - \sum_{j=1}^{m} \alpha_j(s)I_j$$

and, if there exists $(I - \widetilde{F}'(y))^{-1}$, we have

$$[(I - F'(x)]^{-1}(w)](s) = z(s) = w(s) + \sum_{j=1}^{m} \alpha_j(s)I_j. \tag{17}$$

Besides, the integrals I_j can be calculated independently of y. To do this, we multiply equality (17) by $\psi(s)\beta_i(s)\frac{(-1)}{y(s)^2}$ and we integrate in the s variable the equality obtained. So, we have

$$\int_0^1 \psi(s)\beta_i(s)\frac{(-1)}{y(s)^2}z(s)ds = \int_0^1 \psi(s)\beta_i(s)\frac{(-1)}{y(s)^2}w(s)ds + \sum_{j=1}^{m} I_j \int_a^b \psi(s)\beta_i(s)\frac{(-1)}{y(s)^2}\alpha_j(s)ds,$$

and then

$$I_i - \sum_{j=1}^{m}\left(\int_0^1 \psi(s)\beta_i(s)\frac{(-1)}{y(s)^2}\alpha_j(s)\,ds\right)I_j = \int_0^1 \psi(s)\beta_i(s)\frac{(-1)}{y(s)^2}w(s)\,ds.$$

Now, if we denote

$$a_{ij}(y) = \int_0^1 \psi(s)\beta_i(s)\frac{(-1)}{y(s)^2}\alpha_j(s)\,ds \quad \text{and} \quad b_i(y) = \int_0^1 \psi(s)\beta_i(s)\frac{(-1)}{y(s)^2}w(s)\,ds,$$

we obtain the following linear system of equations

$$I_i - \sum_{j=1}^{m} a_{ij}(y)I_j = b_i(y), \quad i = 1, \ldots, m. \tag{18}$$

This system has a unique solution if

$$(-1)^m \begin{vmatrix} a_{11}(y)-1 & a_{12}(y) & a_{13}(y) & \dots & a_{1m}(y) \\ a_{21}(y) & a_{22}(y)-1 & a_{23}(y) & \dots & a_{2m}(y) \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1}(y) & a_{m2}(y) & a_{m3}(y) & \dots & a_{mm}(y)-1 \end{vmatrix} \neq 0.$$

Then, we assume 1 is not an eigenvalue of the matrix $(a_{ij}(y))$. Thus, if $I_1, I_2, \dots, I_m$ is the solution of system (18), we can define

$$[(I - F^{'}(y))^{-1}(w)](s) = w(s) + \sum_{j=1}^{m} \alpha_j(s) I_j,$$

and the iteration (8) can be applied, whose convergence was established in [8].

Now, we assume that $y^*(s)$ is a solution of equation (3), if we denote a solution of $\widetilde{F}(y) = 0$ by $\widetilde{y}(s)$ and we now look for it by means of iterative scheme (8), applying the above procedure for separable kernels, we obtain a sequence $\{\widetilde{y}_n\}$ that, under some conditions (see [9]), converges to $\widetilde{x}(s)$ with cubical order. But, as

$$\|y^*(s) - \widetilde{y}_n(s)\| \leq \|y^*(s) - \widetilde{y}(s)\| + \|\widetilde{y}(s) - \widetilde{y}_n(s)\|,$$

for obtaining a suitable error it is necessary that the quantity $\|y^*(s) - \widetilde{y}(s)\|$ is sufficiently small. Obviously, this depends of the value $\|\mathcal{R}(\theta, s, t)\|$. If, for example, $\mathcal{K}(s,t)$ is sufficiently derivable in some argument , we can apply the Taylor series to calculate the approximation given in (14) and then the error made by the Taylor series will allow us to establish how much $\{\widetilde{y}_n\}$ approaches to y^*. Improving this approach will depend, in general, on the number of Taylor's development terms.

Therefore, having into account the previous reasoning, it is clear that if we want apply the iterative scheme (8) for non-separable kernels working in the infinite dimensional case, we must modify this iterative scheme, as we can see in the next section.

2.2 Solving the Chandrasekhar integral equation using the iterative scheme (9)

As is clear, the application of method (9) has its advantages and disadvantages. On the one hand, as a drawback, let us observe that this Newton-type method loses the cubic convergence that method (8) has. On the other hand, method (9) directly approximates a solution of the equation $F(x) = 0$ while method (8) approximates a solution of the equation $\widetilde{F}(x) = 0$. Note that the proximity of both equations is essential to obtain a good approximation through method (8) and to achieve this it will be necessary to consider a high number of terms of the Taylor development (14), which produces a significant increase in operational cost of process.

For calculating the inverse of the operator $A(y)$, let us observe that the iterative Newton-type process built ad hoc to approximate a solution of the equation $F(y) = 0$

requires the construction of the operator $A(y)$. Now, from (10), it is clear that $A(y) = I - \widetilde{F}'(y)$, then the calculation of $A(y)^{-1}$ and, therefore of $[I - \widetilde{F}'(y)]^{-1}$, has already been developed in section 2.1.2.

To continue, for describing the algorithm that we use for approximating the solution of equation (1) we use a quadrature formula to approximate the integral of this equation,

$$\int_0^1 \phi(t)\, dt \simeq \sum_{k=1}^{\ell} \gamma_k \phi(\theta_k), \tag{19}$$

where the ℓ, nodes $\{\theta_k\}$ and weights $\{\gamma_k\}$ are known.

With this notation, we obtain the following algorithm to apply the iterative scheme (9). By using the procedure indicated in Section 2.1.2 to obtain $[A(y)]^{-1}$, we consider a starting guess $y_n(s) \in \mathcal{C}[0,1]$, for $n \geqslant 0$, and apply the following algorithm:

1. **First step:**

$$w(y_n)(s) = [I - F](y_n)(s) = y_n(s) - 1 + \int_0^1 \frac{s}{s+t}\psi(t)\frac{1}{y_n(t)}dt.$$

2. **Second step:** Calculate the following integrals for $i, j = 0, ..., m$.

$$a_{ij}(y_n) = \int_0^1 \psi(s)\beta_i(s)\frac{(-1)}{y_n(s)^2}\alpha_j(s)\, ds$$

and

$$b_i(y_n) = \int_0^1 \psi(s)\beta_i(s)\frac{(-1)}{y_n(s)^2}w(s)\, ds.$$

3. **Third step:** To obtain I_j with $j = 0, 1..., m$, we solve the following linear system

$$I_i - \sum_{j=1}^{m} a_{ij}(y_n)I_j = b_i(y_n), \quad i = 1, \ldots, m.$$

4. **Fourth step:** The step iteration for iterative method (9) is

$$z_n(s) = y_n(s) - w(y_n)(s) - \sum_{j=0}^{m} a_{ij}(y_n)I_j.$$

5. **Fifth step:** For the second step for iterative method (9).

$$w(z_n)(s) = G(z_n(s)) = z_n(s) - 1 + \int_0^1 \frac{s}{s+t}\psi(t)\frac{1}{z_n(t)}dt.$$

6. **Sixth:** Calculate the following integrals

$$\hat{b}_i(z_n) = -\int_0^1 \frac{i}{i+ms}\psi(s)\frac{1}{z_n(s)^2}w(z_n)(s)ds,$$

7. **Seventh step:** To obtain $\hat{I}_j$, with $j = 0, 1, 2, ..., m$, we solve the following linear system

$$\hat{I}_i - \sum_{j=0}^{m} a_{ij}(y_n)\hat{I}_j = \hat{b}_i(z_n), \quad i = 1, 2, ..., m.$$

8. **Eighth step:** Calculate

$$y_{n+1}(s) = z_n(s) - w(z_n)(s) - \sum_{j=0}^{m} a_{ij}(y_n)\hat{I}_j.$$

Note that the two linear systems that we have to solve to apply method (9) have the same system matrix associated with them.

3 Numerical results

In order to compare our results with the ones obtained in [15] for Example 1, we consider $\Psi = \frac{1}{2}\Psi_0$ with $\Psi_0 = 0 : 0.05 : 1$ and run the algorithms for solving the integral equation and obtain the H-function by using iterative schemes (8) and (9) and taking starting guess $x_0(s) = 1$ and $x_0(s) = s$.

Notice that, when we apply iterative algorithm (8) we solve the approximated problem $\widetilde{F}(x) = 0$ given by (15), but when we apply iterative method (9) we solve the problem $F(x) = 0$ approximating $I - F'(x)$ by (10), obtaining respectively the solutions $\widetilde{y_n}(x)$ and $y_n(x)$.

In all the computations we use Matlab 2019 working with variable precision arithmetic with 100 digits. The stopping criteria it is that the distance between two iterations be smaller than 10^{-30}. For obtaining the integrals that appear in second and successive steps of the algorithm described in section 2.2 we use Gauss-Legendre formula with 8 nodes.

In Table 1 we can see the values of the obtained solution $1/\widetilde{y}_n(s)$ or $1/y_n(s)$ in the interval $[0, 1]$, when we consider approximations of the non separable kernel of order $m = 2$ and $m = 3$ for value of $psi_0 = 0.1$. The last column of this Table is the obtained value for Chandrasekhar integral equation in paper [6]. As one can compare the values obtained are very similar in all cases, but we have to point out that when we use iterative scheme (9) the similarity it is bigger.

In Table 2 first two rows we can see the number of iterations needed to reach a tolerance of 10^{-30} and the residual error when we have approximated the Kernel by a second order polynomials $m = 2$ and $m = 3$ and in 4 we have these values by considering starting

guess $x_0(s) = s$.Once we have solved the problem we calculate numerically the following integral by Gauss-Legendre formula

$$\int_0^1 x(s)dx$$

and denote it by I_{app} also we compare it with the exact value given by:

$$I_{ex} = \frac{2}{\Psi_0}[1 - (1 - \Psi_0)^{1/2}].$$

These both results are very similar and in the tables we truncate the decimals, so, we can appreciate in last row the real difference between them. We notice that when we solve the problem $F(x) = 0$ by approximating the derivative of $I - F(x)$, that is we use (**??**) we obtain better results than when we solved the approximated problem $\widetilde{F}(x) = 0$, that is, using (8).

Now, in order to complete our numerical study we need perform all the computations but considering a better Taylor approximation of orders $m = 4$ and $m = 5$. In Tabla 3 on can check by the results that the approximations improves but the computational cost it is quite important so we conclude that with low order Taylor approximation degree the procedure presented works with high precision.

On the other side, it is also mandatory check the behavior of algorithms when we change the starting guess, so we can check in Tablas 4 and 5 the obtained results for starting guess $y_0 = 1$. The results confirm the conclusions made in the above paragraph, moreover in this case the number of iteration needed decrease in all cases.

4 Conclusions

In this paper we use the well known iterative Newton's method of two step with frozen derivative for approximating, with high precision, the solution of the nonlinear Chandrasekhar integral equation. It is important to point out that we do not discretize the problem and we work in the infinite dimensional space of the continuous functions in the a closed interval where the equation it is defined. So, we are able to obtain continuous functions that approximate the H-functions that are the solution of this useful Chandrasekhar integral equation.

We follow this procedure, we solve a nonlinear equation with separable kernel and define an adequate nonlinear operator between Banach spaces that approximates the first problem. Second one, we introduce an approximation for the inverse of the Fréchet derivative that appears in the Newton's iterative scheme for solving non linear equations.

Moreover, we perform a numerical experience in order to compare our results with previous ones published showing that are competitive.

	(8) $m=2$	(9) $m=2$	(8) $m=3$	(9) $m=3$	x(s)
x	x(s)				
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.0500	1.0045	1.0078	1.0050	1.0078	1.0078
0.1000	1.0085	1.0124	1.0093	1.0124	1.0124
0.1500	1.0121	1.0158	1.0131	1.0158	1.0158
0.2000	1.0153	1.0186	1.0163	1.0186	1.0186
0.2500	1.0182	1.0210	1.0192	1.0210	1.0210
0.3000	1.0207	1.0230	1.0216	1.0230	1.0230
0.3500	1.0230	1.0248	1.0237	1.0248	1.0248
0.4000	1.0250	1.0263	1.0255	1.0263	1.0263
0.4500	1.0268	1.0277	1.0271	1.0277	1.0277
0.5000	1.0285	1.0289	1.0285	1.0289	1.0289
0.5500	1.0300	1.0300	1.0296	1.0300	1.0300
0.6000	1.0315	1.0311	1.0306	1.0311	1.0311
0.6500	1.0329	1.0320	1.0313	1.0320	1.0320
0.7000	1.0342	1.0328	1.0319	1.0328	1.0328
0.7500	1.0356	1.0336	1.0324	1.0336	1.0336
0.8000	1.0371	1.0344	1.0326	1.0344	1.0344
0.8500	1.0387	1.0350	1.0325	1.0350	1.0350
0.9000	1.0405	1.0357	1.0322	1.0357	1.0357
0.9500	1.0425	1.0363	1.0316	1.0363	1.0363
1.0000	1.0447	1.0368	1.0307	1.0368	1.0368

Table 1: Evaluating the solutions: $\widetilde{x}_n(s) = 1/\widetilde{y}_n(s)$ and $x_n(s) = 1/y_n(s)$,

ψ_0		(8) $m=2$	(9) $m=2$	(8) $m=3$	(9) $m=3$
	k	6	10	6	10
0.05	Inc	4.1413e-88	1.0672e-34	5.9905e-69	2.4548e-36
	p	3.0001	1.0000	3.0000	1.0000
	I_{app}	1.0128	1.0128	1.0119	1.0128
	I_{ex}	1.0128	1.0128	1.0128	1.0128
	$\lvert I_{ex} - I_{app}\rvert$	1.6691e-05	1.9523e-41	8.7299e-04	2.6095e-43
	k	5	9	5	9
0.1	Inc	9.1302e-38	4.6929e-32	8.6552e-41	1.4402e-33
	p	3.0001	1.0000	2.9999	1.0000
	I_{app}	1.0264	1.0263	1.0245	1.0263
	I_{ex}	1.0263	1.0263	1.0263	1.0263
	$\lvert I_{ex} - I_{app}\rvert$	7.2146e-05	3.8864e-38	1.8331e-03	6.8638e-40
	k	5	10	5	9
0.15	Inc	1.0954e-34	4.0526e-34	3.6434e-36	6.6822e-31
	p	2.9998	1.0000	2.9997	1.0000
	I_{app}	1.0408	1.0406	1.0377	1.0406
	I_{ex}	1.0406	1.0406	1.0406	1.0406
	$\lvert I_{ex} - I_{app}\rvert$	1.7610e-04	8.5491e-40	2.8941e-03	8.1031e-37
	k	5	10	5	10
0.2	Inc	4.9141e-31	7.9844e-32	2.1653e-32	2.6753e-33
	p	2.9993	1.0000	2.9994	1.0000
	I_{app}	1.0561	1.0557	1.0517	1.0557
	I_{ex}	1.0557	1.0557	1.0557	1.0557
	$\lvert I_{ex} - I_{app}\rvert$	3.4111e-04	3.4313e-37	4.0725e-03	6.5561e-39
	k	6	11	6	10
0.25	Inc	2.8584e-86	6.5017e-34	2.4130e-90	2.0828e-31
	p	3.0000	1.0000	3.0000	1.0000
	I_{app}	1.0724	1.0718	1.0664	1.0718
	I_{ex}	1.0718	1.0718	1.0718	1.0718
	$\lvert I_{ex} - I_{app}\rvert$	5.8359e-04	5.0420e-39	5.3889e-03	9.1573e-37

Table 2: Numerical results starting by $y_0(s) = s$ with $m = 2, 3$

ψ_0		(8) $m=4$	(9) $m=4$	(8) $m=5$	(9) $m=5$
	k	5	8	5	8
0.05	Inc	4.1199e-46	6.0175e-33	5.0555e-47	1.0794e-33
	p	3.0002	1.0000	2.9998	1.0000
	I_{app}	1.0128	1.0128	1.0124	1.0128
	I_{ex}	1.0128	1.0128	1.0128	1.0128
	$\lvert I_{ex} - I_{app}\rvert$	8.1132e-06	4.0532e-40	4.6774e-04	4.9408e-41
	k	5	9	5	9
0.1	Inc	6.4801e-40	1.2937e-34	1.0988e-40	1.7360e-35
	p	3.0005	1.0000	3.0002	1.0000
	I_{app}	1.0264	1.0263	1.0254	1.0263
	I_{ex}	1.0263	1.0263	1.0263	1.0263
	$\lvert I_{ex} - I_{app}\rvert$	3.5013e-05	3.9001e-41	9.8238e-04	3.5488e-42
	k	5	9	5	9
0.15	Inc	6.2426e-35	6.5212e-32	1.4968e-35	9.2521e-33
	p	3.0003	1.0000	3.0000	1.0000
	I_{app}	1.0407	1.0406	1.0391	1.0406
	I_{ex}	1.0406	1.0406	1.0406	1.0406
	$\lvert I_{ex} - I_{app}\rvert$	8.5316e-05	5.0172e-38	1.5514e-03	4.8110e-39
	k	5	10	5	10
0.2	Inc	5.0408e-31	1.8129e-34	1.3947e-31	1.9351e-35
	p	2.9998	1.0000	2.9996	1.0000
	I_{app}	1.0559	1.0557	1.0535	1.0557
	I_{ex}	1.0557	1.0557	1.0557	1.0557
	$\lvert I_{ex} - I_{app}\rvert$	1.6495e-04	2.8300e-40	2.1840e-03	2.0336e-41
	k	6	10	6	10
0.25	Inc	8.6743e-86	1.3652e-32	2.1872e-87	1.5915e-33
	p	3.0000	1.0000	3.0000	1.0000
	I_{app}	1.0721	1.0718	1.0689	1.0718
	I_{ex}	1.0718	1.0718	1.0718	1.0718
	$\lvert I_{ex} - I_{app}\rvert$	2.8161e-04	3.8415e-38	2.8915e-03	3.0009e-39

Table 3: Numerical results starting by $y_0(s) = s$ with $m = 4, 5$

ψ_0		(8) $m=2$	(9) $m=2$	(8) $m=3$	(9) $m=3$
	k	4	6	3	6
0.05	Inc	3.4521e-96	2.3852e-31	3.5422e-31	7.5506e-32
	p	3.0001	0.9999	2.9982	1.0000
	I_{app}	1.0128	1.0128	1.0124	1.0128
	I_{ex}	1.0128	1.0128	1.0128	1.0128
	$\lvert I_{ex} - I_{app}\rvert$	8.1132e-06	1.6338e-38	4.6774e-04	3.4812e-39
	k	4	7	4	7
0.1	Inc	1.2278e-79	3.2759e-33	3.5542e-81	1.1810e-33
	p	3.0001	0.9999	3.0000	1.0000
	I_{app}	1.0264	1.0263	1.0254	1.0263
	I_{ex}	1.0263	1.0263	1.0263	1.0263
	$\lvert I_{ex} - I_{app}\rvert$	3.5013e-05	1.0018e-39	9.8238e-04	2.4220e-40
	k	4	7	4	7
0.15	Inc	1.1044e-69	9.7148e-31	2.8522e-71	4.8438e-31
	p	3.0001	0.9998	3.0000	1.0000
	I_{app}	1.0407	1.0406	1.0391	1.0406
	I_{ex}	1.0406	1.0406	1.0406	1.0406
	$\lvert I_{ex} - I_{app}\rvert$	8.5316e-05	7.5819e-37	1.5514e-03	2.5248e-37
	k	4	8	4	8
0.2	Inc	2.0577e-62	1.9962e-33	4.6909e-64	8.1694e-34
	p	3.0001	0.9999	3.0000	1.0000
	I_{app}	1.0559	1.0557	1.0535	1.0557
	I_{ex}	1.0557	1.0557	1.0557	1.0557
	$\lvert I_{ex} - I_{app}\rvert$	1.6495e-04	3.1357e-39	2.1840e-03	8.5937e-40
	k	4	8	4	8
0.25	Inc	1.3442e-56	1.3709e-31	2.6720e-58	5.4792e-32
	p	3.0001	1.0000	3.0000	1.0000
	I_{app}	1.0721	1.0718	1.0689	1.0718
	I_{ex}	1.0718	1.0718	1.0718	1.0718
	$\lvert I_{ex} - I_{app}\rvert$	2.8161e-04	3.8716e-37	2.8915e-03	1.0338e-37

Table 4: Numerical results starting by $y_0(s) = 1$ with $m = 2, 3$

ψ_0		(8) $m=4$	(9) $m=4$	(8) $m=5$	(9) $m=5$
	k	4	6	3	6
0.05	Inc	3.4521e-96	2.3852e-31	3.5422e-31	7.5506e-32
	p	3.0001	0.9999	2.9982	1.0000
	I_{app}	1.0128	1.0128	1.0124	1.0128
	I_{ex}	1.0128	1.0128	1.0128	1.0128
	$\lvert I_{ex} - I_{app}\rvert$	8.1132e-06	1.6338e-38	4.6774e-04	3.4812e-39
	k	4	7	4	7
0.1	Inc	1.2278e-79	3.2759e-33	3.5542e-81	1.1810e-33
	p	3.0001	0.9999	3.0000	1.0000
	I_{app}	1.0264	1.0263	1.0254	1.0263
	I_{ex}	1.0263	1.0263	1.0263	1.0263
	$\lvert I_{ex} - I_{app}\rvert$	3.5013e-05	1.0018e-39	9.8238e-04	2.4220e-40
	k	4	7	4	7
0.15	Inc	1.1044e-69	9.7148e-31	2.8522e-71	4.8438e-31
	p	3.0001	0.9998	3.0000	1.0000
	I_{app}	1.0407	1.0406	1.0391	1.0406
	I_{ex}	1.0406	1.0406	1.0406	1.0406
	$\lvert I_{ex} - I_{app}\rvert$	8.5316e-05	7.5819e-37	1.5514e-03	2.5248e-37
	k	4	8	4	8
0.2	Inc	2.0577e-62	1.9962e-33	4.6909e-64	8.1694e-34
	p	3.0001	0.9999	3.0000	1.0000
	I_{app}	1.0559	1.0557	1.0535	1.0557
	I_{ex}	1.0557	1.0557	1.0557	1.0557
	$\lvert I_{ex} - I_{app}\rvert$	1.6495e-04	3.1357e-39	2.1840e-03	8.5937e-40
	k	4	8	4	8
0.25	Inc	1.3442e-56	1.3709e-31	2.6720e-58	5.4792e-32
	p	3.0001	1.0000	3.0000	1.0000
	I_{app}	1.0721	1.0718	1.0689	1.0718
	I_{ex}	1.0718	1.0718	1.0718	1.0718
	$\lvert I_{ex} - I_{app}\rvert$	2.8161e-04	3.8716e-37	2.8915e-03	1.0338e-37

Table 5: Numerical results starting by $y_0(s) = 1$ with $m = 4, 5$

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