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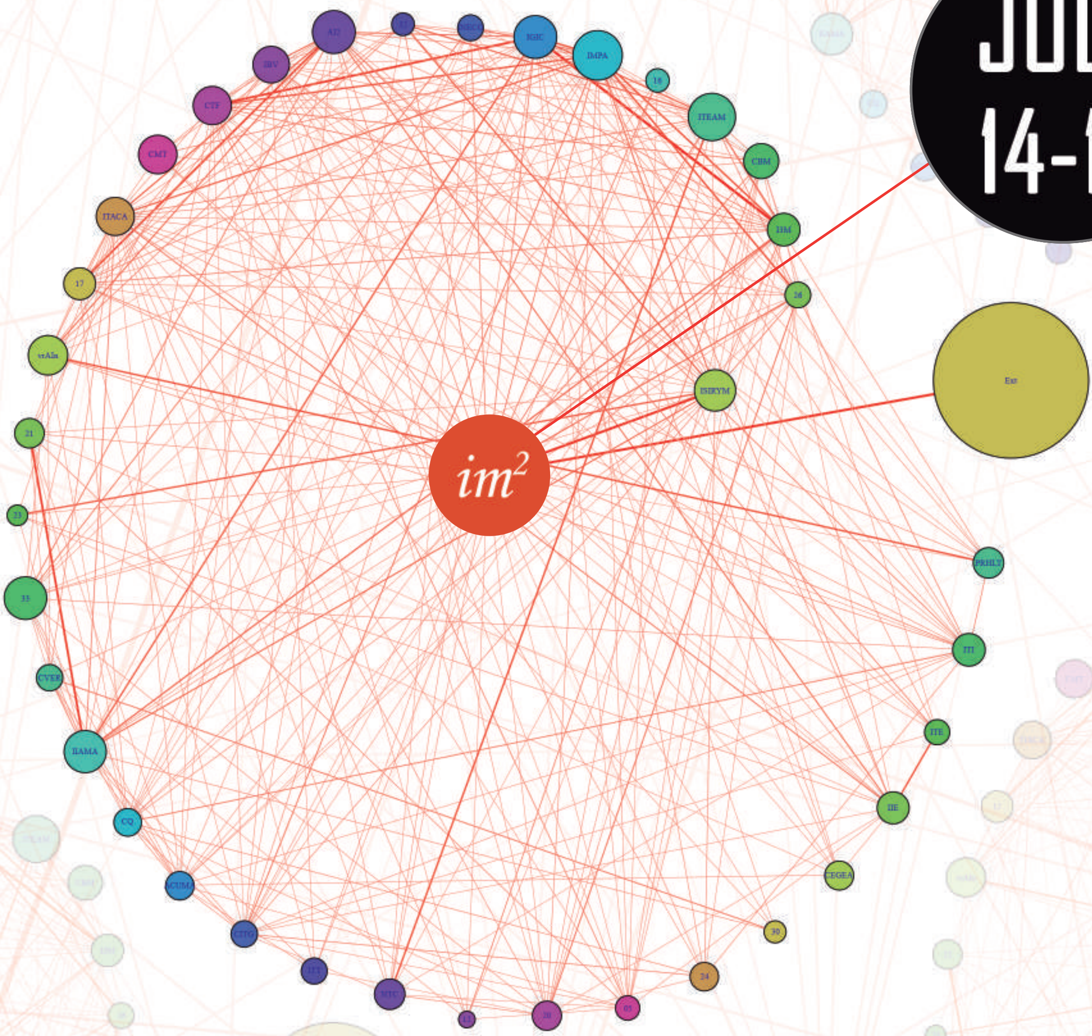

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Conformable fractional iterative methods for solving nonlinear problems

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1 Introduction

Fractional derivatives have attracted increasing interest from the first steps of fractional calculus in the XVII-th century up to now, when they have become an excellent tool to model physical phenomena in porous media, for example.

However, fractional derivatives have been not been widely used, in general, to develop iterative schemes for solving nonlinear problems. Recently, in [5], two fractional Newton-type methods were proposed,

$$x_{k+1} = x_k - \left(\Gamma(\alpha+1)\frac{f(x_k)}{cD^{\alpha}_{a^+}f(x_k)}\right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots \tag{1}$$

and

$$x_{k+1} = x_k - \left(\Gamma(\alpha+1)\frac{f(x_k)}{D^{\alpha}_{a^+}f(x_k)}\right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots \tag{2}$$

with employ Caputo and Riemann-Liouville derivatives respectively. The order of convergence of these methods is $\alpha+1$. Let us denote them as CFN and R-LFN, respectively. Let us remark that, when $\alpha = 1$, we obtain the classical Newton's method for each case above.

The use of Caputo and Riemann-Liouville fractional derivatives require the evaluation of special functions as Gamma and Mittag-Leffler functions, which both involve a high computational cost to compute these fractional derivatives. Theoretically, the order of convergence of these methods tends to be quadratic when $\alpha \approx 1$, but in the practice, the approximated computational order of convergence (ACOC, see [5]) is linear if α is different from 1.

In order to design a new iterative scheme able to avoid these problems, we introduce the conformable fractional derivative. It can be seen in [1,2] that the left conformable fractional derivative starting from a of a function $f : [a, \infty) \longrightarrow \mathbb{R}$ of order $\alpha \in (0, 1]$, being $\alpha, a, x \in \mathbb{R}$, is defined as

$$(T^a_\alpha f)(x) = \lim_{\varepsilon \longrightarrow 0} \frac{f(x + \varepsilon(x-a)^{1-\alpha}) - f(x)}{\varepsilon}. \tag{3}$$

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Let us remark that, if f is differentiable, then $(T_\alpha^a f)(x) = (x-a)^{1-\alpha} f'(x)$ and if f is α-differentiable in (a,b), for some $b \in \mathbb{R}$, then $(T_\alpha^a f)(a) = \lim_{x\longrightarrow a^+}(T_\alpha^a f)(x)$.
The left conformable fractional derivative holds the property of integer derivatives, $T_\alpha^a C = 0$, being C a constant. Conformable derivative is the most natural definition of fractional derivative, also, it does not require the evaluation of special functions, which involves a low computational cost compared with existing fractional derivatives.
In [6], a Taylor power series of $f(x)$ is provided, where the conformal derivative is defined at a different point from where it is evaluated, which is a key fact in the convergence analysis of our proposed scheme.

Theorem 1 (Theorem 4.1, [6])**.** *Let $f(x)$ be an infinitely α-differentiable function for $\alpha \in (0,1]$, at the neighborhood of a_1 with conformable derivative starting from a. The fractional power series for $f(x)$ is:*

$$f(x) = f(a_1) + \frac{(T_\alpha^a f)(a_1)\delta_1}{\alpha} + \frac{(T_\alpha^a f)^{(2)}(a_1)\delta_2}{2\alpha^2} + R_2(x, a_1, a), \tag{4}$$

being $\delta_1 = H^\alpha - L^\alpha$, $\delta_2 = H^{2\alpha} - L^{2\alpha} - 2L^\alpha\delta_1, \dots,$ and $H = x - a$, $L = a_1 - a$.

It is easy to prove that $\delta_2 = \delta_1^2$, $\delta_3 = \delta_1^3$, etc.
In next Section, the conformable fractional Newton-type method is obtained from Taylor power expansion (4).

2 Methods

To obtain a fractional Newton-type method from (4), let us regard the approximation of this Taylor power series to order one evaluated at the solution $\bar{x}$,

$$f(x) \approx f(\bar{x}) + \frac{(T_\alpha^a f)(\bar{x})\delta_1}{\alpha}. \tag{5}$$

As it is known that $f(\bar{x}) = 0$, and $\delta_1 = H^\alpha - L^\alpha$, being $H = x - a$ and $L = a_1 - a$ $(a_1 = \bar{x})$, expression (5) can be rewritten as

$$f(x) \approx \frac{(T_\alpha^a f)(\bar{x})}{\alpha}\left[(x-a)^\alpha - (\bar{x}-a)^\alpha\right]. \tag{6}$$

Now, $(\bar{x}-a)^\alpha$ can be isolated as

$$(\bar{x}-a)^\alpha \approx (x-a)^\alpha - \alpha\frac{f(x)}{(T_\alpha^a f)(\bar{x})}. \tag{7}$$

So, from $(\bar{x}-a)^\alpha$, $\bar{x}$ can be estimated as

$$\bar{x} \approx a + \left((x-a)^\alpha - \alpha\frac{f(x)}{(T_\alpha^a f)(\bar{x})}\right)^{1/\alpha}. \tag{8}$$

Regarding the iterates x_k and x_{k+1} as approximations of the solution $\bar{x}$, we obtain the Conformable fractional Newton-type method as

$$x_{k+1} = a + \left((x_k-a)^\alpha - \alpha\frac{f(x_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0, 1, 2, \dots \tag{9}$$

Let us call this method TFN.
In the next result, the order of convergence of this method is stated. This is the first optimal fractional method according to Kung and Traub's conjecture (see [6]).

Theorem 2. *Let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function in the interval D containing the zero $\bar{x}$ of $f(x)$. Let $(T_\alpha^a f)(x)$ be the conformable fractional derivative of $f(x)$ starting from a, with order $\alpha \in (0,1]$. Let us suppose $(T_\alpha^a f)(x)$ is continuous and not null at $\bar{x}$. If an initial approximation x_0 is sufficiently close to $\bar{x}$, then the local order of convergence of the conformable fractional Newton-type method*

$$x_{k+1} = a + \left((x_k - a)^\alpha - \alpha \frac{f(x_k)}{(T_\alpha^a f)(x_k)} \right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots$$

is at least 2, being $0 < \alpha \leq 1$, and the error equation is

$$e_{k+1} = \alpha(\bar{x} - a)^{\alpha - 1} C_2 e_k^2 + O\left(e_k^3\right),$$

being $C_j = \dfrac{1}{j!\alpha^j} \dfrac{(T_\alpha^a f)^{(j)}(\bar{x})}{(T_\alpha^a f)(\bar{x})}$ for $j = 2, 3, 4, \ldots$

Once the method has been designed and its convergence analyzed, it is necessary to test its performance on some nonlinear problems.

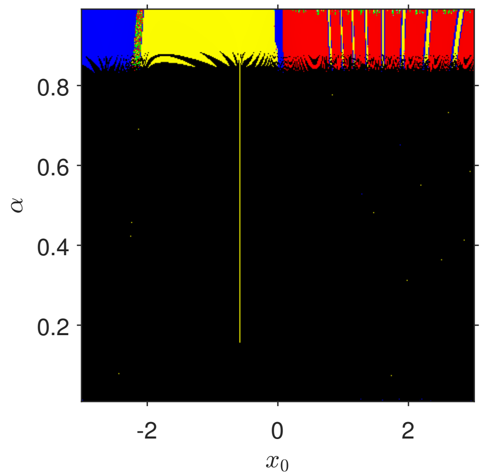
3 Results

In this section, we make some numerical tests on nonlinear equations in order to check its efficiency and reliability. We compare methods CFN and TFN with the classical Newton-Raphson scheme is made (when $\alpha = 1$).
For our test, we use Matlab R2019b with double precision arithmetics, $|f(x_{k+1})| < 10^{-8}$ or $|x_{k+1} - x_k| < 10^{-8}$ as stopping criterium, and a maximum of 500 iterations. For CFN method we use $a = 0$, the program made in [8] for computing the Gamma function, and the code provided by Igor Podlubny in Mathworks for the calculation of Mittag-Leffler function. For TFN method we consider $a = -10$. The initial estimation used is the same in each procedure.
Our test function is $f_1(x) = -12.84x^6 - 25.6x^5 + 16.55x^4 - 2.21x^3 + 26.71x^2 - 4.29x - 15.21$ with roots $\bar{x}_1 = 0.82366 + 0.24769i$, $\bar{x}_2 = 0.82366 - 0.24769i$, $\bar{x}_3 = -2.62297$, $\bar{x}_4 = -0.584$, $\bar{x}_5 = -0.21705 + 0.99911i$ and $\bar{x}_6 = -0.21705 - 0.99911i$. In Table 1, we can see that TFN method requires less iterations than CFN method for the same values of α, even less than classical Newton's scheme when $\alpha \leq 0.4$. It can also be observed that ACOC is 1 if $\alpha \neq 1$ in CFN method, whereas ACOC keeps being 2 or even greater than 2 if $\alpha \neq 1$ in TFN method.

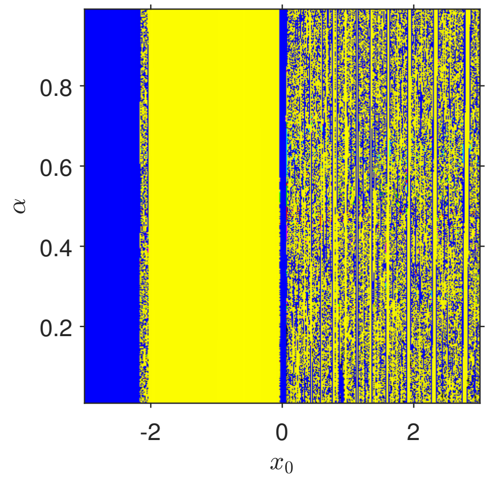
	CFN method					TFN method				
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	iter	ACOC	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	iter	ACOC
1	$\bar{x}_3$	$4.16 \cdot 10^{-12}$	$3.47 \cdot 10^{-8}$	11	2.00	$\bar{x}_3$	$4.16 \cdot 10^{-12}$	$3.47 \cdot 10^{-8}$	11	2.00
0.9	$\bar{x}_3$	$7.96 \cdot 10^{-5}$	$8.11 \cdot 10^{-9}$	68	0.98	$\bar{x}_3$	$6.18 \cdot 10^{-13}$	$7.17 \cdot 10^{-9}$	11	2.00
0.8	$\bar{x}_1$	$1.94 \cdot 10^{-5}$	$9.99 \cdot 10^{-9}$	123	0.99	$\bar{x}_3$	$4.18 \cdot 10^{-12}$	$1.41 \cdot 10^{-9}$	11	2.00
0.7	$\bar{x}_2$	$1.1 \cdot 10^{-14}$	$9.94 \cdot 10^{-9}$	389	1.00	$\bar{x}_3$	$1.6 \cdot 10^{-12}$	$2.67 \cdot 10^{-10}$	11	2.00
0.6	-	-	-	500	-	$\bar{x}_3$	$1.6 \cdot 10^{-12}$	$4.81 \cdot 10^{-11}$	11	2.00
0.5	-	-	-	500	-	$\bar{x}_3$	$2.26 \cdot 10^{-12}$	$8.3 \cdot 10^{-12}$	11	2.00
0.4	-	-	-	500	-	$\bar{x}_3$	$2.91 \cdot 10^{-9}$	$8.89 \cdot 10^{-7}$	10	2.01
0.3	-	-	-	500	-	$\bar{x}_3$	$4.62 \cdot 10^{-10}$	$3.54 \cdot 10^{-7}$	10	2.01
0.2	-	-	-	500	-	$\bar{x}_3$	$7.36 \cdot 10^{-11}$	$1.38 \cdot 10^{-7}$	10	2.00
0.1	-	-	-	500	-	$\bar{x}_3$	$2.26 \cdot 10^{-12}$	$5.27 \cdot 10^{-8}$	10	2.00

Table 1: CFN and TFN results for $f_1(x)$ with initial estimation $x_0 = -2.2$

In order to check also the stability of fractional Newton-type methods, we analyze the dependence on initial estimates by using convergence planes as defined in [9] and used in [3] and [5].
To construct the convergence planes, we regard the initial estimates in horizontal axis and values of $\alpha \in (0,1]$ in vertical axis. Each color represents a different solution found, and it is painted in black when no solution was found in 500 iterations. Each plane is made with a 400×400 grid, and tolerance of 0.001.



(a) CFN, $-3 \leq x_0 \leq 3$, 13.73% convergence



(b) TFN, $-3 \leq x_0 \leq 3$, 99.92% convergence

Figure 1: Convergence planes of CFN and TFN on $f_1(x)$

In Figure 1 we see that TFN method has a much higher percentage of convergence than CFN method, and the convergence is guaranteed even for lower values of α for a wide range of initial estimations.

4 Conclusions

As far as we know, we have proposed the first optimal fractional Newton-type method, by using Conformable derivatives. The fractional derivative used has the most natural definition, so, in this method is not required the evaluation of special functions, which involves a low computational cost compared with the existing fractional Newton-type methods. Also, the order of convergence of this method is quadratic, unlike the existing ones. Numerical tests were made, and the dependence on initial estimates was analyzed, confirming the theory. It can be concluded that this method shows a better numerical behavior than fractional Newton-type methods previously proposed, even than classical Newton-Raphson method in some cases. It was also observed that is possible to obtain both, real and complex roots, with real initial estimates, and that it is possible to get different roots not only by choosing a different initial estimate, but also by choosing a different value of α.

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